Anderson transitions and wave function multifractality
Part II

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Plan (tentative)

- Quantum interference, localization, and field theories of disordered systems
  - diagrammatics; weak localization; mesoscopic fluctuations
  - field theory: non-linear $\sigma$-model
  - quasi-1D geometry: exact solution, localization
  - RG, metal-insulator transition, criticality

- Wave function multifractality
  - wave function statistics in disordered systems
  - multifractality of critical wave functions
  - properties of multifractality spectra: average vs typical, possible singularities, relations between exponents, surface/corner multifractality

- Systems and models
  - Anderson transition in $D$ dimensions
  - Power-Law Random Banded Matrix model: 1D system with $1/r$ hopping
  - mechanisms of criticality in 2D systems
  - quantum Hall transitions in normal and superconducting systems

Wave function statistics

distribution function $\mathcal{P}(|\psi^2(r)|)$, correlation functions $\langle |\psi^2(r_1)\psi^2(r_2)| \rangle$ etc.

perturbative (diagrammatic) approach not sufficient

Field-theoretical method: $\sigma$-model Wegner 79, Efetov 82 (SUSY)

$$S[Q] \propto \int d^d r \text{Str}[ - D(\nabla Q(r))^2 - 2i\omega \Lambda Q(r) ] \quad Q^2(r) = 1$$

$Q \in \{\text{sphere } \times \text{ hyperboloid}\}$ “dressed” by Grassmannian variables

$\sigma$-model contains all the diffuson-cooperon diagrammatics + much more (strong localization; Anderson transition & RG; non-perturbative effects)

- zero mode ($Q = \text{const}$) $\longrightarrow$ RMT distribution $\mathcal{P}(|\psi^2(r)|)$
  
  $\psi(r)$ — uncorrelated Gaussian random variables

- diffusive modes $[\Pi_D(r_1, r_2)]$ $\longrightarrow$ deviations from RMT, long-range spatial correlations

parameter $g = \frac{D/L^2}{\Delta} \equiv \frac{\text{Thouless energy}}{\text{level spacing}} = \frac{G}{e^2/h}$ dimensionless conductance

$g \gg 1$: metal $\quad g \ll 1$: strong localization

quasi-1D: $g = \xi/L$, $\xi$ — localization length
Experiment:
Wave function statistics in disordered microwave billiards

Kudrolli, Kidambi, Sridhar, PRL 95
Wave function statistics

Distribution \( \mathcal{P}(t) \) of \( t = V|\psi^2(r)| \)

Metallic samples (dimensionless conductance \( g \gg 1 \)):

- **Main body of the distribution:**

  \[
  \mathcal{P}(t) = e^{-t} \left[ 1 + \frac{\kappa}{2} (2 - 4t + t^2) + \ldots \right] \quad (U)
  \]

  \[
  \mathcal{P}(t) = \frac{e^{-t/2}}{\sqrt{2\pi t}} \left[ 1 + \frac{\kappa}{2} \left( \frac{3}{2} - 3t + \frac{t^2}{2} \right) + \ldots \right] \quad (O)
  \]

  \( \kappa = \Pi_D(r, r) \propto 1/g \ll 1 \quad \text{(classical return probability)} \)

- **Asymptotic tail:**

  \[
  \mathcal{P}(t) \propto \begin{cases} 
  \exp\{-\ldots \sqrt{t}\} &, & \text{quasi-1D} \\
  \exp\{-\ldots \ln^d t\} &, & d = 2, 3
  \end{cases}
  \]
Wave function statistics: numerical verification

numerics: Uski, Mehlig, Römer, Schreiber 01

quasi-1d, metallic regime, distribution of \( t = V |\psi^2(r)| \)

"body": \( \frac{1}{g} \) corrections to RMT

"tail": \( \propto \exp(-\ldots \sqrt{t}) \)

Physics of the slowly decaying "tail": anomalously localized states
Anomalously localized states: imaging

numerics: Uski, Mehlig, Schreiber ’02

spatial structure

of an anomalously localized state (ALS):

$$\langle |\psi(r)|^2 \delta(V|\psi(0)|^2 - t) \rangle$$

with $t$ atypically large

ADM ’97

ALS determine asymptotic behavior of distributions of various quantities (wave function amplitude, local and global density of states, relaxation time, . . . )

Altshuler, Kravtsov, Lerner, Fyodorov, ADM, Muzykantskii, Khmelnitskii, Falko, Efetov . . .
Wave function statistics: numerical verification. 2D.

