Nouveaux paradigmes en dynamique de populations hétérogènes : Modélisation trajectorielle, Agrégation, et Données empiriques

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1 Introduction

2 Birth Death Swap system (with N. El Karoui)

3 How can a cause-of-death reduction be compensated in presence of heterogeneity? (with H. Labit-Hardy, S. Arnold and N. El Karoui)

4 Inextricable complexity of longevity evolution and modeling challenges (with N. El Karoui and K. Hadji)
Diverging trends in longevity documented at multiple levels:

- Countries with similar mortality experience until the 80s now diverge
  (Gaps in female life expectancy at 50 in 10 high-income countries: ≤ 1 year in 1980, ≥ 5 years in 2007, source: HMD).

- Widening of socioeconomic and geographical mortality inequalities
  (Gap in male life expectancy at 65 between higher managerial and routine occupations (England Wales): 2.4 years 1982-1986, 3.9 years 2007-2011, ONS)

Comment of National Research Council (2011):

“What is perhaps more surprising is that large differences [...] began relatively abruptly around 1980, and that it has taken so long for this divergence to be recognized and analyzed.”
These diverging trends are becoming a key issue for many organizations:


- Insurance and pension funds: interest in understanding risks linked to heterogeneity (Report on longevity basis risk (2014)).

Modeling challenges

- Numerous methodological problems.

- New questions cannot be answered by
  - traditional “macro” demographic point of view,
  - standard mortality models (Lee Carter (1992) Cairns et al. (2006)).

Need for finer grained models, capable of integrating the population heterogeneity and its evolution.
How can we model heterogeneity? How does heterogeneity impact aggregated “macro” dynamics produced by a finer grained modeling?

- Complex modeling issue which has not been fully formalized mathematically.
- Population dynamics framework following advances in mathematical biology and ecology (Méléard et al., Auger et al.)

Outline of the presentation:

1. Stochastic modeling of composition changes and aggregation
2. Empirical viewpoint on the impact of heterogeneity on aggregated indicators
3. Interdisciplinary analysis of the evolution of longevity and associated modeling challenges
1 Introduction

2 Birth Death Swap system (with N. El Karoui)
   - Setup
   - Pathwise representation of BDS systems
   - Averaging result and aggregation in the presence of two timescales

3 How can a cause-of-death reduction be compensated in presence of heterogeneity? (with H. Labit-Hardy, S. Arnold and N. El Karoui)

4 Inextricable complexity of longevity evolution and modeling challenges (with N. El Karoui and K. Hadji)
Population process

Population structured in \( p \) subgroups: population process 
\[(Z_t) = ((Z^i_t)_{i=1}^p)\] counting the number of individuals in each subgroup.

Aggregated population: \( Z^\mathbb{N}_t = \sum_{i=1}^p Z^i_t \) = size of the population.

Notations

Canonical basis \( (e_i)_{1\leq i \leq p}, \ e_i = (0,..,1_i,0,..) \).

\( y^\mathbb{N} = <y, 1> = \sum_{1\leq i \leq p} y_i \).

Random structure

A given filtered probability space \( (\Omega, \mathcal{G}, P, (\mathcal{G}_t)) \).

All components \( Z^i \in \mathbb{N} \) are adapted and càdlàg
Two kinds of events can occur:

- **Demographic events:**
  - Birth/Entry in a subgroup $i$.
  - Death in a subgroup $i$.

- **Swap events:** moves of an individual from a subgroup $i$ to $j$. 
Birth, Death and Swap events

Each event is characterized by a particular jump in the population:

- Jumps function $\phi : \mathcal{J}$ (set of all events) $\rightarrow \mathbb{N}^p$. 

\[
\begin{align*}
\phi(2, \infty) &= -e_2 \\
\phi(\infty, 1) &= e_1 \\
\phi(2, 3) &= e_3 - e_2
\end{align*}
\]
Each event is characterized by a particular jump in the population:

- **Jumps function** \( \phi : \mathcal{J} \) (set of all events) \( \rightarrow \mathbb{N}^p \).
  - If a swap event occurs at \( t \): \( \Delta Z_t = \phi(i, j) = e_j - e_i \).
  - Birth event: \( \Delta Z_t = \phi(\infty, i) = e_i \), Death: \( \Delta Z_t = \phi(i, \infty) = -e_i \).
Each type of event $\gamma \in \mathcal{J}$ is associated with the process $N_\gamma^\gamma$:

$$N_t^\gamma = \sum_{0 < s \leq t} \mathbb{1}_{\{\Delta Z_s = \phi(\gamma)\}}$$

Jumps counting system of the population: multivariate counting process $N = (N^\gamma)_{\gamma \in \mathcal{J}}$ (jump measure).

