Exercise Sheet: Markov Chains

Otherwise mentioned, we make use of the following notation: \((\Omega, \mathcal{F}, (\mathcal{F})_n, \mathbb{P})\) is a filtered space, on which the Markov chain \(X = (X_n, n \geq 0)\) is defined. \(X\) takes values in the finite \(\text{countable}\) space \(S\). For \(i, j \in S\) we let \(p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i)\) and we denote by \(\mathbb{P} = (p_{ij})_{i,j \in S}\) the transition matrix (or kernel when \(S\) is infinite) of the chain. For \(\theta\) a probability measure on \(S\) we let \(\mathbb{P}^{\theta}\) denote the probability measure under which \(\mathbb{P}^{\theta}[X_0 = i] = \theta(i), \forall i \in S\) (and for \(i \in S\), we let \(\mathbb{P}_i = \mathbb{P}^\delta_i\) for shorthand). Finally we let \(T_j = \inf\{n \geq 0 : X_n = j\}\) the entrance time of the chain at \(j\), and \(T_j^+ = \inf\{n > 0 : X_n = j\}\) the hitting time of \(j\) by our chain.

1 Simple examples of finite state space Markov chains.

Exercise 1 Two State Chain.

Let \(S := \{1, 2\}\), fix \(p \in [0, 1], q \in [0, 1]\) and set
\[
P := \begin{pmatrix} p & 1 - p \\ 1 - q & q \end{pmatrix}.
\]

1. Under which conditions on \(p, q\) is the chain irreducible and aperiodic? What happens when it is not irreducible, or when it is periodic?

In these degenerate cases, which states are transient? recurrent? Can you express the set of stationary distributions?

2. In this question we assume that \(p, q\) are such that the chain is aperiodic, irreducible. Find all stationary distributions.

3. Express \(E_1[T_j^+]\), \(E_2[T_1]\)?

4. Express \(P_1[X_n = j]\) in terms of \(n\), for \(j = 1, 2\).

Exercise 2 Tennis game.

We shall remind the simple rule that a tennis player wins a game when he is the first to win at least 4 points, and 2 more than his opponent. In this exercise we will only be interested in the occurrence of one game, and we will stop as soon as one of the two players wins it. We assume in addition that the player serving has a probability \(p \in [0, 1]\) of winning each point played, independently of previous played points.

1. Define a finite state Markov chain modelling the situation. For the sake of simplicity, one can simply draw a diagram.

Note: One should be able to have the state space contain exactly 17 elements.

2. Is the chain irreducible? What are the transient states? the recurrent ones? Does \(\lim_{n \to \infty} X_n\) exists almost surely?

3. Express the probability for the server to win the game in terms of \(p\).

Note: It could ne useful to consider first the case when the game ends in less than 6 points.

Exercise 3 Shuffling cards.

We assume the deck has 52 cards. Notice that a configuration of the deck simply is a permutation of \(\{1, \ldots, 52\}\). In this exercise, we shuffle cards in the following (bad) way: at each time \(n \in \mathbb{N}^*\), one chooses uniformly at random two cards in the deck and exchanges them.
1. Introduce a Markov chain which models this shuffling.
2. Is the chain irreducible? aperiodic? How many stationary distributions are there?
3. Show that the chain is reversible and write the detailed balance equation. What can be deduced about the stationary distribution(s)?

Exercice 4 Urn model.
The following is very close to Ehrenfest’s model. Consider 2m balls, m whites and m blacks, placed in two urns, each containing exactly m balls. At each time $n \in \mathbb{N}^*$, a ball is choosen uniformly at random in each of the two urns, and the two selected balls are exchanged. Let $X_n$ denote the number of black balls in urn 1 just after the $n$-th exhange.

1. Express the transition probabilities of $(X_n, n \geq 0)$. Show there exists a unique stationary distribution $\pi$.
2. Without computation, can you guess the probability that the $j$-th black ball is in urn 1 at a large time? Can you then guess the expression of $\pi$?
3. Show that the chain is reversible and validate/refute the guess of the previous question.

Exercice 5 Transience/Recurrence
Consider $(X_n, n \geq 0)$ a Markov chain on the countable set $S$, let $T_y = \inf\{n > 0 : X_n = y\}$, and, for $k \geq 2$, $T_y^k := \inf\{n > T_y^{k-1} : X_n = y\}$. Recall that $y$ is said recurrent iff $P_y(T_y < \infty) = 1$.

