Radiative Transfer
and Diffusion Approximation

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General setting

• Phase-space density

\[ F \equiv F(t, x, w) \geq 0 \]

where \( t \) is the time, \( x \in \mathbb{R}^D \) the position, and \( w \in \mathbb{R}^N \) a microscopic variable. (For instance, \( w = (\omega, \nu) \) where \( \omega \in S^{D-1} \) is the direction of photons and \( \nu > 0 \) is their frequency).

• We shall study situations where the evolution of \( F \) is governed by an equation of the form

\[
\partial_t F + v(w) \cdot \nabla_x F = C(F)
\]

where \( v(w) \in \mathbb{R}^D \) is the velocity of particles while \( C \) is an operator that acts only on on the \( w \) variable in \( F \).
Thomson scattering

• Radiative intensity $I \equiv I(t, x, \omega, \nu) = ch\nu \psi(t, x, \omega, \nu)$ (where $\psi =$ the number density of photons located at position $x$ at time $t$, traveling in the direction $\omega$ with frequency $\nu$)

• Interaction of radiation with matter: Thomson scattering by free electrons, at low photon energy ($h\nu << m_e c^2$). The Thomson scattering cross-section is known to be

$$\sigma_T(x) = \frac{8\pi}{3} N_e(x) r_e^2$$

where $N_e \equiv N_e(x)$ is the number of electrons per unit volume, and $r_e$ the classical electron radius (defined by $\frac{e^2}{4\pi\epsilon_0 r_e} = m_e c^2$).
• The Thomson scattering operator is given by the following formula that involves Rayleigh’s phase function
\[ \frac{3}{4}(1 + (\omega \cdot \omega')^2) \]

One has
\[
C(I)(x, \omega, \nu) = \sigma_T(x) \int_{S^2} \frac{3(1+(\omega \cdot \omega')^2)}{4} (I(x, \omega', \nu) - I(x, \omega, \nu)) \frac{d\omega'}{4\pi}
\]

• Here, \( w = (\omega, \nu) \) and the radiative transfer equation is
\[
\frac{1}{c} \partial_t I + \omega \cdot \nabla_x I = C(I)
\]
Grey material in radiative equilibrium

• Grey material in LTE: absorption coefficient independent of frequency

\[ \kappa \equiv \kappa(T) \] where \( T \) is the material temperature

• Kirchhoff-Planck relation:

\[
\text{sink term} = \kappa(T)I, \quad \text{source term} = \kappa(T)B_\nu(T)
\]

where \( B_\nu(T) \) is Planck’s black body radiation

\[
B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}
\]

where \( k \) is Boltzmann’s constant, \( h \) is Planck’s constant and \( c \) the speed of light.
• Notice that

\[ \int_0^\infty B_\nu(T) d\nu = aT^4 \]  
with \( a = \frac{2\pi^4 k^4}{15c^2h^3} \)

• In radiative equilibrium

\[ \frac{1}{4\pi} \int_0^\infty \int_{S^2} \kappa(T) I d\omega d\nu = \frac{1}{4\pi} \int_0^\infty \int_{S^2} \kappa(T)B_\nu(T) d\omega d\nu \]

meaning that

\[ aT^4 = \frac{1}{4\pi} \int_0^\infty \int_{S^2} I d\omega \]
Defining the frequency integrated radiative intensity

\[ u(t, x, \omega) = \int_0^\infty I(t, x, \omega, \nu) d\nu \]

the radiative transfer equation is expressed in terms of \( u \) by

\[ \frac{1}{c} \partial_t u + \omega \cdot \nabla_x u = C(u) \]

with

\[ C(u) = \kappa(T) \frac{1}{4\pi} \int_{S^2} (u(t, x, \omega') - u(t, x, \omega)) d\omega' \]

where \( T \) is defined in terms of \( u \) by

\[ aT^4(t, x) = \frac{1}{4\pi} \int_{S^2} u(t, x, \omega) d\omega \]
The diffusion scaling

In all the examples above, we shall be interested in situations where

- the time and space variables are slow variables, while

- the microscopic variable $w$ is a fast variable.

