Characterization of dependence of multidimensional Lévy processes using Lévy copulas

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Abstract

This paper suggests to use Lévy copulas to characterize the dependence among components of multidimensional Lévy processes. This concept parallels the notion of a copula on the level of Lévy measures. As for random vectors, a kind of Sklar’s theorem states that the law of a general multivariate Lévy process is obtained by combining arbitrary univariate Lévy processes with an arbitrary Lévy copula. We construct parametric families of Lévy copulas and prove a limit theorem, which indicates how to obtain the Lévy copula of a multidimensional Lévy process $X$ from the ordinary copulas of the random vectors $X_t$ for fixed $t$.

Key words: Lévy process, copula, limit theorems
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1 Introduction

Copulas allow to separate the dependence structure of a random vector from its univariate margins. Their role is twofold. Firstly, they provide a complete characterization of the possible dependence structures of a random vector with fixed margins. Secondly, they can be used to construct multidimensional distributions with specified dependence from a collection of univariate laws. Despite the presence of a vast body of literature on copulas and, more generally, on the dependence of random vectors (see e.g. Joe (1997)), few efforts have been made to study dependence in the dynamic context of stochastic processes. This paper aims to partially fill this gap by addressing the dependence among components of multivariate Lévy processes.

The first goal of this study is to characterize all $\mathbb{R}^d$-valued Lévy processes $X$ whose components $X^1, \ldots, X^d$ are equal in law to $d$ given univariate Lévy processes $Y^1, \ldots, Y^d$.

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respectively. In principle, the whole distribution of a \(d\)-dimensional Lévy process \(X = (X_t)_{t \in \mathbb{R}_+}\) is determined by the law of \(X_t\) for one fixed \(t\). Therefore, one could describe the dependence among components of \(X\) by the copula \(C_t\) of \(X_t\). However, this copula generally depends on \(t\). Moreover, given \(d\) one-dimensional Lévy processes, it is unclear which copulas \(C_t\) lead to a \(d\)-dimensional Lévy process. It would be more satisfactory to use an alternative characterization which does not involve a specific time and always leads to Lévy processes. Since the law of \(X\) is described in a time-independent fashion in terms of its Lévy-Khintchine triplet \((\alpha, \nu, \gamma)\), it seems natural to use the latter for this purpose.

The location parameter \(\gamma\) has no effect on the dependence between components, so it does not play a role in this context. The dependence structure of the Brownian motion part of the Lévy process is characterized entirely by its covariance matrix \(\alpha\). Since the continuous part and the jump part of \(X\) are independent, it remains to describe the dependence structure of the purely discontinuous part of \(X\).

This is done in Section 3 by introducing the notion of a Lévy copula. In the particular case of two-dimensional processes whose components have atomless Lévy measures and admit only positive jumps, this concept is discussed in Cont & Tankov (2004) but it is introduced here for the first time in the context of general Lévy processes. A version of Sklar’s theorem states that, as for random vectors, the margins and the dependence structure of a Lévy process can be modelled independently (cf. Theorem 3.6). This suggests to construct multidimensional models based on Lévy processes by combining arbitrary one-dimensional margins with a Lévy copula from a parametric family (cf. Section 5).

The second aim of this work is to express special dependence structures of Lévy processes as complete dependence and independence in terms of Lévy copulas. This is done in Section 4 where we also characterize the dependence structure of stable Lévy motions in terms of Lévy copulas.

Our third objective is to construct parametric families of Lévy copulas which may turn out to be useful in applications. In Section 5 we discuss two possible approaches. The first one allows to build families of Lévy copulas in arbitrary dimension where the number of parameters does not depend on the dimension. This is motivated by the observation that typically one does not have enough information about the dependence structure to estimate many parameters or to proceed with a nonparametric approach. The second construction is useful when a more precise vision of dependence is necessary.

In principle the Lévy copula and also the Gaussian copula of the Brownian motion part of a Lévy process \(X\) can be recovered from the ordinary copula of \(X_t\) at small fixed times \(t\). This relation is established in Section 6 by way of limit theorems.

An important field of application of Lévy copulas is mathematical finance. Many problems in this domain require a multivariate model with dependence between components, where jumps in assets are taken into account. While Lévy processes with jumps have been successfully applied by many authors to construct one-dimensional models (cf. e.g. Barndorff-Nielsen (1998), Eberlein (2001), Kou (2002), Madan et al. (1998), Rydberg (1997)), multivariate applications continue to be dominated by Brownian motion (but cf.
2 Preliminaries

In this section we recall a few facts on increasing functions. We set $\mathbb{R} := (-\infty, \infty]$ in this paper and

$$\text{sgn } x := \begin{cases} 
1 & \text{for } x \geq 0 \\
-1 & \text{for } x < 0.
\end{cases}$$

For $a, b \in \mathbb{R}^d$ we write $a \leq b$ if $a_k \leq b_k$, $k = 1, \ldots, d$. In this case, let $(a, b]$ denote a right-closed left-open interval of $\mathbb{R}^d$:

$$(a, b] := (a_1, b_1] \times \cdots \times (a_d, b_d].$$

**Definition 2.1** Let $F : S \to \mathbb{R}$ for some subset $S \subset \mathbb{R}^d$. For $a, b \in S$ with $a \leq b$ and $[a, b] \subset S$, the $F$-volume of $(a, b]$ is defined by

$$V_F((a, b]) := \sum_{u \in \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}} (-1)^{N(u)} F(u),$$

where $N(u) := \# \{k : u_k = a_k\}$.

In particular, $V_F((a, b]) = F(b) - F(a)$ for $d = 1$ and $V_F((a, b]) = F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(b_1, a_2)$ for $d = 2$. If $F(u) = \prod_{i=1}^d u_i$, the $F$-volume of any interval is equal to its Lebesgue measure.

**Definition 2.2** A function $F : S \to \mathbb{R}$ for some subset $S \subset \mathbb{R}^d$ is called $d$-increasing if $V_F((a, b]) \geq 0$ for all $a, b \in S$ with $a \leq b$ and $[a, b] \subset S$.

**Example 2.3** The distribution function $F$ of a random vector $X \in \mathbb{R}^d$ is usually defined by

$$F(x_1, \ldots, x_d) := P[X_1 \leq x_1, \ldots, X_d \leq x_d]$$

for $x_1, \ldots, x_d \in \mathbb{R}$. $F$ is then clearly increasing because

$$V_F((a, b]) = P[X \in (a, b]] \quad (2.1)$$

for every $a, b \in \mathbb{R}^d$ with $a \leq b$.

**Definition 2.4** Let $F : \mathbb{R}^I \to \mathbb{R}$ be a $d$-increasing function such that $F(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \ldots, d\}$. For any non-empty index set $I \subset \{1, \ldots, d\}$, the $I$-margin of $F$ is the function $F^I : \mathbb{R}^I \to \mathbb{R}$, defined by

$$F^I((u_i)_{i \in I}) := \lim_{c \to -\infty} \sum_{(u_j)_{j \in I^c} \in (-c, \infty)^{I^c}} F(u_1, \ldots, u_d) \prod_{j \in I^c} \text{sgn } u_j,$$

where $I^c := \{1, \ldots, d\} \setminus I$. 

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In particular, we have $F^{(1)}(u) = F(u, \infty) - \lim_{c \to -\infty} F(u, c)$ for $d = 2$. To understand the reasoning leading to the above definition of margins, consider the following “alternative” definition of a distribution function of a random vector $X$:

$$
\tilde{F}(x_1, \ldots, x_d) := P[X_1 \in (x_1 \wedge 0, x_1 \vee 0], \ldots, X_d \in (x_d \wedge 0, x_d \vee 0)] \prod_{i=1}^{d} \text{sgn } x_i \quad (2.2)
$$

for $x_1, \ldots x_d \in \mathbb{R}$. This function satisfies Equation (2.1) and can therefore play the role of a distribution function. It is then easy to show that the margins of $\tilde{F}$ (e.g. the distribution functions computed using Equation (2.2) for the components of $X$) are given by

$$
\tilde{F}^{(i)}((x_i)_{i \in I}) = \lim_{c \to -\infty} \sum_{(x_j)_{j \in I^c} \in \{-c, \infty\}^{I^c}} \tilde{F}(x_1, \ldots, x_d) \prod_{i \in I^c} \text{sgn } x_i
$$

$$
= P[X_i \in (x_i \wedge 0, x_i \vee 0], i \in I) \prod_{i \in I} \text{sgn } x_i.
$$

3 Definition of Lévy copulas

As it is explained in the introduction, the dependence structure of a multidimensional Lévy process can be reduced to the Lévy measure and the covariance matrix of the Gaussian part. Since the Lévy measure is a measure on $\mathbb{R}^d$, it is possible to define a suitable notion of a copula. However, one has to take care of the fact that the Lévy measure is possibly infinite with a singularity at the origin.

**Definition 3.1** A function $F : \mathbb{R}^d \to \mathbb{R}$ is called Lévy copula if

1. $F(u_1, \ldots, u_d) \neq \infty$ for $(u_1, \ldots, u_d) \neq (\infty, \ldots, \infty),$
2. $F(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \ldots, d\},$
3. $F$ is $d$-increasing,
4. $F^{(i)}(u) = u$ for any $i \in \{1, \ldots, d\}, u \in \mathbb{R}.$

The next lemma establishes that, similarly to ordinary copulas (cf. Nelsen (1999), Th. 2.10.7), Lévy copulas are Lipschitz continuous.