1. Let $\rho_{xy} = P_x[T_y < \infty]$. Show that $P_x[T_y^k < \infty] = \rho_{xy}\rho_y^{k-1}$.
2. Deduce that $y$ is recurrent iff $\mathbb{E}_y[\sum_{n \geq 0} 1_{X_n = y}] = \infty$.
3. Show that if $x$ is recurrent and $\rho_{xy} > 0$, then $y$ also is recurrent, and $\rho_{xy} = 1$.
4. Deduce that a closed, finite irreducible class can only contain recurrent states.
5. Assume $S = \{1, 2, 3, 4, 5, 6\}$ and

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0.4 & 0.6 & 0 & 0 & 0 & 0 \\
0.3 & 0 & 0.4 & 0.2 & 0.1 & 0 \\
0 & 0 & 0 & 0.3 & 0.7 & 0 \\
0 & 0 & 0 & 0.5 & 0 & 0.5 \\
0 & 0 & 0 & 0 & 0.8 & 0 & 0.2
\end{pmatrix}
$$

What are the recurrent/transient states of the chain?

Exercice 6 Assume here that $S = \{1, 2, 3\}$ with

$$
P := \begin{pmatrix}
0 & 1 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 \\
\frac{1}{3} & \frac{1}{3} & 0
\end{pmatrix}
$$

1. What are the recurrent/transient states of the chain?
2. Compute $\mathbb{E}_x(N_y)$ for any $x, y \in S$, where $N_y$ is the occupation time of $X$ at $y$ (one shall make use of the result of question 1). Express the potential matrix $U := \sum_{n \geq 0} P^n$ (with the slight abuse that some coefficients of $U$ may be infinite).
3. Let $v : x \rightarrow \mathbb{E}_x[T_1]$, where $T_1$ is the entrance time (i.e. time of first visit) of state 1. Show that $v$ is a solution to the equation

$$
v(x) = 1 + \sum_{y \in S} P(x, y)v(y), x \in \{2; 3\}, \quad v(1) = 0,
$$

and compute $v$. 
4. Find an invariant probability measure. Is it unique?

5. Let $T_{[1,2]}$ be the entrance time of $\{1, 2\}$. Find the law of $T_{[1,2]}$ under $P_3$.

6. Show that $E_3(T_{[1,2]}) = E_3(N_3)$.

**Exercise 7** Here, $S = \{1, 2, 3\}$ and

$$P := \begin{pmatrix}
0 & 1 & 1 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
1 & 0 & 0
\end{pmatrix}$$

1. What are the recurrent/transient states of the chain?
2. Find an invariant probability measure. Is it unique?
3. Is the chain reversible?
4. For $x \in E$ find $E_x(S_x)$, where $S_x$ is the hitting time of $x$ (so that in this case, we are looking for the expectation of the return time to $x$).
5. Compute the period of each $x \in E$. What is the limit of $P^n(x, y)$, when $n \to \infty$?

## Mean occupation time.

In this paragraph, we assume $S$ to be finite, and $X$ to be irreducible.

**Exercise 8**

Let $\theta$ a probability on $S$, $T$ a $(\mathcal{F}_n)$-stopping time such that

$$T > 0, \quad E[T] < \infty, \quad P_\theta(X_T \in .) = \theta(.)$$

Define $\rho$ on $S$ by

$$\rho(j) := E_\theta \left[ \sum_{k=0}^{T-1} 1_{X_k = j} \right].$$

Show that $\rho$ is proportional to the unique stationary distribution of the chain. Deduce that for any $j \in S$

$$E_\theta \left[ \sum_{k=0}^{T-1} 1_{X_k = j} \right] = \pi_j E_\theta[T]. \quad (1)$$

**Exercise 9**

Unless otherwise mentioned, $i, j, \ell$ are generic elements of $S$. For any $j \in S, n \in \mathbb{N}$ we let $V(j, n) = \sum_{k=0}^{n-1} 1_{X_k = j}$, that is, the occupation time of state $j$ up to time $n$.

In the following questions, a good idea is to find adequate $\theta, T$ and make use of $(1)$.