Because we are mainly interested in limits leading to diffusion equations, we apply the following change of variables

\[ x \rightarrow \epsilon x , \quad t \rightarrow \epsilon^2 t , \quad \epsilon \ll 1 \]

in the equation governing $F$, while leaving $w$ unchanged.
• The equation governing $F$ in scaled variables becomes
\[ \epsilon^2 \partial_t F_\epsilon + \epsilon v(w) \cdot \nabla_x F_\epsilon = C(F_\epsilon) \]

• One expects that $F_\epsilon$ should behave as an equilibrium, i.e. a solution of
\[ C(F) = 0 \]
that is slowly modulated in $t$ and $x$. More precisely, such equilibria are in general parametrized by a finite number of parameters $\vec{\beta}$:
\[ C(F') = 0 \iff F(w) = \mathcal{E}(\vec{\beta}, w) \]

• A modulated equilibrium is a density of the form
\[ F_\epsilon(t, x, w) = \mathcal{E}(\vec{\beta}(\epsilon^2 t, \epsilon x), w) \]
and our goal is to determine the equation(s) governing $\vec{\beta}$. 
A necessary condition for the diffusion scaling described above to be well adapted to the kinetic model under consideration is that the mean velocity of any population of particles in a state of equilibrium should be 0, i.e. that

\[ \langle v(w) \mathcal{E}(\vec{\beta}, w) \rangle = 0, \quad \text{for each } \vec{\beta}. \]

If this is not the case, then the kinetic model under consideration may still admit solutions near slowly modulated equilibria, however, with a different scaling leading to hyperbolic (instead of parabolic) PDEs. A typical instance of such scalings is

\[ x \to \epsilon x, \quad t \to \epsilon t, \quad \epsilon << 1 \]

and is used in the context of the kinetic theory of gases.
Linear setting

Consider the linear Boltzmann equation

$$\partial_t F + v(w) \cdot \nabla_x F + \mathcal{L}(F) = 0$$

where the linear collision operator $\mathcal{L}$ is defined by

$$\mathcal{L}f(x, w) = \int_{\mathbb{R}^N} k(x, w, w')(f(x, w) - f(x, w'))d\mu(w')$$

In the formula above, $\mu$ is a probability measure on $\mathbb{R}^N$, while $k$ is a function that satisfies the following assumptions:

$$\sup_{x \in \mathbb{R}^D} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} k(x, w, w')^2 d\mu(w)d\mu(w') = K < +\infty$$

and $k(x, w, w') = k(x, w', w) \geq \kappa > 0$
Main properties of $\mathcal{L}$

For each $x \in \mathbb{R}^D$, the operator $\mathcal{L}_x$ defined by

$$\mathcal{L}_x f(w) = \int_{\mathbb{R}^N} k(x, w, w')(f(w) - f(w'))d\mu(w')$$

is self-adjoint on $L^2(\mathbb{R}^N, d\mu)$. It can be put in the form

$$\mathcal{L}_x f(w) = \sigma(x, w)f(w) - \mathcal{K}_x f(w)$$

where

$$\sigma(x, w) = \int_{\mathbb{R}^N} k(x, w, w')d\mu(w') \text{ and}$$

$$\mathcal{K}_x f(w) = \int_{\mathbb{R}^N} k(x, w, w')f(w')d\mu(w')$$
The bound assumed on $k$ implies that

$$\sigma \in L_x^\infty(L^2(d\mu(w))) \text{ with } \sigma(x, w) \geq \kappa > 0$$

while

$$\mathcal{K}_x \text{ is a Hilbert-Schmidt operator on } L^2(\mathbb{R}^N, d\mu(w))$$

Moreover, for each $x \in \mathbb{R}^D$ and each $f \in L^2(\mathbb{R}^N, d\mu)$, one has

$$\int_{\mathbb{R}^D} f(w) \mathcal{L}_x f(w) d\mu(w)$$

$$= \int\int_{\mathbb{R}^N \times \mathbb{R}^N} k(x, w, w')(f(w) - f(w')) f(w) d\mu(w) d\mu(w')$$

$$= \frac{1}{2} \int\int_{\mathbb{R}^N \times \mathbb{R}^N} k(x, w, w')(f(w) - f(w'))^2 d\mu(w) d\mu(w')$$

by the symmetry of $k$ in $w, w'$. 
Hence

\[
\int_{\mathbb{R}^D} f(w) \mathcal{L}_x f(w) d\mu(w)
\geq \frac{\kappa}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} (f(w) - f(w'))^2 d\mu(w) d\mu(w')
\geq \frac{\kappa}{2} \int_{\mathbb{R}^N} \left( f(w) - \int_{\mathbb{R}^N} f(w') d\mu(w') \right)^2 d\mu(w)
\]