**Lemma 3.2** Let $F$ be a Lévy copula and $u, v \in \mathbb{R}^d$. Then

$$
|F(v_1, \ldots, v_d) - F(u_1, \ldots, u_d)| \leq \sum_{i=1}^{d} |v_i - u_i|.
$$

**Proof.** It is easy to see that it suffices to consider the case $u_i v_i \geq 0$ for $i = 1, \ldots, d$. To simplify notation, we suppose that $0 \leq u_i \leq v_i$ for every $i$. The general case can be treated
similarly.

\[
|F(v_1, \ldots, v_d) - F(u_1, \ldots, u_d)|
\]

\[
= |V_F((0, v_1] \times \cdots \times (0, v_d)) - V_F((0, u_1] \times \cdots \times (0, u_d))|
\]

\[
\leq \sum_{i=1}^d \lim_{c \to \infty} V_F((-c, \infty]^{d-1} \times (u_i, v_i] \times (-c, \infty]^{d-i})
\]

\[
= \sum_{i=1}^d \left( F^{(i)}(v_i) - F^{(i)}(u_i) \right)
\]

\[
= \sum_{i=1}^d |v_i - u_i|
\]
as claimed. \hfill \Box

In the sequel, we will need a special interval associated with any \(x \in \mathbb{R}\):

\[
I(x) := \begin{cases} (x, \infty), & x \geq 0, \\ (-\infty, x), & x < 0. \end{cases}
\]

In the same way as the distribution of a random vector can be represented by its distribution function, the Lévy measure of a Lévy process will be represented by its tail integral.

**Definition 3.3** Let \(X\) be a \(\mathbb{R}^d\)-valued Lévy process with Lévy measure \(\nu\). The **tail integral** of \(X\) is the function \(U : (\mathbb{R} \setminus \{0\})^d \to \mathbb{R}\) defined by

\[
U(x_1, \ldots, x_d) := \prod_{i=1}^d \text{sgn}(x_i)\nu \left( \prod_{j=1}^d I(x_j) \right).
\]

Since the tail integral is only defined on \((\mathbb{R} \setminus \{0\})^d\), it does not determine the Lévy measure uniquely (unless we know that the latter does not charge the coordinate axes). However, we will see that the Lévy measure is completely determined by its tail integral and all its marginal tail integrals.

**Definition 3.4** Let \(X\) be a \(\mathbb{R}^d\)-valued Lévy process and let \(I \subset \{1, \ldots, d\}\) non-empty. The **I-marginal tail integral** \(U^I\) of \(X\) is the tail integral of the process \(X^I := (X^i)_{i \in I}\). To simplify notation, we denote one-dimensional margins by \(U_i := U^{(i)}\).

**Lemma 3.5** Let \(X\) be a \(\mathbb{R}^d\)-valued Lévy process. Its marginal tail integrals \(\{U^I : I \subset \{1, \ldots, d\}\}\) non-empty are uniquely determined by its Lévy measure \(\nu\). Conversely, its Lévy measure is uniquely determined by the set of its marginal tail integrals.

**Proof.** \(\Rightarrow\): By Proposition 11.10 in Sato (1999), the Lévy measure of \(X^I\) is given by

\[
\nu^I(A) = \nu(\{x \in \mathbb{R}^d : (x_i)_{i \in I} \in A \setminus \{0\}\}), \quad A \in \mathcal{B}(\mathbb{R}^{|I|})
\]
for any non-empty $I \subset \{1, \ldots, d\}$. Moreover, the marginal tail integrals are uniquely determined by the marginal Lévy measures.

$\Leftarrow$: It is sufficient to prove that $\nu((a, b])$ is completely determined by the tail integrals for any $a, b \in \mathbb{R}^d$ with $a \leq b$ and $0 \notin (a, b]$. We prove by induction on $k = 0, \ldots, d$ that $\nu^I(\prod_{i \in I}(a_i, b_i])$ is determined by the tail integrals for any $a, b \in \mathbb{R}^d$ such that $a \leq b$ and $a_i b_i \leq 0$ for at most $k$ indices and any non-empty $I \subset \{1, \ldots, d\}$ with $0 \notin \prod_{i \in I}(a_i, b_i]$. If $k = 0$, Definitions 3.3 and 3.4 entail that

$$
\nu^I \left( \prod_{i \in I}(a_i, b_i] \right) = (-1)^{|I|} \nu_{U_I} \left( \prod_{i \in I}(a_i, b_i] \right).
$$

Let $a, b \in \mathbb{R}^d$ such that $a_i b_i \leq 0$ for at most $k$ indices. For ease of notation we suppose that $a_ib_i \leq 0$ for $i = 1, \ldots, k$. Let $I \subset \{1, \ldots, d\}$ non-empty with $0 \notin \prod_{i \in I}(a_i, b_i]$. By induction hypothesis, $\nu^I(\prod_{i \in I}(a_i, b_i])$ is uniquely determined if $k \notin I$. Suppose that $k \in I$. If $a_k = 0$, then

$$
\nu^I \left( \prod_{i \in I}(a_i, b_i] \right) = \lim_{a_{i=1}^d = 0} \nu^I \left( \prod_{i \in I, i<k}(a_i, b_i] \times (a, b_k] \times \prod_{i \in I, i>k}(a_i, b_i] \right)
$$

and the right-hand side is uniquely determined by the induction hypothesis. If $a_k \neq 0$, then

$$
\nu^I \left( \prod_{i \in I}(a_i, b_i] \right) = \nu_I \setminus \{k\} \left( \prod_{i \in I \setminus \{k\}}(a_i, b_i] \right)

- \lim_{\beta \searrow \infty} \nu^I \left( \prod_{i \in I, i<k}(a_i, b_i] \times (\beta, c] \times \prod_{i \in I, i>k}(a_i, b_i] \right)

- \lim_{\epsilon \searrow \infty} \nu^I \left( \prod_{i \in I, i<k}(a_i, b_i] \times (c, a_k] \times \prod_{i \in I, i>k}(a_i, b_i] \right),
$$

which is uniquely determined as well.

The following theorem is our first main result. It explains the relation between Lévy copulas and Lévy processes. It may be called Sklar’s theorem for Lévy copulas.

**Theorem 3.6**

1. Let $X = (X^1, \ldots, X^d)$ be a $\mathbb{R}^d$-valued Lévy process. Then there exists a Lévy copula $F$ such that the tail integrals of $X$ satisfy:

$$
U^I((x_i)_{i \in I}) = F^I((U_i(x_i))_{i \in I})
$$

for any non-empty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$. The Lévy copula $F$ is unique on $\prod_{i=1}^d \text{Ran} U_i$.

2. Let $F$ be a $d$-dimensional Lévy copula and $U_i, i = 1, \ldots, d$ tail integrals of real-valued Lévy processes. Then there exists a $\mathbb{R}^d$-valued Lévy process $X$ whose components
have tail integrals $U_1, \ldots, U_d$ and whose marginal tail integrals satisfy Equation (3.1) for any non-empty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$. The Lévy measure $\nu$ of $X$ is uniquely determined by $F$ and $U_i, i=1, \ldots, d$.

**Proof.** I. Denote the Lévy measure of $X$ and $X^1, \ldots, X^d$ by $\nu$ and $\nu_1, \ldots, \nu_d$, respectively. For the purposes of this proof we set for $x \in \mathbb{R}$, $i=1, \ldots, d$,

$$
\hat{U}_i(x) := \begin{cases} U_i(x) & \text{for } x \neq 0 \text{ and } x \neq \infty, \\ 0 & \text{for } x = \infty, \\ \infty & \text{for } x = 0 
\end{cases}
$$

and

$$
\Delta U_i(x) := \begin{cases} \lim_{x \uparrow \infty} U_i(x) - U_i(x) & \text{for } x \neq 0 \text{ and } x \neq \infty, \\ 0 & \text{for } x = \infty \text{ or } x = 0. 
\end{cases}
$$

Let $m$ be the measure on $(\mathbb{R}^d \setminus \{0\}) \times [0, 1]^d \times \mathbb{R}$ defined by

$$
m := \nu^* \otimes \lambda_{(0,1)^d} \otimes \epsilon_0 + \sum_{i=1}^d \epsilon_{(0,0,0,\ldots,0)} \otimes \epsilon_{(0,0,0,\ldots,0)} \otimes \lambda_{[\nu((0,\infty)),\infty) \cup (-\infty,-\nu((\infty,0)))},
$$

where $\nu^*$ is the extension of $\nu$ to $\mathbb{R}^d \setminus \{0\}$, i.e. $\nu^*(A) := \nu(A \cap \mathbb{R}^d)$. Let

$$
g_i : \mathbb{R} \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}, \quad (x, y, z) \mapsto \hat{U}_i(x) + y \Delta U_i(x) + z
$$

and define a measure $\tilde{m}$ on $\mathbb{R}^d \setminus \{\infty, \ldots, \infty\}$ via

$$
\tilde{m}(B) := m(\tilde{g}^{-1}(B))
$$

with

$$
\tilde{g}(x_1, \ldots, x_d, y_1, \ldots, y_d, z) := (g_1(x_1, y_1, z), \ldots, g_d(x_d, y_d, z)).
$$

