Random Polymer Models

Disorder and Localization Phenomena

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A Polymer Close to a Membrane

Does the polymer stick to the membrane?
A Polymer Close to a Membrane

Does the polymer stick to the membrane?

- For localization: contact energy (inhomogeneities?)
- For delocalization: entropy (fluctuation freedom)
A Copolymer Near a Selective Interface

+ → hydrophobic (monomer)
− → hydrophilic

Does the polymer localize at the interface?
A Copolymer Near a Selective Interface

+ −→ hydrophobic (monomer)
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Does the polymer localize at the interface?

Possible relevant factors (in random order):

- Distribution of charges ($\pm$)
- Charge asymmetry (quantity or interaction)
- Polymer intrinsic properties
DNA Unbinding (and More)

Reduced models (Peyrard-Bishop, Poland-Scheraga)
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Reduced models (Peyrard-Bishop, Poland-Scheraga)

General framework:

- pinning of chains on *defect* structures
- The *defect* structure is (approximately) linear
- It may (or may not) be possible to go around the *defect*
On page 685 of *Walks, Walls, Wetting, and Melting* [M. E. Fisher, *JSP* (1984)] we read:

"In fact, there is a rather simple but general mathematical mechanism which underlies a broad class of exactly soluble one-dimensional models which display phase transitions. This mechanism does not seem to be as well appreciated as it merits and it operates in a number of applications we wish to discuss."
On page 685 of *Walks, Walls, Wetting, and Melting* [M. E. Fisher, JSP (1984)] we read:

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Fisher’s *broad class of models* is precisely the class we are focusing on if we limit ourselves to *homogeneous models* (and we start from there).

We aim at treating *inhomogeneous (disordered) models*. 
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From my web-page → research (.../pub/RPMay30.pdf.gz)
The Basic (Pinning) Model

Free process: Symmetric Random Walk \( \{S_n\}_n \)

\[
S_0 = 0, \quad S_n = \sum_{j=1}^{n} X_j, \\
\{X_j\}_j \text{ IID with } P(X_1 = x) = \frac{1}{3}, \quad x \in \{-1, 0, +1\}.
\]
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\( \{ X_j \} \) IID with \( P( X_1 = x ) = 1/3, \quad x \in \{-1, 0, +1\} \).

Coupling parameter: \( \beta \in \mathbb{R} \)

\[
\frac{dP^c_{N,\beta}}{dP}(S) = \frac{1}{Z^c_{N,\beta}} \exp \left( \beta \sum_{n=1}^{N} 1_{S_n=0} \right) 1_{S_N=0}.
\]
In this case: \( N = 16 \) and the energetic contribution is \( 6\beta \).

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\]
The Basic Pinning Model

Notation and facts:

- $\tau_0 = 0$ and $\tau_n = \inf\{m > \tau_{n-1} : S_m = 0\}$ if $\tau_{n-1} < \infty$
- $(\tau_n = \infty$ otherwise).
The Basic Pinning Model

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- By the (strong) Markov property \( \eta := \{\eta_i\}_{i \in \mathbb{N}} \), \( \eta_i = \tau_i - \tau_{i-1} \), is IID.
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- \( K(n) := P(\tau_1 = n). \) Well known that \( K(n) \overset{n \to \infty}{\sim} c_K n^{-3/2} \) (\( c_K > 0 \)). Moreover \( \sum_n K(n) = 1 \) (i.e. \( S \) is recurrent).
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2. By the (strong) Markov property \( \eta := \{\eta_i\}_{i \in \mathbb{N}}, \)
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\( K(n) \overset{\text{n} \to \infty}{\sim} c_K n^{-3/2} \) (\( c_K > 0 \)). Moreover \( \sum_n K(n) = 1 \)
(i.e. \( S \) is recurrent).

4. For us \( \tau := \{\tau_n\}_n \) is also a random subset of \( \mathbb{N} \). So
expressions like \( N \in \tau \) and \( \tau \cap A \) are meaningful.
The Basic Pinning Model

Call $F(\beta)$ the only solution of

$$\sum_{n} K(n) \exp(-F(\beta)n) = \exp(-\beta),$$

when such a solution exists, that is for $\beta \geq 0$, and $F(\beta) := 0$ otherwise.
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Note that:

- $F(\beta) > 0$ for $\beta > 0$.

- $F : \mathbb{R} \to [0, \infty)$ is non decreasing and $C^0$ (in fact: analytic on $\mathbb{R} \setminus \{0\} : \beta_c = 0$).
The Basic Pinning Model

Call $F(\beta)$ the only solution of

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$$

when such a solution exists, that is for $\beta \geq 0$, and $F(\beta) := 0$ otherwise.

Proposition 1. For every $\beta$

$$
F(\beta) = \lim_{N \to \infty} \frac{1}{N} \log Z_{N,\beta}^c.
$$
Proof of Prop. 1

The fundamental formula is

\[ Z_{N,\beta}^c = \sum_{n=1}^{N} \sum_{\ell \in \mathbb{N}^n : \sum_{j=1}^{n} \ell_j = N} \prod_{j=1}^{n} \exp(\beta) K(\ell_j). \]
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Set \( \tilde{K}_\beta(n) := \exp(\beta) K(n) \exp(-F(\beta)n) \). Then

\[ Z_{N,\beta}^c = \exp(F(\beta)N) \sum_{n=1}^{N} \sum_{\ell \in \mathbb{N}^n: \sum_{j=1}^{n} \ell_j = N} \prod_{j=1}^{n} \tilde{K}_\beta(\ell_j). \]
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Proof of Prop. 1

With $\tilde{P}_\beta$ the law of the renewal $\tau$ on $\mathbb{N} \cup \{\infty\}$ with inter-arrival law $\tilde{K}_\beta(\cdot)$:

$$Z_{N,\beta}^c = \exp \left( F(\beta)N \right) \tilde{P}_\beta(N \in \tau),$$
Proof of Prop. 1

With $\tilde{P}_\beta$ the law of the renewal $\tau$ on $\mathbb{N} \cup \{\infty\}$ with inter-arrival law $\tilde{K}_\beta(\cdot)$:

$$Z_{N,\beta}^c = \exp (F(\beta)N) \tilde{P}_\beta(N \in \tau),$$

so

$$Z_{N,\beta}^c \leq \exp (F(\beta)N),$$

and it suffices to show that

$$\lim \inf_{N \to \infty} \frac{1}{N} \log \tilde{P}_\beta(N \in \tau) \geq 0.$$
Proof of Prop. 1

If $\beta > \beta_c$: $m_{\tilde{K}_\beta} := \sum_n n\tilde{K}_\beta(n) < \infty$ and (by the Renewal Theorem) $\tilde{P}_\beta(N \in \tau)^{N \to \infty} \sim 1/m_{\tilde{K}_\beta} > 0$. 
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If $\beta < \beta_c$: use $\tilde{P}_\beta(N \in \tau) \geq \tilde{K}_\beta(N) = \exp(\beta)K(N)$, which is bounded below by $\text{const.}/N^{3/2}$. 
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- If $\beta < \beta_c$: use $\tilde{P}_\beta(N \in \tau) \geq \tilde{K}_\beta(N) = \exp(\beta)K(N)$, which is bounded below by $\text{const.} / N^{3/2}$.

