BOOTSTRAP AND JAMMING PERCOLATION

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1. Introduction

The aim of these notes is to review some results for Bootstrap Percolation (BP) [1–8] and the more recently introduced Jamming Percolation (JP) models [9–13].

Let us start by presenting two examples to explain what is (and what is not) bootstrap and jamming percolation. Take a square box and fill it with insulating and metal balls, then shake it in order to reach a random mixture. A natural question arises: Does the overall box behave like a conductor or an insulator? In other words: Does there exist a group of metal balls touching one another and reaching the opposite sides of the box? the answer depends on the size of the box and on the fraction of metal vs insulating balls: the higher this fraction the higher the probability of an overall conductive behavior. The simplest model for this is Site Percolation (SP): a lattice model in which each site is independently occupied or empty with probability \( p \) and \( 1 - p \) (metal and conducting balls) and the focus is on the properties of occupied clusters, i.e. group of nearest neighbour occupied sites. SP tells us that, in the limit of large box size, a cluster of metal balls which spans or percolates the box occurs if and only if \( p > p_c \). This means that a sharp change in the global behavior of the system occurs upon a small variation of the density: a phase transition occurs at \( p_c \). This is a second-order transition: the continuously changing order parameter, \( \theta(p) \), is the probability that the origin belongs to an infinite cluster or, which is the same thanks to translation invariance, the density of the infinite cluster itself. Indeed, the main appeal of SP relies on being the simplest model which displays a second-order phase transition. Thus, the natural issues of estimating the value of \( p_c \) and determining the behavior of \( \theta(p) \) as well as the other cluster properties around criticality can be analyzed by using standard tools for second-order transitions, in particular renormalization group theory. From this we know that \( \theta(p) \) and the other cluster properties scale as power laws, \( \theta(p) = (p - p_c)^{\beta} \), and their finite size scaling has the form \( \theta(p, L) = L^{-\beta/\nu}F[(p - p_c)L^{1/\nu}] \) where \( \beta \) is a quantity dependent critical exponent and \( \nu \) is the exponent of the power law divergence of the incipient cluster size. Thanks to the knowledge of finite size effects, one can extrapolate via numerical simulations the value of \( p_c \) and of all critical exponents. Finally it is important to recall that there is a large degree of universality for SP models: \( p_c \) is model dependent but the critical exponents depend only on the spatial dimension of the lattice. This means that
the occurrence and the character of the transition is not influenced by microscopic
details.

Now, instead of the mixture of conducting and isolating balls, put inside the
square box just one kind of balls without filling it completely. Then put the box
on a horizontal plane, shake it in order to start again from a random configuration
and let the system evolve with a dynamics in which a given ball can move only
if “there is enough empty space around”. You could either imagine a determin-
istic newtonian dynamics, e.g. balls interacting via a hard sphere potential, or a
stochastic dynamics, e.g. the motion of a suspension of colloidal particles which
is driven by the thermal agitation of the solvent. The natural question which now
arises is the following: Is the whole system mobile or does there exist a frozen
backbone which does not evolve during dynamics? In other words: if we focus
on a specific ball will it always move (provided we wait long enough) or could it
belong to a finite fraction of balls which are forever blocked? Of course, the an-
swer depends again on the density of particles and on the size of the box. But now,
as we will see, it also crucially depends on the microscopic details, namely on
the specific meaning of the words “enough empty space” when defining the dy-
namics. Turning to real systems, numerical simulations for sphere packings [14]
and experiments for continuum particle systems [15] show that there is a packing
density where a transition occurs. This can roughly be regarded as the transi-
tion from a fluid to an amorphous solid and should correspond to the formation
of a frozen backbone for the system. The onset of this transition is sharp and
displays a peculiar mixture of first-order and critical character. The average co-
ordination number of a particle is discontinuous but displays a singularity (as
for a continuous transition) when the critical density is approached from above.
Also, a singularity in the typical relaxation time occurs when the transition is
approached from below. Similar mixed first-order and continuous properties are
observed in other systems of very different microscopic nature where upon tun-
ing a proper external parameter dynamics is slowed down leading ultimately to
the formation of an amorphous solid. These include supercooled liquids when
temperature is lowered until the formation of a glass phase [16] and non-thermal
systems such as vibrated granular materials [17]. The understanding of these
phenomena, which are usually referred to as glass or jamming transitions, still
leaves many unsettled and fascinating questions for condensed matter physicists.
From the theoretical point of view, the first natural step is to devise a microscopic
model simple enough to be analyzed and displaying a dynamical arrest with this
mixture of first order and critical features. Another property one would like to
reproduce is the unconventional scaling of relaxation times which for molecular
fragile liquids diverge faster than power law, a very popular fit being the Vogel
Fulcher form: \( \log \tau \simeq 1/T - T_c \).