Finally, let $F$ be given by

$$
F(u_1, \ldots, u_d) := \tilde{m} \left( \prod_{i=1}^d (u_i \wedge 0, u_i \vee 0) \right) \prod_{i=1}^d \text{sgn } u_i
$$

for $(u_1, \ldots, u_d) \in \mathbb{R}^d$. Properties 1 and 2 in Definition 3.1 are obvious. From the fact that $\tilde{m}$ is a positive measure it follows immediately that $F$ is $d$-increasing. Let $I \subset \{1, \ldots, d\}$, $(u_i)_{i \in I} \in \mathbb{R}^I$. For ease of notation, we consider only the case of non-negative $u_i$. The
general case follows analogously. By definition of $F$ we have

$$F^I((u_i)_{i \in I}) = \lim_{c \to -\infty} \sum_{(u_j)_{j \in I^c} \in (-c, \infty)^I} F(u_1, \ldots, u_d) \prod_{j \in I^c} \text{sgn} u_j$$

$$= \tilde{m} \left( \prod_{i \in I} (0, u_i] \times \mathbb{R}^d \right)$$

$$= m \left( \left\{ (x_1, \ldots, x_d, y_1, \ldots, y_d, z) \in (\mathbb{R}^d \setminus \{0\}) \times [0,1]^d \times \mathbb{R} : \right. \right.$$ 

$$\left. \hat{U}_i(x_i) + y_i \Delta U_i(x_i) + z \in (0, u_i] \text{ for } i \in I \right\} \right)$$

If $I = \{i\}$, then the definition of $m$ implies that this equals

$$(\nu_i \otimes \lambda|_{\{0,1\}}) \left( \{(x,y) \in \mathbb{R} \times [0,1] : \hat{U}_i(x) + y \Delta U_i(x) \in (0, u_i]\} \right)$$

$$\cup \left( u_i - \nu_i((0,\infty)) \right) 1_{\{u_i > \nu_i((0,\infty))\}}.$$

Introducing $x^* := \inf \{x \geq 0 : \hat{U}_i(x) + \Delta U_i(x) \leq u_i\}$, this can be expressed as

$$\nu_i((x^*, \infty)) + (u_i - \hat{U}_i(x^*)) 1_{\{x^* \neq 0\}} + (u_i - \nu_i((0,\infty))) 1_{\{x^* = 0\}} = u_i,$$

i.e. property 4 in Definition 3.1 is met.

Now, let $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$. Again, we consider only the case where all the $x_i$ are nonnegative. Then

$$F^I((U_i(x_i))_{i \in I}) = m \left( \left\{ (\bar{x}_1, \ldots, \bar{x}_d, y_1, \ldots, y_d, z) \in \mathbb{R}^I \times [0,1]^d \times \mathbb{R} : \right. \right.$$ 

$$\left. \hat{U}_i(\bar{x}_i) + y_i \Delta U_i(\bar{x}_i) + z \in (0, U_i(x_i)] \text{ for } i \in I \right\} \right)$$

$$= \nu \left( \prod_{i \in I} (x_i, \infty) \times \mathbb{R}^d \right)$$

$$= \nu^I \left( \prod_{i \in I} (x_i, \infty) \right)$$

$$= U^I((x_i)_{i \in I})$$

as claimed. The uniqueness statement follows from (3.1) and Lemma 3.2.

2. Since $F$ is $d$-increasing and continuous (by Lemma 3.2), there exists a unique measure $\mu$ on $\mathbb{R}^d \setminus \{\infty, \ldots, \infty\}$ such that $V_F((a,b]) = \mu((a,b])$ for any $a,b \in \mathbb{R}^d \setminus \{\infty, \ldots, \infty\}$ with $a \leq b$. (see Kingman & Taylor (1966), Section 4.5). For a one-dimensional tail integral $U(x)$, we define

$$U^{-1}(u) = \begin{cases} \inf \{x > 0 : u \geq U(x)\}, & u \geq 0 \\ \inf \{x < 0 : u \geq U(x)\} \land 0, & u < 0. \end{cases}$$
Let \( \nu' := f(\mu) \) be the image of \( \mu \) under

\[
f : (u_1, \ldots, u_d) \mapsto (U_1^{-1}(u_1), \ldots, U_d^{-1}(u_d))
\]

and let \( \nu \) be the restriction of \( \nu' \) to \( \mathbb{R}^d \setminus \{0\} \). We need to prove that \( \nu \) is a Lévy measure and that its marginal tail integrals \( U^I_\nu \) satisfy

\[
U^I_\nu((x_i)_{i \in I}) = F^I((U_i(x_i))_{i \in I})
\]

for any non-empty \( I \subset \{1, \ldots, d\} \) and any \( (x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I \). Suppose for ease of notation that \( x_i > 0, i \in I \). Then

\[
U^I_\nu((x_i)_{i \in I}) = \nu(\{\xi \in \mathbb{R}^d : \xi_i \in (x_i, \infty), i \in I\}) = \mu(\{u \in \mathbb{R}^d : U_i^{-1}(u_i) \in (x_i, \infty), i \in I\}) = \mu(\{u \in \mathbb{R}^d : 0 < u_i < U(x_i), i \in I\}).
\]

By Lemma 3.2 we have \( \mu(\{u \in \mathbb{R}^d : u_i = U(x_i)\}) = 0 \) for \( i \in I \). Therefore,

\[
U^I_\nu((x_i)_{i \in I}) = \mu(\{u \in \mathbb{R}^d : 0 < u_i \leq U(x_i), i \in I\}) = F^I((U_i(x_i))_{i \in I}).
\]

This proves in particular that the one-dimensional marginal tail integrals of \( \nu \) equal \( U_1, \ldots, U_d \).

Since the marginals \( \nu_i \) of \( \nu \) are Lévy measures on \( \mathbb{R} \), we have \( \int (x_i^2 \wedge 1) \nu_i(dx_i) < \infty \) for \( i = 1, \ldots, d \). This implies

\[
\int (|x|^2 \wedge 1) \nu(dx) \leq \sum_{i=1}^d \int (x_i^2 \wedge 1) \nu_i(dx)
\]

\[
= \sum_{i=1}^d \int (x_i^2 \wedge 1) \nu_i(dx_i) < \infty
\]

and hence \( \nu \) is a Lévy measure on \( \mathbb{R}^d \). The uniqueness of \( \nu \) follows from the fact that it is uniquely determined by its marginal tail integrals (cf. Lemma 3.5).

\[
\square
\]

**Definition 3.7** We call any Lévy copula as in Statement 1 of Theorem 3.6 a Lévy copula of the Lévy process \( X \).

Lévy copulas are not limited to Lévy processes. A large class of Markov processes or even semimartingales behaves locally as a Lévy process in the sense that its dynamics can be described by a drift rate, a covariance matrix, and a Lévy measure, which may all change randomly through time (cf. e.g. Jacod & Shiryaev (2003), II.2.9, II.4.19). Therefore, the notion of Lévy copula could naturally be extended to these more general classes of processes.
4 Examples of Lévy copulas

We start by considering extreme cases of dependence, namely complete dependence and independence. To this end, we need the following simple criterion for independence of Lévy processes.

**Lemma 4.1** The components $X^1, \ldots, X^d$ of a $\mathbb{R}^d$-valued Lévy process $X$ are independent if and only if their continuous martingale parts are independent and the Lévy measure $\nu$ is supported by the coordinate axes. The measure $\nu$ is then given by

$$\nu(B) = \sum_{i=1}^{d} \nu_i(B_i) \quad \forall B \in \mathcal{B}(\mathbb{R}^d),$$

where for every $i$, $\nu_i$ denotes the Lévy measure of $X^i$ and

$$B_i = \{ x \in \mathbb{R} : (0, \ldots, 0, x, 0, \ldots, 0) \in B \}.$$

**Proof.** Since the continuous martingale part and the jump part of $X$ are independent, we can assume without loss of generality that $X$ has no continuous martingale part, i.e. its characteristic triplet is given by $(0, \nu, \gamma)$.

$\Longleftarrow$: Suppose $\nu$ is supported by the coordinate axes. Then we have $\nu(B) = \sum_{i=1}^{d} \nu_i(B_i)$, $B \in \mathcal{B}(\mathbb{R}^d)$ for some measures $\nu_i$. Proposition 11.10 in Sato (1999) implies that these measures coincide with the margins of $\nu$, i.e. $\nu_i = \nu$, $i = 1, \ldots, d$. Using the Lévy-Khintchine formula for the process $X$, we obtain

$$E[e^{i(u, X_t^i)}] = \exp \left( t \left( i\langle \gamma, u \rangle + \int_{\mathbb{R}^d \setminus \{0\}} (e^{i(u, x)} - 1 - i\langle u, x \rangle 1_{\{|x| \leq 1\}}) \nu(dx) \right) \right).$$

$$= \exp \left( t \sum_{k=1}^{d} \left( i\gamma_k u_k + \int_{\mathbb{R}^d \setminus \{0\}} (e^{i(u_k x_k)} - 1 - iu_k x_k 1_{\{|x_k| \leq 1\}}) \nu_k(dx_k) \right) \right).$$

$$= \prod_{k=1}^{d} E[e^{i(u_k X_t^k)}],$$

which shows that the components of $X$ are independent Lévy processes.