$N$ has the $\mathcal{G}_t$-multivariate intensity $\mu = (\mu^\gamma)_{\gamma \in \mathcal{J}}$ iff $N_t^\gamma - \int_0^t \mu_s^\gamma ds$ is a $\mathcal{G}_t$-local martingale ($E[N_{t+dt}^\gamma - N_t^\gamma | \mathcal{G}_t] \simeq \mu_t^\gamma dt$).

Population decomposition:

Population process can be expressed as a linear function of $N$:

$$Z_t = Z_0 + \sum_{\gamma \in \mathcal{J}} \phi(\gamma) N_t^\gamma = Z_0 + \phi \odot N_t.$$

The size of the population process is $Z_t^b = Z_0^b + N_t^{b,\bar{b}} - N_t^{d,\bar{b}}$. 
Birth Death Swap system (BDSs)

**Goal:** Build a population process $Z = Z_0 + \phi \otimes N$ from a jumps counting system $N = (N^\gamma)$ characterized by a non-linear class of multivariate intensity $\mu(t, Z_0 + \phi \otimes N_t) = (\mu^\gamma(\omega, t, Z_t))_{\gamma \in \mathcal{J}}$.

### Intensity properties

- Predictable intensity functional $\mu(t, z)$.
- **Support condition** (no death or move from an empty class):
  \[
  \mu^{i\beta}(t, z) \mathbf{1}_{\{z^i = 0\}} \equiv 0 \quad \forall i \in \mathcal{J}_p, \beta \in \mathcal{J}^{(i)}.
  \]
- A BDS system with intensity functional $\mu$ as above is a triplet $(Z_0, N, Z = Z_0 + \phi \otimes N)$.
- Example of birth intensity: $\mu^{b,i}(\omega, t, z) = b_t^i(\omega)z^i$. 
Thinning permits to represent jump processes as strong solutions of stochastic differential systems (SDS) driven by Poisson measures:


- Birth Death Swap systems:
  - Representation with the jumps counting system $\mathbf{N} \Rightarrow$ point of view of multivariate counting processes.
  - Construction by strong domination: existence and uniqueness of BDS systems under weaker conditions $+$ “free” tightness.
Thinning of Poisson measure

- **Space-time $G_t$ Poisson measure** $Q(dt, d\theta)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ of $\sigma$-finite $G_t$-intensity $dt \otimes d\theta$. **No** increasing enumeration of jumps times.
*Space-time $G_t$ Poisson measure* $Q(dt, d\theta)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ of $\sigma$-finite $G_t$-intensity $dt \otimes d\theta$. *No* increasing enumeration of jumps times.

*Cox processes*: given a predictable intensity process $(\lambda_t)_{t \geq 0}$,
Thinning of Poisson measure

- **Space-time** $G_t$ **Poisson measure** $Q(dt, d\theta)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ of $\sigma$-finite $G_t$-intensity $dt \otimes d\theta$. No increasing enumeration of jumps times.

- **Cox processes**: given a predictable intensity process $(\lambda_t)_{t \geq 0}$,

\[ N_t^\lambda = \int_0^t \int_{\mathbb{R}^+} 1_{]0,\lambda_s]}(\theta) Q(ds, d\theta) \]

is a **counting process** of $G_t$-intensity $\lambda_t$. 
Thinning of Poisson measure

- **Space-time** $\mathcal{G}_t$ **Poisson measure** $Q(dt, d\theta)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ of $\sigma$-finite $\mathcal{G}_t$-intensity $dt \otimes d\theta$. No increasing enumeration of jumps times.

- **Cox processes**: given a predictable intensity process $(\lambda_t)_{t \geq 0}$, 
  \[ N^\lambda_t = \int_0^t \int_{\mathbb{R}^+} 1_{[0, \lambda_s]}(\theta) Q(ds, d\theta) \] 
  is a counting process of $\mathcal{G}_t$-intensity $\lambda_t$.