In the course of this lecture we will see that in a sense BP is the easiest discrete
model one could imagine to describe the constrained dynamics of the balls in the box. However BP fails to provide a jamming transition: a frozen cluster in the thermodynamic limit never occurs on any finite dimensional hypercubic lattice (square lattice in $d = 2$, cubic lattice in $d = 3$, ...). On the other hand, if one keeps the lattice size finite and raises the density, a sharp onset which is very reminiscent of a first-order phase transition occurs at a (size dependent) density threshold. This metastability phenomenon is due to the fact that proper large and rare critical droplets of empty sites are necessary in order to make the system mobile. In turn, this induces time scales which diverge very rapidly when the density is raised towards one.

We will explain how, by a different choice of the microscopic dynamics, it is possible to construct models which instead display a jamming transition on finite dimensional lattices [9–13]: a frozen cluster occurs in the thermodynamic limit for $p \geq p_c$ with $0 < p_c < 1$. These are what we call Jamming Percolation (JP) models. Here we define the simplest JP model, the so called Spiral Model (SM) [11, 12], and we provide the tools which allow to identify and analyze its transition. This has a character which is different from SP and all standard percolation transitions: the density of the frozen cluster is discontinuous at $p_c$, namely the clusters are compact rather than fractal at criticality, and the size of the incipient cluster diverges when $p \nearrow p_c$ faster than any power law. Furthermore, since the transition is not second order, we do not have the large degree of universality which holds for SP: the existence and character of the transition (and not only its location) depends on the specific choice of the microscopic details. However, we will see that it is possible to identify a class of models which give rise to a jamming transition and belong to the same universality class.

2. Bootstrap Percolation (BP)

2.1. Definition and main results

Consider a $d$-dimensional lattice and fix a parameter $m$, $1 < m < 2d$. Let each site be independently occupied or empty with probability $p$ and $1 - p$, as for SP. The objects of interest for BP are blocked (or frozen) clusters with respect to the dynamics in which the “enough empty space” required to move a particle corresponds to (at least) $m$ empty nearest neighbours. More precisely, blocked clusters are sets of occupied sites each of which has less than $m$ empty neighbours (more than $2d - m$ occupied neighbours). Denoting by $R(L, p)$ the probability of finding the cluster on a lattice of linear size $L$ with filled boundary conditions, it

\footnote{Unless otherwise stated, we always refer to filled boundary conditions for finite volume lattices (since with empty b.c. blocked clusters do not occur).}
is immediate to check that $R(L, 0) = 0$, $R(L, p)$ increases with $p$ and $R(L, 1) = 1$. Does there exists a non trivial critical density, $0 < p_c < 1$, such that in the thermodynamic limit a blocked cluster never occurs ($\lim_{L \to \infty} R(L, p) = 0$) for $p < p_c$, while it always occurs ($\lim_{L \to \infty} R(L, p) = 1$) for $p > p_c$? And, if such transition takes place, is it second-order (as for SP) or first-order? namely, is the density of the frozen cluster, $\Phi(p)$, zero or greater than zero at $p_c$? How do $\Phi(p)$ and the other cluster properties scale around criticality?

Before answering to these questions let us explain more in detail what are blocked clusters and give a receipt which allows to identify them. Focus for simplicity on the case of a two-dimensional lattice with $m = 2$. It is immediate to check that the configuration of Fig 2.1 a) does not contain a blocked cluster. This is also true for any configuration which does not have occupied clusters spanning the lattice: the external particles are not blocked and starting from them one can subsequently unblock the whole cluster. Consider instead the configurations in Fig. 2.1 b) and c) which contain a spanning cluster: do they also contain a blocked cluster? As one can directly check the answer is no for b) and yes for c): the existence of a spanning cluster is a necessary but not sufficient condition for a blocked cluster. How can we decide whether a blocked cluster occurs or not? Should we check all subsets of nearest neighbouring particles and verify whether there existst (at least) one in which all particles are blocked? after some thought you can realize that the following pruning or leaf removal algorithm does the job in a polynomial time. Take the initial configuration and remove all particles that have at least two empty neighbours. Then repeat the procedure on the new configuration and continue this culling of particles until reaching a stable configuration: either one which is completely empty or one that contains only groups of particles each of which have less than two empty neighbors. The first result means that the initial configuration had no blocked cluster, the second one that a blocked cluster was present. Furthermore the blocked cluster coincides with the set of particles which remain in the final configuration. The search for a receipt to identify blocked clusters has thus lead us to an alternative definition for BP (which is indeed the more common one). BP is a cellular automaton with a discrete time deterministic dynamics in which at each time unit the configuration is updated according to a local and translation invariant rule: an empty site remains empty and a filled one gets emptied if and only if it has at least $m$ empty nearest neighbors. The questions on the existence and character of the transition can be rephrased by asking which is the threshold, $p_c$, below which the final config-

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The convention we choose (empty sites being stable) is the one that is usually adopted in physical literature, while in mathematical literature the inverse one is considered. Of course the two models can easily be mapped one into the other. Another warn concerns the meaning of the parameter $m$: here $m$ is the minimal number of vacancies required for not being blocked, in other works it is the minimal number of nearest neighbour occupied site which do block (our $2d - m$)
uration is completely empty and whether the final density at criticality is zero or positive. Note that now we can also ask dynamical questions, for example which is the velocity of convergence to the empty lattice for $p < p_c$.

