Une introduction à la théorie des Structures de Régularité

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7 novembre 2014, LPMA
Les probas du vendredi

Motivations

Let us consider a SPDE driven by a space-time white noise ξ, e.g.

$$\text{(KPZ)} \qquad \partial_t u = \partial_{xx} u + (\partial_x u)^2 + \xi, \qquad t > 0,\ x \in \mathbb{R}.$$

It is well known that this equation is ill-posed, since the solution u is expected to be no more than $\frac{1}{2}$-Hölder in space.

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The theory of Regularity Structures [Hairer 14] gives a framework to study *renormalized products of random Schwartz distributions*.

A related point of view concerns the following *stability* problem: find topologies on ξ and u such that

1. (Probabilistic step) one can find ξ_ε smooth and converging to the space-time white noise as $\varepsilon \to 0$
2. (Analytic step) $\xi \mapsto u$ is continuous.

Classical Monomials

We use variables $x, y, z \in \mathbb{R}^d$, $d \geq 1$. For $k \in \mathbb{N}^d$ we define monomials

$$x^k = x_1^{k_1} \cdots x_d^{k_d}, \qquad |k| := k_1 + \cdots + k_d.$$

With this notation, in what follows one can almost forget that x, y, z are not one-dimensional.

A classical polynomial $p = p(y)$ in $y \in \mathbb{R}^d$ is a linear combination of monomials in y.

To each point $x \in \mathbb{R}^d$ we can associate the finite Taylor expansion of p around x

$$p(y) = \sum_{k \in \mathbb{N}^d} a_k(x)\,(y - x)^k$$

where

$$a_k(x) = \frac{1}{k!}\frac{d^k p}{dx^k}(x) = \frac{1}{k_1!}\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{1}{k_d!}\frac{\partial^{k_d}}{\partial x_d^{k_d}} p(x).$$

The Π_x operators

We introduce symbols $X_1, \ldots, X_d$ and

$$X^k := X_1^{k_1} \cdots X_d^{k_d}, \qquad k \in \mathbb{N}^d$$

and the evaluation operators

$$\Pi_x X^k(y) = (y-x)^k, \qquad x, y \in \mathbb{R}^d.$$

We notice that

$$|\Pi_x X^k(y)| = |(y-x)^k| \leq \|y-x\|^{|k|}.$$

$|k| = k_1 + \cdots + k_d \geq 0$ is the degree (or homogeneity) of X^k and this has both an algebraic and analytic interpretation.

Abstract Taylor sums

Given a polynomial $p(y)$, we write a function (an abstract Taylor sum)

$$x \mapsto P(x) = \sum_{k \in \mathbb{N}^d} a_k(x)\, X^k, \qquad x \in \mathbb{R}^d,$$

and then we can recover p from the Taylor sum P

$$(\Pi_x P(x))\,(y) = \sum_{k \in \mathbb{N}^d} a_k(x)\,(y-x)^k = p(y)$$

(note that these are in fact finite sums).

The Γ_{xz} operators

The Taylor series of the function $y \mapsto (y-z)^k$ around the fixed base point x is

$$(y-z)^k = (y-x+x-z)^k$$

i.e.

$$(y-z)^k = \Pi_z X^k(y) = \Pi_x \left[(X+x-z)^k \right] (y)$$

or

$$\Pi_z X^k = \Pi_x \left[(X+x-z)^k \right].$$

This gives a rule to transform a classical Taylor series centered at z into one centered at x, with the definition

$$\Gamma_{xz} X^k = (X+x-z)^k = \sum_{i=0}^{k} \binom{k}{i} (x-z)^{k-i} X^i.$$

This definition satisfies the simple properties $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$, $\Gamma_{xx} = Id$,

$$\Pi_z = \Pi_x \Gamma_{xz}, \quad |\Gamma_{xz} X^k - X^k| < k, \quad \|\Gamma_{xz} X^k - X^k\|_i \leq C \|x-z\|^{k-i}.$$

Classical polynomials

Given a classical monomial $y \mapsto y^k$, we can associate to each x its abstract Taylor sum around x

$$F(x) = (X + x)^k = \Gamma_{x0} X^k = \Gamma_{x0} F(0).$$

We obtain that $F : \mathbb{R}^d \mapsto \langle X^k, k \geq 0 \rangle$ is the Taylor series of a (classical) polynomial $p(\cdot)$ if and only if

$$F(x) - \Gamma_{xz} F(z) \equiv 0, \quad x, z \in \mathbb{R}^d,$$

and in this case

$$F(x) = \sum_{k \in \mathbb{N}^d} \frac{1}{k!} \frac{d^k p(x)}{dx^k} X^k.$$

In particular $\Pi_x F(x)(y) \equiv \Pi_z F(z)(y) = p(y)$ for all x, y, z.