- **Marked random measure**: $Q^\lambda(dt, d\theta) = 1_{[0, \lambda_t]} Q(dt, d\theta)$ \( \Rightarrow \) Jump times can be enumerated increasingly (same jump times than $N^\lambda$).
Markov pure Birth process

- When $\lambda_t$ is a functional of $N^\lambda_t \Rightarrow$ SDE driven by $Q$.
- Markov Pure Birth counting process:

$$dG^b_t = Q(dt, [0, Kg(n_0 + G^b_{t-})])$$

(2)

- **Feller non-explosion criterion:** $\sum_{n=1}^{\infty} \frac{1}{g(n)} = \infty \Rightarrow$ existence and uniqueness of (2).
- **Marked Birth measure:**

$$Q^g(dt, d\theta) = 1_{[0, Kg(n_0 + G^b_{t-})]} Q(dt, d\theta).$$

(same jump times than $G^b$).
Let $\alpha(\omega, t, n) \leq Kg(n)$, existence and uniqueness of (non-exploding) solution of

$$dN_t^\alpha = Q(dt, \alpha(\omega, t, n_0 + N_{t_0}^\alpha)) \quad ? \quad (3)$$

- **Step 1**: replace $Q$ by $Q^g(dt, d\theta)$ in (3):

$$d\tilde{N}_t^\alpha = Q^g(dt, \alpha(\omega, t, \tilde{N}_{t_0}^\alpha)). \quad (4)$$

- Jumps times of $Q^g(dt, d\theta)$ can be enumerated increasingly $\Rightarrow (4)$.

- **Step 2**: show equivalence between (3) and (4).

**Stronger result**: $N^\alpha$ is strongly dominated by $G^b (N^\alpha < G^b)$ i.e. all jumps times of $N^\alpha$ are jumps times of $G^b$. 
Markov birth process can be replaced by a non-exploding multivariate counting process solution of:

\[ d\bar{Y}_{t}^{\beta} = \bar{Q}(dt, [0, \bar{p}(t, \bar{y}_{0} + \bar{Y}_{t}^{\beta})]) \]

**Theorem**

Let \( \bar{\alpha} \) be a multivariate intensity functional such that:

\[ \alpha^i(t, \tilde{y}) \leq \beta^i(t, \bar{y}), \quad \forall 1 \leq i \leq \rho, \quad \tilde{y} \leq \bar{y} \in \mathbb{N}^\rho. \]

Then, for all \( \tilde{y}_0 \leq \bar{y}_0 \), there exists a unique (non-exploding) solution of

\[ d\bar{Y}_{t}^{\alpha} = \bar{Q}(dt, [0, \bar{\alpha}(t, \bar{y}_0 + \bar{Y}_{t}^{\alpha})]) \]  \hspace{1cm} (5)

Furthermore, \( \bar{Y}^{\alpha} \) is strongly dominated by \( \bar{Y}^{\beta} \): \( \bar{Y}^{\alpha} < \bar{Y}^{\beta} \).
Markov birth process can be replaced by a non-exploding multivariate counting process solution of:

$$d \tilde{Y}_t^\beta = \tilde{Q}(dt, [0, \tilde{\beta}(t, \tilde{y}_0 + \tilde{Y}_t^\beta)])$$

**Theorem**

Let $\tilde{\alpha}$ be a multivariate intensity functional such that:

$$\alpha^i(t, \tilde{y}) \leq \beta^i(t, \tilde{y}), \quad \forall 1 \leq i \leq \rho, \quad \tilde{y} \leq \tilde{y} \in \mathbb{N}^\rho.$$

Then, for all $\tilde{y}_0 \leq \tilde{y}_0$, there exists a unique (non-exploding) solution of

$$d \tilde{Y}_t^\alpha = \tilde{Q}(dt, [0, \tilde{\alpha}(t, \tilde{y}_0 + \tilde{Y}_t^\alpha)])$$

(5)

Furthermore, $\tilde{Y}^\alpha$ is strongly dominated by $\tilde{Y}^\beta$: $\tilde{Y}^\alpha \prec \tilde{Y}^\beta$.

**Corollary:** existence of multivariate Cox-Birth processes of intensity $k_t g(y^\alpha)$ with $(k_t)$ a predictable locally bounded process and $g = (g^z)$ with $g^z$ verifying Feller criterion.
Existence and uniqueness of BDS equation

- Driving $p(p + 1)$ independent Poisson measures $Q = (Q^\gamma)_{\gamma \in \mathcal{J}}$.
- BDS multivariate differential system associated with intensity $\mu(t, z) = (\mu^\gamma(t, z))_{\gamma \in \mathcal{J}}$:

\[ dN_t = Q(dt, [0, \mu(t, Z_{t-})]), \quad Z_t = Z_0 + \phi \otimes N_t. \quad (6) \]

- Idea: control birth part $N^b$ of $N$ with Cox-Birth domination assumption:

\[ \mu^b(\omega, t, z) \leq k_t g(z^b). \quad (7) \]

Proposition

Assume that $\mu^s(t, K)$ and $\mu^d(t, K)$ are locally bounded for any $K \geq 0$. Under the Cox Birth assumption (7), there exists a unique solution of (6), strongly dominated by a non-exploding Cox-birth process $G$ ($N \prec G$).
**Hyp:** intensity of swap events $\sim O(\frac{1}{\epsilon}) >>$ demographic events $\sim O(1)$.