![Fig. 1. a) and b): configurations without blocked clusters; c) configuration with a blocked cluster](image)

At first sight you could think that the occurrence of a transition at a finite $p_c$ follows by a simple extension of the arguments that work for SP. Let us explain why this is not the case focusing again on the case $d = 2, m = 2$. Since the existence of blocked cluster requires a spanning cluster, as for SP we can conclude that $p_c > 0$. Indeed, if $p$ is too low, the cost for a given path being occupied overcomes the entropic factor given by the number of all possible paths going from the origin to infinity, thus $\Phi(p)$ equals zero. On the other hand, since for BP a spanning cluster is not necessarily a blocked one, Peierls type contour arguments leading to $p_c < 1$ for standard percolations, here cannot be applied. Indeed the contrary can be proved: for $m = 2, d = 2$ $p_c = 1$ [3]. Moreover, on any $d$-dimensional lattice and for any choice of $m$, BP transition never occurs at a non trivial $p_c$: either $p_c = 0$ (for $m > d$) or $p_c = 1$ (for $m \leq d$) [5]. The first result is an easy consequence of the fact that, if $m > d$, it is possible to construct frozen clusters with a finite number of particles. These will always be present in the thermodynamic limit $\forall p > 0$, therefore $p_c = 0$. Before explaining the result $p_c = 1$ for the more interesting cases $m \leq d$, we wish to recall that prior to above rigorous results the phase transition scenario for BP was not at all clear. The technique that had been used to measure $p_c$ relied (as for SP) on finite size scaling. From numerical simulations the value of the concentration at which a system of size $L$ contains with probability $0.5$ a blocked cluster, $p_{0.5}(L)$, was evaluated. Then $p_c$ was extrapolated by using $p_{0.5}(L) - p_c \propto f(L)$ with $f(L) = L^{-1/\nu}$, getting $p_c \simeq 0.965$ for $d = 2, m = 2$ and $p_c \simeq 0.896$ for $d = 3, m = 3$ [2]. These incorrect numerical estimates of $p_c$ were due to the fact that the correct finite size scaling have an unconventional slow decay, $f(L) \propto \ln L$ for $d = 2, m = 2$ [4] and $f(L) \propto \ln \ln L$ for $d = 3, m = 3$ [7].

2.2. *Unstable voids: $p_c = 1$ and finite size effects for $d = 2, m = 2$*

The fact that for $d = 2, m = 2$ the critical density equals one is due to a mechanism pointed out for the first time by Straley and proved by van Enter [3]: large
voids are unstable. What does it mean? Instead of giving the original argument, we will present here a modified one [4, 5] which, as we shall explain in next section, can be readily extended to higher dimensions. Focus on a configuration which has a $2 \times 2$ empty square centered at the origin and, on any subsequent shell, at least one empty site on each side as in Fig. 2.2 a). If this construction is continued up to the boundary there cannot be a blocked cluster. This can be checked starting from the first internal shell (of linear size $2 + 2$) and removing the particles which are surrounded by the dotted circle in Fig. 2.2 a), i.e. those that are adjacent to the vacancy and to the empty $2 \times 2$ nucleus. Then we can subsequently erase all sites in the first shell, thus expanding the empty square of size $2 \times 2$ to one of size $4 \times 4$. The procedure can then be performed analogously in the next shell and so on, until emptying the whole lattice. The probability of the empty square centered in the origin plus (at least) the minimal vacancies on all shells, $E(p, L) = (1 - p)^4 \prod_{j} (1 - p^j)^4$, converges to a positive value when $L \to \infty$ for any $p < 1$. Thus, by the ergodic theorem, in the thermodynamic limit BP reaches an empty configuration with unit probability, namely $p_c = 1$. This roughly corresponds to the fact that, since there are $O(L^2)$ possible positions for the empty nucleus plus vacancies and a finite (though small) probability that this occurs on a given site, in the limit $L \to \infty$ we will always find (at least) one such structure from which we can perform the emptying procedure, i.e. $\lim_{L \to \infty} R(L, p) = 0 \forall p, 1$. On the other hand, if we fix $L$ and raise the density towards one, all lattice will be occupied: $\lim_{p \to 1} R(L, p) = 1$. What does it happen if we let simultaneously $L \to \infty$ and $p \to 1$? There exists a crossover length, $L_c \simeq \exp(e/(1 - p))$ with $c = \pi^2/9$ [4, 6], such that $R(L, p)$ goes to one (to zero) if we take $L \to \infty$ and $p \to 1$ with $L < L_c(p)$ ($L > L_c(p)$). In other words, $L_c$ is the threshold value such that for lattice size $L > L_c$, a critical droplet from which we can empty the whole lattice typically occurs, while for $L < L_c$ blocked clusters typically occur. Therefore $L_c$ can be regarded as the size of the incipient blocked cluster and corresponds to the typical time needed to converge to the empty configuration (see [5] for a nice renormalization argument which connects $L_c$ to the first time at which the origin gets emptied in the BP procedure). We stress that large corrections to the asymptotic value of $L_c$ occur. It has