$\Longrightarrow$: Define a measure $\tilde{\nu}$ on $\mathbb{R}^d$ by $\tilde{\nu}(B) := \sum_{i=1}^{d} \nu_i(B_i)$, where $\nu_i$ is the Lévy measure of $X^i$ and $B_i$ is as above. It is straightforward to verify that $\tilde{\nu}$ is a Lévy measure. Since the components of $X$ are independent, the Lévy-Khintchine formula applied separately to each component of $X$ yields

$$E[e^{i(u, X_t^i)}] = \exp \left( t \left( i\langle \gamma, u \rangle + \int_{\mathbb{R}^d \setminus \{0\}} (e^{i(u, x)} - 1 - i\langle u, x \rangle 1_{\{|x| \leq 1\}}) \tilde{\nu}(dx) \right) \right).$$

From the uniqueness of the Lévy-Khintchine representation we conclude that $\tilde{\nu}$ is the Lévy measure of $X$. \qed
Lemma 4.2 The components \( X^1, \ldots, X^d \) of a \( \mathbb{R}^d \)-valued Lévy process \( X \) are independent if and only if their Brownian motion parts are independent and if the tail integrals of the Lévy measure satisfy \( U^I((x_i)_{i \in I}) = 0 \) for all \( I \subset \{1, \ldots, d\} \) with \( \text{card} \, I \geq 2 \) and all \( (x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I \).

**Proof.** If \( I \subset \{1, \ldots, d\} \) with \( \text{card} \, I \geq 2 \) and \( (x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I \). Then the components of \((X^i)_{i \in I}\) are independent as well. Applying Lemma 4.1 to this process, we conclude, using Equation (4.1), that \( U^I((x_i)_{i \in I}) = 0 \).

Let \( \nu \) be defined by Equation (4.1), where \( \nu_i \) is the Lévy measure of \( X^i \) for \( i = 1, \ldots, d \). Then all marginal tail integrals of \( \nu \) coincide with those of the Lévy measure of \( X \). Therefore, \( \nu \) is the Lévy measure of \( X \) (cf. Lemma 3.5), which entails by Lemma 4.1 that \( X^1, \ldots, X^d \) are independent. \( \square \)

We are now ready to characterize the Lévy copula corresponding to independent components.

Theorem 4.3 The components \( X^1, \ldots, X^d \) of a \( \mathbb{R}^d \)-valued Lévy process \( X \) are independent if and only if their Brownian motion parts are independent and if it has a Lévy copula of the form

\[
F_\pm(x_1, \ldots, x_d) := \sum_{i=1}^d x_i \prod_{j \neq i} 1_{(\infty)}(x_j) \tag{4.2}
\]

**Proof.** It is straightforward to see that Equation (4.2) defines a Lévy copula.

\( \Leftarrow: \) Let \( I \subset \{1, \ldots, d\} \) with \( \text{card} \, I \geq 2 \). Definition 4.2 entails that \( F^I((u_i)_{i \in I}) = 0 \) for all \( (u_i)_{i \in I} \in \mathbb{R}^I \). Therefore, \( U^I((x_i)_{i \in I}) = 0 \) for all \( (x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I \) by (3.1). From Lemma 4.2 we conclude that \( X^1, \ldots, X^d \) are independent.

\( \Rightarrow: \) We have \( U^I((x_i)_{i \in I}) = F^I((U_i(x_i))_{i \in I}) = 0 \) for all \( I \subset \{1, \ldots, d\} \) with \( \text{card} \, I \geq 2 \) and all \( (x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I \) (cf. Lemma 4.2). This implies that \( F \) is a Lévy copula for \( X \). \( \square \)

For the characterization of complete dependence we need a number of definitions. First of all recall that a subset \( S \) of \( \mathbb{R}^d \) is called ordered if, for any two vectors \( u, v \in S \), either \( u_k \leq v_k, k = 1, \ldots, d \) or \( u_k \geq v_k, k = 1, \ldots, d \). Similarly, \( S \) is called strictly ordered if, for any two different vectors \( u, v \in S \), either \( u_k < v_k, k = 1, \ldots, d \) or \( u_k > v_k, k = 1, \ldots, d \). In the following definition and below we set

\[
K := \{ x \in \mathbb{R}^d : \text{sgn} \, x_1 = \ldots = \text{sgn} \, x_d \}.
\]

**Definition 4.4** Let \( X \) be a \( \mathbb{R}^d \)-valued Lévy process. Its jumps are said to be completely dependent or comonotonic if there exists a strictly ordered subset \( S \subset K \) such that \( \Delta X_t := X_t - X_{t-} \in S, t \in \mathbb{R}_+ \) (except for some null set of paths).

Clearly, an element of a strictly ordered set is completely determined by one coordinate only. Therefore, if the jumps of a Lévy process are completely dependent, the jumps of all components can be determined from the jumps of any single component. If the Lévy process
in question has no continuous martingale part, then the trajectories of all components can be determined from the trajectory of any component, which indicates that Definition 4.4 is a reasonable dynamic notion of complete dependence for Lévy processes. The condition $\Delta X_t \in K$ means that if the components of a Lévy process are comonotonic, they always jump in the same direction.

For any $\mathbb{R}^d$-valued Lévy process $X$ with Lévy measure $\nu$ and for any $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ the number of jumps in the time interval $[0, t]$ with sizes in $B$ is a Poisson random variable with parameter $t\nu(B)$. Therefore, Definition 4.4 can be restated equivalently as follows:

**Definition 4.5** Let $X$ be a $\mathbb{R}^d$-valued Lévy process with Lévy measure $\nu$. Its jumps are said to be **completely dependent** or comonotonic if there exists a strictly ordered subset $S$ of $K$ such that $\nu(\mathbb{R}^d \setminus S) = 0$.

The following theorem characterizes complete jump dependence in terms of Lévy copulas.

**Theorem 4.6** Let $X$ be a $\mathbb{R}^d$-valued Lévy process whose Lévy measure is supported by an ordered set $S \subset K$. Then the complete dependence Lévy copula given by

$$F\|_K(x_1, \ldots, x_d) := \min(|x_1|, \ldots, |x_d|) \mathbb{1}_K(x_1, \ldots, x_d) \prod_{i=1}^d \text{sgn} x_i$$

(4.3)

is a Lévy copula of $X$.

Conversely, if $F\|_K$ is a Lévy copula of $X$, then the Lévy measure of $X$ is supported by an ordered subset of $K$. If, in addition, the tail integrals $U_i$ of $X^i$ are continuous and satisfy $\lim_{x \to 0} U_i(x) = \infty$, $i = 1, \ldots, d$, then the jumps of $X$ are completely dependent.

The proof is based on the following representation of an ordered set as a union of a strictly ordered set and countable many segments that are parallel to some coordinate axis.

**Lemma 4.7** Let $S \subset \mathbb{R}^d$ be ordered. It can be written as

$$S = S^* \cup \bigcup_{n=1}^{\infty} S_n,$$

(4.4)

where $S^* \subset \mathbb{R}^d$ is strictly ordered and for every $n$, $S_n \subset \mathbb{R}^d$ and there exist $k(n)$ and $\xi(n)$ such that $x_{k(n)} = \xi(n)$ for all $x \in S_n$.

**Proof.** For the purposes of this proof we define the length of an ordered set $S'$ by $|S'| := \sup_{a, b \in S'} \sum_{i=1}^d (a_i - b_i)$. Let

$$S(\xi, k) = \{x \in \mathbb{R}^d : x_k = \xi \} \cap S.$$

(4.5)

Firstly, we want to prove that there is at most a countable number of such segments with non-zero length. Fix $\varepsilon > 0$ and $k \in \{1, \ldots, d\}$. Consider $N$ different segments $S_i = S(\xi_i, k)$, $i = 1, \ldots, N$ with length $\geq \varepsilon$. Since the $S_i$ are different, the $\xi_i$ must be different from each
other as well and we can suppose without loss of generality that \( \xi_i < \xi_{i+1} \) for all \( i \). Then \( \pi_i \leq \pi_{i+1} \) for all \( i \), where \( \pi_i \) and \( \pi_{i+1} \) are the upper and the lower bounds of \( S_i \). Since all \( S_i \) are subsets of \( S \), which is an ordered set, this implies that \( \bigcup_{i=1}^N S_i \geq N\epsilon \). Therefore, for all \( A > 0 \) and for all \( \epsilon > 0 \), the set \([-A, A]^d \) contains a finite number of segments of type (4.5) with length greater or equal to \( \epsilon \). This means that there is at most a countable number of segments of non-zero length, which we denote by \( S_n, n \in \mathbb{N} \).

Now let \( S^* = S \setminus \bigcup_{n=1}^\infty S_n \). \( S^* \) is ordered because it is a subset of \( S \). Let \( x, y \in S^* \). If \( x_k = y_k \) for some \( k \), then either \( x \) and \( y \) are the same or they are in some segment of type (4.5) hence not in \( S^* \). Therefore, either \( x_k < y_k \) for every \( k \) or \( x_k > y_k \) for every \( k \), which entails that \( S^* \) is strictly ordered and we have obtained the desired representation for \( S \). \( \square \)

**Proof of Theorem 4.6.** We start by proving that \( F \parallel \) is indeed a Lévy copula in the sense of Definition 3.1. Properties 1 and 2 are obvious. To show property 3, introduce a positive measure \( \mu \) on \( \mathbb{R}^d \) by

\[
\mu(B) = \lambda(\{x \in \mathbb{R} : (x, \ldots, x) \in B\}), \quad B \in \mathcal{B}(\mathbb{R}^d),
\]

where \( \lambda \) denotes Lebesgue measure on \( \mathbb{R} \). Then \( V_{F_{\lambda}}((a, b]) = \mu((a, b]) \) for any \( a \leq b \), and therefore \( F \parallel \) is \( d \)-increasing. The margins of \( F \) have the same form as \( F \), namely

\[
F_{\parallel}(x) = \min_{i \in I} |x_i| 1_{((-1, \ldots, -1), (1, \ldots, 1))}(\sgn x_i) \prod_{i \in I} \sgn x_i. \tag{4.6}
\]

Therefore, the one-dimensional margins satisfy \( F^{(i)}(u) = u \).

