The Bottom-Up Top-Down Puzzle Solved

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In [1], we introduce a common shock model of portfolio credit risk where one can build a consistent picture of bottom up defaults that are also manageable in a top down aggregate loss space. In this sense this model solves the bottom-up top-down puzzle [2], which the CDO industry had been trying to do for a long time and basically failed. Then the CDO market died and the problem remained standing.

In our model, defaults are the consequence of some "shocks" associated with groups of obligors. We define the following pre-specified set of groups

$$\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\},$$

where $I_1, \ldots, I_m$ are subsets of $N = \{1, \ldots, n\}$, and each group I_j contains at least two obligors or more. The shocks are divided in two categories: the "idiosyncratic" shocks associated with singletons $\{1\}, \ldots, \{n\}$ can only trigger the default of name $1, \ldots, n$ individually, while the "systemic" shocks associated with multi-name groups $I_1, \ldots, I_m$ may simultaneously trigger the default of all names in these groups. Note that several groups I_j may contain a given name i, so that only the shock occurring first effectively triggers the default of that name. As a result, when a shock associated with a specific group occurs at time t, it only triggers the default of names that are still alive in that group at time t. In the following, the elements Y of $\mathcal{Y}$ will be used to designate shocks and we let $\mathcal{I} = (I_l)_{1 \le l \le m}$ denote the pre-specified set of multi-name groups of obligors. Shock intensities $\lambda_Y(t, \mathbf{X}_t)$ will be specified later in terms of a Markovian factor process $\mathbf{X}_t$. Letting $\Lambda_t^Y = \int_0^t \lambda_Y(s, \mathbf{X}_s)ds$, we define

$$\tau_Y(t) = \inf\{t > 0; \Lambda_t^Y > E_Y\}, \tag{1}$$

where the random variables E_Y are i.i.d. and exponentially distributed with parameter 1. For every obligor i we let

$$\tau_i = \min_{\{Y \in \mathcal{Y};\, i \in Y\}} \tau_Y(t), \tag{2}$$

which defines the default time of obligor i in the common shocks model. The model filtration is given as $\mathbb{F} = \mathbb{X} \vee \mathbb{H}$, the filtration generated by the factor process $\mathbf{X}$ and the point process $\mathbf{H} = (H^i)_{1 \le i \le n}$ with $H_t^i = \mathbb{1}_{\tau_i \le t}$.

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This model can be viewed as a doubly stochastic (via the stochastic intensities Λ^Y) and dynamized (via the introduction of the filtration $\mathbb{F}$) generalization of the Marshall-Olkin model [3]. The purpose of the factor process $\mathbf{X}$ is to more realistically model diffusive randomness of credit spreads. Figure 1 shows one possible defaults path in our model with $n = 5$ and

$$\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}.$$

The inner oval shows which common-shock happened and caused the observed default scenarios at successive default times. At the first instant, default of name 2 is observed as the consequence of the idiosyncratic shock $\{2\}$. At the second instant, names 4 and 5 have defaulted simultaneously as a consequence of the systemic shock $\{4, 5\}$. At the fourth instant, the systemic shock $\{2, 3, 4\}$ triggers the default of name 3 alone as name 2 and 4 have already defaulted. At the fifth instant, default of name 1 alone is observed as the consequence of the systemic shock $\{1, 2\}$.

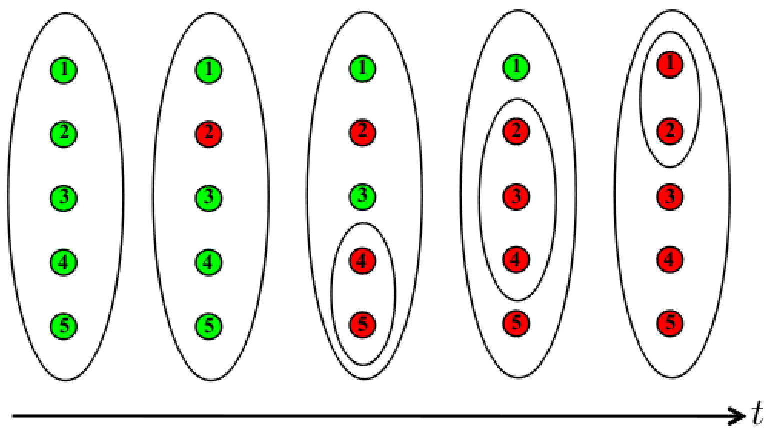

Figure 1: One possible defaults path in a model with $n = 5$ and $\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}$.