1. $E_\theta[T_j^+] = 1/\pi_j$; and if $j \neq i$, $E_\theta[V(j, T_i^+)] = \frac{\pi_i}{\pi_j}$.
2. if $j \neq i$, $E_\theta[V(j, T_j)] = \pi_j (E_\theta[T_j] + E_\theta[T_j])$.
3. Deduce that if $j \neq i$, $P_i(T_j < T_j^+) = \left( \frac{\pi_i}{\pi_j} \left( E_\theta[T_j] + E_\theta[T_i] \right) \right)^{-1}$.
4. if $i \neq \ell, j \neq \ell$, $E_\ell[V(j, T_\ell)] = \pi_j \left( E_\ell T_\ell + E_\ell[T_j] - E_\ell[T_i] \right)$.
5. Deduce that if $i \neq \ell, j \neq \ell$, $P_i[T_j < T_\ell] = \frac{E_i T_j + E_j[T_j] - E_\ell[T_j]}{E_i[T_j] + E_j[T_j]}$. 

\[ \]
Exercice 10 Asssume in addition to this exercise $X$ is aperiodic. We will admit that $Z_{ij} = \sum_{n=0}^{\infty} (p_{ij}^{(n)} - \pi_j)$ is well-defined for any $i,j \in S$ (you can show it using the fact that the convergence of the distribution of $X_n$ towards $\pi$ is exponentially fast). Notice moreover that $\sum_{j \in S} Z_{ij} = 0$. 

Again by making use of (1)) show the following assertions:

1. $Z_{ii} = \pi_i \mathbb{E}_n[T_i]$. Hint: Consider, for fixed $n_0 \in \mathbb{N}$ the first hitting time of $i$ after $n_0$.
2. $Z_{jj} = \pi_j \mathbb{E}_n[T_j]$. Hint: For fixed $n_0 \in \mathbb{N}$, consider the first hitting time of $j$ after $T_i + n_0$, and use a result of the previous exercise.
3. Deduce that $\sum_{j \in S} \pi_j \mathbb{E}_n[T_j] = \sum_{j \in S} Z_{jj}$.
4. $\mathbb{E}_n[V(j, T_i)] = \frac{\pi_i}{\pi_j} Z_{ii} - Z_{ij}$.

Exercice 11 Application: finding a prescribed word in a random sequence of letters.

To simplify a bit we consider here an alphabet of 2 letters $\{0, 1\}$. Each letter in the sequence is chosen, independently of the previous ones, with probability 1/2 (in other words, we are looking at the classical infinite Heads and Tails game). Let $U_0, U_1, \ldots$ denote the outcome of the successive picks, $(\Omega, \mathcal{F}, \mathbb{P})$ the associated probability space and $\mathcal{F}_n = \sigma(U_0, \ldots, U_{n+1})$.

The prescribed word is a fixed word of $n$ letters.

Note: It is in fact extremely easy to generalize the method to any finite alphabet. In particular, it is a good exercise to recover the result of Williams book for a 26 letters alphabet and the prescribed word ABRACADABRA.

1. Let $X_p = \{U_p, \ldots, U_{p+n-1}\}$. Show that $(X_p)_{p \in \mathbb{N}}$ is an irreducible Markov chain on the finite state space $S = \{0, 1\}^n$. What is the unique stationary distribution?
2. Let $n = 12$, $i = \{1, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1\}$. What is $Z_{ii}$? Deduce the mean hitting time of $i$.
3. If $j = \{0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1\}$, what are $Z_{jj}, Z_{ij}, Z_{ji}$? Deduce $\mathbb{P}_\pi(T_i < T_j)$, and compute an approximate value of that quantity.
4. Can you find three words $i, j, \ell$ such that
   $\min\{\mathbb{P}_\pi(T_i < T_j), \mathbb{P}_\pi(T_j < T_\ell), \mathbb{P}_\pi(T_\ell < T_i)\} > 1/2$  

3 Random walk on a finite graph.

Consider in this graph a finite, unoriented, connected graph $G = (V, E)$, where $V$ is the set of vertices and $E$ the set of unoriented edges between certain pairs of vertices of the graph. We assume in addition that the graph has no multiple edges, however we authorize self-loops. In particular $\#E \leq \#V(\#V - 1)/2 + \#V$.

The edges of the graph are weighted, i.e. an application

$$W : \begin{cases} V \times V & \to \mathbb{R}_+ \\ (u, v) & \to w_{u,v} \end{cases}$$

is given, with the convention that $w_{u,v} = w_{v,u}$ and $w_{u,u} = 0$ when there is no edge linking $u$ and $v$.

We also define, for $u \in V$, the positive real $w_u := \sum_{v \in V} w_{u,v}$. Finally the total weight of the edges of the graph is denoted $w := \sum_{u \in V} w_u$.