Numerics: Uski, Mehlig, Römer, Schreiber ’01

“body” of the distribution:

\[ (1/g) \ln(L/l) \text{ corrections} \]

“tail” \( \propto \exp(-\ldots \ln^2 t) \)

Falko, Efetov ’95

\[ \rightarrow \text{precursors of Anderson criticality} \]
Multifractality at the Anderson transition

\[ P_q = \int d^d r |\psi(r)|^{2q} \quad \text{inverse participation ratio} \]

\[ \langle P_q \rangle \sim \begin{cases} 
L^0 & \text{insulator} \\
L^{-\tau_q} & \text{critical} \\
L^{-d(q-1)} & \text{metal} 
\end{cases} \]

\[ \tau_q = d(q-1) + \Delta_q \equiv D_q(q-1) \quad \text{multifractality} \]

normal anomalous \hspace{1cm} \Delta_0 = \Delta_1 = 0

Wave function correlations at criticality:

\[ L^{2d} \langle |\psi^2(r)\psi^2(r')| \rangle \sim \left( |r - r'|/L \right)^{-\eta}, \quad \eta = -\Delta_2 \]

\[ L^{d(q_1+q_2)} \langle |\psi^{2q_1}(r_1)\psi^{2q_2}(r_2)| \rangle \sim L^{-\Delta_{q_1} - \Delta_{q_2}} (|r_1 - r_2|/L)^{\Delta_{q_1+q_2} - \Delta_{q_1} - \Delta_{q_2}} \]

many-points correlators \hspace{1cm} \langle |\psi^{2q_1}(r_1)\psi^{2q_2}(r_2) \ldots \psi^{2q_n}(r_n)| \rangle \quad \text{— analogously}

different eigenfunctions:

\[ L^{2d} \langle |\psi^2_i(r)\psi^2_j(r')| \rangle \]
\[ L^{2d} \langle \psi_i(r)\psi^*_j(r)\psi^*_i(r')\psi_j(r') \rangle \sim \left( \frac{|r - r'|}{L_\omega} \right)^{-\eta} \]

\[ \omega = \epsilon_i - \epsilon_j \quad L_\omega \sim (\rho \omega)^{-1/d} \quad \rho = \text{DoS} \quad |r - r'| < L_\omega \]
Multifractality and the field theory

\[ \Delta_q - \text{scaling dimensions of operators} \quad \mathcal{O}^{(q)} \sim (Q\Lambda)^q \]

\[ d = 2 + \epsilon: \quad \Delta_q = -q(q - 1)\epsilon + O(\epsilon^4) \quad \text{Wegner '80} \]

- Infinitely many operators with negative scaling dimensions
- \( \Delta_1 = 0 \quad \leftrightarrow \quad \langle Q \rangle = \Lambda \quad \text{naive order parameter uncritical} \)

Transition described by an order parameter function \( F(Q) \)

\[ \text{Zirnbauer 86, Efetov 87} \]

\[ \leftrightarrow \quad \text{distribution of local Green functions and wave function amplitudes} \]

\[ \text{ADM, Fyodorov '91} \]
Singularity spectrum

\[ \tau_q \rightarrow \text{Legendre transformation} \]
\[ \tau_q = q\alpha - f(\alpha) , \quad q = f'(\alpha) , \quad \alpha = \tau_q' \]
\[ \rightarrow \text{singularity spectrum } f(\alpha) \]

\[ \mathcal{P}(\psi^2) \sim \frac{1}{|\psi^2|} L^{-d+f\left(-\frac{\ln |\psi^2|}{\ln L}\right)} \quad \text{wave function statistics} \]