The BDS system now depends on a small parameter $\epsilon$:

$$Z^\epsilon_t = Z_0 + \phi^s \odot N^s,\epsilon_t + N^b,\epsilon_t - N^d,\epsilon_t,$$

$$dN^s,\epsilon_t = Q^s(dt, [0, \frac{1}{\epsilon} \mu^s(t, Z^\epsilon_t)]), \quad dN^{\text{dem},\epsilon}_t = Q^{\text{dem}}(dt, [0, \mu^{\text{dem}}(t, Z^\epsilon_t)]).$$

- $N^s,\epsilon$ is a "fast" counting system of intensity functional $\frac{1}{\epsilon} \mu^s(t, z)$: explosion when $\epsilon \to 0$.
- **But** $N^{\text{dem},\epsilon}$ only depends on $\epsilon$ through $Z^\epsilon$ and is strongly dominated by a multivariate counting process which doesn’t depend on $\epsilon$.

$$\forall \epsilon > 0, \quad N^{\text{dem},\epsilon} < G^{\text{dem}}.$$
Convergence of demographic process

- State space of $\mathbf{N}^{\text{dem},\epsilon}$: $\mathcal{A}^{2p}$ subspace of Skorohod space $D(\mathbb{R}^+, \mathbb{N}^{2p})$ of functions whose components have unit and no common jumps.

- **Consequence of strong domination**: $(\mathbf{N}^{\text{dem},\epsilon})$ is tight in $\mathcal{A}^{2p}$.

**Identification of limit points of $(\mathbf{N}^{\text{dem},\epsilon})$**

- $\mathcal{G}_t$-local martingale $\mathbf{N}_t^{\text{dem},\epsilon} = \int_0^t \mathbf{\mu}^{\text{dem}}(\omega, s, Z_{s-}^\epsilon) ds$.

- Due to explosion of swap events, $Z^\epsilon$ cannot converge as processes.

- Weaker framework: $Z^\epsilon$ is seen as an $\mathbb{N}^p$-valued random variable on

  $$(\Omega \times \mathbb{R}^+, \mathcal{G} \otimes \mathcal{B}(\mathbb{R}^+), \mathbb{P} \otimes \text{Leb}) \quad Z^\epsilon(\omega, s) = Z_s^\epsilon(\omega).$$

- $(Z^\epsilon, h^b(\omega, s))$ uniformly dominated by $G^{b, h}(\omega, s) \Rightarrow$ Tightness.
Stable convergence

- Here: $\mu^{\text{dem}}(\omega, t, z)$. Need convergence of random functionals preserving martingale properties $\Rightarrow$ Stable convergence
Stable convergence

- Deterministic intensity functional (Markov framework) \( \Rightarrow \) Averaging result of Kurtz (1992).
- Here: \( \mu^{\text{dem}}(\omega, t, z) \). Need convergence of random functionals preserving martingale properties \( \Rightarrow \text{Stable convergence} \)

Let \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\) be a Polish space. **Enlarged space:**
\[
(\bar{\Omega}, \bar{\mathcal{G}}) = (\Omega \times \mathcal{X}, \mathcal{G} \otimes \mathcal{B}(\mathcal{X}))
\]

**Space of rules** \(\mathcal{R}(\mathcal{P}, \mathcal{X})\): probability measure \(\mathcal{R}(d\omega, d\chi)\) of marginal \(\mathcal{P}\) on \((\Omega, \mathcal{G})\):
\[
\mathcal{R}(d\omega, d\chi) = \Gamma(\omega, d\chi)\mathcal{P}(d\omega).
\]

**Stable convergence**: convergence of rules for the **Space of test functions** \(\mathcal{C}_{bmc}(\Omega \times \mathcal{X})\) of bounded \(\bar{\mathcal{G}}\)-measurable \(H(\omega, \chi)\) continuous in \(\chi\) for any \(\omega\).
Stable convergence of random variables

- Rule associated with a r.v $Y(\omega) \in \mathcal{X}$: $R^Y(H) = E[H(\cdot, Y)]$.