- If $\beta = \beta_c$: $\tilde{K}_\beta(\cdot) = K(\cdot)$ and $\tilde{P}_\beta(N \in \tau) = P(N \in \tau) \geq K(N) \geq \text{const.} / N^{3/2}$. 
Proof of Prop. 1

If $\beta > \beta_c$: $m_{\tilde{K}_\beta} := \sum_n n\tilde{K}_\beta(n) < \infty$ and (by the Renewal Theorem) $\tilde{P}_\beta(N \in \tau) \overset{N \to \infty}{\sim} 1/m_{\tilde{K}_\beta} > 0$.

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The proof is complete. \qed
A Further Look at the Proof

We have in fact proven the sharp estimate

\[ Z_{N,\beta}^c = \frac{(1 + o(1))}{m \tilde{K}_\beta} \exp (N \mathcal{F}(\beta)), \]

for every \( \beta \).
A Further Look at the Proof

We have in fact proven the sharp estimate

\[ Z_{N,\beta}^c = \frac{(1 + o(1))}{m \tilde{K}_\beta} \exp(NF(\beta)), \]

for every \( \beta \).

This can be improved for \( \beta < \beta_c \)

\[ Z_{N,\beta}^c = (c_\beta + o(1))N^{-3/2}, \]

and at criticality:

\[ Z_{N,\beta_c}^c = (c_0 + o(1))N^{-1/2}. \]

We'll come back to this.
What about the trajectories?

By convexity the contact fraction is:

$$\partial_\beta F(\beta) = \lim_{N \to \infty} \frac{1}{N} \partial_\beta \log Z_{N,\beta}^c = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{N,\beta}^c \left[ \sum_{n=1}^{N} 1_{S_n=0} \right],$$

if $\partial_\beta F(\beta)$ exists.
What about the trajectories?

By convexity the contact fraction is:

\[
\partial_\beta F(\beta) = \lim_{N \to \infty} \frac{1}{N} \partial_\beta \log Z_N^{c,\beta} = \lim_{N \to \infty} \frac{1}{N} E_N^{c,\beta} \left[ \sum_{n=1}^{N} 1_{S_n=0} \right],
\]

if \( \partial_\beta F(\beta) \) exists.

Therefore:

- The contact fraction is zero for \( \beta < \beta_c \).
- The contact fraction is positive for \( \beta > \beta_c \).

So the localization can be characterized by \( F(\beta) > 0 \).
A basic (penetrable) wall model

Coupling parameter: $h \in [0, \infty)$

$$\frac{dP_{N,\lambda}^{c,+}}{dP}(S) = \frac{1}{Z_{N,h}^{c,+}} \exp \left( -h \sum_{n=1}^{N} 1_{S_n < 0} \right) 1_{S_N = 0}. $$
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\]

One sees immediately that \( F(h) = 0 \) and that

\[
-\partial_h F(h) = \lim_{N \to \infty} \frac{1}{N} E_{N,h}^{c,+} \left[ \sum_{n=1}^{N} 1_{S_n < 0} \right],
\]

so the walk spends (essentially) all its time in the upper half plane. Keyword: Entropic Repulsion.
A basic (penetrable) wall model

For a closer look let us write

\[ Z_{N,h}^{c,+} = \sum_{n=1}^{N} \sum_{\ell \in \mathbb{N}^n: \sum_{j=1}^{n} \ell_j = N} \prod_{j=1}^{n} \frac{1}{2} (\exp(-h(\ell_j - 1)) + 1) K(\ell_j). \]
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If we set \( K_h(\ell_j) \) and \( h > 0 \) we have \( \sum_n K_h(n) < 1 \).
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$$ Z_{N,h}^{c,+} = \sum_{n=1}^{N} \sum_{\ell \in \mathbb{N}^n: \sum_{j=1}^{n} \ell_j = N} \prod_{j=1}^{n} \frac{1}{2} \left( \exp(-h(\ell_j - 1)) + 1 \right) K(\ell_j). $$

If we set $K_h(\ell_j)$ and $h > 0$ we have $\sum_n K_h(n) < 1$.

This strongly suggests that, as $N \to \infty$, $P_{N,h}^{c,+}$ converges to the terminating renewal $\tau$ with inter-arrival distribution $K_h(\cdot)$ and results like ($L$ large)

$$ \lim_{N \to \infty} P_{N,h}^{c,+} (\tau \cap (L, N - L) = \emptyset) = 1 - o_L(1). $$
If $\tau$ is a general renewal with inter-arrivals taking values in $\mathbb{N}$ we introduce $\mathcal{N}_N(\tau) = |\tau \cap \{1, 2, \ldots, N\}|$ and

$$\frac{dP_{N,\beta}^c}{dP}(\tau) = \frac{1}{Z_{N,\beta}^c} \exp(\beta \mathcal{N}_N(\tau)) 1_{N \in \tau}.$$
The General Homogeneous Model

If $\tau$ is a general renewal with inter-arrivals taking values in $\mathbb{N}$ we introduce $\mathcal{N}_N(\tau) = |\tau \cap \{1, 2, \ldots, N\}|$ and

$$
\frac{dP_{N,\beta}^c}{dP}(\tau) = \frac{1}{Z_{N,\beta}^c} \exp \left( \beta \mathcal{N}_N(\tau) \right) 1_{N \in \tau}.
$$

We focus on inter-arrivals with law

$$
K(n) = \frac{L(n)}{n^{1+\alpha}}, \quad n = 1, 2, \ldots
$$

with $L(\cdot)$ slowly varying and $\alpha \geq 0$. We set $\Sigma_K := \sum_{n \in \mathbb{N}} K(n)$,

$K(\infty) := 1 - \Sigma_K$ and $m_K := \sum_{n \in \mathbb{N} \cup \{\infty\}} nK(n)$. 
The General Homogeneous Model

Call $F(\beta)$ the only solution of

$$\sum_{n} K(n) \exp(-F(\beta)n) = \exp(-\beta),$$

when such a solution exists, that is for $\beta \geq \beta_c = - \log \Sigma_K$, and $F(\beta) := 0$ otherwise.
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and $F(\beta) := 0$ otherwise.

Therefore:

- $F(\beta) > 0$ for $\beta > \beta_c.$
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and $F(\beta) := 0$ otherwise.