To verify: calculate moments \( \langle P_q \rangle \) and use saddle-point method

\[ \langle P_q \rangle = L^d \langle |\psi^{2q}| \rangle \sim \int d\alpha L^{-q\alpha+f(\alpha)} , \quad \alpha = -\ln |\psi^2|/\ln L \]

\[ L^{f(\alpha)} - \text{measure of the set of points where } |\psi|^2 \sim L^{-\alpha} \]

- \( \tau_q \) – non-decreasing, convex (\( \tau_q' \geq 0, \tau_q'' \leq 0 \)), with \( \tau_0 = -d, \tau_1 = 0 \)
- \( f(\alpha) \) – convex (\( f''(\alpha) \leq 0 \)), defined on \( \alpha \geq 0 \), maximum \( f(\alpha_0) = d \).

Statistical ensemble \( \rightarrow f(\alpha) \) may become negative
Multifractal wave functions at the Quantum Hall transition
Role of ensemble averaging: Average vs typical spectra

\[ P_{q}^{\text{typ}} = \exp(\ln P_{q}) \sim L^{-\tau_{q}^{\text{typ}}}, \]

\[ \tau_{q}^{\text{typ}} = \begin{cases} 
q\alpha_{-}, & q < q_{-} \\
\tau_{q}, & q_{-} < q < q_{+} \\
q\alpha_{+}, & q > q_{+} 
\end{cases} \]

Singularity spectrum \( f^{\text{typ}}(\alpha) \) is defined on \([\alpha_{+}, \alpha_{-}]\), where it is equal to \( f(\alpha) \).

**IPR distribution:**

\( \mathcal{P}(P_{q}/P_{q}^{\text{typ}}) \) is scale-invariant at criticality

Power-law tail at large \( P_{q}/P_{q}^{\text{typ}} \):

\[ \mathcal{P}(P_{q}/P_{q}^{\text{typ}}) \propto (P_{q}/P_{q}^{\text{typ}})^{-1-x_{q}} \]

tail exponent \( x_{q} \) \( \begin{cases} 
= 1, & q = q_{\pm} \\
> 1, & q_{-} < q < q_{+} \\
< 1, & \text{otherwise} 
\end{cases} \)

\[ x_{q} \tau_{q}^{\text{typ}} = \tau_{q} x_{q} \]
Dimensionality dependence of multifractality

RG in $2 + \epsilon$ dimensions, 4 loops, orthogonal and unitary symmetry classes

Wegner ’87

\[ \Delta_q^{(O)} = q(1 - q)\epsilon + \frac{\zeta(3)}{4} q(q - 1)(q^2 - q + 1)\epsilon^4 + O(\epsilon^5) \]

\[ \Delta_q^{(U)} = q(1 - q)(\epsilon/2)^{1/2} - \frac{3}{8} q^2(q - 1)^2 \zeta(3)\epsilon^2 + O(\epsilon^{5/2}) \]

$\epsilon \ll 1 \quad \longrightarrow \quad$ weak multifractality

\[ \tau_q \simeq d(q - 1) - \gamma q(q - 1), \quad \Delta_q \simeq \gamma q(1 - q), \quad \gamma \ll 1 \]

\[ f(\alpha) \simeq d - \frac{(\alpha - \alpha_0)^2}{4(\alpha_0 - d)}; \quad \alpha_0 = d + \gamma \]

$\gamma = \epsilon$ (orthogonal); $\gamma = (\epsilon/2)^{1/2}$ (unitary)

\[ q_\pm = \pm (d/\gamma)^{1/2} \]
Dimensionality dependence of multifractality: IPR distribution