![Fig. 2. a)The empty nucleus plus the minimal vacancies required for emptying; b) and c) Site $x$ and its NE,SE,NW,SW neighbours. For the Spiral Model $x$ is blocked in b) and unblocked in c)](image-url)
indeed been estimated that a linear size of the order $10^{47}$ [18] should be con-
gerated to see the true asymptotic limit. Therefore, while on the one hand numerical
simulations should be taken with utmost care when extrapolating asymptotic re-
sults (recall the initial incorrect prediction of $p_c < 1$), the same should be done
when one uses the rigorous asymptotic results for applications to real systems.

2.3. Generalisation to $d > 2$, $m \leq d$

The result $p_c = 1$ can immediately be extended to the cases $d > 2$, $m = 2$ by “monotonicity” arguments, i.e. by using the fact that a site has more neighbours
then in the 2-dimensional case and yet the condition for being emtpiable is the same. On the other hand, extensions to the cases $d > 2$, $m \neq 2$ are not immediate
and rely on the following observation [5]. Consider the case $d = 3, m = 3$ and focus on a configuration which contains a $\ell \times \ell \times \ell$ empty cube around the
origin. What is the minimal requirement we should now impose to expand this
cube of one step? After some thought you should realize that it is sufficient to
require that each two-dimensional face of the cube is “emptiable” with respect
to BP dynamics with parameters $d = 2, m = 2$. The probability of the one step
expansion for the cube is therefore $O(E(p, \ell)^4)$, with $E(p, \ell)$ converging to one
and defined in previous section. Since $\prod_{i=1}^{\infty} E(p, i)$ converges to a finite value of
order $1/\exp \exp(c/(1-p))$, we conclude again that $p_c = 1$. The crossover length
now scales as $[7, 8] L_c \simeq \exp \exp(c/(1-p))$, i.e. roughly as exponential of the one
for $d = 2, m = 2$ (the exact value of $c$ has not been established). All other cases
can be treated analogously iterating from models in lower dimension, obtaining
the results $p_c = 1 [5]$ and $L_c \propto \exp^{\exp(m-1)(c/1 - p^{d-m+1})} [7, 8]$, where $\exp^{\exp(s)}$
stands for the exponential iterated $s$ times.

2.4. Other applications and general graphs

The results in above sections imply that, as far as the search of a finite dimen-
sional model displaying a jamming transition is concerned, BP is unsuccessful.
However, BP has been successfully used to model other systems (see references
in [1]) both in physics and in different fields such as biology (infection mod-
els), geology (crack formation models) and more recently in computer science.
For the latter application occupied sites can be regarded as the functional units
of memory arrays [19] or computer networks [20] and the study of BP on the
 correspondingly graphs is relevant to analyze stability against random damage,
namely to find the minimal value of connectivity which is necessary to main-
tain a proper level of inner communication and data mirroring. In this context
the problem is usually referred to as k-core percolation [21], where the k-core
of a graph is the maximal subgraph with degree at least $k$ for each edge. Note
that the graph which are relevant for these applications are not necessarily the
hypercubic d-dimensional lattices discussed so far. Thus it is important to know that for BP on a mean field lattice a transition can occur at a non trivial critical density [10, 20–22]. For example on a Bethe lattice (a random regular graph), where an analytical solution can be obtained thanks to the local tree-like structure, a transition occurs at \( p_c < 1 \) for \( 1 < m \leq k \), where \( k + 1 \) is the connectivity. This (except for the choice \( m = k \)) has mixed first order and critical features: the giant frozen cluster appears discontinuously and the critical length diverges as power law.

3. Jamming Percolation (JP)

As explained in the introduction, Jamming Percolation models [9–13] have been introduced with the aim of finding a jamming transition, namely a first-order critical transition on a finite dimensional lattice. Here we will review the easiest example of a JP model, namely the two-dimensional model which has been introduced in [11, 12] and dubbed Spiral Model (SM). For SM the existence of a jamming transition has been rigorously proved [11, 12] and the exact value of \( p_c \) has been identified: \( p_c \) coincides with the critical threshold of directed site percolation (DP) in two dimensions, \( p_{c \text{DP}} \approx 0.705 \). The discontinuity of the density of the frozen cluster, \( \Phi(p) \), and the faster then power law divergence of the crossover length, \( L_c(p) \), have also been proved [12] modulo the standard conjecture on the existence of two different correlation lengths for DP [23].