Proposition 2. For every $\beta$

$$F(\beta) = \lim_{N \to \infty} \frac{1}{N} \log Z_{c}^{N,\beta}.$$
The General Homogeneous model

Critical behavior:

**Theorem 3.** For every $\alpha \geq 0$ and every $L(\cdot)$ there exists a slowly varying function $\hat{L}(\cdot)$ such that for $\delta > 0$

$$F(\beta_c + \delta) = \delta^{1/\min(1,\alpha)} \hat{L}(1/\delta).$$

In particular if $\sum_{n \in \mathbb{N}} nK(n) < \infty$ then

$$\lim_{\delta \searrow 0} \hat{L}(1/\delta) = \Sigma K / \sum_{n \in \mathbb{N}} nK(n).$$
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*In particular if $\sum_{n \in \mathbb{N}} nK(n) < \infty$ then*

$$\lim_{\delta \searrow 0} \hat{L}(1/\delta) = \Sigma_K / \sum_{n \in \mathbb{N}} nK(n).$$

Therefore the transition is of $k^{th}$ order if $\alpha \in (1/k, 1/(k - 1))$. 
The General Homogeneous model

\[ \beta_c \]

\[ \alpha > 1 \]
The General Homogeneous Model

Sharp estimates on the partition function [Deuschel, G., Zambotti PTRF2005], [Caravenna, G., Zambotti, EJP2006]:

Theorem 4. As $N \to \infty$ we have:

1. (The localized regime.) If $\beta > \beta_c$ then

\[ Z_{N,\beta}^c \sim \frac{1}{m \tilde{K}_\beta} \exp(\mathcal{F}(\beta)N). \]

2. (The strictly delocalized regime.) If $\beta < \beta_c$ then

\[ Z_{N,\beta}^c \sim \frac{\exp(\beta)}{(1 - \varrho)^2} K(N), \]

where $\varrho := \exp(\beta) \Sigma_K (< 1)$. 
The General Homogeneous model

Theorem 4 (Cont.)

3. (The critical regime. ) If we set $\overline{K}(n) := \sum_{j > n} K(j)$ and $m_N := \sum_{n=0}^{N} \overline{K}(n)$ then $N \mapsto m_N$ is increasing and, if $\alpha \geq 1$, slowly varying and

$$Z_{N,\beta_c}^c \sim \sum K \times \begin{cases} 
1/m_N & \text{if } \alpha \geq 1, \\
\alpha \sin(\pi \alpha) N^{\alpha-1}/(\pi L(N)) & \text{if } \alpha \in (0, 1).
\end{cases}$$

- If $\sum_n nK(n) < \infty$ then $m_N$ may be substituted by $m_\infty$. If $K(\infty) = 0$ then $m_\infty$ is the mean of the inter-arrival.

- Subtle result in the critical regime [R. A. Doney (1997)]. Open for $\alpha = 0$. 
The General Homogeneous model

Why are we so much interested in sharp estimates on $Z_{N,\beta}^c$?
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Sharp estimates $\iff$ Sharp path behavior
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Sharp estimates $\iff$ Sharp path behavior

**Proposition 5.** For every $\beta$ the sequence $\left\{ P_{N,\beta}^c \right\}_N$ converges weakly to $\tilde{P}_\beta$, the law of the renewal $\tilde{\tau}_\beta$ with inter-arrival $\tilde{K}_\beta(\cdot)$.

- $\tilde{\tau}_\beta$ is positive recurrent if $\beta > \beta_c$.
- $\tilde{\tau}_\beta$ is terminating if $\beta < \beta_c$.
- $\tilde{\tau}_\beta$ is recurrent if $\beta = \beta_c$. 

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The General Homogeneous Model

Proof of Prop. 5: \( \ell_1, \ell_2 \in \mathbb{N} \) and \( q := \mathbf{P}_{N,\beta}^c (\tau_1 = \ell_1, \tau_2 = \ell_2) \)

\[
q = \frac{e^\beta K(\ell_1) e^\beta K(\ell_2) Z_{N-\ell_1-\ell_2,\beta}^c}{Z_{N,\beta}^c} = \tilde{K}_\beta(\ell_1) \tilde{K}_\beta(\ell_2) R_N,
\]

with

\[
R_N := \exp((\ell_1 + \ell_2) F(\beta)) \frac{Z_{N-\ell_1-\ell_2,\beta}^c}{Z_{N,\beta}^c} = \frac{\tilde{\mathbf{P}}_\beta(N - \ell_1 - \ell_2 \in \tau)}{\tilde{\mathbf{P}}_\beta(N \in \tau)},
\]

and \( R_N \xrightarrow{N \to \infty} 1 \) by the sharp estimates.

\[ \square \]

Scaling limits: \( \tau^{(N)} := \frac{\tau}{N} \cap [0, 1] \). Particularly interesting: critical case.
M. E. Fisher’s approach considers the Laplace transform of $Z_{N,\beta}^c$:

$$\Theta(\mu) := \sum_N \exp(-\mu N) Z_{N,\beta}^c,$$

which can be computed explicitly. By super-additive arguments it is easy to show that

$$\lim_N (1/N) \log Z_{N,\beta}^c$$

exists and this immediately implies that this limit coincides with $\inf \{\mu : \Theta(\mu) < \infty\}$. 
M. E. Fisher’s approach considers the Laplace transform of $Z^c_{N,\beta}$:

$$\Theta(\mu) := \sum_{N} \exp(-\mu N) Z^c_{N,\beta},$$

which can be computed explicitly. By super-additive arguments it is easy to show that

$$\lim_{N} (1/N) \log Z^c_{N,\beta}$$

exists and this immediately implies that this limit coincides with $\inf \{ \mu : \Theta(\mu) < \infty \}$.

Extracting sharp estimates requires Tauberian arguments.

[Erdös, Pollard, Feller, Port, Garsia, Lamperti,...(1949 onwards)]
A Copolymer Near a Selective Interface

+ $\rightarrow$ hydrophobic (monomer)
$- \rightarrow$ hydrophilic

Water

Oil

Contact Energy

Interface

Does the polymer localize at the interface?
A Copolymer Near a Selective Interface

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Does the polymer localize at the interface?

Possible relevant factors (in random order):

1. Distribution of charges (±)
2. Charge asymmetry (quantity or interaction)
3. Polymer intrinsic properties
Weakly Inhomogeneous Models

Periodic charge sequence \( \omega := \{ \omega_n \}_n : \)

\[ \omega_n \in \mathbb{R} \quad \text{and} \quad \omega_{n+T} = \omega_n, \]

for some \( T \in \mathbb{N} \) and every \( n \).
Weakly Inhomogeneous Models

Periodic charge sequence $\omega := \{\omega_n\}_n$:

$$\omega_n \in \mathbb{R} \quad \text{and} \quad \omega_{n+T} = \omega_n,$$

for some $T \in \mathbb{N}$ and every $n$.