\( \Rightarrow \): Let \( x \in (0, \infty)^d \). Clearly, \( U(x) \leq U_k(x_k) \) for any \( k \). On the other hand, since \( S \) is an ordered set, we have

\[
\{y \in \mathbb{R}^d : x_k \leq y_k\} \cap S = \{y \in \mathbb{R}^d : x \geq y\} \cap S
\]

for some \( k \). Indeed, suppose that this is not so. Then there exist points \( z^1, \ldots, z^d \in S \) and indices \( j_1, \ldots, j_d \) such that \( z^k_k \geq x_k \) and \( z^k_{j_k} < x_{j_k} \) for \( k = 1, \ldots, d \). Choose the greatest element among \( z^1, \ldots, z^d \) (this is possible because they all belong to an ordered set) and call it \( z^k \). Then \( z^k_{j_k} < x_{j_k} \). However, by construction of \( z^1, \ldots, z^d \) we also have \( z^k_{j_k} \geq x_{j_k} \), which is a contradiction to the fact that \( z^k \) is the greatest element. Therefore,

\[
U(x) = \min(U_1(x_1), \ldots, U_d(x_d)).
\]

Similarly, it can be shown that for every \( x \in (-\infty, 0)^d \),

\[
U(x) = (-1)^d \min(|U_1(x_1)|, \ldots, |U_d(x_d)|).
\]

Since \( U(x) = 0 \) for any \( x \notin K \), we have shown that

\[
U(x) = F_{\parallel}(U_1(x_1), \ldots, U_d(x_d))
\]
for any $x \in (\mathbb{R} \setminus \{0\})^d$. Since the marginal Lévy measures of $X$ are also supported by non-decreasing sets and the margins of $F_I$ have the same form as $F_{II}$, we have

$$U_I((x_i)_{i \in I}) = F_{I}((U_i(x_i))_{i \in I})$$

(4.7)

for any $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$.

$\iff$: Let $S := \text{supp} \nu$. Let us first show that $S \subset K$. Suppose that this is not so. Then there exists $x \in S$ such that for some $m$ and $n$, $x_m < 0$ and $x_n > 0$ and for every neighbourhood $N$ of $x$, $\nu(N) > 0$. This implies that $U^{(m,n)}_{\{x_m/2, x_n/2\}} > 0$, which contradicts Equation (4.7).

Suppose now that $S$ is not an ordered set. Then there exist two points $u, v \in S$ such that $u_m > v_m$ and $u_n < v_n$ for some $m$ and $n$. Moreover, we can have either $u_i \geq 0$ and $v_i \geq 0$ for all $i$ or $u_i \leq 0$ and $v_i \leq 0$ for all $i$. Suppose that $u_i \geq 0$ and $v_i \geq 0$, the other case being analogous. Let $x = \frac{u + v}{2}$. Since $u, v \in S$, we have $\nu(\{z \in \mathbb{R}^d : z_m < x_m, z_n \geq x_n\}) > 0$ and $\nu(\{z \in \mathbb{R}^d : z_m \geq x_m, z_n < x_n\}) > 0$. However

$$\nu(\{z \in \mathbb{R}^d : z_m < x_m, z_n \geq x_n\}) = U_n(x_n) - U^{(m,n)}_{\{x_m, x_n\}}(x_n) = U_n(x_n) - \min(U_m(x_m), U_n(x_n))$$

and

$$\nu(\{z \in \mathbb{R}^d : z_m \geq x_m, z_n < x_n\}) = U_m(x_m) - \min(U_m(x_m), U_n(x_n)),$$

which is a contradiction because these expressions cannot be simultaneously positive.

For the last assertion, we assume that the tail integrals $U_i$ of $X^i$ are continuous and satisfy $\lim_{x \to 0} U_i(x) = \infty$, $i = 1, \ldots, d$. It suffices to show that $\nu(S_n) = 0$ for any $n$ in decomposition (4.4). If $\xi(n) \neq 0$, then

$$\nu(S_n) = \lim_{\varepsilon \downarrow 0} (U_{k(n)}(\xi(n) - \varepsilon) - U_{k(n)}(\xi(n))) = 0,$$

because $U_{k(n)}$ is continuous. Suppose now that $\xi(n) = 0$. Since $S_n$ does not reduce to a single point, we must have either $x_m > 0$ or $x_m < 0$ for some $x \in S_n$ and some $m$. Suppose that $x_m > 0$, the other case being analogous. Since $S$ is ordered, we have

$$\nu(\{x \in \mathbb{R}^d : x_{k(n)} \geq \varepsilon\} \cap S) \leq \nu(\{\xi \in \mathbb{R}^d : \xi_m \geq x_m\} \cap S) < \infty$$

uniformly in $\varepsilon > 0$. This implies $\lim_{x \to 0} U_{k(n)}(x) = \infty$ in contradiction to $\lim_{x \to 0} U_{k(n)}(x) = \infty$. Hence, $\xi(n) > 0$ for any $n$. Therefore, $\nu(\mathbb{R}^d \setminus S^*) = 0$ and the proof is completed. \( \square \)

Lévy copulas provide a simple characterization of possible dependence patterns of multidimensional stable Lévy motions.

**Theorem 4.8** Let $X$ be a $\mathbb{R}^d$-valued Lévy process and let $\alpha \in (0, 2)$. $X$ is $\alpha$-stable if and only if its components $X^1, \ldots, X^d$ are $\alpha$-stable and if it has a Lévy copula $F$ that is a homogeneous function of order 1, i.e.

$$F(ru_1, \ldots, ru_d) = r F(u_1, \ldots, u_d)$$

(4.8)

for any $r > 0$, $u_1, \ldots, u_d \in \mathbb{R}^d$. 

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Proof. $\Rightarrow$: The Lévy measure of a one-dimensional $\alpha$-stable distribution has a density given by
\[ x \mapsto \frac{A}{x^{1+\alpha}}1_{\{x>0\}} + \frac{B}{|x|^{1+\alpha}}1_{\{x\leq 0\}} \]
for some $A \geq 0$ and $B \geq 0$ (Theorem 14.3 in Sato (1999)). Consequently, three situations are possible for any $i = 1, \ldots, d$, namely $\text{Ran} U_i = (-\infty, 0]$ (only negative jumps), $\text{Ran} U_i = [0, \infty)$ (only positive jumps), or $\text{Ran} U_i = \mathbb{R} \setminus \{0\}$ (jumps of both signs). We exclude the trivial case of a component having no jumps at all. Let $I_1 = \{i : \text{Ran} U_i = (-\infty, 0]\}$ and $I_2 = \{i : \text{Ran} U_i = [0, \infty)\}$. For any $i$, let $\tilde{X}^i$ be a copy of $X^i$, independent of $X$ and of $\tilde{X}^k$ for $k \neq i$. Define a $\mathbb{R}^d$-valued Lévy process $\tilde{X}$ by
\[ \tilde{X}^i = \begin{cases} X^i, & i \notin I_1 \cup I_2 \\ X^i - \tilde{X}, & i \in I_1 \cup I_2. \end{cases} \]
Denote by $\tilde{\nu}$ the Lévy measure of $\tilde{X}$, by $\tilde{U}$ its tail integral, and by $\tilde{F}$ its Lévy copula. From the construction of $\tilde{X}$ it follows that
\[ U^I((x_i)_{i \in I}) = \tilde{U}^I((x_i)_{i \in I}) \tag{4.9} \]
for any $I \subset \{1, \ldots, d\}$ with $\text{card } I \geq 2$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$. The process $\tilde{X}$ is clearly $\alpha$-stable and each component of this process has jumps of both signs (i.e. $\text{Ran} \tilde{U}_i = \mathbb{R} \setminus \{0\}$). By Theorem 14.3 in Sato (1999) we have
\[ \tilde{\nu}(B) = r^\alpha \tilde{\nu}(rB) \tag{4.10} \]
for any $B \in B(\mathbb{R}^d)$ and for any $r > 0$. Therefore,
\[ \tilde{U}^I((x_i)_{i \in I}) = r^\alpha \tilde{U}^I((r x_i)_{i \in I}) \tag{4.11} \]
for any nonempty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$. By Theorem 3.6 this implies
\[ \tilde{F}^I((u_i)_{i \in I}) = r^{-1} \tilde{F}^I((r u_i)_{i \in I}) \]
for any $(u_1, \ldots, u_d) \in (\mathbb{R} \setminus \{0\})^d$. Therefore (4.8) holds for $\tilde{F}$. It remains to prove that $\tilde{F}$ is also a Lévy copula of $X$. Indeed, let $I \subset \{1, \ldots, d\}$ nonempty and $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$. Two situations are possible:
\begin{itemize}
  \item $\tilde{U}_i(x_i) = U(x_i)$ for every $i \in I$. Then (4.9) implies that Equation (3.1) holds with $F$ replaced by $\tilde{F}$.
  \item $U_k(x_k) = 0$ for some $k \in I$. Then $\tilde{F}^I((U_i(x_i))_{i \in I}) = 0$, but on the other hand $|U^I((x_i)_{i \in I})| \leq |U_k(x_k)| = 0$ and (3.1) holds again with $F$ replaced by $\tilde{F}$.
\end{itemize}
$\Leftarrow$: Since $X$ has $\alpha$-stable margins and a homogeneous Lévy copula, its marginal tail integrals satisfy (4.11). From the proof of Lemma 3.5 it follows that the Lévy measure of every set of the form $[a, b]$ can be expressed as a limit of linear combinations of tail integrals. Therefore, the Lévy measure of $X$ has the property (4.10). We conclude from Theorem 14.3 in Sato (1999) that $X$ is $\alpha$-stable. □
5 Parametric families of Lévy copulas

The following result is analogous to the Archimedean copula construction (cf. e.g. Nelsen (1999)). It allows to obtain parametric families of Lévy copulas in arbitrary dimension, where the number of parameters does not depend on the dimension.