Denoting \( \langle \cdot \rangle \) the average with respect to \( \mu \), i.e.

\[
\langle \phi \rangle = \int_{\mathbb{R}^N} \phi(w) d\mu(w)
\]

one has

\[
\langle f \mathcal{L}_x f \rangle \geq \frac{\kappa}{2} \| f - \langle f \rangle \|_{L^2(\mathbb{R}^N, d\mu)}^2
\]
In particular \( \ker \mathcal{L}_x \) is reduced to the (one-dimensional) space of constant functions, and

\[
\mathcal{L}_x \text{ is self-adjoint and Fredholm on } L^2(\mathbb{R}^N, d\mu)
\]

Henceforth, we assume that

\[ v \in L^2(\mathbb{R}^N, d\mu) \text{ and that } \langle v \rangle = 0, \text{ i.e. that } v \in (\ker \mathcal{L}_x)^\perp \]

hence there exists a unique \( \beta \equiv \beta(x, w) \) such that

\[
\mathcal{L}_x \beta(x, w) = v(w) \text{ and } \langle \beta(x, \cdot) \rangle = 0
\]

In particular, \( \beta \) satisfies the bound

\[
\| \beta(x, \cdot) \|_{L^2(\mathbb{R}^N, d\mu)} \leq \frac{2}{\kappa} \| v \|_{L^2(\mathbb{R}^N, d\mu)}
\]
We shall henceforth describe 3 different methods for deriving the diffusion equation from the scaled linear Boltzmann equation

• the moment method

• the relative entropy method

• the Hilbert expansion method

We shall conclude with a discussion of their respective merits
The moment method

• Start from the scaled, linear Boltzmann equation

\[ \epsilon^2 \partial_t F_\epsilon + \epsilon v(w) \cdot \nabla_x F_\epsilon + \mathcal{L}(F_\epsilon) = 0, \quad x \in \mathbb{R}^D, \; w \in \mathbb{R}^N, \; t > 0 \]

\[ F_\epsilon(0, x, w) = F^{\text{in}}(x) \]

• Multiplying the above equation by \( F_\epsilon \) and integrating in \( x \) and \( w \) leads to

\[ \frac{d}{dt} \int_{\mathbb{R}^D} \frac{1}{2} \langle F_\epsilon^2 \rangle dx + \frac{1}{\epsilon^2} \int_{\mathbb{R}^D} \frac{\kappa}{2} \| F_\epsilon - \langle F_\epsilon \rangle \|^2_{L^2(\mathbb{R}^N, d\mu)} dx \leq 0 \]

which implies the following a priori bounds

\[ \| F_\epsilon(t) \|_{L^2(dx d\mu)} \leq \| F^{\text{in}} \|_{L^2(dx)} \]

\[ \| F_\epsilon - \langle F_\epsilon \rangle \|_{L^2(dtdx d\mu)} \leq \frac{1}{\kappa} \epsilon \| F^{\text{in}} \|_{L^2(dx)} \]
By the Banach-Alaoglu theorem, modulo extraction of a subsequence

\[ F_\epsilon \rightharpoonup F \text{ in } L_t^\infty(L^2(dx d\mu)) \text{ weak-*} \]
\[ \frac{1}{\epsilon}(F_\epsilon - \langle F_\epsilon \rangle) \rightharpoonup Q \text{ in } L^2(dt dx d\mu) \]

in particular, the second limit implies that

\[ F_\epsilon \rightharpoonup F \equiv F(t, x) \text{ in } L_t^\infty(L^2(dx d\mu)) \text{ weak-*} \]
Averaging the scaled linear Boltzmann equation in \( w \) leads to the continuity equation

\[
\partial_t \langle F_\epsilon \rangle + \text{div} \left( x \left( v_\epsilon (F_\epsilon - \langle F_\epsilon \rangle) \right) \right) = 0
\]

so that

\[
\partial_t \langle F_\epsilon \rangle = O(1) \text{ in } L^2(dt; H^{-1}(\mathbb{R}^D))
\]

while

\[
\langle F_\epsilon \rangle = O(1) \text{ in } L^\infty(dt; L^2(\mathbb{R}^D))
\]

By Ascoli’s theorem, one also has

\[
\langle F_\epsilon(t, x, \cdot) \rangle \rightarrow F \equiv F(t, x) \text{ in } H^{-1}(\mathbb{R}^D_x) \text{ uniformly in } t \in [0, T]
\]
Passing to the limit in the continuity equation above shows that

$$\partial_t F + \text{div}_x \langle vQ \rangle = 0, \quad F\big|_{t=0} = F^{in}$$

It remains to identify $Q$.