A copolymer model: $S$ is the simple random walk ($N \in 2\mathbb{N}$), $\lambda, h \geq 0$, $\omega_n = (-1)^n$ (so $T = 2$) and $\text{sign}(0) = +1$

$$\frac{dP^c_{N,\omega,\lambda,h}}{dP}(S) = \frac{1}{Z^c_{N,\omega,\beta}} \exp \left( \lambda \sum_{n=1}^{N} (\omega_n + h) \text{sign}(S_n) \right) 1_{S_N=0}.$$

[Garel, Monthus, Orland (2000)], [Sommer, Daoud (1996)],...
Weakly Inhomogeneous Copolymer

\[
S_n = \lambda \sum_{n=1}^{N} (\omega_n + h) \text{sign} (S_n), \quad \text{sign}(0) = +1
\]
Weakly Inhomogeneous Copolymer

Observe that:

\[
\lambda \sum_{n=1}^{N} (\omega_n + h) \text{sign} (S_n) = \\
-2\lambda \sum_{n=1}^{N} (\omega_n + h) \mathbf{1}_{\text{sign}(S_n=-1)} + \lambda \sum_{n=1}^{N} (\omega_n + h).
\]

So: the copolymer is equivalent to a pinning model with

\[
\exp(\tilde{\beta}) := \sum_n \frac{1 + \exp(+2\lambda - 2\lambda h n)}{2} K(n),
\]

and \( K(n) \) replaced by the probability \( \tilde{K}(n) := \exp(-\tilde{\beta}) \frac{1 + \exp(+2\lambda - 2\lambda h n)}{2} K(n) \).
Weakly Inhomogeneous Copolymer

From this computation we extract the phase diagram:

\[ h \]
\[ \lambda \]

[D]-model

[SD]-model

\[ \mathcal{D} \]

\[ \mathcal{L} \]
Weakly Inhomogeneous Copolymer

More general case:

- $T > 2$. 
- Periodic pinning and periodic copolymer interaction.
Weakly Inhomogeneous Copolymer

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Possible strategy of solution: decimation (of step $T$).

Problem: the model does not renormalize to a pinning.
Weakly Inhomogeneous Copolymer

More general case:

- \( T > 2 \).

- Periodic pinning and periodic copolymer interaction.

Possible strategy of solution: decimation (of step \( T \)).

Problem: the model does not renormalize to a pinning.

[Bolthausen, G. AAP2005]: an expression for the free energy (LD arguments).
[Caravenna, G., Zambotti2005,2006]: Renewal theory approach (sharp estimates and path behavior). The underlying contact site process is a Markov renewal \( \rightarrow \) richer phenomenology.
Strongly Inhomogeneous Models

Program:

- Quenched disordered models: definitions.
Strongly Inhomogeneous Models

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Strongly Inhomogeneous Models

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Strongly Inhomogeneous Models

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- A smoothing mechanism.
Strongly Inhomogeneous Models

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A number of questions can be treated for copolymers and pinning at the same time. But on some other issues pinning and copolymer models have sharply different behaviors.
Disordered Models: Definition

Free process: \((\tau, s)\) couple of independent processes s.t.

- \(\tau\) renewal process on \(\mathbb{N} \cup \{\infty\}\) with inter-arrival law \(K(\cdot)\) \((K(n) = L(n)/n^{1+\alpha})\);
- \(s = \{s_j\}_{j \in \mathbb{N}}\) IID with \(P(s_1 = \pm 1) = 1/2\);

\(S\): walk s.t. \(\{n \in \mathbb{N} \cup \{0\} : S_n = 0\} = \tau \setminus \{\infty\}\) and \(\text{sign}(S_n) = s_{\tau_j}, j = \inf\{i : \tau_i \geq n\}\).
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Parameters: $v$

Charges $\omega \in \mathbb{R}^N$ (centered quenched disorder)

$$\frac{d\mathbb{P}_{N,\omega,v}^c}{d\mathbb{P}}(S) = \frac{1}{Z_{N,\omega,v}^c} \exp \left( \mathcal{H}_{N,\omega}^v(S) \right) \mathbf{1}_{\{S_N = 0\}}.$$
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Parameters: $\nu$

Charges $\omega \in \mathbb{R}^\mathbb{N}$ (centered quenched disorder)

$$\frac{d\mathbb{P}^f_{N,\omega,\nu}}{d\mathbb{P}}(S) = \frac{1}{Z^c_{N,\omega,\nu}} \exp \left( \mathcal{H}^\omega_{N,\omega}(S) \right).$$
Disordered Models: Definition

Copolymer: \( \mathbf{v} = (\lambda, h), \lambda, h \geq 0 \)

\[
\mathcal{H}_{N,\omega}^{\lambda,h}(S) = \lambda \sum_{n=1}^{N} (\omega_n + h) \text{sign} (S_n)
\]
Disordered Models: Definition

Copolymer: \( \psi = (\lambda, h), \lambda, h \geq 0 \)

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\mathcal{H}_{N,\omega}^{\lambda,h}(S) = \lambda \sum_{n=1}^{N} (\omega_n + h) \text{sign} (S_n) = \lambda \sum_{j=1}^{N_N(\tau)} s_j \sum_{n=\tau_{j-1}+1}^{\tau_j} (\omega_n + h).
\]
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\]

Pinning at a defect line: \( \underline{v} = (\beta, h), \beta \geq 0, h \in \mathbb{R} \)

\[
\mathcal{H}_{N,\omega}^{\beta,h}(S) = \sum_{n=1}^{N} (\beta \omega_n - h) \mathbf{1}_{\{S_n=0\}}
\]
Disordered Models: Definition

Copolymer: $\nu = (\lambda, h), \lambda, h \geq 0$

$$\mathcal{H}_{N,\omega}^{\lambda,h}(S) = \lambda \sum_{n=1}^{N} (\omega_n + h) \text{sign}(S_n) = \lambda \sum_{j=1}^{\mathcal{N}_N(\tau)} s_j \sum_{n=\tau_{j-1}+1}^{\tau_j} (\omega_n + h).$$

Pinning at a defect line: $\nu = (\beta, h), \beta \geq 0, h \in \mathbb{R}$

$$\mathcal{H}_{N,\omega}^{\beta,h}(S) = \sum_{n=1}^{N} (\beta \omega_n - h) 1\{S_n=0\} = \sum_{n \in \tau: 1 \leq n \leq N} (\beta \omega_n - h).$$
Choice of the disorder

Random (quenched):
\{\omega_n\}_n \text{ IID (law } \mathbb{P}\text{) such that}

- \(M(t) := \mathbb{E}[\exp(t\omega_1)] < \infty\) for \(|t|\text{ small}
- \(\mathbb{E}[\omega_1] = 0\text{ (let us say } \omega_1 \sim -\omega_1\))

Without loss of generality: \(\mathbb{E}[\omega_1^2] = 1\).

\[\Box\]

Stochastic Processes in Mathematical Physics, Firenze June 19–23, 2006 – p.34/65
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Without loss of generality: \( \mathbb{E}[\omega_1^2] = 1. \)

Copolymer: \( \lambda, h \geq 0, \Delta_n := (1 - \text{sign}(S_n))/2 \in \{0, 1\} \)

\[
\mathcal{H}_{N,\omega}^{\lambda,h}(S) = -2\lambda \sum_{n=1}^{N} (\omega_n + h) \Delta_n + c_N(\omega)
\]
The Free Energy

Theorem 5. The limit of the sequence of r.v.’s \( \left\{ \frac{1}{N} \log Z_{N,\omega,v}^a \right\}_N \) exists \( \mathbb{P}(d\omega) \)-a.s. and in \( L^1(\mathbb{P}) \), it is \( \mathbb{P}(d\omega) \)-a.s. equal to a constant (self-averaging!) that we denote by \( F(v) \).
The Free Energy

**Theorem 5.** The limit of the sequence of r.v.’s $\left\{ \frac{1}{N} \log Z_{N,\omega,v}^a \right\}_N$ exists $\mathbb{P}(d\omega)$–a.s. and in $L^1(\mathbb{P})$, it is $\mathbb{P}(d\omega)$–a.s. equal to a constant (self-averaging!) that we denote by $F(v)$.