In the following we will sketch the arguments which lead to the above results providing the tools one should use to analyze this transition and explaining the underlying mechanism: it is the consequence of two perpendicular directed percolation processes which together can form a compact network of frozen directed paths at criticality. From this discussion it will emerge how one can modify the microscopic rules without loosing the jamming transition. This is relevant since the transition is not second order, thus we do not have the large degree of universality with respect to microscopic details which holds for Site Percolation. This should not come as a surprise: we already know that for BP (which can be regarded as a microscopic modification of SM) a transition does not occur. The extension and universality of the jamming percolation transition of SM remain fundamental questions to be investigated. However, as it has been discussed in [11], it is possible to identify a class of rules which give rise to a jamming transition and belong to the same universality class of SM: as \( p \neq p_c \) the divergence of the incipient frozen cluster follows the same scaling. A model that belongs to this class is for example the Knight model defined in [9]. Note that in general, at variance with SM, it will not be possible to determine analytically the exact value...
of \( p_c \). It is therefore important to analyze finite size effects and give a proper receipt which allows to obtain a reliable estimate of \( p_c \) from numerical simulations (since, as for BP, convergence to the asymptotic results can be extremely slow). For an extended discussion on this we refer to [11], where for example the value of the critical density for the Knight model has been derived, \( p_c^{\text{Knight}} \approx 0.635 \). Note that this differs from the original conjecture \( p_c^{\text{Knight}} = p_c^{\text{DP}} \) [9] which was due to the overlooking of some blocked structures [13].

3.1. The simplest example: the Spiral Model (SM)

When defining the new rules we should keep in mind the lessons we learnt from BP. One the one hand, in order to have \( p_c < 1 \), blocked clusters should occur with higher probability than for BP. This means that we should allow a larger variety of shapes: blocked clusters should “bend more” than BP ones. At the same time we should not let them “bend too much”; if they can close on themselves forming finite blocked clusters we would get \( p_c = 0 \), as for BP with \( m > d \). Is it possible to realize such a compromise?

Consider a square lattice and, for each site \( x \), define among its first and second neighbours the couples of its North-East (NE), South-West (SW), North-West (NW) and South-East(SE) neighbours as in Fig.2.2 b) and c), namely NE= \((x + e_1, x + e_1 + e_2)\), SW= \((x - e_2, x - e_1 - e_2)\), NW= \((x - e_1, x - e_1 + e_2)\) and SE= \((x + e_1, x + e_1 - e_2)\), where \(e_1\) and \(e_2\) are the coordinate unit vectors. The update rule for the Spiral Model is the following: empty site \( s \) remain empty (as for BP), occupied sites get emptied if both its NE and/or both its SW neighbours are empty and both its SE and/or both its NW neighbours are empty too (see Fig. 2.2 b) and c) for examples in which the constraint for \( x \) is satisfied and not satisfied, respectively). These rules can be rephrased by saying that at least one among the four sets \( \text{NE} \cup \text{SE}, \text{SE} \cup \text{SW}, \text{SW} \cup \text{NW} \) and \( \text{NW} \cup \text{NE} \) should be completely empty. Frozen or blocked clusters are, as for BP, the groups of particles that can never be erased under the iteration of the update rule. Again: Which is the critical density \( p_c \) above which a frozen cluster occurs? What is the value of the density of the frozen cluster, \( \Phi(p) \), at \( p_c \)? How does the size of the the incipient blocked cluster scales when \( p \uparrow p_c \)?

3.2. SM: Occurrence of blocked clusters for \( p > p_c^{\text{DP}} \)

As for BP, arguments analogous to those for SP can be applied only to conclude that \( p_c > 0 \). In order to establish \( p_c < 1 \) we will identify a set of configurations which contain a blocked cluster and prove that they cover the configuration space for \( p > p_c^{\text{DP}} \), therefore \( p_c \leq p_c^{\text{DP}} < 1 \).

Let us start by recalling the definition and a few basic results on DP (see e.g. [23]). Take a square lattice and put two arrows going out from each site \( x \)
towards its neighbours in the positive coordinate directions, \(x + e_1\) and \(x + e_2\). On this directed lattice a continuous percolation transition occurs at a non-trivial critical density \(p_{DP}^c \approx 0.705\) (a percolating cluster is now one which spans the lattice following the direction of the arrows). This transition is second order, as for SP, but belongs to a different universality class. In particular, due the anisotropy of the lattice, the typical sizes of the incipient percolating cluster in the parallel \((e_1 + e_2)\) and transverse \((e_1 - e_2)\) directions diverge with different exponents, \(\xi_\parallel \approx (p_{DP}^c - p)^{-\nu_\parallel}\) and \(\xi_\parallel \approx \xi_z\) with \(\nu_\parallel \approx 1.74\) and \(z \approx 1.58\).