Write the scaled Boltzmann equation multiplied by $\epsilon$ in the form

$$\epsilon \partial_t F_\epsilon + v(w) \cdot \nabla_x F_\epsilon + L \left( \frac{1}{\epsilon} (F_\epsilon - \langle F_\epsilon \rangle) \right) = 0$$

and letting $\epsilon \to 0$ shows that

$$v(w) \cdot \nabla_x F(t, x) + LQ = 0, \quad \langle Q \rangle = 0$$

Remembering the function $\beta \equiv \beta(x, w) \in (\ker L_x)^\perp$ that satisfies the equation $L_x \beta(x, w) = v(w)$, we see that

$$Q(t, x, w) = -\beta(x, w) \cdot \nabla_x F(t, x)$$
Collecting the information above, we have proved

**Theorem.** Let $F^{in} \equiv F^{in}(x) \in L^2(\mathbb{R}^D)$, and let $F_\epsilon$ be for each $\epsilon > 0$ the solution of the scaled linear Boltzmann equation. In the limit as $\epsilon \to 0$, $F_\epsilon \rightharpoonup F \equiv F(t, x)$ in $L^\infty_t(L^2(dx d\mu))$ weak-*\textsuperscript{−}, where $F$ is the solution of the diffusion equation

$$\partial_t F - \text{div}_x(\Sigma(x) \nabla_x F) = 0, \quad F\big|_{t=0} = F^{in}.$$ 

The diffusion matrix is given by

$$\Sigma_{ij}(x) = \langle v_i \beta_j(x, \cdot) \rangle$$

and satisfies

$$\Sigma(x) = \Sigma(x)^T > 0.$$
Properties of the diffusion matrix

Let $\xi \in \mathbb{R}^D \setminus \{0\}$; then

$$\Sigma_{ij}(x)\xi_i\xi_j = \langle \phi(x, \cdot) L \phi(x, \cdot) \rangle \geq 0 \text{ with } \phi(x, w) = \xi \cdot \beta(x, w)$$

Since $L$ is nonnegative self-adjoint, for each $x \in \mathbb{R}^D$, the matrix $\Sigma(x)$ is symmetric and nonnegative.

Furthermore, since $\beta(x, \cdot) \perp \ker L$ for each $x \in \mathbb{R}^D$, $\phi(x, \cdot) \perp \ker L$ and therefore

$$\Sigma_{ij}(x)\xi_i\xi_j \geq \frac{\kappa}{2} \|\phi(x, \cdot)\|_{L^2(d\mu(w))} > 0$$

unless $\xi = 0$. Hence $\Sigma(x)$ is positive definite for each $x \in \mathbb{R}^D$. 
In many physical situations, the integral kernel \( k \equiv k(x, w, w') \) satisfies
\[
k(x, R \cdot w, R \cdot w') = k(x, w, w'), \quad R \in O_D(\mathbb{R})
\]
for some action of the orthogonal group \( O_D(\mathbb{R}) \) on \( \mathbb{R}^N \) such that
\[
v(R \cdot w) = Rv(w)
\]
while
\[
\text{the measure } \mu \text{ is invariant under } w \mapsto R \cdot w
\]
(remember that \( w \) may include a direction variable, and additional variables such as the frequency; the isometry \( R \) acts only on the former variable).
In that case, one can show that

\[ \beta(x, w) = b(x, |w|)v(w) \text{ with } b(x, |w|) \in \mathbb{R} \]

and the diffusion matrix is scalar:

\[ \Sigma(x) = d(x)I \text{ with } d(x) = \int b(x, |w|)|v(w)|^2d\mu(w) \]

This how one proves for instance that the viscosity or thermal conductivity of a perfect monatomic gas computed from kinetic theory are scalars.
The relative entropy method

**Idea:** estimate the relative entropy of $F_\varepsilon$ w.r.t. the solution of the limiting diffusion equation. In the linear case

$$H(F_\varepsilon|\rho) = \frac{1}{2} \int_{\mathbb{R}^D} \langle (F_\varepsilon - \rho)^2 \rangle dx$$