**Stable convergence of** $(Y_n(\omega))$ **to a rule** $R$ ("2 viewpoints"): 
Stable convergence of random variables

- Rule associated with a r.v $Y(\omega) \in \mathcal{X}$: $R^Y(H) = E[H(., Y)]$.

**Stable convergence of** $(Y_n(\omega))$ to a rule $R$ (**2 viewpoints**):

1. **Convergence of the given space:**
   - **View 1** $R^{Y_n}(H) = E[H(., Y_n)] \to E[\Gamma(H)]$ ($= R[H(\omega, \chi)]$).

2. **Realization of stable limit on** $(\Omega \times \mathcal{X}, \bar{\mathcal{G}}, R)$ with $\Gamma(\omega, \chi) = \chi$:
   - **View 2** $E[H(., Y^n)] \to R[H(., \Gamma)]$, $\forall H \in \mathcal{C}_{bmc}(\Omega \times \mathcal{X})$. 

Fundamental property:

If $Y_n$ converges weakly, there exists a subsequence of $Y_n$ converging stably.
Stable convergence of random variables

- Rule associated with a r.v \( Y(\omega) \in \mathcal{X} \): \( R^Y(H) = E[H(\cdot, Y)] \).

**Stable convergence of** \((Y_n(\omega))\) **to a rule** \( R \) (**2 viewpoints**):

1. Convergence of the given space:
   - View 1 \( R^Y_n(H) = E[H(\cdot, Y_n)] \rightarrow E[\Gamma(H)] = R[H(\omega, \chi)] \).

2. Realization of stable limit on \((\Omega \times \mathcal{X}, \bar{G}, R)\) with \( \Upsilon(\omega, \chi) = \chi \):
   - View 2 \( E[H(\cdot, Y^n)] \rightarrow R[H(\cdot, \Upsilon)], \quad \forall H \in \mathcal{C}_{bmc}(\Omega \times \mathcal{X}). \)

**Fundamental property:** If \((Y_n)\) converges weakly, there exists a subsequence of \((Y_n)\) converging stably.
Application of the stable convergence

In the following, all convergences are “up to a subsequence”.

Three applications of the stable convergence:

\[ \left( \begin{array}{c} \mathbb{N}_t^{\text{dem}, \epsilon} - \int_0^t \mu^{\text{dem}}(\omega, s, Z_s^\epsilon) \, ds \\ \end{array} \right) \]

\( (i) \) with View 1: stable convergence of \( Z^\epsilon(\omega, s) \) to a limit rule

\[ \tilde{\Gamma}(\omega, s, dz) P(d\omega) \, ds \Rightarrow \]

\[ \mathbb{E}[\mathbb{1}_B \int_0^t \mu^{\text{dem}}(\omega, s, Z_s^\epsilon(\omega)) \, ds] \quad \xrightarrow{\epsilon \to 0} \quad \mathbb{E}[\mathbb{1}_B \int_0^t \tilde{\Gamma}(\omega, s, \mu^{\text{dem}}(\omega, s, .)) \, ds], \quad \forall B \in \mathcal{G}. \]
Stable convergence of the demographic process

\[ N_{t}^{\text{dem},\epsilon} - \int_{0}^{t} \mu_{\text{dem}}(\omega, s, Z_{s}^{\epsilon}) ds \]

\( (i) \) with View 2, realization of stable limits of \( (N_{t}^{\text{dem},\epsilon}) \):

- Extended space: \( (\tilde{\Omega},(\tilde{G}_{t})) = (\Omega \times \mathcal{A}^{2p},(G_{t} \otimes \mathcal{F}_{t}^{A})) \).

- If \( R(d\omega,d\alpha) \) is a limit rule and \( \tilde{N}_{t}^{\text{dem}}(\omega, \alpha) = \alpha \in \mathcal{A}^{2p} \),

\[ E[\mathbb{1}_{B} h(N_{t}^{\text{dem},\epsilon})] \xrightarrow{\epsilon \to 0} R[\mathbb{1}_{B} h(\tilde{N}_{t}^{\text{dem}})], \quad \forall \ B \in G, \ h \in C_{cb}(\mathcal{A}^{2p}). \]
Stable convergence of the demographic process

\[
\begin{align*}
\mathbb{N}^{\text{dem}, \epsilon}_t & - \int_0^t \mu^{\text{dem}}(\omega, s, Z^\epsilon_s) ds \\
& \quad (i) (ii)
\end{align*}
\]

2. (i) with View 2, realization of stable limits of \((\mathbb{N}^{\text{dem}, \epsilon})\):

- Extended space: \((\tilde{\Omega}, (\mathcal{G}_t)) = (\Omega \times \mathcal{A}^{2p}, (\mathcal{G}_t \otimes \mathcal{F}_t^{\mathcal{A}}))\).