**Theorem 5.1** Let \( \varphi : [-1, 1] \to [-\infty, \infty] \) be a strictly increasing continuous function with \( \varphi(1) = \infty, \varphi(0) = 0, \) and \( \varphi(-1) = -\infty, \) having derivatives of orders up to \( d \) on \((-1, 0)\) and \((0, 1)\), and satisfying

\[
\frac{d^d \varphi(e^x)}{dx^d} \geq 0, \quad \frac{d^d \varphi(-e^x)}{dx^d} \leq 0, \quad x \in (-\infty, 0).
\]

(5.12)

Let

\[
\bar{\varphi}(u) := 2^{d-2}(\varphi(u) - \varphi(-u))
\]

for \( u \in [-1, 1] \). Then

\[
F(u_1, \ldots, u_d) := \varphi \left( \prod_{i=1}^{d} \bar{\varphi}^{-1}(u_i) \right)
\]

defines a Lévy copula.

**Proof.** Firstly, note that \( \bar{\varphi} \) is a strictly increasing continuous function from \([-1, 1]\) to \([-\infty, \infty]\), satisfying \( \bar{\varphi}(1) = \infty \) and \( \bar{\varphi}(-1) = -\infty \), which means that \( \bar{\varphi}^{-1} \) exists for all \( u \in \mathbb{R} \) and \( F \) is well defined. Properties 1 and 2 of Definition 3.1 are clearly satisfied. For \( k = 1, \ldots, d \) and \( u_k \in \mathbb{R} \) we have

\[
F^{(k)}(u_k) = \lim_{c \to -\infty} \sum_{(u_i)_{i \neq k} \in \{-c, \infty\}^{d-1}} F(u_1, \ldots, u_d) \prod_{i \neq k} \text{sgn} u_i
\]

\[
= \sum_{(u_i)_{i \neq k} \in \{-\infty, \infty\}^{d-1}} \varphi \left( \bar{\varphi}^{-1}(u_k) \prod_{i \neq k} \text{sgn} u_i \right) \prod_{i \neq k} \text{sgn} u_i
\]

\[
= \sum_{i=0}^{d-1} \binom{d-1}{i} (-1)^i \varphi \left( \bar{\varphi}^{-1}(u_k)(-1)^i \right)
\]

\[
= 2^{d-2} \left( \varphi(\bar{\varphi}^{-1}(u_k)) - \varphi(-\bar{\varphi}^{-1}(u_k)) \right) = u_k,
\]

which proves property 4. It remains to show that \( F \) is \( d \)-increasing. Since \( \bar{\varphi}^{-1} \) is increasing, it suffices to show that \((u_1, \ldots, u_d) \mapsto \varphi(\prod_{i=1}^{d} u_i)\) is \( d \)-increasing on \((-1, 1)^d\). Since \( \varphi(\prod_{i=1}^{d} u_i) = \varphi(\prod_{i=1}^{d} \left| u_i \right| \prod_{i=1}^{d} \text{sgn} u_i) \), it suffices to prove that both \((u_1, \ldots, u_d) \mapsto \varphi(\prod_{i=1}^{d} u_i)\) and \((u_1, \ldots, u_d) \mapsto -\varphi(-\prod_{i=1}^{d} u_i)\) are \( d \)-increasing on \([0, 1]^d\) or, equivalently, on \((0, 1)^d\) (since \( \varphi \) is continuous). The first condition of (5.12) implies that

\[
\frac{\partial^d \psi(z_1, \ldots, z_d)}{\partial z_1 \ldots \partial z_d} \geq 0
\]

on \((-\infty, 0)^d\) for \( \psi(z_1, \ldots, z_d) := \varphi(e^{z_1 + \ldots + z_d}) \). From Definition 2.1 it follows easily that

\[
V_\psi(B) = \int_B \frac{\partial^d \psi(z_1, \ldots, z_d)}{\partial z_1 \ldots \partial z_d} dz_1 \ldots dz_d.
\]
Therefore, $\psi$ is increasing on $(-\infty, 0)^d$, which implies that $(u_1, \ldots, u_d) \mapsto \varphi(\prod_{i=1}^d u_i)$ is $d$-increasing on $(0, 1)^d$. The second condition of (5.12) entails similarly that $(u_1, \ldots, u_d) \mapsto -\varphi(-\prod_{i=1}^d u_i)$ is $d$-increasing on $(0, 1)^d$ as well. 

**Remark.** Condition (5.12) is satisfied in particular if for any $k = 1, \ldots, d$,

$$
\frac{d^k \varphi(u)}{du^k} \geq 0, \quad u \in (0, 1) \quad \text{and} \quad (-1)^k \frac{d^k \varphi(u)}{du^k} \leq 0, \quad u \in (-1, 0).
$$

**Example 5.2** Let

$$
\varphi(x) := (1 - \log |x|)^{-1/d} 1_{\{x > 0\}} - (1 - \eta)(1 - \log |x|)^{-1/d} 1_{\{x < 0\}}
$$

with $\vartheta > 0$ and $\eta \in (0, 1)$. Then

$$
\bar{\varphi}(x) = 2^{d-2} (-\log |x|)^{-1/\vartheta} \text{sgn } x,
$$

$$
\bar{\varphi}^{-1} (u) = \exp (-|2^{2-d} u|^{-\vartheta}) \text{sgn } u,
$$

and therefore

$$
F(u_1, \ldots, u_d) = 2^{2-d} \left( \sum_{i=1}^d |u_i|^{-\vartheta} \right)^{-1/\vartheta} \left( \eta 1_{\{u_1 \cdots u_d \geq 0\}} - (1 - \eta) 1_{\{u_1 \cdots u_d < 0\}} \right) \quad (5.13)
$$

defines a two-parameter family of Lévy copulas. It resembles the Clayton family of ordinary copulas. $F$ is in fact a Lévy copula for any $\vartheta > 0$ and any $\eta \in [0, 1]$. The role of the parameters is easiest to analyze in the case $d = 2$, when (5.13) becomes

$$
F(u, v) = (|u|^{-\vartheta} + |v|^{-\vartheta})^{-1/\vartheta} \left( \eta 1_{\{uv \geq 0\}} - (1 - \eta) 1_{\{uv < 0\}} \right). \quad (5.14)
$$

From this equation it is readily seen that the parameter $\eta$ determines the dependence of the *sign* of jumps: when $\eta = 1$, the two components always jump in the same direction, and when $\eta = 0$, positive jumps in one component are accompanied by negative jumps in the other and vice versa. For intermediate values of $\eta$, positive jumps in one component can correspond to both positive and negative jumps in the other component. The parameter $\vartheta$ is responsible for the dependence of absolute values of jumps in different components. In particular, $F$ converges to the independence Lévy copula (4.2) if $\eta = 1$ and $\vartheta \to 0$. Similarly, $F$ tends to the Lévy copula of complete dependence (4.3) for $\eta = 1$ and $\vartheta \to \infty$.

Theorem 5.1 can be used to construct parsimonious models of dependence. This is typically useful when one has little information about the dependence structure of the problem. If a more precise vision is necessary, a possible strategy is to model the dependence of the Lévy measure separately in each orthant. The dependence in a given orthant can be modelled using a positive Lévy copula (cf. Definition 5.8 in Cont & Tankov (2004)). A *positive Lévy copula* is a function $F : [0, \infty]^d \to [0, \infty]$, having properties 1–3 of Definition 3.1
and satisfying $F(u_1, \ldots, u_d) = u_k$ if $u_i = \infty$ for all $i \neq k$. It is easily seen that a positive Lévy copula can be extended to a Lévy copula in the sense of Definition 3.1 by setting $F(x_1, \ldots, x_d) = 0$ if $x_k < 0$ for some $k$. A $d$-dimensional Lévy copula can be constructed from $2^d$ positive Lévy copulas (one for each orthant) as follows:

**Theorem 5.3** For each $\{\alpha_1, \ldots, \alpha_d\} \in \{-1, 1\}^d$ let $g^{(\alpha_1, \ldots, \alpha_d)}(u) : [0, \infty] \to [0, 1]$ be a nonnegative, increasing function satisfying

$$
\sum_{\alpha \in \{-1, 1\}^d \text{ with } \alpha_k = -1} g^{(\alpha_1, \ldots, \alpha_d)}(u) = 1 \quad \text{and} \quad \sum_{\alpha \in \{-1, 1\}^d \text{ with } \alpha_k = 1} g^{(\alpha_1, \ldots, \alpha_d)}(u) = 1
$$

for all $u \in [0, \infty]$ and all $k \in \{1, \ldots, d\}$. Moreover, let $F^{(\alpha_1, \ldots, \alpha_d)}$ be a positive Lévy copula that satisfies the following continuity property at infinity: for all $I \subset \{k : \alpha_k = -1\}$, $(u_i) \in I^c \subset [0, \infty]^d$ we have

$$
\lim_{\{u_i\} \in I^c \to (\infty, \ldots, \infty)} F^{(\alpha_1, \ldots, \alpha_d)}(u_1, \ldots, u_d) = F^{(\alpha_1, \ldots, \alpha_d)}(v_1, \ldots, v_d),
$$

where $v_i = u_i$ for $i \in I^c$ and $v_i = \infty$ otherwise. Then

$$
F(u_1, \ldots, u_d) := F^{(\text{sgn } u_1, \ldots, \text{sgn } u_d)}(\left| u_1 \right| g^{(\text{sgn } u_1, \ldots, \text{sgn } u_d)}(\left| u_1 \right|), \ldots, \left| u_d \right| g^{(\text{sgn } u_1, \ldots, \text{sgn } u_d)}(\left| u_d \right|)) \prod_{i=1}^d \text{sgn } u_i
$$

defines a Lévy copula.