Mildenberger, Evers, ADM ’02

IPR distribution in 3D and 4D
3D: $L = 8, 11, 16, 22, 32, 44, 64, 80$
4D: $L = 8, 10, 12, 14, 16$

variance $\sigma_q$ of distribution $\mathcal{P}(\ln P_q)$
$2 + \epsilon$ with $\epsilon = 0.2, 1$ (analytically),
3D, 4D (numerically)

one-loop results in $d = 2 + \epsilon$:

$$\sigma_q \approx 0.00387 \quad \text{(periodic b.c.)}$$

\[
\sigma_q = 8\pi^2 a \epsilon^2 q^2 (q - 1)^2 , \quad |q| \ll q_+ ,
\]

\[
\sigma_q \approx x_q^{-1} = q / q_+ , \quad q > q_+
\]
Dimensionality dependence of multifractality spectra

Analytics (2 + ε, one-loop) and numerics

$$\tau_q = (q - 1)d - q(q - 1)\epsilon + O(\epsilon^4)$$

$$f(\alpha) = d - (d + \epsilon - \alpha)^2/4\epsilon + O(\epsilon^4)$$

- $d = 4$ (full)
- $d = 3$ (dashed)
- $d = 2 + \epsilon$, $\epsilon = 0.2$ (dotted)
- $d = 2 + \epsilon$, $\epsilon = 0.01$ (dot-dashed)

Inset: $d = 3$ (dashed)

vs. $d = 2 + \epsilon$, $\epsilon = 1$ (full)

Mildenberger, Evers, ADM ’02
Singularities in multifractal spectra: Termination and freezing

(a) no singularities    (b) termination    (c) freezing
Relations between multifractal exponents

Non-linear $\sigma$-model

distribution of local Green function $G_R(r, r) / \pi \langle \rho \rangle = u - i\tilde{\rho}$

$$P(u, \tilde{\rho}) = \frac{1}{2\pi\tilde{\rho}} P_0 \left( (u^2 + \tilde{\rho}^2 + 1) / 2\tilde{\rho} \right)$$

symmetry of LDOS distribution: $P_\rho(\tilde{\rho}) = \tilde{\rho}^{-3} P_\rho(\tilde{\rho}^{-1})$

ADM, Fyodorov ’94 ($\beta = 2$)

Recently:

more complete derivation ($\beta = 1, 2, 4$) via relation to a scattering problem:

system with a channel attached at a point $r$

S-matrix $S = (1 - iK) / (1 + iK)$

$K = \frac{1}{2} V^\dagger G_R(r, r) V = u - i\tilde{\rho}$

distribution $P(S)$ invariant with respect to the phase $\theta$ of $S = \sqrt{r} e^{i\theta}$.

Fyodorov, Savin, Sommers ’04-05
Relations between multifractal exponents (cont’d)

ADM, Fyodorov, Mildenberger, Evers ’06

LDOS distribution in $\sigma$-model + universality

$\longrightarrow$ exact symmetry of the multifractal spectrum:

$$\Delta_q = \Delta_{1-q} \quad f(2d - \alpha) = f(\alpha) + d - \alpha$$

Consequence: assuming no singularities in $f(\alpha)$ spectrum,
its support is bounded by the interval $[0, 2d]$
Relations between multifractal exponents (cont’d):
Anderson transition in symplectic class in 2D

Mildenberger, Evers ’07

Symmetry of the spectrum: \( \Delta_q = \Delta_{1-q} \)

Non-parabolicity of the spectrum

\[
\delta(q) \equiv \frac{\Delta_q}{q(1-q)} \neq \text{const}
\]

Conformal invariance

2D \leftrightarrow \text{quasi-1D strip}

\[
\Lambda_c = \frac{1}{\pi \delta_0}
\]

\( \delta_0 \approx 0.172 \), \( \Lambda_c \approx 1.844 \)

\( \rightarrow \pi \delta_0 \Lambda_c = 0.999 \pm 0.003 \)

Related work:

Obuse, Subramaniam, Furusaki, Gruzberg, Ludwig ’07
Relations between multifractal exponents (cont’d)

Wigner delay time: \( t_W = \partial \theta(E)/\partial E \)