Back to the Spiral Model, let us consider the directed lattice that is obtained from the square lattice putting two arrows from each site towards its NE neighbours, as in Fig. 3.2 a). This lattice is equivalent to the one of DP, simply tilted and squeezed. Therefore, for \(p > p_{DP}^c\), there exists a cluster of occupied sites which spans the lattice following the direction of the arrows (cluster inside the continuous line in Fig. 3.2 a)). We denote by NE-SW clusters the occupied sets which follow the arrows of such lattice and NW-SW clusters those that follow instead the arrows drawn starting from each site towards its NW neighbours. Consider now a site in the interior of a spanning NE-SW cluster, e.g. site \(x\) in the Fig 3.2a): by definition there is at least one occupied site in both its NE and SW neighbouring couples, therefore \(x\) is blocked with respect to the update rule of SM. Thus, the presence of the DP cluster implies a blocked cluster and \(p_c \leq p_{DP}^c\) follows. Note that these results would remain true also for a different updating rule with the milder requirement that only at least one among the two couples of NE and SW sites is completely empty (and no requirement on the NW-SE direction). As we will see, the coexistence of the constraint in the NE-SW and NW-SE directions will be crucial to find a discontinuous transition for SM, otherwise we would have a standard DP-like continuous transition.

![Fig. 3. a) The directed lattice obtained drawing arrows from each site towards its NE neighbours. Particles inside the continuous line belong to a spanning cluster on the directed lattice and form a blocked cluster. b) A non spanning NE-SW cluster blocked by T-junctions with NW-SE clusters.](image)

### 3.3. SM: Absence of blocked clusters for \(p < p_{DP}^c\)

Before proving that below \(p_{DP}^c\) blocked clusters do not occur, a few remarks are in order. If instead of SM we were considering the milder rules described at the end of previous section, the result would follow immediately since the presence
of a blocked cluster would imply the existence of a DP one. On the other hand
for SM rules, since blocking can occur along either the NE-SW or the NW-SE
direction (or both), a directed path implies a blocked cluster but the converse is
not true. This is because a NE-SW non spanning cluster can be blocked if both its
ends are blocked by a T-junction with NW-SE paths, as shown in Fig. 3.2 b). By
using such T-junctions it is also possible to construct frozen clusters which do not
contain a percolating cluster neither in the NE-SW direction nor in the NW-SE
one; all NE-SW (NW-SE) clusters are finite and are blocked at both ends by T
junctions by finite NW-SE (NE-SW) ones (see figure 3.2 b)). As we will show
in section 3.4 these T-junctions are crucial to make the behavior of the transition
for SM very different from DP transition, although they share the same critical
density. This also means that the fact that spanning DP clusters do not occur for
$p < p_{c,DP}$ is not sufficient to conclude that also blocked clusters do not occur.

What strategy could we use? Recalling BP results, a possible idea is to search for
proper unstable voids from which we can iteratively empty the whole lattice. Of
course, since we already know that blocked clusters occur when $p \geq p_{c,DP}$, here
something should prevent this unstable voids to expand at high density.

Consider the region $Q_\ell$ inside the continuous line in Fig.3.3, namely a “square”
of linear side $\sqrt{2}\ell$ tilted of 45 degrees with respect to the coordinate axis and with
each of the four vertexes composed by two sites. If $Q_\ell$ is empty and the four sites
external and adjacent to each vertex denoted by $*$ in Fig.3.3 are also empty, then
it is possible to enlarge the empty region $Q_\ell$ to $Q_{\ell+1}$. Indeed, as can be directly
checked, all the sites external to the top right side can be subsequently emptied
starting from the top one and going downwards. For the sites external to the other
three sides of $Q_\ell$ we can proceed analogously, some care is only required in deciding whether to start from top sites and go downwards or bottom ones and go upwards. Therefore we can expand $Q_\ell$ of one step provided all the four * sites are empty or can be emptied after some iterations of the dynamics. Let us focus one of these * sites, e.g. the left one, $x_L$ in Fig. 3.3. As it can be proved by an iterative procedure (see [12]), in order for $x_L$ not to be emptiable there should exist a NE-SW cluster which spans the square $S_\ell$ of size $\ell$ containing the top left part of $Q_\ell$ (region inside the dashed line in Fig.3.3). This is due to the fact that any directed path in the NW-SE direction can be unblocked starting from the empty part of $S_\ell$ below the diagonal in the $e_1 + e_2$ direction. Therefore in order to block $x_L$ we need a NE-SW cluster which is at least of length $\ell$ since it can be supported from NW-SE clusters only outside $S_\ell$ (see Fig.3.3). Therefore, for large $\ell$, the cost for a one step expansion of $Q_\ell$ is proportional to the probability of not finding such a DP path, $1 - 4 \exp(-c\ell/\xi)$. The probability that the emptying procedure can be continued up to infinity is bounded from below by the product of these single step probabilities which goes to a strictly positive value for $p < p_{c, DP}$ since $\xi < \infty$. Note that, as we already knew from the results of section 3.2, this is not true for $p > p_{c, DP}$: the presence of long DP paths prevents the expansion of voids. As for BP, by the ergodic theorem we conclude that in the thermodynamic limit with probability one the final configuration is completely empty, therefore $p_c \leq p_{c, DP}$. This, together with the result of section 3.2, yields $p_c = p_{c, DP}$.