1 Calibration Results

Table 1 summarizes the calibration results obtained with this model (using piecewise-constant intensities), for two different quotation dates and two different CDS indices under the constraint that the model perfectly reproduces each individual CDS curve of the corresponding index at these two dates. Even better fits can be obtained by resorting to random recoveries specifications. The calibrated model can then be used for any bottom-up dynamical portfolio credit purpose, in particular, valuation and hedging of counterparty risk on credit derivatives (see [4]).

Table 1: CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0, 3]$ spread is quoted in %. All maturities are for five years.

CDX 2007-12-17

CDO tranche	$[0, 3]$	$[3, 7]$	$[7, 10]$	$[10, 15]$	$[15, 30]$
Market spread	48.07	254.0	124.0	61.00	41.00
Model spread	48.07	254.0	124.0	61.00	38.94
Absolute error in bp	0.010	0.000	0.000	0.000	2.061
Relative error in %	0.0001	0.000	0.000	0.000	5.027

iTraxx Europe 2008-03-3

CDO tranche	$[0, 3]$	$[3, 6]$	$[6, 9]$	$[9, 12]$	$[12, 22]$
Market spread	40.15	479.5	309.5	215.1	109.4
Model spread	41.68	429.7	309.4	215.1	103.7
Absolute error in bp	153.1	49.81	0.0441	0.0331	5.711
Relative error in %	3.812	10.39	0.0142	0.0154	5.218

2 Min-Variance Hedging

In this section we present some numerical results illustrating performance of the min-variance hedging strategies given in [1]. This will be done in the setup of the constant recoveries model calibrated on CDX.NA.IG Series 9 data set of December 17, 2007 (see previous section for calibration results).

Figure 2 displays the nominal exposure for the d most riskiest CDSs when hedging one unit of nominal exposure. Furthermore, Table 2 displays the names and sizes of the 3-year CDS spreads used in the hedging strategy.

Table 2: The names and CDS spreads (in bp) of the six riskiest obligors used in the hedging strategy displayed by Figure 2.

Company (Ticker)	CCR-HomeLoans	RDN	LEN	SFI	PHM	CTX
3-year CDS spread	1190	723	624	414	404	393

Each plot in Figure 2 should be interpreted as follows: in every pair (x, y) the x-component represents the size of the 3-year CDS spread at the hedging time $t = 0$ while the y-component is the corresponding nominal CDS-exposure using the d riskiest CDSs. The graphs are ordered from top to bottom, where the top panel corresponds to hedging with the $d = 3$ riskiest CDS and the bottom panel corresponds to hedging with the $d = 6$ riskiest names. Note that the x-axes are displayed from the riskiest obligor to the safest. Thus, hedge-sizes y for riskier CDSs are aligned to the left in each plot while y-values for safer CDSs are consequently displayed more to the right. In doing this, going from the top to the bottom panel consists in observing the effect of including new safer names from the right part of the graphs. We have connected the pairs (x, y) with lines forming graphs that visualizes possible trends of the min-variance hedging strategies for the d most riskiest CDSs.

For example, when the three riskiest names are used for hedging (top left panel), we observe that the amount of nominal exposure in hedging instruments decreases with the degree of subordination, i.e., the [0-3%] equity tranche requires more nominal exposure in CDSs than the upper tranches. Note moreover that the min-variance hedging portfolio contains more CDSs on names with lower spreads. When lower-spread CDSs are added in the portfolio, the picture remains almost