Let $\rho$ satisfy

$$\partial_t \rho = \text{div}_x (\Sigma(x) \nabla_x \rho)$$

then

$$\frac{d}{dt} H(F_\varepsilon|\rho) = -\frac{1}{\varepsilon^2} \int \langle F_\varepsilon L F_\varepsilon \rangle dx - \frac{d}{dt} \int \langle F_\varepsilon \rangle \rho dx - \int (\Sigma \nabla_x \rho | \nabla_x \rho)$$
Letting $\rho \equiv 0$ in the above identity gives the same a priori estimates as in the moment method, so that, modulo extraction of a subsequence

$$F_\epsilon \rightharpoonup F \equiv F(t, x) \text{ in } L^\infty_t(L^2(dx d\mu)) \text{ weak-}^*$$

while $Q_\epsilon = \frac{1}{\epsilon}(F_\epsilon - \langle F_\epsilon \rangle) \rightharpoonup Q$ in $L^2(dt dx d\mu)$

and, by the same argument as before

$$Q(t, x, w) = -\beta(x, w) \cdot \nabla_x F(t, x)$$

Then, the first term on the r.h.s. of the formula for $\frac{d}{dt}H(F_\epsilon|\rho)$ satisfies

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^2} \int_0^t \int \langle F_\epsilon \mathcal{L} F_\epsilon \rangle dx ds = \lim_{\epsilon \to 0^+} \int_0^t \int \langle Q_\epsilon \mathcal{L} Q_\epsilon \rangle dx ds$$

$$\geq \int_0^t \int \langle Q \mathcal{L} Q \rangle dx ds = \int_0^t \int (\Sigma \nabla_x F \mid \nabla_x F') dx ds$$

by convexity and weak convergence
Write the second term on the r.h.s. of the formula for \( \frac{d}{dt} H(F_\epsilon|\rho) \) as

\[
\frac{d}{dt} \int \rho \langle F_\epsilon \rangle dx = - \int (\Sigma \nabla x \rho | \nabla x \langle F_\epsilon \rangle) dx + \int \left( \nabla x \rho \left| \frac{1}{\epsilon} \langle vF_\epsilon \rangle \right. \right) dx
\]

The last integral above is recast as

\[
\int \left( \nabla x \rho \left| \frac{\beta}{\epsilon} \mathcal{L} F_\epsilon \right. \right) dx = \int \langle (\beta \cdot \nabla x \rho)(v \cdot \nabla x \langle F_\epsilon \rangle) \rangle dx
\]

\[
+ \epsilon \int \langle (\beta \cdot \nabla x \rho)(v \cdot \nabla x G_\epsilon + \partial_t F_\epsilon) \rangle dx
\]

After integrating over \([0, t]\), the first term on the last r.h.s. is precisely

\[
\int_0^t \int (\Sigma \nabla x \rho | \nabla x \langle F_\epsilon \rangle) dx ds \to \int_0^t \int (\Sigma \nabla x \rho | \nabla x F) dx ds
\]
• The second term on that same r.h.s. is a remainder

\[-\epsilon \int_0^t \int \langle \partial_t + v \cdot \nabla_x \rangle \langle \beta \cdot \nabla_x \rho \rangle G_{\epsilon} \rangle dx + \epsilon \left[ \int \langle \beta \cdot \nabla_x \rho \rangle F_{\epsilon} \rangle dx \right]_0^t = O(\epsilon)\]

provided we know that \( \beta \equiv \beta(x, w) \) is smooth in \( x \), by parabolic regularity.

• Putting together all the terms in the formula for \( \frac{d}{dt} H(F_{\epsilon} | \rho) \) and integrating over \([0, t]\), we find that

\[
\overline{\lim_{\epsilon \to 0}} H(F_{\epsilon}(t) | \rho) \leq H \left( F_{\epsilon}^{in} \big| \rho \big|_{t=0} \right) \\
- \int_0^t \int (\Sigma \nabla_x(\langle F \rangle - \rho) | \nabla_x(\langle F \rangle - \rho)) dx \\
- \left( \overline{\lim_{\epsilon \to 0}} \int_0^t \int \langle Q_{\epsilon} \mathcal{L} Q_{\epsilon} \rangle dx ds - \int_0^t \int \langle Q \mathcal{L} Q \rangle dx ds \right)
\]
• By letting $\rho\big|_{t=0} = F^{in}$ in the inequality above, we conclude that