Relation between wave function \((P_y)\) and delay time \((P_W)\) distribution in the \(\sigma\)-model:

\[
P_W(\tilde{t}_W) = \tilde{t}_W^{-3}P_y(\tilde{t}_W^{-1}) \quad \tilde{t}_W = t_W\Delta/2\pi
\]

Ossipov, Fyodorov ’05

+ universality

\[\rightarrow\] exact relation between multifractal exponents

for closed (wave functions, \(\tau_q\)) and open (delay times, \(\gamma_q\)) systems

\[
\gamma_q = \tau_{1+q}
\]

ADM, Fyodorov, Mildenberger, Evers ’06
**Surface multifractality**

**Subramaniam, Gruzberg, Ludwig, Evers, Mildenberger, ADM ’06**

Critical fluctuations of wave functions at surface: new set of exponents

\[ L^{d-1} \langle |\psi(r)|^{2q} \rangle \sim L^{-\tau_q^s} \]

\[ \tau_q^s = d(q - 1) + q\mu + 1 + \Delta_q^s \]

Weak multifractality (2 + \( \epsilon \) or 2D):

\[ \gamma = (\beta\pi g)^{-1} \ll 1 \]

\[ \tau_q^b = 2(q - 1) + \gamma q(1 - q) \]

\[ \tau_q^s = 2(q - 1) + 1 + 2\gamma q(1 - q) \]

\[ f^b(\alpha) = 2 - (\alpha - 2 - \gamma)^2/4\gamma \]

\[ f^s(\alpha) = 1 - (\alpha - 2 - 2\gamma)^2/8\gamma \]

Studied numerically for a variety of critical systems: PRBM, IQHE, SQHE, 2D symplectic
Corner multifractality

Obuse, Subramaniam, Furusaki, Gruzberg, Ludwig ’07

Conformal invariance

\[ \Delta^\theta_q = \frac{\pi}{\theta} \Delta^s_q \]

\[ f_\theta(\alpha^\theta_q) = \frac{\pi}{\theta} [f_s(\alpha^s_q) - 1] \]

\[ \alpha^\theta_q - 2 = \frac{\pi}{\theta} [\alpha^s_q - 2] \]
Power-law random banded matrix model (PRBM)

Anderson transition: dimensionality dependence:
\( d = 2 + \epsilon \): weak disorder/coupling \( d \gg 1 \): strong disorder/coupling

Evolution from weak to strong coupling – ?

PRBM \( \text{ADM, Fyodorov, Dittes, Quezada, Seligman '96} \)

\( N \times N \) random matrix \( H = H^\dagger \)

\[ \langle |H_{ij}|^2 \rangle = \frac{1}{1 + |i - j|^2/b^2} \]

\[ \longleftrightarrow \text{1D model with } 1/r \text{ long range hopping} \]

Critical for any \( b \) \( \longrightarrow \) family of critical theories!

\( b \gg 1 \) analogous to \( d = 2 + \epsilon \) \( b \ll 1 \) analogous to \( d \gg 1 \) (?)

Analytics:
\( b \gg 1 \): \( \sigma \)-model RG
\( b \ll 1 \): real space RG

Numerics:
efficient in a broad range of \( b \)

Evers, ADM ’01
Weak multifractality, $b \gg 1$

supermatrix $\sigma$-model

$$S[Q] = \frac{\pi \rho \beta}{4} \text{Str} \left[ \pi \rho \sum_{rr'} J_{rr'} Q(r) Q(r') - i\omega \sum_r Q(r)\Lambda \right].$$

In momentum $(k)$ space and in the low-$k$ limit:

$$S[Q] = \beta \text{Str} \left[ -\frac{1}{t} \int \frac{dk}{2\pi} |k| Q_k Q_{-k} - \frac{i\pi \rho \omega}{4} Q_0\Lambda \right]$$

DOS \quad \rho(E) = (1/2\pi^2 b)(4\pi b - E^2)^{1/2}, \quad |E| < 2\sqrt{\pi b}

coupling constant \quad \frac{1}{t} = (\pi/4)(\pi \rho)^2 b^2 = (b/4)(1 - E^2/4\pi b)