- If \(R(d\omega, d\alpha)\) is a limit rule and \(\tilde{\mathbb{N}}^{\text{dem}}(\omega, \alpha) = \alpha \in \mathcal{A}^{2p}\),

\[
\mathbb{E}[\mathbb{1}_B h(\mathbb{N}^{\text{dem}, \epsilon})] \quad \underset{\epsilon \to 0}{\longrightarrow} \quad \mathbb{R}[\mathbb{1}_B h(\tilde{\mathbb{N}}^{\text{dem}})], \quad \forall \ B \in \mathcal{G}, \ h \in C_{cb}(\mathcal{A}^{2p}).
\]

3. Joint stable convergence of \((\mathbb{N}^{\text{dem}, \epsilon}, Z^\epsilon) ((i) + (ii))\):

- Limit rules:

\[
R^*(d(\omega, s), d(\alpha, z)) = R(d\omega, d\alpha) \Gamma(\omega, s, \alpha, dz).
\]
Theorem

Let \((R, \Gamma)\) be a limit point of \((N_{\text{dem}, \epsilon}, Z_{\epsilon})\). Then, for all subsequences converging stably to \((R, \Gamma)\), \((N_{\text{dem}, \epsilon})\) converges stably to the canonical demographic process \(\bar{N}_{\text{dem}}(\omega, \alpha) = \alpha\) on \((\bar{\Omega}, \bar{\mathcal{G}}, R)\), and:

\[
\bar{N}_{\text{dem}} \text{ has the } (R, \bar{\mathcal{G}}_t)\text{-compensator } \int_0^t (\Gamma(s, \bar{N}_{\text{dem}}, \mu_{\text{dem}}))^\circ ds, \quad (9)
\]

with \((\Gamma(\cdot, t, \bar{N}_{\text{dem}}, \mu_{\text{dem}}))^\circ, \text{ the } (R, \bar{\mathcal{G}}_t)\text{-optional projection of process}
\]
t \rightarrow \Gamma(\cdot, t, \bar{N}_{\text{dem}}, \mu_{\text{dem}}).
Particular case: deterministic swap intensity function $\mu^s(z)$.

$$Z_t^\epsilon = Z_0 + \phi^s \odot N_t^{s,\epsilon} + N_t^{b,\epsilon} - N_t^{d,\epsilon},$$

$$dN_t^{s,\epsilon} = Q^s(dt, \left]0, \frac{1}{\epsilon} \mu^s(Z_{t-}^\epsilon)\right]), \quad dN_t^{\text{dem},\epsilon} = Q^{\text{dem}}(dt, \left]0, \mu^{\text{dem}}(\omega, t, Z_{t-}^\epsilon)\right]).$$

- Pure swap processes $X$ of $G_t$-intensity $\mu^s$:
  - Population with NO demographic events.
  - Constant size: $X_0^b = d \Rightarrow X_t = d$ ($X \in \mathcal{U}_d$, populations of size $d$).
- Deterministic intensity $\mu^s \Rightarrow X$ continuous time Markov chain.

Assumption: $\forall d \in \mathbb{N}$, the swap process restricted to $\mathcal{U}_d$ admits a unique stationary distribution $(\pi(d, dx))_{x \in \mathcal{U}_d}$. 
Convergence of the demographic system

Hyp: pure swap on \( \mathcal{U}_d \) admits a unique stationary distribution \((\pi(d, dx))_{x \in \mathcal{U}_d}\).

Theorem

The family of demographic counting systems \((\mathbf{N}_{\text{dem}, \epsilon})\) converges weakly in \( \mathcal{A}^{2p} \) to the jumps counting system \( \mathbf{\bar{N}}_{\text{dem}} = (\mathbf{\bar{N}}^b, \mathbf{\bar{N}}^d) \) of a true multi-type Birth-Death process, defined on an extension \((\bar{\Omega}, (\bar{G}_t), R)\) of \((\Omega, (G_t), P)\), and such that:

\[
\mathbf{\bar{N}}_{\text{dem}} t \text{ has the } \bar{G}_t \text{ intensity } \pi(\bar{Z}_t^b, \mu^{\text{dem}}(t, \cdot)), \quad \bar{Z}_s^b = \mathbf{\bar{N}}_t^b - \bar{\mathbf{\bar{N}}}_t^d.
\]