Generalized monomials

We want to add to this classical framework *random (Schwartz) distributions*.

We consider a class of symbols $\mathcal{T} \supseteq \{X^k, k \geq 0\}$ representing generalized monomials. We associate to each $\tau \in \mathcal{T}$ a *degree* or *homogeneity* :

$$\forall \tau \in \mathcal{T}, \qquad |\tau| \in \mathbb{R},$$

with the assumption that $|X^k| = |k| = k_1 + \cdots + k_d$.

We want to endow $\mathcal{T}$ with operators (Π_x, Γ_{xz}) such that:

1. Π_x associates to each symbol $\tau \in \mathcal{T}$ a (random) distribution on $\mathbb{R}^d$ whose local regularity around each point x is no worse than the homogeneity $|\tau|$; informally:

$$|\Pi_x \tau(y)| \leq C\|y - x\|^{|\tau|}, \qquad x, y \in \mathbb{R}^d$$

2. Γ_{xz} is a rule to transform an abstract Taylor sum around z in one around x.

General models

A model of $\mathcal{T}$ is given by a couple (Π_x, Γ_{xz}) such that

1. for all x, $\Pi_x : \mathcal{T} \mapsto \mathcal{S}'(\mathbb{R}^d)$ and for all $\varphi \in C_c^\infty(\mathbb{R}^d)$
$$|\Pi_x \tau(\varphi_{x,\delta})| \leq C\delta^{|\tau|},$$
where $\varphi_{x,\delta}(y) := \delta^{-d}\varphi\left(\frac{y-x}{\delta}\right)$.
2. for all x, y, z, $\Gamma_{xz} : \langle \mathcal{T} \rangle \mapsto \langle \mathcal{T} \rangle$ is such that $\Gamma_{xx} = Id$, $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and
$$|\Gamma_{xz}\tau - \tau| < |\tau|, \qquad \|\Gamma_{xz}\tau - \tau\|_\ell \leq C\|z-x\|^{|\tau|-\ell}, \ \ell < |\tau|.$$
3. for all x, z: $\Pi_z = \Pi_x \Gamma_{xz}$.

Hölder functions

Back to the classical situation, $f \in C^\gamma$, $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$, iff

$$f(y) = \sum_{i \le \gamma} \frac{f^{(i)}(x)}{i!} (y-x)^i + r(x,y), \qquad |r(x,y)| \le C \|y-x\|^\gamma.$$

If we define $F(x) = \sum_{i \le \gamma} \frac{f^{(i)}(x)}{i!} X^i$, then

$$F(x) - \Gamma_{xz} F(z) = \sum_{i \le \gamma} \frac{X^i}{i!} \left(f^{(i)}(x) - \sum_{j \le \gamma - i} \frac{f^{(i+j)}(z)}{j!} (x-z)^j \right)$$

and in particular $f \in C^\gamma$ iff $f^{(i)} \in C^{\gamma - i}$ for all $i \le \gamma$, i.e.

$$\|F(x) - \Gamma_{xz} F(z)\|_i \le C \|x - z\|^{\gamma - i}.$$

In this case

$$f(x) = \Pi_x F(x)(x), \qquad (\textit{reconstruction})$$

$$f(y) - \Pi_x F(x)(y) \neq 0, \qquad \Pi_x F(x) \neq \Pi_z F(z).$$

Functional norm

In the general case, for $\gamma > 0$ we say that $F \in \mathcal{D}^\gamma$ if F takes values in the linear span of the symbols with homogeneity $< \gamma$ and for all $\beta < \gamma$

$$\|F(x) - \Gamma_{xz} F(z)\|_\beta \leq C\|x - z\|^{\gamma-\beta}$$

where $\|\cdot\|_\beta$ is the norm of the projection onto the span of the symbols with homogeneity equal to β.

This innocent-looking condition can be in practice very complicated, because of the presence of the Γ operators. It is a notion of Hölder regularity in this setting of generalized monomials.

This definition is inspired by Gubinelli's theory of controlled rough paths.

The reconstruction theorem

Our starting problem was to associate to a function f a Taylor expansion $F(x)$ around each point x.