3.4. SM: Discontinuity of the transition

In the two previous sections we have shown that the percolation transition due to the occurrence of a frozen backbone for the Spiral Model occurs at $p_{c, DP}$. We will now explain why it is qualitatively different from DP and any conventional percolation transition: the density of the frozen cluster is discontinuous, $\Phi(p_{c, DP}) > 0$, namely the frozen structures are compact rather than fractal at criticality, and their typical size increases faster than any power law for $p \rightarrow p_{c, DP}$.

In order to prove discontinuity we construct a set of configurations, $B$, for which the origin is occupied and that cannot be unblocked under the iteration of the update rule. The probability of this set, $P_B$, gives a lower bound for $\Phi(p)$, thus it is sufficient to show that $P_B(p_{c, DP}) > 0$ to conclude $\Phi(p_{c, DP}) > 0$. To construct $B$ we make use of the T-junctions which have been introduced in Section 3.3. Consider a configuration in which the origin belongs to a NE-SW path.

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3The event that a directed spanning cluster does not occur when expanding from $\ell$ to $\ell + 1$ and from $\ell + 1$ and $\ell + 2$ are not independent since $S_\ell$ and $S_{\ell+1}$ do intersect (and the same is true on the following steps). However these events are positively correlated, therefore the joint probability is bounded from below by the product of single step probabilities.
of length $\ell_0/2$: this occurs with probability $q_0 > 0$. Now focus on the infinite sequence of pairs of rectangles of increasing size $\ell_i \times \ell_i/12$ with $\ell_1 = \ell_0$, $\ell_i = 2^{i-2}$ and intersecting as in Fig. 5 a). A configuration belongs to $\mathcal{B}$ if each of these rectangles with long side along the NE-SW (NW-SE) diagonal contains a NE-SW (NW-SE) percolating path (dotted lines in Fig. 5 b)). This implies that an infinite backbone of particles containing the origin (cluster inside the continuous line in Fig. 5 c)) survives thanks to the T-junctions among paths in intersecting rectangles. Therefore $\Phi(p) > q_0 \prod_{i=1}^{\infty} P(\ell_i)^2$, where $P(\ell_i)$ is the probability that a rectangle of size $\ell_i \times 1/12\ell_i$ with short side in the transverse direction is spanned by a DP cluster. Recall that there is a parallel and a transverse length for DP with different exponents, i.e. a cluster of parallel length $\ell$ has typically transverse length $\ell^z$. Let us divide the $\ell_i \times 1/12\ell_i$ rectangle into $\ell_i^{1-z}$ slices of size $\ell_i \times 1/12\ell_i$. For each slice the probability of having a DP cluster along the parallel direction at $\rho_{DP}$ is order unity. Thus, the probability of not having a DP cluster in each of the slice is $1 - P(\ell_i) = O[\exp(-c\ell_i^{1+z})]$. From this result and above inequality we get $\Phi(p_{DP}) > 0$. Therefore the infinite cluster of jamming percolation is “compact”, i.e. of dimension $d = 2$ at the transition. Note that to obtain discontinuity two ingredients of the SM rules are crucial: the existence of two transverse blocking directions each with an underlying percolation transition and the anisotropy of these transitions. Indeed, anisotropy is necessary to obtain that the probability that the above rectangles are spanned converges to one exponentially fast as their size is increased. In turn, this is necessary to get a finite probability for the construction which freezes the origin.

Fig. 5. a) The sequence of intersecting rectangles. b) Dotted non straight line stand for NE-SW (NW-SE) clusters spanning the rectangles c) Frozen structure containing the origin

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4 Again, since the rectangles do intersect the events that they are spanned are not independent. However these are positively correlated and the lower bound follows.
3.5. SM: Dynamical correlation length

In the previous sections we have analyzed the behavior of SM in the thermodynamic limit. We now turn to the finite volume behavior and establish that the typical size, $L_c$, below which frozen clusters occur on finite lattices, namely the size of the incipient blocked cluster, diverges faster than any power law when $p > p_c^{DP}$: $\log L_c(p) \simeq k(p - p_c^{DP})^{-\mu}$, where $\mu = \nu(1 - 1/z) \simeq 0.64$. We sketch separately the arguments leading to lower and upper bounds for $L_c$.