$\rightarrow$ \quad weak multifractality

$$\tau_q \simeq (q - 1)(1 - qt/8\pi \beta), \quad q \ll 8\pi \beta/t$$

$$E = 0, \quad \beta = 1 \quad \rightarrow \quad \Delta_q = \frac{1}{2\pi b} q(1 - q)$$
Multifractality in PRBM model: analytics vs numerics

**Numerics:**  
- $b = 4, 1, 0.25, 0.01$

**Analytics:**  
- $b \gg 1$ (σ–model RG), $b \ll 1$ (real-space RG)
Scale-invariant IPR distribution

IPR variance

\[
\text{var}(P_q) / \langle P_q \rangle^2 = q^2(q - 1)^2 / 24\beta^2 b^2 , \quad q \ll q_+(b) \equiv (2\beta\pi b)^{1/2}
\]

IPR distribution function for \( P_q / \langle P_q \rangle - 1 \ll 1 \):

\[
\mathcal{P}(\tilde{P}) = e^{-\tilde{P} - C} \exp(-e^{-\tilde{P} - C}) , \quad \tilde{P} = \left[ \frac{P_q}{\langle P_q \rangle} - 1 \right] \frac{2\pi\beta b}{q(q - 1)}
\]

\( C \simeq 0.5772 \) – Euler constant

\( P_q / \langle P_q \rangle - 1 \gg 1 \): power-law tail

\[
\mathcal{P}(P_q) \sim (P_q / \langle P_q \rangle)^{-1-x_q} , \quad x_q = 2\pi\beta b / q^2 , \quad q^2 < 2\pi\beta b
\]
Scale-invariant IPR distribution (cont’d)

Distribution $\mathcal{P}(\ln P_2)$ for $b = 1$ and $L = 256, 512, 1024, 2048, 4096$

Distribution $\mathcal{P}(\tilde{P}_q)$ at $b = 4$
for $q = 2$ (○), 4 (□), and 6 (◇)
Solid line — analytical result
for $q \ll q_+(b) = (8\pi)^{1/2} \simeq 5$.

Inset: Power-law asymptotics of $\mathcal{P}(\tilde{P}_4)$.
Power-law exponent: numerically $x_4 = 1.7$, analytically ($b \gg 1$): $x_4 = \pi/2$
Strong multifractality, $b \ll 1$

Real-space RG:

- start with diagonal part of $\hat{H}$: localized states with energies $E_i = H_{ii}$
- include into consideration $H_{ij}$ with $|i - j| = 1$

Most of them irrelevant, since $|H_{ij}| \sim b \ll 1$, while $|E_i - E_j| \sim 1$

Only with a probability $\sim b$ is $|E_i - E_j| \sim b$

$\longrightarrow$ two states strongly mixed ("resonance") $\longrightarrow$ two-level problem

$$\hat{H}_{two\text{-level}} = \begin{pmatrix} E_i & V \\ V & E_j \end{pmatrix}; \quad V = H_{ij}$$

New eigenfunctions and eigenenergies:

$$\psi^{(+)} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}; \quad \psi^{(-)} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$

$$E_{\pm} = (E_i + E_j)/2 \pm |V|\sqrt{1 + \tau^2}$$

$$\tan \theta = -\tau + \sqrt{1 + \tau^2} \text{ and } \tau = (E_i - E_j)/2V$$

- include into consideration $H_{ij}$ with $|i - j| = 2$
- ...
Evolution equation for IPR distribution ("kinetic eq." in "time" $t = \ln r$):

$$\frac{\partial}{\partial \ln r} f(P_q, r) = \frac{2b}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sin^2 \theta \cos^2 \theta}$$

$$\times \left[ -f(P_q, r) + \int dP_q^{(1)} dP_q^{(2)} f(P_q^{(1)}, r) f(P_q^{(2)}, r) \right.$$