Several "different" proofs: for example observe that

$$\log Z_{N,\omega}^c \geq \log Z_{M,\omega}^c + \log Z_{N-M,\omega}^c$$

for $M = 0, \ldots, N$. So $\left\{ \mathbb{E} \log Z_{N,\omega}^c \right\}_N$ is superadditive, so

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}^c = \sup_N \frac{1}{N} \mathbb{E} \log Z_{N,\omega}^c.$$ 

A concentration bound suffices to conclude. □
Free energy approach

The free energy:

$$\lim_{N \to \infty} \frac{1}{N} \log Z_{N,\omega,v}^f = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \left[ \exp \left( \mathcal{H}_{N,\omega}^{\lambda,h}(S) \right) \right] = F(\lambda, h)$$

$$\mathbb{P}(d\omega) - \text{a.s. and in } L^1(\mathbb{P}).$$

Identification of the free energy of the delocalized state:

$$\mathbb{E} \left[ \exp \left( \mathcal{H}_{N,\omega}^{\lambda,h}(S) \right) \right] \geq \mathbb{E} \left[ \exp \left( -2\lambda \sum_{n=1}^{N} (\omega_n + h)\Delta_n \right) \mid S_n > 0 \text{ for } n = 1, 2, \ldots, N \right]$$
Free energy approach

The free energy:

$$\lim_{N \to \infty} \frac{1}{N} \log Z_{N,\omega,v}^f = \lim_{N \to \infty} \frac{1}{N} \log E \left[ \exp \left( \mathcal{H}_{N,\omega}^{\lambda,h} (S) \right) \right] = F(\lambda, h)$$

\(\mathbb{P}(d\omega)\)-a.s. and in \(L^1(\mathbb{P})\).

Identification of the free energy of the delocalized state:

$$E \left[ \exp \left( \mathcal{H}_{N,\omega}^{\lambda,h} (S) \right) \right] \geq \mathbb{P} (\cdot) \approx 1/N^\alpha$$

$$E \left[ 1 ; S_n > 0 \text{ for } n = 1, 2, \ldots, N \right] = \exp(o(N))$$
Free energy approach

The free energy:

$$\lim_{N \to \infty} \frac{1}{N} \log Z_{N,\omega,v}^f = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \left[ \exp \left( \mathcal{H}_{N,\omega}^{\lambda,h}(S) \right) \right] = F(\lambda, h) \geq 0$$

$$\mathbb{P}(d\omega)$$–a.s. and in $L^1(\mathbb{P})$.

Identification of the *free energy of the delocalized state*:

$$\mathbb{E} \left[ \exp \left( \mathcal{H}_{N,\omega}^{\lambda,h}(S) \right) \right] \geq \mathbb{P} (\cdot) \approx 1/N^\alpha$$

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Localized and Delocalized regions

\[ \mathcal{D} = \{ \underline{v} : F(\underline{v}) = 0 \} \]

\[ \mathcal{L} = \{ \underline{v} : F(\underline{v}) > 0 \} \]
Localized and Delocalized regions

\[ \mathcal{D} = \{ v : F(v) = 0 \} \]
\[ \mathcal{L} = \{ v : F(v) > 0 \} \]

The analysis (not obvious!):

- Understanding the phase diagram
- What does \( v \in \mathcal{L} \) (or \( \mathcal{D} \)) mean for the polymer trajectories?
Localized and Delocalized regions

\[ \mathcal{D} = \{ \nu : F(\nu) = 0 \} \]
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The analysis (not obvious!):

- Understanding the phase diagram
- What does \( \nu \in \mathcal{L} \) (or \( \mathcal{D} \)) mean for the polymer trajectories?

Chronological math approach:
Path analysis \( \longrightarrow \) free energy approach
[S]:=[Sinai TPA93] \( \longrightarrow \) [BdH97]:=[Bolthausen, den Hollander AP97]
Pinning: Phase Diagram

\[
\text{Energy} = \sum_{n \in \tau \cap [1, N]} (\beta \omega_n - h)
\]

There exists \( h_c(\cdot) \) (convex, increasing) separating \( \mathcal{L} \) and \( \mathcal{D} \).

- The upper bound is the annealed bound.
- \( h_{LB}(\cdot) \geq 0 \) is easy. \( h_{LB}(\cdot) > 0 \): [Alexander, Sidoravicius 05], [G]
Copolymer: Phase Diagram

\[ h_c(\lambda) \]

\[ \text{Energy} = \lambda \sum_{n=1}^{N} (\omega_n + h) 1_{\text{sign}(S_n) = -1} \]

[S], [BdH], [BG]:=[Bodineau and G. JSP04], [GT05]:=[G., Toninelli PTRF05]
Copolymer: Phase Diagram

[S], [BdH], [BG]:=[Bodineau and G. JSP04], [GT05]:=[G., Toninelli PTRF05]
($\sum_n K(n) = 1$)

As $\lambda \downarrow 0$: $h_c(\lambda) = m_c \lambda + o(\lambda)$, $m_c \in [1/(1 + \alpha), 1]$. Recently: $m_c \geq 1/\sqrt{1 + \alpha}$ if $\alpha \geq 1$. 
Phase diagram estimates

\[ h^{(m)}(\lambda) := \frac{1}{2m\lambda} \log M(2m\lambda), \quad h^{(1/(1+\alpha))}(\lambda) \leq h_c(\lambda) \leq h^{(1)}(\lambda) \]
The physical literature is split: $\alpha = 1/2$ and $L(n) \overset{n \to \infty}{\sim} c > 0$
For $h^{(1)}(\cdot) = h_{c}(\cdot)$: [Garel et al. 89], [Maritan, Trovato 99] (Replica, Gauss case)
Phase diagram estimates

The physical literature is split: \( \alpha = 1/2 \) and \( L(n)^n \sim c > 0 \)

For \( h^{(1)}(\cdot) = h_c(\cdot) \): [Garel et al. 89], [Maritan, Trovato 99] (Replica, Gauss case)

For \( h^{(2/3)}(\cdot) = h_c(\cdot) \):

- [Stepanov et al. 98] (Replica, Gauss)
- [Monthus 00] (Fisher’s renormalization scheme)
What Does the Computer Say?