Consider the set of NE-SW and NW-SE paths of length $s$ intersecting as in Fig. 3.3b). As can be directly checked, this structure can be emptied only starting from its border since each finite directed path terminates on both ends with T-junctions with a path in the transverse direction. Therefore, if the structure is continued up to the border of the lattice it is completely frozen. Furthermore, a similar frozen backbone exists also if one (or more) of the finite paths is displaced inside an adjacent rectangular region of size $s \times s/6$, as shown in Fig. 3.3b). Therefore the probability that there exists a frozen cluster, $1 - R(L, \rho)$, is bounded from below by the probability that each of the $O(L^2)$ rectangles contains at least one path connecting its short sides. This leads to $R(L, \rho) \leq (L/s)^2 \exp(-cs^{1-z})$ since the probability of not having a DP path in a region $s \times s/6$ is $\simeq \exp(-cs^{1-z})$ as long as $s \leq \xi_l$. Thus $\lim_{L \to \infty, p \to p_c^{DP}} R(L, \rho) = 0$ for $L/\xi_l \exp(c\xi_l^{1-z}) \to 0$, therefore $\log L_c \geq k_l(p - p_c^{DP})^{-\mu}$.

On the other hand, in order to establish an upper bound on $L_c$, we determine the lattice size $L$ above which an unstable voids which can be expanded until emptying the whole lattice typically occur. From the results in Section 3.3 it is easy to see that the probability of expanding an empty nucleus to infinity is dominated by the probability of expanding it up to $\ell = \xi_l$. Indeed, above this size the probability of an event which prevents expansion is exponentially suppressed. Therefore, considering the $O(L/\xi_l)^2$ possible positions for a region that it is guaranteed to be emptyable up to size $\xi_l$, we get that the probability that a lattice of linear size $L$ is emptyable is roughly bounded as $R(L, \rho) \geq L^2 \delta$, where $\delta$ is the probability that a small empty nucleus can be expanded until size $\xi_l$. In the emptying procedure described in Section 3.3 we expanded of one step at a time the region $Q_\ell$. If we require instead that on a region of size $O(\xi_l)$ around each corner there is not a spanning DP cluster, we can expand directly to $\xi_l$ getting $\delta \geq \exp(-c\xi_l^{1-z})$. This, together with above inequality, yields $\log L_c \leq k_u(p - p_c^{DP})^{-\mu}$.
4. Related stochastic models

In the above sections we studied the Spiral Model and showed that this, at variance with BP, is a JP model, namely a cellular automaton with a jamming transition at a non trivial \( p_c \). Starting from this result it is possible [9] to define a stochastic lattice gas with Glauber dynamics (i.e. one in which elementary moves are birth and death of particles) which displays a purely dynamical transition with the same mixed first-order and critical character. This is done by considering the correspondent kinetically constrained model (KCM), namely a stochastic lattice gas in which a site is filled at rate \( p \) and emptied at rate \( 1 - p \) provided the surrounding configuration satisfies the constraint that is required in the SM model in order to empty the same site if it is occupied. Otherwise, if this constraint is not satisfied, both the birth and death rates are zero. As an immediate consequence of SM percolation transition, an ergodicity breaking transition occurs for this stochastic model at \( p_c^{DP} \). Furthermore, the discontinuous character of SM transition implies a discontinuous jump of the order parameter \( q_{EA} = \langle \eta_x(t)\eta_x(0) \rangle_{c} \). Indeed, when raising the density towards \( p_c^{DP} \) the correlation curves clearly display a two step relaxation with this developing (discontinuous) plateau. The length of the plateau, namely the relaxation time \( \tau(p) \) to equilibrium, diverges faster than any power law as \( p / p_c^{DP} \) with a form similarly to the Vogel Fulcher law for supercooled liquids. In particular in [24] it has been proved that \( \tau \) diverges at least as \( L_c \). One can also construct a kinetically constrained model with Kawasaki dynamics (evolution is now a sequence of particle jumps) which displays the same behavior: the rate of a given jump is different from zero if SM constraints is satisfied both in the initial and final position of the particle \(^5\). So, back to example discussed in in the introduction, this models provide a possible choice of the “enough empty space” a particle should find in order to move which gives a jamming transition at a finite critical density. Of course much work remains to be done to investigate the connection of this and similar models with real systems undergoing jamming and glass transition, such as hard spheres or colloidal suspensions. This would require on the one hand analysing the extension and universality of this jamming percolation transition. A partial step in this direction has been undertaken in [11], where a class of rules which belong to the same universality class of SM has been identified. It remains to be understood if one can devise different (possibly more isotropic) rules displaying a jamming transition with a different scaling for the size of the incipient

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3The KCM corresponding to BP constraints with Glauber or Kawasaki dynamics have also been analyzed. They are the so called Friedrickson Andersen [25] and Kob Andersen [26], respectively. In both cases a dynamical transition does not occur. The result for Glauber dynamics follows immediately from the fact that a jamming transition does not occur for BP. For Kawasaki dynamics additional arguments are required [27].
cluster as $p \nearrow p_c$. Also, for the connection with real systems it would be important to analyze the effect of constraint-violating processes and to compare the geometrical and statistical properties of the clusters of JP models with those that can be measured for the real systems.

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References