$$\times \delta(P_q - P_q^{(1)} \cos^{2q} \theta - P_q^{(2)} \sin^{2q} \theta) \left. \right]$$

$$\rightarrow \text{ evolution equation for } \langle P_q \rangle: \quad \frac{\partial \langle P_q \rangle}{\partial \ln r} = -2b T(q) \langle P_q \rangle \quad \text{with}$$

$$T(q) = \frac{1}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sin^2 \theta \cos^2 \theta} (1 - \cos^{2q} \theta - \sin^{2q} \theta) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(q - 1/2)}{\Gamma(q - 1)}$$

$$\rightarrow \text{ multifractality } \langle P_q \rangle \sim L^{-\tau_q}, \quad \tau_q = 2b T(q)$$

This is applicable for $q > 1/2$

For $q < 1/2$ resonance approximation breaks down; use $\Delta_q = \Delta_{1-q}$
Strong multifractality, \( b \ll 1 \) (cont’d)

**T(q) asymptotics:**

\[
T(q) \simeq -1/[\pi(q - 1/2)] , \quad q \to 1/2 ;
\]

\[
T(q) \simeq (2/\sqrt{\pi})q^{1/2} , \quad q \gg 1
\]

**Singularity spectrum:**

\[
f(\alpha) = 2bF(A) ; \quad A = \alpha/2b , \quad F(A) - \text{Legendre transform of } T(q)
\]

**F(A) asymptotics:**

\[
F(A) \simeq -1/\pi A , \quad A \to 0 ;
\]

\[
F(A) \simeq A/2 , \quad A \to \infty
\]
Multifractality in PRBM model: analytics vs numerics

**Numerics:** $b = 4, 1, 0.25, 0.01$

**Analytics:** $b \gg 1$ (σ–model RG), $b \ll 1$ (real-space RG)
Critical level statistics

Two-level correlation function:

\[
R_2^{(c)}(s) = \langle \rho \rangle^{-2} \langle \rho (E - \omega / 2) \rho (E + \omega / 2) \rangle - 1 ; \quad \rho (E) = V^{-1} \text{Tr} \delta (E - \hat{H})
\]

\[
s = \omega / \Delta, \quad \Delta = 1 / \langle \rho \rangle V - \text{mean level spacing}
\]

Level number variance:

\[
\langle \delta N(\mathcal{E})^2 \rangle = \int_{-\langle N(\mathcal{E}) \rangle}^{\langle N(\mathcal{E}) \rangle} ds (\langle N(\mathcal{E}) \rangle - |s|) R_2^{(c)}(s)
\]

Spectral compressibility \( \chi \):

\[
\langle \delta N^2 \rangle \simeq \chi \langle N \rangle
\]

unitary symmetry \((\beta = 2)\):

RMT: \( R_2^{(c)}(s) = \delta (s) - \sin^2 (\pi s) / (\pi s)^2 \), \( \chi = 0 \)

Poisson: \( R_2^{(c)}(s) = \delta (s) \), \( \chi = 1 \)

Criticality: intermediate scale-invariant statistics, \( 0 < \chi < 1 \)
Critical level statistics in PRBM ensemble

- $b \gg 1 \quad \rightarrow \quad \sigma$-model \quad $\rightarrow$

\[ R_{2}^{(c)}(s) = \delta(s) - \frac{\sin^{2}(\pi s)}{(\pi s)^{2}} \frac{(\pi s/4b)^{2}}{\sinh^{2}(\pi s/4b)} \quad (\beta = 2) , \quad \chi \simeq 1/2\pi\beta b \]

- $b \ll 1 \quad \rightarrow \quad$ real-space RG \quad $\rightarrow$

\[ R_{2}^{(c)}(s) = \delta(s) - \text{erfc}(|s|/2\sqrt{\pi}b) , \quad \chi \simeq 1 - 4b , \quad (\beta = 1) \]

\[ R_{2}^{(c)}(s) = \delta(s) - \exp\left(-\frac{s^{2}}{2\pi b^{2}}\right) , \quad \chi \simeq 1 - \pi\sqrt{2}b , \quad (\beta = 2) \]
Disordered electronic systems: Symmetry classification