- **Corollary:** The aggregated processes \( Z^{\epsilon, \bar{\eta}} \) converge to the true Birth-Death process \( \bar{Z}^\eta \) of intensity functionals:

\[
\lambda^b(t, n) = \pi(n, \mu^{b, \bar{\eta}}(t, \cdot)), \quad \lambda^d(t, n) = \pi(n, \mu^{d, \bar{\eta}}(t, \cdot)).
\]

- **Averaging effect:** aggregated intensities depend non-linearly of the number of individuals in the population.
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2 Birth Death Swap system (with N. El Karoui)

3 How can a cause-of-death reduction be compensated in presence of heterogeneity? (with H. Labit-Hardy, S. Arnold and N. El Karoui)
   - Data: Population by Index of deprivation
   - Population dynamics model
   - Numerical results

4 Inextricable complexity of longevity evolution and modeling challenges (with N. El Karoui and K. Hadji)
Forecasting heterogeneous mortality rates


- Issue: consistency of sub-national and national estimates/forecasts.

- Other difficulty: interpreting targets set by institutions (Department of Health, WHO).

- **Our approach**: take into account all population data.

How changes in the socioeconomic composition of the population affect aggregated indicators such as the life expectancy? Could we miss a cause-of-death reduction in presence of heterogeneity?
Data

- **Two databases:**
  - 1981-2007: Department of Applied Health Research, UCL.
- English cause-specific number of deaths and mid-year population estimates per socioeconomic circumstances, age and gender.
Two databases:
- 1981-2007: Department of Applied Health Research, UCL.

English cause-specific number of deaths and mid-year population estimates per socioeconomic circumstances, age and gender.

Socioeconomic circumstances are measured by the Index of multiple deprivation (IMD), based on the postcode of individuals.

- Small areas (LSOA) are ranked based on seven broad criteria: income, employment, health, education, barriers to housing and services, living environment and crime.
- This ranking permits to divide the population in 5 quintiles with about same number of individuals in each quintile.
Population composition, 2015

Age-pyramids by IMD quintile, 2015

(a) Most deprived quintile ($)  
Median age: 33y

- Baby-boom cohort less deprived than younger/older cohorts.

(b) Least deprived quintile ($$$$$)  
Median age: 44.2y
Changes in the population composition

**Figure:** Composition of males age classes in years *1981, 1990, 2005, 2015.*

- Decrease of deprivation over time for older age classes, (IMD 1+2: 28% → 46%).
- Increase of deprivation for younger age classes, (IMD 1+2: 36% → 31%).
Mortality by deprivation

Figure: Males average annual mortality improvement rates by age

- Over the period 2001-2015 gap in mortality improvement rates increased significantly for ages above 60.
Heterogeneous population dynamics

- **Simple** population dynamics framework to illustrate different impacts of heterogeneity on the aggregated mortality.

- **Deterministic** evolution of each subgroup is described by a McKendrick (1926) - Von Foerster (1959).

Equation for each gender $\epsilon = m$ or $f$ and subgroup:

- **Aging law:** $(\partial_a + \partial_t)g_j^\epsilon(a, t) = -\mu_j^\epsilon(a, t)g_j^\epsilon(a, t)$
- **Birth law:** $g_j^\epsilon(0, t) = \int_0^{a^\dagger} p^\epsilon g_j^f(a, t)b_j(a, t)da$
- **Initial Pyramid:** $g_j^\epsilon(a, 0)$

- Short term behavior: depends on the initial population.

- Long term behavior: characterized by birth and survival functions.
\textbf{Aggregated population}

- Aggregated population:
  \begin{itemize}
    \item \( g^e(a, t) = \sum_{j=1}^{P} g_j^e(a, t) \)
    \item Aging law: \((\partial_a + \partial_t)g^e(a, t) = -d^e(a, t)g^e(a, t)\)
  \end{itemize}

- Aggregated death rate:
  \begin{itemize}
    \item Weighted sum of the subpopulations death rates:
      \[ d^e(a, t) = \sum_j w_j^e(a, t) \mu_j^e(a, t), \quad w_j^e(a, t) = \frac{g_j^e(a, t)}{g^e(a, t)} \]
    \item \(d\) depends non-linearly on the population inputs: \(g_j^0, \mu_j, \text{ and } b_j\).
  \end{itemize}

- Even with time-independent rates \(\mu_j^e(a, \xi)\)

  \(\Rightarrow\) the aggregate death rate \(d^e(a, t)\) depends on time, due to changes of composition of the heterogeneous population.
Numerical results

Two applications:

- Short-term: impact of the initial pyramid heterogeneity.