Note: An unweighted graph can be defined as a weighted graph of which every edge has weight 1. The random walk on $(G, W)$ is the Markov chain $X$ with state space $V$ which transition probabilities are

$$p_{uv} = \frac{w_{u,v}}{w_u}.$$

1. Check that our assumptions guarantee that the Markov chain $X$ is irreducible.
2. Set $\pi_v := w_v / w$. Show that for any $(u, v) \in V^2$, $\pi_u p_{uv} = \pi_v p_{vu}$.
3. Deduce that $\pi$ is the unique stationary distribution of the chain and that the latter is reversible.

Exercice 13 Chessboard.
Consider a king on a square of a chessboard, elsewhere empty. At each time step, move the king at random by choosing uniformly one of its legal moves.

1. How many time steps do we expect to wait until the king reaches its original position?
   Indication: One will show that the king is in fact performing a random walk on a finite unweighted graph. One shall separate different cases according to the king’s initial position on the chessboard.
2. Suppose the king is started at the top left corner (denoted $i$). What is the expected occupation time at the center of the chessboard (the center being the 4 central squares) before the king gets back to its initial position? If $j$ denotes one of the central squares, does one have $E_j[T_i] = E_i[T_j]$?

4 Some examples of Markov chains on a countable space

Exercice 14 Consider a sequence $(e_n)_{n \geq 0}$ of nonnegative real random variables. This sequence could for example model the life lengths of a sequence of lightbulbs. More precisely the lightbulbs are indexed by $\{0, 1, ...,\}$, and $e_n$ is the lifetime of the $n$th bulb.

We further define the successive record times:

$\tau_0 := 0$, $\tau_{n+1} := \inf\{k > \tau_n : e_k > e_{\tau_n}\}, n \geq 0$;

and the corresponding record life lengths:

$Z_n := e_{\tau_n}, n \geq 0$.

In this exercise, we will consider $(e_n) = (X_n)$ where the Markov chain $(X_n, n \geq 0)$ is defined as follows: $p \in (0, 1)$ is fixed, $q := 1 - p$, and $(X_n, n \geq 0)$ has transition kernel

$Q(x, y) = q^{y-x-1}p, x, y \in \mathbb{N}^*$.

1. Show that under $\mathbb{P}_x$, the sequence $(X_n)_{n \geq 0}$ is independent, and for every $n \geq 1$, $X_n \sim \text{Geom}(p)$.
2. Let $x \in \mathbb{N}^*$ be fixed. Compute

$\mathbb{P}_x(X_1 \leq x, ..., X_{k-1} \leq x, X_k > y), \quad k, y \in \mathbb{N}^*, y \geq x$.

3. Let $\tau := \inf\{n \geq 1 : X_n > X_0\}, \inf \emptyset := +\infty$. Compute

$\mathbb{P}_x(\tau = k, X_k > y), \quad k, y \in \mathbb{N}^*, y \geq x$.

Show that, under $\mathbb{P}_x$, $\tau$ is a geometric variable. What is its parameter? Show that $X_\tau$ has the same law as $x + X_1$. Under $\mathbb{P}_x$, is the pair $(\tau, X_\tau)$ independent?

4. Show that $\pi(x) = q^{x-1}p, x \in \mathbb{N}^*$, is the unique stationary distribution for $Q$. Is the chain positive recurrent? Show that $\mathbb{P}_\pi(\tau < \infty) = 1$, but $\mathbb{E}_\pi(\tau) = +\infty$. 


5. Show that \( \tau \) is a stopping time. Notice that for \( \tau_n, n \geq 1 \) defined as above we have 
\[
\tau_{n+1} = \tau_n + \tau \circ \theta_n.
\]
6. Show that, under \( P_x \), \( (Z_n)_{n \geq 0} \) is a Markov chain taking values in \( \mathbb{N}^* \) with respect to the filtration \( (\mathcal{F}_n) \). Compute its initial distribution, and its transition kernel.
7. For \( f : \mathbb{N}^* \to \mathbb{R} \) bounded, compute \( \mathbb{E}[f(Z_{n+1} - Z_n) | \mathcal{F}_n] \). Show that, under \( P_x \), the sequence \( (Z_n - Z_{n-1})_{n \geq 1} \) is i.i.d.
8. For any \( x \in \mathbb{N}^* \), compute, under \( P_x \), the almost sure limit \( P_x \) of \( Z_n / n \).