[Caravenna, G., Gubinelli JSP06]

- Statistical test based on concentration and super-additivity: $h_c(\cdot) > h^{(2/3)}(\cdot)$
- $h'_c(0) \approx 0.83 \pm 0.01$

[Garel, Monthus 05]
Testing for Localization

Basic ingredients:

- Super-additivity of \( \left\{ \mathbb{E} \log Z_{N,\omega,v}^c \right\}_N \) implies:

  \[ v \in \mathcal{L} \iff \text{there exists } N \text{ s.t. } \mathbb{E} \log Z_{N,\omega,v}^c > 0. \]
Testing for Localization

Basic ingredients:

- Super-additivity of \( \left\{ \mathbb{E} \log Z_{N,\omega,v}^c \right\}_N \) implies:

  \[ v \in \mathcal{L} \iff \text{there exists } N \text{ s.t. } \mathbb{E} \log Z_{N,\omega,v}^c > 0. \]

- For a wide class of r.v.'s \( \omega_1 \) we have the concentration bound: there exist \( c_1, c_2 > 0 \) such that for every convex \( G : \mathbb{R}^k \to \mathbb{R} \) satisfying \( |G(r) - G(r')| \leq C|r - r'| \) we have that for every \( t \)

  \[ \mathbb{P} \left( |G(\omega) - \mathbb{E}G(\omega)| \geq t \right) \leq c_1 \exp \left( -c_2 t^2 / C^2 \right). \]
In our case $k = nN$ and $G(\omega) := \frac{1}{n} \sum_{j=1}^{n} \log Z_{N,\theta j-1}\omega$. Now $C^2 = 16\lambda^2 N/n$ and

$$\mathbb{P} \left( \frac{1}{n} \sum_{j=1}^{n} \log Z_{N,\theta j-1}\omega - \mathbb{E} \left[ \log Z_{N,\omega} \right] \geq t \right) \leq c_1 \exp \left( -c_2 nt^2/(16\lambda^2 N) \right).$$
Testing for Localization

In our case $k = nN$ and $G(\omega) := \frac{1}{n} \sum_{j=1}^{n} \log Z_{N,\theta j \omega}^c$. Now $C^2 = 16\lambda^2 N/n$ and

$$\mathbb{P}\left( \frac{1}{n} \sum_{j=1}^{n} \log Z_{N,\theta j \omega}^c - \mathbb{E} \left[ \log Z_{N,\omega}^c \right] \geq t \right) \leq c_1 \exp \left( -c_2 n t^2 / (16\lambda^2 N) \right).$$

Therefore:

- fix $N$ and make the hypothesis that $\mathbb{E}[\log Z_{N,\omega}^c] \leq 0$.
- Compute $n$ independent samples of $\log Z_{N,\omega}^c$ (comp. time $O(nN^{3/2})$) and compute the mean $\hat{u}$ of the results.
- If $\hat{u} > 0$ we refuse the hypothesis (therefore: localization!) with a level of error smaller than $c_1 \exp \left( -c_2 n \hat{u}^2 / (16\lambda^2 N) \right)$. 
A quick look at the paths

*Real question:* what do the paths do?

- The localized phase $\sim$ Large Deviations regime
  
  [S], [Albeverio, Zhou JSP96]
  
  [Biskup, den Hollander AAP99] $\leftarrow$ Gibbs measure viewpoint
  
  [G., Toninelli Alea06] $\leftarrow$ two distinct correlation lengths
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  - If \( \underline{v} \in \hat{\mathcal{D}} \implies \) zero density \((o(N))\) of visits to the interface
    - [G., Toninelli PTRF05]: \( O(\log N) \) and \( O(1) \) results
  - If \( \underline{v} \in \partial \mathcal{D} \): \( O(N^{2/3}) \) visits to the interface [Toninelli JSP06]
A quick look at the paths

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The delocalized phase $\sim$ Moderate Deviations

If $\nu \in \mathcal{D}$ $\Longrightarrow$ zero density ($o(N)$) of visits to the interface

[G., Toninelli PTRF05]: $O(\log N)$ and $O(1)$ results

If $\nu \in \partial \mathcal{D}$: $O(N^{2/3})$ visits to the interface [Toninelli JSP06]

Expected in $\mathcal{D}$ (?): scaling toward Brownian meander
Theorem 6. Besides the standard assumptions on $\omega$, let us assume that there exist positive constants $c_1$ and $c_2$ s.t.

$$\mathbb{P} \left( G(\omega_1, \ldots, \omega_k) - \mathbb{E} G(\omega_1, \ldots, \omega_k) > t \right) \leq c_1 \exp \left( -c_2 t^2 \right),$$

for every $t \geq 0$ and every (convex) $G : \mathbb{R}^k \rightarrow \mathbb{R}$ such that $|G(x) - G(y)| \leq |x - y|$. Then for every $(\beta, h) \in \mathcal{D}$ there exist two positive constants $c$ and $q$ such that

$$\mathbb{E} \mathbb{P}_{N, \omega}^{\alpha} \left( \mathcal{N}_N(\tau) \geq n \right) \leq \exp \left( -cn \right),$$

both for $\alpha = c$ and $\alpha = f$ and for every $N$ and $n \geq q \log N$. 

On the Delocalized phase [GT05]
Proof of Theorem 6

Set

$$F_{N,\omega}^{a}(\beta, h; n) := \frac{1}{N} \log Z_{N,\omega}^{a}(\mathcal{N}_{N}(\tau) = n),$$

and

$$F_{N,\omega}^{a}(\beta, h) := \frac{1}{N} \log Z_{N,\omega}^{a}.$$
Proof of Theorem 6

Set

\[ F^{a}_{N,\omega}(\beta, h; n) := \frac{1}{N} \log Z^{a}_{N,\omega}(\mathcal{N}_{N}(\tau) = n), \]

and \( F^{a}_{N,\omega}(\beta, h) := (1/N) \log Z^{a}_{N,\omega}. \)

Important because:

\[ P^{c}_{N,\omega}(\mathcal{N}_{N}(\tau) \geq m) = \frac{1}{Z^{c}_{N,\omega}} \sum_{n \geq m} Z^{c}_{N,\omega}(\mathcal{N}_{N}(\tau) = n). \]
Proof of Theorem 6

Set

\[ F^a_{N,\omega}(\beta, h; n) := \frac{1}{N} \log Z^a_{N,\omega}(\mathcal{N}_N(\tau) = n), \]

and \( F^a_{N,\omega}(\beta, h) := (1/N) \log Z^a_{N,\omega}. \)

We have:

**Lemma 7.** Same assumptions as in Th. 6. We have

\[ \mathbb{P} \left( F^a_{N,\omega}(\beta, h; n) - \mathbb{E} \left[ F^a_{N,\omega}(\beta, h; n) \right] \geq u \right) \leq c_1 \exp \left( -\frac{c_2 u^2 N^2}{\beta^2 n} \right), \]

for every \( N \in \mathbb{N}, n \in \{0, 1, \ldots, N\} \) and every \( u \geq 0. \)
Proof of Theorem 6

Set
\[ F^a_{N,\omega}(\beta, h; n) := \frac{1}{N} \log Z^a_{N,\omega}(\mathcal{N}_N(\tau) = n), \]
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**Lemma 7.** *Same assumptions as in Th. 6. We have*

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*for every* \( N \in \mathbb{N}, n \in \{0, 1, \ldots, N\} \) *and every* \( u \geq 0. \)

If \( n = N \) we recover the standard fluctuation/concentration bound, but for small \( n \) the bound improves substantially.
Proof of Theorem 6

Choose \((\beta, h) \in \bar{D}\) and \(\varepsilon > 0\) s.t. \((\beta, h - \varepsilon) \in D\).