Altland, Zirnbauer ’97

Conventional (Wigner-Dyson) classes

<table>
<thead>
<tr>
<th>T</th>
<th>spin rot.</th>
<th>chiral</th>
<th>p-h</th>
<th>symbol</th>
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<td>GOE</td>
<td>+</td>
<td>+</td>
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<td>GUE</td>
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Chiral classes

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<td>ChOE</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>ChUE</td>
<td>-</td>
<td>+/-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>ChSE</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

Bogoliubov-de Gennes classes

<table>
<thead>
<tr>
<th>T</th>
<th>spin rot.</th>
<th>chiral</th>
<th>p-h</th>
<th>symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>CI</td>
</tr>
<tr>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>C</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>DIII</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>D</td>
</tr>
</tbody>
</table>

$H = \begin{pmatrix} 0 & t \\ t^\dagger & 0 \end{pmatrix}$

$H = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix}$
Mechanisms of Anderson criticality in 2D

“Common wisdom”: all states are localized in 2D

In fact, in 9 out of 10 symmetry classes the system can escape localization!

→ variety of critical points

Mechanisms of delocalization & criticality in 2D:

• broken spin-rotation invariance → antilocalization, metallic phase, MIT classes AII, D, DIII

• topological term \( \pi_2(\mathcal{M}) = \mathbb{Z} \) (quantum-Hall-type) classes A, C, D: IQHE, SQHE, TQHE

• topological term \( \pi_2(\mathcal{M}) = \mathbb{Z}_2 \) classes AII, CII

• chiral classes: vanishing \( \beta \)-function, line of fixed points classes AIII, BDI, CII

• Wess-Zumino term (random Dirac fermions, related to chiral anomaly) classes AIII, CI, DIII
Multifractality at the Quantum Hall critical point

Evers, Mildenberger, ADM ’01

important for identification of the CFT of the Quantum Hall critical point

\[ f(\alpha) \approx 2 - \frac{(\alpha - \alpha_0)^2}{4(\alpha_0 - 2)}, \quad \Delta_q \approx (\alpha_0 - 2)q(1-q) \quad \text{with} \quad \alpha_0 - 2 = 0.262 \pm 0.003 \]

\[ \rightarrow \text{spectrum is parabolic with a high (1\%) accuracy:} \]

recent data, still higher accuracy (unpublished): non-parabolicity is for real!
Multifractal wave functions at the Quantum Hall transition
Spin quantum Hall effect

- disordered $d$-wave superconductor (class C):
  - charge not conserved but spin conserved
- time-reversal invariance broken:
  - $d_{x^2-y^2} + id_{xy}$ order parameter
  - strong magnetic field
  - Haldane-Rezayi $d$-wave paired state of composite fermions at $\nu = 1/2$

$\rightarrow$ SQH plateau transition: spin Hall conductivity quantized

$$j^Z_x = \sigma_{xy}^s \left( - \frac{dB^z(y)}{dy} \right)$$

Model: $SU(2)$ modification of the Chalker-Coddington network

Kagalovsky, Horovitz, Avishai, Chalker ’99 ; Senthil, Marston, Fisher ’99
Spin quantum Hall effect (cont’d)

Similar to IQH transition but:

- DoS critical \( \rho(E) \propto E^\mu \)
- mapping to percolation: analytical evaluation of
  - DOS exponent \( \mu = 1/7 \)
  - localization length exponent \( \nu = 4/3 \)
  - lowest multifractal exponents: \( \Delta_2 = -1/4, \Delta_3 = -3/4 \)
- numerics: analytics confirmed
  - multifractality spectrum: \( \Delta_q, f(\alpha) \) not parabolic

Gruzberg, Ludwig, Read ’99; Beamond, Cardy, Chalker ’02; Evers, Mildenberger, ADM ’03