What about the inverse problem? Given a Taylor expansion $F(x)$ around each point x, can we find a function/distribution f which has this expansion up to a remainder?

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What about the inverse problem? Given a Taylor expansion $F(x)$ around each point x, can we find a function/distribution f which has this expansion up to a remainder?

This is the content of the cornerstone of the theory:

Reconstruction Theorem. *For all $\gamma > 0$ and $F \in \mathcal{D}^\gamma$ there exists a unique $\mathcal{R}F \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$|\mathcal{R}F(y) - \Pi_x F(x)(y)| \leq C(x)\|x-y\|^\gamma$$

or, more precisely, such that

$$|\mathcal{R}F(\varphi_{x,\delta}) - \Pi_x F(x)(\varphi_{x,\delta})| \leq C(x)\delta^\gamma.$$

Multiplication of modelled distributions

Consider $F_i \in \mathcal{D}^{\gamma_i}$, $\gamma_i > 0$, $i = 1, 2$:

$$F_i(x) = \sum_{j=1}^{N_i} a_i^j(x)\, \tau_i^j, \qquad \tau_i^j \in \mathcal{T}, \quad |\tau_i^j| \geq \alpha_i.$$

By the reconstruction theorem, $f_i := \mathcal{R}F_i \in \mathcal{S}'(\mathbb{R}^d)$ satisfies

$$f_i(y) = \sum_{j=1}^{N_i} a_i^j(x)\, \Pi_x \tau_i^j(y) + r_i(x, y)$$

where $|r_i(x, y)| \leq C(x)\|x - y\|^{\gamma_i}$ is a remainder.

Question: Can we define the product of f_1 and f_2?

Multiplication of Taylor sums

For $F_i \in \mathcal{D}^{\gamma_i}$ and $f_i := \mathcal{R}F_i \in \mathcal{S}'(\mathbb{R}^d)$, $j = 1, 2$,

$$F_i(x) = \sum_{j=1}^{N_i} a_i^j(x)\, \tau_i^j, \qquad f_i(y) = \sum_{j=1}^{N_i} a_i^j(x)\, \Pi_x \tau_i^j(y) + r_i(x, y),$$

$|r_i(x, y)| \leq C(x)\|x - y\|^{\gamma_i}$. Formal pointwise multiplication yields

$$(f_1 f_2)(y) = \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} a_1^{j_1}(x) a_2^{j_2}(x)\, \Pi_x \tau_1^{j_1}(y) \Pi_x \tau_2^{j_2}(y)$$

$$+ r_1(x, y) \Pi_x F_2(x)(y) + r_2(x, y) \Pi_x F_1(x)(y) + r_1 r_2(x, y).$$

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$$+ r_1(x,y)\Pi_x F_2(x)(y) + r_2(x,y)\Pi_x F_1(x)(y) + r_1 r_2(x,y).$$

Now $|r_1(x,y)\Pi_x F_2(x)(y)| \le \|x-y\|^{\gamma_1+\alpha_2}$. Therefore, setting $\gamma = (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1)$, we have

$$(f_1 f_2)(y) = \sum_{|\tau_i^1| + |\tau_i^2| < \gamma} a_i^1(x) a_i^2(x)\, \Pi_x \tau_1^{j_1}(y) \Pi_x \tau_2^{j_2}(y) + r(x,y)$$

with $|r(x,y)| \le C(x)\|x-y\|^{\gamma}$.

Multiplication of Taylor sums

This suggests the following definition:

$$(F_1F_2)(x) = \sum_{|\tau_i^1|+|\tau_i^2|<\gamma} a_i^1(x)a_i^2(x)\,\tau_i^1\tau_i^2$$

and, by the reconstruction theorem, if $F_1F_2 \in \mathcal{D}^\gamma$ and $\gamma > 0$ then $f_1 * f_2 := \mathcal{R}(F_1F_2)$ satisfies

$$(f_1 * f_2)(y) = \sum_{|\tau_i^1|+|\tau_i^2|<\gamma} a_i^1(x)a_i^2(x)\,\Pi_x(\tau_1^{j_1}\tau_2^{j_2})(y) + r(x,y)$$

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Two important comments are necessary:

1. we must assume that a product is defined on $\mathcal{T}$
2. if $\Pi_x\tau_1$ and $\Pi_x\tau_2$ are genuine distributions, their pointwise product is ill-defined in general and therefore Π_x can fail to be multiplicative.