Super-additivity guarantees:

\[
(\beta, h) \in D \iff \mathbb{E} \left[ F_{N,\omega}(\beta, h) \right] \leq 0 \text{ for every } N \in \mathbb{N}.
\]
Proof of Theorem 6

Choose \((\beta, h) \in \mathcal{D} \) and \(\varepsilon > 0\) s.t. \((\beta, h - \varepsilon) \in \mathcal{D}\).

Super-additivity guarantees:

\[
(\beta, h) \in \mathcal{D} \iff \mathbb{E} [F^c_{N,\omega}(\beta, h)] \leq 0 \text{ for every } N \in \mathbb{N}.
\]

For every \(\omega\) we have the equivalence

\[
F^c_{N,\omega}(\beta, h;n) \geq -\frac{1}{2}\varepsilon n/N \iff F^c_{N,\omega}(\beta, h - \varepsilon;n) \geq \frac{1}{2}\varepsilon n/N,
\]

and \(F^c_{N,\omega}(\beta, h - \varepsilon;n) \leq F^c_{N,\omega}(\beta, h - \varepsilon)\).
Choose \((\beta, h) \in \bar{D}\) and \(\varepsilon > 0\) s.t. \((\beta, h - \varepsilon) \in D\).

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\]

and \(F^c_{N,\omega}(\beta, h - \varepsilon; n) \leq F^c_{N,\omega}(\beta, h - \varepsilon)\).

Note that \(\mathbb{E} F^c_{N,\omega}(\beta, h - \varepsilon) \leq 0\).
Therefore (Lemma 7)

$$
P \left( F_{N,\omega}^c (\beta, h; n) \geq -\frac{1}{2} \varepsilon n/N \right) \leq \nabla$$

$$\leq \mathbb{P} \left( F_{N,\omega}^c (\beta, h - \varepsilon; n) - \mathbb{E} \left[ F_{N,\omega}^c (\beta, h - \varepsilon; n) \right] \geq \frac{1}{2} \varepsilon n/N \right)$$

$$\leq c_1 \exp \left( -c_2 n \left( \frac{\varepsilon}{2|\beta|} \right)^2 \right).$$
Proof of Theorem 6

Therefore (Lemma 7)

\[
P \left( F_{N,\omega}^c (\beta, h; n) \geq -\frac{1}{2} \varepsilon n / N \right) \leq \\
P \left( F_{N,\omega}^c (\beta, h - \varepsilon; n) - \mathbb{E} \left[ F_{N,\omega}^c (\beta, h - \varepsilon; n) \right] \geq \frac{1}{2} \varepsilon n / N \right) \\
\leq c_1 \exp \left( -c_2 n \left( \frac{\varepsilon}{2|\beta|} \right)^2 \right).
\]

With \( E_m := \{ \omega : \exists n \geq m \text{ s.t. } F_{N,\omega}^c (\beta, h; n) \geq -\varepsilon n / 2N \} \), we have

\[
P (E_m) \leq \sum_{n \geq m} P \left( F_{N,\omega}^c (\beta, h; n) \geq -\frac{1}{2} \varepsilon n / N \right) \leq C_1^{-1} \exp(-C_1 m),
\]

for every \( m \) and some \( C_1 > 0 \).
Proof of Theorem 6

Choose now $\omega \in E_m^c$. We have ($\alpha > \alpha^+$):

$$P_{N,\omega}^c (\mathcal{N}_N(\tau) \geq m) = \frac{1}{Z_{N,\omega}^c} \sum_{n \geq m} Z_{N,\omega}^c (\mathcal{N}_N(\tau) = n)$$

$$\leq C_2 N^{1+\alpha} \exp (-\beta \omega_N + h) \sum_{n \geq m} \exp (-\varepsilon n/2)$$

$$\leq C_3 N^{1+\alpha} \exp (-\beta \omega_N + h) \exp (-\varepsilon m/2),$$

and by taking the expectation we conclude. \qed
Localization and Smoothing

Program:

- Copolymer model: lower bound on critical curve.
- All models: smoothing inequality.
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- All models: smoothing inequality.

Common idea: role of rare stretches of charges
Phase diagram estimates

\[ h^{(m)}(\lambda) := \frac{1}{2m\lambda} \log M(2m\lambda), \quad h^{(1/(1+\alpha))}(\lambda) \leq h_c(\lambda) \leq h^{(1)}(\lambda) \]
Phase diagram estimates

The physical literature is split: $\alpha = 1/2$ and $L(n) \overset{n \to \infty}{\sim} c > 0$

For $h^{(1)}(\cdot) = h_c(\cdot)$: [Garel et al. 89], [Maritan, Trovato 99] (Replica, Gauss case)
The physical literature is split: $\alpha = 1/2$ and $L(n)^n \sim c > 0$

For $h^{(1)}(\cdot) = h_c(\cdot)$: [Garel et al. 89], [Maritan, Trovato 99] (Replica, Gauss case)

For $h^{(2/3)}(\cdot) = h_c(\cdot)$:

- [Stepanov et al. 98] (Replica, Gauss)
- [Monthus 00] (Fisher’s renormalization scheme)
What Does the Computer Say?

[Caravenna, G., Gubinelli JSP06]

- Statistical test based on concentration and super-additivity: \( h_c(\cdot) > h^{(2/3)}(\cdot) \)
- \( h'_c(0) \approx 0.83 \pm 0.01 \)

[Garel, Monthus 05]
Rare Stretch Strategy

We want to prove:

\[ h^{(m)}(\lambda) := \frac{1}{2m\lambda} \log M(2m\lambda), \quad h^{(1/(1+\alpha))}(\lambda) \leq h_c(\lambda) \]  

(BG04)

Picture to keep in mind: \( N \rightarrow \infty \) and then \( \ell \rightarrow \infty \)
Proof of inequality (BG04)

\[ S_n \]

with \( Q_j := \sum_{n=(j-1)\ell+1}^{j\ell} (\omega_j + h) \). Note that \( \{Q_j\}_j \) is IID and

\[ p(\ell) := \mathbb{P}(Q_1 \leq -q\ell) \xrightarrow{\ell \to \infty} \exp(-\ell \Sigma(-q - h)), \]

with \( \Sigma(\cdot) \) the Cramér LD functional. \( \{G_j\}_j \) IID too.