**Exercise 15 Renewal chain**

Consider \( (X_n, n \geq 0) \) a Markov chain with \( S = \mathbb{N} \), with transition kernel

\[
Q(x, y) := \begin{cases} 
 f(y + 1) & \text{if } x = 0, y \geq 0 \\
 1 & \text{if } x > 0, y = x - 1 \\
 0 & \text{otherwise,}
\end{cases}
\]

where \( f \) is a probability distribution on \( \mathbb{N}^* \), \( f : \mathbb{N}^* \to (0, 1) \), \( \sum_y f(y) = 1 \), such that \( f(y) > 0 \) for any \( y \in \mathbb{N}^* \). Let \( S_0 := 0, S_{n+1} := \inf \{ i > S_n : X_i = 0 \} \), the successive visits to state 0.

1. Show that \( P_0(S_1 = n) = f(n), n \geq 1 \). Deduce a classification of states into recurrent and transient classes.
2. Show that the measure \( \lambda \) on \( \mathbb{N} \)

\[
\lambda(x) := \sum_{y=x+1}^{\infty} f(y), x \in \mathbb{N}
\]

is \( Q \)-invariant and that any invariant measure is proportional to \( \lambda \).
3. Find a necessary and sufficient condition for \( X \) to be positive recurrent. Show there exists a unique stationary distribution for \( X \) iff

\[
m := \sum_{n \in \mathbb{N}} nf(n) < +\infty.
\]

We assume \( m < +\infty \) in the sequel.
4. Compute \( \lim_{n \to +\infty} P_x(X_n = y) \), for any \( x, y \in \mathbb{N} \).
5. Define \( u(n) := P_0(X_n = 0) \). Show that \( \{ X_0 = X_n = 0 \} = \bigcup_{z \geq 0} \{ X_0 = X_n = 0, S_1 = z \} \) and deduce that

\[
u(n) = \sum_{z=1}^{n} f(z) u(n - z) = (f \star u)(n); n \geq 1.
\]
6. Let \( t_i := S_i - S_{i-1}, i \geq 1 \). Show that, under \( P_0 \), \( (t_i)_{i \geq 1} \) is an i.i.d. sequence, then compute \( P_0(t_i = n), n \geq 1 \).
7. Show that \( P_0(S_i = n) = f^{*i}(n) \), where \( f^{*i} = f \star ... \star f \) is, for \( i \geq 1 \), the \( i \)th convolution of \( f \) with itself.
8. Show that \( \{ X_n = 0 \} = \bigcup_{i \in \mathbb{N}} \{ S_i = n \} \) and deduce

\[
u(n) = \sum_{i=1}^{\infty} f^{*i}(n), n \geq 1.
\]

9. Deduce from the preceding questions a renewal theorem : if \( u \) is defined by (2), then

\[
\lim_{n \to +\infty} u(n) = \frac{1}{m}.
\]
Exercice 16 Suppose \((X_n)_{n \geq 0}\) is an adapted process taking values in \(S = \mathbb{N}^*\) such that \(X_0 = 1\) p.s., and for any \(n \geq 0\):

\[
P(X_{n+1} - X_n = 1 \mid \mathcal{F}_n) = \exp(-X_n), \quad P(X_{n+1} - X_n = 0 \mid \mathcal{F}_n) = 1 - \exp(-X_n).
\]

1. Show that \((X_n)_{n \geq 0}\) is a Markov chain and compute its transition kernel \((Q(x, y) : x, y \in \mathbb{N}^*)\).

2. Show that a.s. \(X_\infty := \lim_{n \to \infty} X_n\) exists. Compute \(E(X_{n+1} \mid \mathcal{F}_n)\) and show by contradiction that \(E(X_\infty) = +\infty\).

3. For any \(n \geq 0\) compute \(P(X_n = 1)\).

4. For any \(n \geq m \geq 0\) compute \(P(X_n = X_m \mid \mathcal{F}_m)\).

5. For any \(m \geq 0\) compute

\[
\lim_{n \to \infty} P(X_n = X_m).
\]

6. Deduce that

\[
P\left(\liminf_{n \to \infty} (X_{n+1} = X_n)\right) = 0, \quad P\left(\limsup_{n \to \infty} (X_{n+1} = X_n + 1)\right) = 1
\]

and conclude that \(P(X_\infty = +\infty) = 1\).