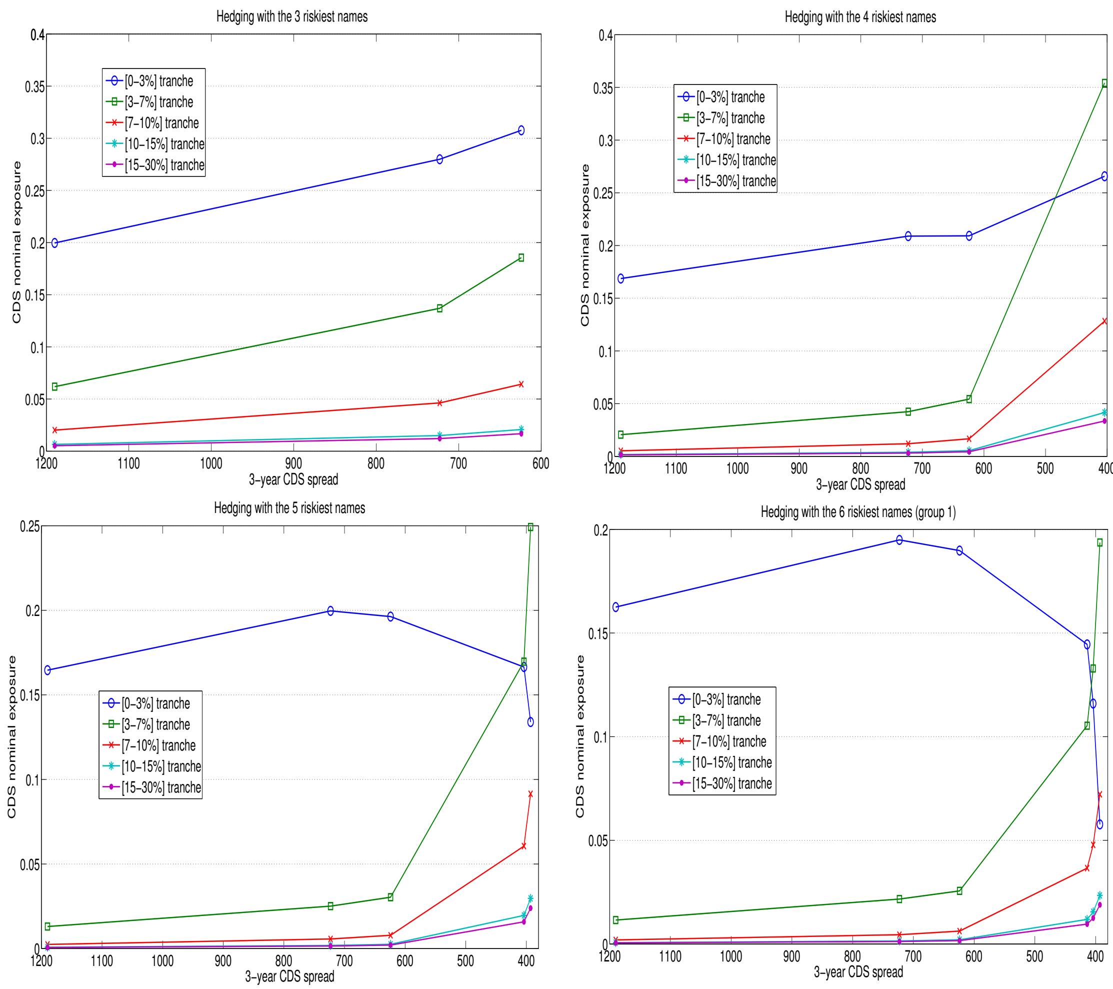

Figure 2: Min-variance hedging strategies associated with the d riskiest CDSs, $d = 3, 4, 5, 6$ for one unit of nominal exposure of different CDO tranches in a model calibrated to market spreads of CDX.NA.IG Series 9 on December 17, 2007.

the same for the 3 riskiest names. For the remaining safer names however, the picture depends on the characteristics of the tranche. For the [0-3%] equity tranche, the quantity of the remaining CDSs required for hedging sharply decrease as additional safer names are added. One possible explanation is that adding too many names in the hedging strategy will be useless when hedging the equity tranche. This is intuitively clear since one expects that the most riskiest obligors will default first and consequently reduce the equity tranche substantially, explaining the higher hedge-ratios for riskier names, while it is less likely that the more safer names will default first and thus incur losses on the first tranche which explains the lower hedge ratios for the safer names. We observe the opposite trend for the senior (safer) tranches: adding new (safer) names in the hedging portfolio seems to be useful for "non equity" tranches since the nominal exposure required for these names increases when they are successively added.

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