**Indicators** at time $t$ and age $a$:

- **Period life expectancy**: average lifetime remaining for an imaginary individual living in the mortality conditions of year $t \Rightarrow $ Period index.
- **Cohort life expectancy**: average length of life remaining for individuals born at time $t-a \Rightarrow $ “Real expectation”.
Example of scenario illustrating impact of changes of demographic rates:

- Period or Cohort indicators capture effects of different nature.
- Difficulty of interpreting the data at the aggregated level with combined changes of different nature.

- Synthetic population composed of the most and least deprived IMD quintile (illustrative purpose).
- Baseline (“neutral”) scenario:
  - Death rates: fixed death rates in each population (input year 2015).
  - Birth rates same birth rates in each population (England, 2015).
  - Initial pyramid our aim is to limit influence of initial pyramids evolution: Same initial pyramid $\Rightarrow$ Stable pyramids.
Scenario 1a: cause of death reduction

Figure: Reduction of mortality rates from cardiovascular disease (CVD)

- Reduction of 10% of cause of death mortality from CVD in the most deprived subpopulation over a period of 30 years, starting at $t = 40$.
- Period life expectancy more adapted to detect mortality reduction.
Scenario 1b: change in birth rates

Figure: “Reverse” Cohort effect

- Cohort effect: increase of 20% of birth rates over the period [0, 20].
- Cohort life expectancy more adapted to detect changes in composition.
Combined reduction of mortality from CVD and reverse cohort effect.

Figure: Male period life expectancy at 25

- When the population heterogeneity is not taken into account, impacts of cause-of-death mortality changes can be compensated and/or misinterpreted.
1 Introduction

2 Birth Death Swap system (with N. El Karoui)

3 How can a cause-of-death reduction be compensated in presence of heterogeneity? (with H. Labit-Hardy, S. Arnold and N. El Karoui)

4 Inextricable complexity of longevity evolution and modeling challenges (with N. El Karoui and K. Hadji)
15+ fields involve in the study of human longevity ⇒ necessity to understand this mass of interdisciplinary information for a relevant mathematical modeling.

1 Understanding and explaining the “second demographic transition”
   ▶ Underlying factors responsible for the SES gradient in mortality.

2 The historic demographic transition
   ▶ Impact of macro environment from a historical perspective.

3 Micro demographic models: Microsimulation and Agent Based Models (Van Imhoff and Post(1998), Billari (2006), Morand et al. (2010), Li and O'Donoghue (2013), Zinn (2017)...)
What does heterogeneity means?

- Longstanding research on the relationship socioeconomic status (SES)/mortality (Villermé (1830), General Register Office (1851)).

- Three broad categories of risk factors: material, behavioral and psychosocial risk factors (Cutler (2006), Elo (2009)).

Each producing different type of modeling:

- **State variables**: e.g. absolute (material explanations) vs relative measures (psychosocial risk factors).

- **Micro-Macro interactions**:
  - Material factors: concave relationship between income and mortality (Preston (1975)) \(\Rightarrow\) impact of income inequalities at macro level.
  - Psychosocial factors: impact of income inequalities at micro level (Wilkinson and Pickett (2009)).
Life course of individuals are embedded in a local (neighborhood, city...) and global context (state).

- Major role of public health in the **historical reduction of mortality**.
- Important delays between scientific findings and public action (ex: more than 10 international conferences over 50 years (1851-1903) to obtain tangible results against cholera epidemics, **Huber (2006)**).

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**Modeling challenges** Take into account the macro environment:

- Include these timescales between discovery and management of health hazards (50 years for asbestos, **Cicolella (2010)**).
- Public health measures can increase inequalities ⇒ behavioral risk factors (ex: link between smoking and lung cancer ⇒ SES gradient in smoking behavior, **Link (2008)**).
Conclusion and perspectives

Study of heterogeneity and impact on aggregated variables from different angles. Perspectives:

- **Make connections** between results of different approaches.
- **Probabilistic modeling:**
  - Take into account the age-structure of the population in BDSs.
  - Pathwise comparison of population dynamics.
  - Stochastic age groups: measure the rejuvenation populations.
- **Empirical viewpoint:** include data on internal/external migrations.
  - Use population dynamics framework to test consistency of sub-national and national mortality forecast.

Dynamic modeling as an experimenting tool

- Analytical and simulation framework to test existing theories, as it is often “difficult, expensive, and ethically challenging to alter individual behavior”.
Thank You For Your Attention