**Proof.** Properties 1 and 2 of Definition 3.1 are obvious. Property 3 follows after observing that $u \mapsto u g^{(\alpha_1, \ldots, \alpha_d)}(u)$ is increasing on $[0, \infty]$ for any $\{\alpha_1, \ldots, \alpha_d\} \in \{-1, 1\}^d$. To prove property 4, note that

$$
F^{(\alpha_1, \ldots, \alpha_d)}(\left| u_1 \right| g^{(\alpha_1, \ldots, \alpha_d)}(\left| u_1 \right|), \ldots, \left| u_d \right| g^{(\alpha_1, \ldots, \alpha_d)}(\left| u_d \right|)) = \left| u_k \right| g^{(\alpha_1, \ldots, \alpha_d)}(\left| u_k \right|)
$$

for any $\{\alpha_1, \ldots, \alpha_d\} \in \{-1, 1\}^d$ and any $\{u_1, \ldots, u_d\} \in \mathbb{R}^d$ with $u_i = \infty$ for all $i \neq k$. Therefore,

$$
F^{(k)}(u) = \begin{cases} 
\sum_{\alpha \in \{-1, 1\}^d \text{ with } \alpha_k = 1} u g^{(\alpha_1, \ldots, \alpha_d)}(u) \prod_{i=1}^d \alpha_i \prod_{j \neq k} \alpha_j & \text{if } u \geq 0 \\
\sum_{\alpha \in \{-1, 1\}^d \text{ with } \alpha_k = -1} \left| u \right| g^{(\alpha_1, \ldots, \alpha_d)}(\left| u \right|) \prod_{i=1}^d \alpha_i \prod_{j \neq k} \alpha_j & \text{if } u < 0
\end{cases}
$$

$$
= u.
$$

**Example 5.4** Let

$$
g^{(\alpha_1, \ldots, \alpha_d)}(u) = \begin{cases} 
1 & \text{for } \alpha_1 = \cdots = \alpha_d \\
0 & \text{otherwise}.
\end{cases}
$$

Then the Lévy copula $F$ in Theorem 5.3 satisfies $F(u_1, \ldots, u_d) = 0$ if $u_i u_j < 0$ for some $i, j$. This means that the Lévy measure is supported by the positive and the negative orthant: either all components of the process jump up or all components jump down.
6 Limit theorems

In this section, we explore the relation between the Lévy copula $F$ of a Lévy process $X$ and the (ordinary) copula $C_t$ of its distribution at a given time $t$. It turns out that in all points where the Lévy copula is unique (cf. Theorem 3.6), it is completely determined by the limiting behavior of $C_t$ as $t \to 0$. Moreover, we shall see that it is only the behavior of $C_t$ in the corners of its domain of definition (which is $[0, 1]^d$) that matters.

A copula is a function $C : [0, 1]^d \to [0, 1]$ such that $C$ is $d$-increasing, $C(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for some $i$, and $C(u_1, \ldots, u_d) = u_k$ if $u_i = 1$ for all $i \neq k$. Let $X = (X^1, \ldots, X^d)$ be an $\mathbb{R}^d$-valued random variable with distribution function $F(u_1, \ldots, u_d) := \Pr[X^1 \leq u_1, \ldots, X^d \leq u_d]$ and marginal distribution functions $H_i(x) := \Pr[X^i \leq x]$. The copula of $X$ or the copula of $H$ is any copula $C$ such that

$$C(H_1(x_1), \ldots, H_d(x_d)) = H(x_1, \ldots, x_d)$$

for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$. The survival function of $X$ is defined by

$$\overline{F}(x_1, \ldots, x_d) = \Pr[X^1 > x_1, \ldots, X^d > x_d],$$

and the survival copula $\overline{C}$ of $X$ is a copula that relates the survival function of $X$ to its marginal survival functions:

$$\overline{C}(\overline{F}(x_1), \ldots, \overline{F}(x_d)) = \overline{F}(x_1, \ldots, x_d)$$

for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$. It is easy to verify that any copula of $X$ is also a survival copula of $-X$ and vice versa.

Theorem 6.1 Let $X$ be a $\mathbb{R}^d$-valued Lévy process with marginal tail integrals $U_1, \ldots, U_d$, and denote by $F$ its Lévy copula in the sense of Theorem 3.6. Denote by $C_{t}^{(\alpha_1, \ldots, \alpha_d)} : [0, 1]^d \to [0, 1]$ an (ordinary) copula of $(-\alpha_1 X^1_t, \ldots, -\alpha_d X^d_t)$ (or, equivalently, a survival copula of $(\alpha_1 X^1_t, \ldots, \alpha_d X^d_t)$) for $t \in (0, \infty)$, $\alpha \in \{-1, 1\}^d$. Then

$$F(u_1, \ldots, u_d) = \lim_{t \to 0} \frac{1}{t} C_{t}^{(\text{sgn} \alpha_1 u_1, \ldots, \text{sgn} \alpha_d u_d)}(t|u_1|, \ldots, t|u_d|) \prod_{i=1}^{d} \text{sgn} \alpha_i. \quad (6.1)$$

for any $(u_1, \ldots, u_d) \in \prod_{i=1}^{d} \text{Ran} U_i$.

Proof. Step 1: By Nelsen (1999), Th. 2.10.7 and Lemma 3.2 above, copulas and Lévy copulas are Lipschitz with a common Lipschitz constant. Consequently, it suffices to prove the assertion for $(u_1, \ldots, u_d) \in \prod_{i=1}^{d} \text{Ran} U_i$.

Step 2: It suffices to prove the assertion for nonnegative $u_1, \ldots, u_d$. Indeed, let $(u_1, \ldots, u_d) \in \prod_{i=1}^{d} \text{Ran} U_i$ and $\alpha_i := \text{sgn} u_i$, $i = 1, \ldots, d$. Let $\tilde{X} := (\alpha_1 X^1, \ldots, \alpha_d X^d)$ and denote
by $\tilde{F}$ a Lévy copula of $\tilde{X}$ and by $\tilde{U}$ and $\tilde{U}_i$ its tail integrals. Theorem 3.6 and Lemma 3.2 imply that

$$U(x_1, \ldots, x_d) = \lim_{\xi_i |x_i| = 1, \ldots, d} \tilde{U}(\alpha_1 \xi_1, \ldots, \alpha_d \xi_d) \prod_{i=1}^d \alpha_i$$

$$= \lim_{\xi_i |x_i| = 1, \ldots, d} \tilde{F} \left( \tilde{U}_1(\alpha_1 \xi_1), \ldots, \tilde{U}_d(\alpha_d \xi_d) \right) \prod_{i=1}^d \alpha_i$$

$$= \tilde{F} \left( \lim_{\xi_i |x_i| = 1, \ldots, d} \tilde{U}_1(\alpha_1 \xi_1), \ldots, \lim_{\xi_i |x_i| = 1, \ldots, d} \tilde{U}_1(\alpha_1 \xi_1) \right) \prod_{i=1}^d \alpha_i$$

$$= \tilde{F}(\alpha_1 U_1(x_1), \ldots, \alpha_d U_d(x_d)) \prod_{i=1}^d \alpha_i.$$ for any $x_1, \ldots, x_d \in \mathbb{R} \setminus \{0\}$. Therefore,

$$F(u_1, \ldots, u_d) = \tilde{F}(|u_1|, \ldots, |u_d|) \prod_{i=1}^d \alpha_i = \lim_{c \to \infty} \frac{1}{t} C_t(\alpha_1, \ldots, \alpha_d) (t|u_1|, \ldots, t|u_d|) \prod_{i=1}^d \alpha_i,$$

if the assertion holds for the Lévy process $\tilde{X}$ and $(|u_1|, \ldots, |u_d|) \in \prod_{i=1}^d \text{Ran} \tilde{U}_i$.