We restrict \( Z_{N,\omega} \) to the trajectories on the figure (lower bound).
Proof of inequality (BG04)

In one excursion the contribution to $\log Z_{N,\omega}$ is:

$$\log K \left( G_j \frac{\ell}{p(\ell)} \right) + \log \left( K(\ell) \exp(2\lambda q\ell) \right) \approx -(1+\alpha) \Sigma (-q-h)\ell + 2\lambda q\ell.$$
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We can still optimize over $q$: we gain if

$$0 < \text{gain} := \sup_q (2\lambda q - (1 + \alpha)\Sigma(-q - h))$$

$$= -2\lambda h + (1 + \alpha) \sup_q (\left((-2\lambda/(1 + \alpha))q - \Sigma(q)\right))$$

$$= -2\lambda h + (1 + \alpha) \log M \left(-2\lambda/(1 + \alpha)\right),$$

obtaining, for $\ell$ sufficiently large, $\log Z_{N,\omega} \geq N p(\ell) \text{gain}/2$. 

▽
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Since $\text{gain} > 0 \iff h < h^{1/(1+\alpha)}(\lambda)$ we are done. □
On the Regularity of the Free Energy

Theorem 8. \([G. \text{ and Toninelli Alea06}] \quad \mathcal{F}(\cdot) \in C^\infty \text{ in } \mathcal{L}.

The proof is based on estimates on decay of correlations of local observables and on \([\text{von Dreifus, Klein and Fernando Perez CMP95}]

Intriguing issues: Griffiths singularities?
Regularity at criticality

Stated for pinning \((\beta \to \lambda)\) for copolymers

**Theorem 9.** [G. and Toninelli, CMP06 and PRL06]. For a wide class of disorders (including \(\omega_1\) bounded and \(\omega_1 \sim \mathcal{N}(0, 1)\)), for every \(\beta > 0\) there exists \(C'\) such that for every \(h\)

\[
f(\beta, h) \leq (1 + \alpha)C' (h - h_c(\beta))^2.
\]
Regularity at criticality

Stated for pinning ($\beta \rightarrow \lambda$ for copolymers)

**Theorem 9.** [G. and Toninelli, CMP06 and PRL06]. For a wide class of disorders (including $\omega_1$ bounded and $\omega_1 \sim \mathcal{N}(0, 1)$), for every $\beta > 0$ there exists $C'$ such that for every $h$

$$F(\beta, h) \leq (1 + \alpha)C' (h - h_c(\beta))^2.$$ 

Possibly more transparent when written as

$$0 \leq F(\beta, h) - F(\beta, h_c(\beta)) \leq (1 + \alpha)C' (h - h_c(\beta))^2,$$

(the result is non trivial only for $h < h_c(\beta)$.)
Regularity at criticality

Stated for pinning ($\beta \to \lambda$ for copolymers)

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Rephrasing once more:

$F(\lambda, \cdot)$ is $C^{1,1}$ at $h_c(\beta) \implies$ the transition is at least of second order (almost third...)


Smoothing?

Is this a smoothing/rounding effect? [Aizenman, Wehr CMP 90]
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We need to precise:

- the pure (i.e. non disordered) model
- the regularity of $F$ for the pure model
Smoothing?

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We need to precise:

- the pure (i.e. non disordered) model
- the regularity of $F$ for the pure model

The pure models are:

- Pinning: set $\beta = 0 \longrightarrow$ just reward $-h$ per visit. Explicitly solvable (seen).
- Copolymer: periodic ($T$) distribution of charges. (Almost) explicitly solvable too.
Smoothing! (Sometimes)

In all cases:

\[ F_{\text{pure}}(h) \xrightarrow{h \sim h_c} (h_c - h)^{1/\min(\alpha,1)} \hat{L}(h_c - h), \]
Smoothing! (Sometimes)

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Disordered above / pure below

Smoothing: like \( \alpha = 1/2 \)

0 \ldots 1/4 1/3 \( \alpha = 1/2 \) 4th 3rd order
\ldots \quad \text{second order} \quad \alpha = 1 \quad \text{first order}
Smoothing! (Sometimes)

In all cases:

\[ F^{\text{pure}}(h) \overset{h/h_c}{\sim} (h_c - h)^{1/\min(\alpha,1)} \hat{L}(h_c - h), \]

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Harris (pinning): noise is relevant if \( \alpha > 1/2 \) (irrelevant if \( \alpha < 1/2 \))
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[Forgacs et al. 1986], [Derrida et al. 1992] ... (!)
Smoothing! (Sometimes)

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\[ F_{\text{pure}}(h) \sim (h - h_c)^{1/\min(\alpha,1)} \hat{L}(h_c - h), \]

Disordered above / pure below

Smoothing: like \( \alpha = 1/2 \)

Harris (pinning): noise is relevant if \( \alpha > 1/2 \) (irrelevant if \( \alpha < 1/2 \))

[Cule, Hwa 97], [Tang, Chaté 01], [Garel, Monthus 05], [Coluzzi 05]

(Biomath: DNA denaturation, Poland-Scheraga model)
Rare stretch strategy (again)

On the proof of:

$$F(\lambda, h) \leq (1 + \alpha)C(\beta)(h - h_c(\beta))^2$$

(GT06)

Picture to keep in mind: $N \to \infty$ and then $\ell \to \infty$
On the proof of the smoothing inequality (Th. 9)

Restricted set-up: $\omega_1 \sim \mathcal{N}(0, 1)$

and only pinning model (energy= $\sum_{n \in \tau \cap [1,N]} (\beta \omega_n - h)$).
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Note that, observing $\frac{1}{\ell} \sum_{n=1}^{\ell} \omega_n \geq \delta/\beta$ implies for $\ell$ large (roughly) observing $\omega_j \sim \mathcal{N}(\delta/\beta,1)$ for $j = 1, \ldots, \ell$ and therefore

$$\log Z_{\ell,\omega,\beta,h} \approx \ell F(\beta, h - \delta).$$
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\[ \log Z_{\ell, \omega, \beta, h} \approx \ell F(\beta, h - \delta). \]

Precisely:

\[
\lim_{\ell \to \infty} \frac{1}{\ell} \log \mathbb{P} \left( \log Z_{\ell, \omega, \beta, h} \geq \ell F(\beta, h - \delta)/2 \right) \geq -\frac{1}{2} \left( \frac{\delta}{\beta} \right)^2.
\]
On the proof of the smoothing inequality (Th. 9)

Now apply the rare stretch strategy:

\[ F(\beta, h) \geq p(\ell) \left( \frac{1}{2} F(\beta, h - \delta) - 2(1 + \alpha) \left( \frac{\delta}{\beta} \right)^2 + o_\ell(1) \right). \]
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Set \( h = h_c(\beta) \), so that the LHS is zero and we obtain that for every \( \ell \)

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By letting \( \ell \to \infty \) we obtain the smoothing inequality. \( \square \)

Obs.: no CLT and no boundary! Rather: LDs and flexibility.
THE END