Abstract multiplication of symbols

We need a product on $\mathcal{T}$: we assume that

1. $(\tau_1, \tau_2) \mapsto \tau_1 \tau_2 \in \mathcal{T}$ is associative and commutative
2. $|\tau_1 \tau_2| = |\tau_1| + |\tau_2|$
3. $\Gamma_{xz}(\tau_1 \tau_2) = (\Gamma_{xz}\, \tau_1)(\Gamma_{xz}\, \tau_2)$.

However this product is only defined at the abstract level. If $\Pi_x \tau_i$ are genuine distributions, then $\Pi_x \tau_1(y) \Pi_x \tau_2(y)$ is in general ill-defined, even if one tries by regularization.

What is needed is a well-defined model (Π_x, Γ_{xy}) on $\mathcal{T}$. In particular we need to *postulate* $\Pi_x(\tau_1 \tau_2)$, which could in general fail to be multiplicative. In the applications, it will be necessary to renormalize some of these products.

Examples

We show the explicit form of some elements of the theory. The class of abstract monomials $\mathcal{T}$ contains

$$\{X^k, \Xi, \Xi X^k, \mathcal{I}(\Xi), \mathcal{I}_1(\Xi), \Xi\mathcal{I}(\Xi), (\mathcal{I}_1(\Xi))^2, \ldots\}.$$

We consider a regularized space-time white noise $\xi_\varepsilon := \rho_\varepsilon * \xi$ and the heat kernel G.

We are going to build a model $(\Pi^\varepsilon_x, \Gamma^\varepsilon_{xz})$ which is given by polynomial functions of ξ_ε and is therefore random.

To begin with, we set

$$\Pi^\varepsilon_x X^k(y) = (y-x)^k, \qquad \Gamma^\varepsilon_{xz} X^k = (X + x - z)^k$$

$$\Pi^\varepsilon_x \Xi(y) = \xi_\varepsilon(y), \qquad \Gamma^\varepsilon_{xz} \Xi = \Xi$$

Examples

$$\Pi_x^\varepsilon \,\Xi X^k(y) = \xi_\varepsilon(y)(y-x)^k, \qquad \Gamma_{xz}^\varepsilon \,\Xi X^k = \Xi(X+x-z)^k$$

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$$\Pi_x^\varepsilon\,\mathcal{I}(\Xi)(y) = G*\xi_\varepsilon(y) - G*\xi_\varepsilon(x),$$

$$\Gamma_{xz}^\varepsilon\,\mathcal{I}(\Xi) = \mathcal{I}(\Xi) + (G*\xi_\varepsilon(x) - G*\xi_\varepsilon(z))1$$

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$$\Pi_x^\varepsilon \, \Xi\mathcal{I}(\Xi)(y) = \xi_\varepsilon(y)(G * \xi_\varepsilon(y) - G * \xi_\varepsilon(x)),$$
$$\Gamma_{xz}^\varepsilon \, \Xi\mathcal{I}(\Xi) = \Xi(G * \xi_\varepsilon(x) - G * \xi_\varepsilon(z))$$

For instance, $\mathbb{E}(\Pi_x^\varepsilon \, \Xi\mathcal{I}(\Xi)(y)) \sim \rho_\varepsilon * G * \rho_\varepsilon(0) \sim \varepsilon^{-1}$. Therefore this product needs to be renormalized by subtracting its mean.

SPDEs

Let us now consider a SPDE driven by space-time white noise ξ, e.g.

$$(\text{KPZ}) \qquad \partial_t u = \partial_{xx} u + (\partial_x u)^2 + \xi, \qquad t > 0,\ x \in \mathbb{R}.$$

The theory of regularity structures allows to "solve" this equation as follows:

1. One finds an appropriate set of generalized monomials $\mathcal{T}$
2. ξ is replaced by a regularized version $\xi_\varepsilon = \rho_\varepsilon * \xi$
3. one constructs an appropriate model $(\Pi^\varepsilon, \Gamma^\varepsilon)$
4. the solution u^ε has a generalized Taylor series U^ε in $\mathcal{D}^{3/2+\kappa,\varepsilon}$, $\kappa > 0$
5. the regularized SPDE is written as a fixed point in U^ε in $\mathcal{D}^{3/2+\kappa,\varepsilon}$
6. a renormalization procedure is performed on the model, yielding $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ and $\hat{U}^\varepsilon$, $\hat{u}^\varepsilon$
7. one proves convergence of $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ to $(\hat{\Pi}, \hat{\Gamma})$ and of $\hat{u}^\varepsilon$ to $\hat{u}$, independent of the particular regularization

Conclusions

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