**Step 3:** Let $u_i = U_i(x_i)$ with $x_i > 0$ for $i = 1, \ldots, d$. Choose $x_0 > 0$ small enough such that $\nu([0, x_0] \setminus [-x_0, x_0]^d) > 0$ and let $A := \{y \in \mathbb{R}^d : |y_i| \geq \min(x_0, x_i/2), i = 1, \ldots, d\}$. Choose a continuous mapping $g : \mathbb{R}^d \to [0, 1]$ such that $g(y) = 1$ if $y \in A$ and $g(y) = 0$ in a neighborhood of 0. For $t \in (0, \infty)$ define a probability measure $P_t^g$ on $\mathbb{R}^d$ via

$$P_t^g(B) := \frac{E[1_B(X_t)g(X_t)]}{E[g(X_t)].}$$

Observe that $P_t^g$ is well defined if $\nu \neq 0$, because in this case the support of $P_{X_t}$ is unbounded for any $t > 0$ (cf. Sato (1999), Th. 24.3). Let $c_t := E[g(X_t)]$. Then $P_t^g(B) = \frac{1}{c_t} P_{X_t}(B)$ for any $B \subset B(\mathbb{R}^d)$ with $B \setminus A = \emptyset$. This implies

$$C_t(\overline{H}_{t,1}(y_1), \ldots, \overline{H}_{t,d}(y_d)) = \frac{1}{c_t} C_t(1, \ldots, 1) (c_t \overline{H}_{t,1}(y_1), \ldots, c_t \overline{H}_{t,d}(y_d)).$$

(6.2)

for any $y_i \geq x_i/2$, where $C_t$ denotes a survival copula of $P_t^g$ and $\overline{H}_{t,i}$, $i = 1, \ldots, d$ are the survival functions of the marginals of $P_t^g$.

**Step 4:** Define a probability measure $Q^g$ on $\mathbb{R}^d$ by

$$Q^g(B) := \frac{\int g 1_B d\nu}{\int g d\nu}.$$

Denote by $\overline{C}$ a survival copula of $Q^g$, by $\overline{H}_{t,i}$, $i = 1, \ldots, d$ the marginal survival functions of $Q^g$, and let $c := \int g d\nu$. Then $\overline{H}_{t,i}(x_i) = \frac{g}{c}$ and

$$F(u_1, \ldots, u_d) = c \overline{C} \left( \frac{u_1}{c}, \ldots, \frac{u_d}{c} \right).$$

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follows similarly as (6.2) from the definition of $\bar{C}$ and $F$.

Step 5: By Sato (1999), Cor. 8.9 we have $\int f dP_{X_t} \overset{t \to 0}{\longrightarrow} \int f d\nu$ for any bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$ which vanishes in a neighborhood of 0. This implies $\frac{1}{t} \int f g dP_{X_t} \overset{t \to 0}{\longrightarrow} \int f g d\nu$ and hence

$$\frac{c_t}{ct} \int f dP^g \overset{t \to 0}{\longrightarrow} \int f dQ^g$$

for any bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$. Therefore,

$$\frac{c_t}{ct} P^g \to Q^g$$

weakly for $t \to 0$. In particular $\frac{\bar{P}^g}{c} \to 1$, which implies that $P^g_t \to Q^g$ as well. Therefore, $\bar{C}_{t}(\frac{u_1}{c}, \ldots, \frac{u_d}{c}) \to \bar{C}(\frac{u_1}{c}, \ldots, \frac{u_d}{c})$ for $t \to 0$ because weak convergence implies pointwise convergence of the copulas (cf. Lindner & Szimayer (2003), Th. 2.1). Since $P^g_t \to Q^g$, we have that $\bar{H}_{t,i}$ converges to $\bar{H}$ on a dense set. Hence, there exist $y_{t,i} \to x_i$ with $\bar{H}_{t,i}(y_{t,i}) \to \bar{H}(x_i) = \frac{u_i}{c}$ for $t \to 0$.

Step 6: Together, it follows that

$$F(u_1, \ldots, u_d) = c \bar{C} \left( \frac{u_1}{c}, \ldots, \frac{u_d}{c} \right)$$

$$= \lim_{t \to 0} c \bar{C}_{t} \left( \frac{u_1}{c}, \ldots, \frac{u_d}{c} \right)$$

$$= \lim_{t \to 0} c \bar{C}_{t} \left( \bar{H}_{t,1}(y_{t,1}), \ldots, \bar{H}_{t,d}(y_{t,d}) \right)$$

$$= \lim_{t \to 0} \frac{c}{ct} C_{t}^{(1,\ldots,1)} \left( c_i \bar{H}_{t,1}(y_{t,1}), \ldots, c_i \bar{H}_{t,d}(y_{t,d}) \right)$$

$$= \lim_{t \to 0} \frac{1}{t} C_{t}^{(1,\ldots,1)} (tu_1, \ldots, tu_d).$$

In the third and the last equality we used the fact that copulas are Lipschitz with a common Lipschitz constant (see e.g. Nelsen (1999), Th. 2.10.7).\[\square\]

Remark. The tail copulas $F^t$ of $F$ are obtained accordingly by considering the copulas of $X^t$ instead of $X$.

Example 6.2 Let $X$ be a $\mathbb{R}^d$-valued stable Lévy process with Lévy copula $F$. For any $a > 0$ there exist $b > 0$ and $c \in \mathbb{R}^d$ such that $X_{at} \overset{d}{=} bX_t + ct$, $t \in \mathbb{R}_+$. This implies that the copula of $X_t$ is also the copula of $X_{at}$ for all $t \in \mathbb{R}_+$, $a > 0$. By Theorem 6.1, we have

$$F(u_1, \ldots, u_d) = \lim_{t \to 0} \frac{1}{t} C^{(\text{sgn} u_1, \ldots, \text{sgn} u_d)}(t|u_1|, \ldots, t|u_d|) \prod_{i=1}^d \text{sgn} u_i,$$

for any $(u_1, \ldots, u_d) \in \prod_{i=1}^d \text{Ran} \bar{C}_i$, where $C^{\alpha_1,\ldots,\alpha_d}$ is the survival copula of $(\alpha_1 X_1, \ldots, \alpha_d X_d)$ for any fixed $t > 0$. In particular, this implies that the Lévy copula of a stable Lévy process is a homogeneous function of order 1, i.e. $F(cu_1, \ldots, cu_d) = cF(u_1, \ldots, u_d)$ for all $c > 0$.
The dependence between the components of a Lévy process $X$ is not entirely characterized by the Lévy copula because $X$ may also have a Brownian motion part $B$. Because of the scaling property of Brownian motion, the copula of the random vector $B_t$ does not depend on $t$ (cf. Example 6.2). Since $B_t$ has a multivariate normal distribution, this copula $C^B$ is a Gaussian copula. The following theorem shows that it can also be recovered as a limit of the copulas of $X_t$ for $t \to 0$.

**Theorem 6.3** Let $X$ be a $\mathbb{R}^d$-valued Lévy process and denote by $C^B$ the Gaussian copula of the continuous martingale part $B = (B^1, \ldots, B^d)$ of $X$ (which is possibly 0). For $t > 0$ denote by $C_t : [0, 1]^d \to [0, 1]$ the probabilistic copula of $(-X_t^1, \ldots, -X_t^d)$ (i.e., $C_t = C_{t(1,\ldots,1)}$ in the notation of Theorem 6.1). Then we have

$$C^B(u_1, \ldots, u_d) = \lim_{t \to 0} C_t(u_1, \ldots, u_d)$$

for any $(u_1, \ldots, u_d) \in V$, where $V$ denotes the subset of $[0, 1]^d$ where the Gaussian copula $C^B$ is uniquely defined.

**Remark.** The Gaussian copula $C^B$ is not uniquely defined only if some components of $B$ are zero; otherwise the marginals of $B$ are continuous and therefore the copula is unique.

**Proof.** Choose a sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n \downarrow 0$ for $n \to \infty$. Write $X = B + L$, where $B$ is a Brownian motion and $L$ is a Lévy process without continuous martingale part. Denote by $(0, \nu, \gamma)_t$ the Lévy-Khintchine triplet of $L_t$ relative to the truncation function $x \mapsto x 1_{\{|x| \leq 1\}}$ (cf. e.g. Sato (1999), Def. 8.2). Straightforward calculations yield that the corresponding triplet of $t_n^{-1/2}L_{t_n}$ equals

$$\left(0, \nu^{(n)}t_n, \left(\gamma - \int x 1_{\{|x| < 1\}} \nu(dx)\right) \sqrt{t_n}\right),$$

where $\nu^{(n)}(A) := \int 1_A (t_n^{-1/2} x) \nu(dx)$, $A \in \mathcal{B}(\mathbb{R}^d)$. By dominated convergence we have that

$$\left|\left(\gamma - \int x 1_{\{|x| < 1\}} \nu(dx)\right) \sqrt{t_n}\right| \leq |\gamma| \sqrt{t_n} + \int_{\{|x| \leq 1\}} (|x|^2 \wedge |x| \sqrt{t_n}) \nu(dx) \xrightarrow{n \to \infty} 0$$

and similarly

$$\int |x|^2 1_{\{|x| \leq 1\}} \nu^{(n)}(dx) t_n \xrightarrow{n \to \infty} 0,$$

$$\int g(x) \nu^{(n)}(dx) t_n \xrightarrow{n \to \infty} 0$$

for any bounded function $g : \mathbb{R}^d \to \mathbb{R}$ which vanishes in a neighborhood of 0. From Jacod & Shiryaev (2003), Th. VII.2.9 it follows that $t_n^{-1/2}L_{t_n}$ converges weakly to 0 for $n \to \infty$. Moreover, we have $t_n^{-1/2}B_{t_n} \xrightarrow{d} B_1$ for any $t \in \mathbb{R}_+$. Consequently, we have $t_n^{-1/2}X_{t_n} \xrightarrow{d} B_1$ weakly for $n \to \infty$. From Lindner & Szimayer (2003), Th. 2.1 it follows that the corresponding sequence of copulas converges as well. But note that the copula of $t_n^{-1/2}X_{t_n}$ coincides for any $n \in \mathbb{N}$ with the copula of $X_{t_n}$. Therefore, the copula of $X_{t_n}$ converges to the copula of $B_1$ on the set where the latter is uniquely defined. \qed
References