**Theorem.** Let $F^{in} \equiv F^{in}(x) \in L^2(\mathbb{R}^D)$, and let $F_\epsilon$ be for each $\epsilon > 0$ the solution of the scaled linear Boltzmann equation. Assume that the scattering kernel $k$ is such that $\beta \equiv \beta(x, w)$ is smooth in $x$, so that the diffusion matrix, given by $\Sigma_{ij}(x) = \langle v_i \beta_j(x, \cdot) \rangle$ is smooth. Then

$$F_\epsilon(t, x, w) \to F(t, x) \text{ in } L^2(dx d\mu(w)) \text{ for each } t \geq 0, \text{ and }$$

$$\frac{1}{\epsilon}(F_\epsilon - \langle F_\epsilon \rangle) \to -\beta(x, w) \cdot \nabla_x F(t, x) \text{ in } L^2(dt dx d\mu(w))$$

where $F$ is the solution of the diffusion equation

$$\partial_t F = \text{div}_x (\Sigma(x) \nabla_x F) = 0, \quad F\big|_{t=0} = F^{in}.$$
The Hilbert expansion method

Hilbert’s idea, which he tried on the Boltzmann equation of the kinetic theory of gases, is to seek the solution of the scaled, linear Boltzmann equation as a formal power series in $\epsilon$:

$$F_\epsilon(t, x, w) = \sum_{m \geq 0} \epsilon^m F_m(t, x, w)$$

Inserting this expression for $F_\epsilon$ in the linear Boltzmann equation and balancing order by order, we arrive at the following, infinite hierarchy of equations

- Order $\epsilon^0$: one finds

$$\mathcal{L} F_0 = 0, \text{ hence } F_0 \text{ is constant in } w: F_0 \equiv F_0(t, x)$$
• Order $\epsilon$: one finds

\[ \mathcal{L}F_1(t, x, w) = -v(w) \cdot \nabla_x F_0(t, x) \]

The general solution of this equation is known to be

\[ F_1(t, x, w) = f_1(t, x) - \beta(x, w) \cdot \nabla_x F_0(t, x) \]

• Order $\epsilon^2$: one finds

\[ \mathcal{L}F_2(t, x, w) = -v(w) \cdot \nabla_x F_1(t, x, w) - \partial_t F_0(t, x) \]

This is a Fredholm integral equation; the compatibility condition for $F_2$ to exist is that

\[ \langle \partial_t F_0 + v \cdot \nabla_x F_1 \rangle = 0 \]
• Inserting the formula for $F_1$ in that equality, one finds

$$\partial_t F_0 - \partial_{x_i} (\langle v_i \beta_j \rangle \partial_{x_j} F_0) = 0$$

which is precisely that $F_0$ should satisfy the expected diffusion equation.

• If it does, the equation for $F_2$ is recast as

$$F_2(t, x, w) = f_2(t, x) - (I - P)(v_i \partial_{x_j} (\beta_j \partial_{x_j} F_0(t, x)))$$

where $P$ is the orthogonal projection on constants in $L^2(d\mu(w))$, i.e.

$$P\phi = \langle \phi \rangle$$
An easy induction shows that the \( n \)-th order equation is

\[
\mathcal{L} F_n(t, x, w) = -v(w) \cdot \nabla_x F_{n-1}(t, x, w) - \partial_t F_{n-2}(t, x)
\]

leading to a compatibility condition that is

\[
\langle \partial_t F_{n-2} + v \cdot \nabla_x F_{n-1} \rangle = 0
\]

Hence \( F_n \) is of the form

\[
F_n(t, x, w) = f_n(t, x) + P_n(t, x, w, \partial_x) F_0(t, x)
\]

where \( P_n \) is a differential operator of order \( n \) which has mean 0 in the \( w \) variable. The equation governing \( f_n \) will result from the compatibility condition for the \( n + 2 \)-nd order Fredholm equation.
In general, one does not seek to prove that the series converge. An alternative method for proving the diffusion limit relies on using truncated Hilbert expansions.

Let then $F$ be the solution of the diffusion equation

$$\partial_t F = \text{div}_x (\sum(x) \nabla_x F) , \quad F\big|_{t=0} = F^{in}$$

Define $F_0 = F$, and $F_1$ and $F_2$ by the same formulas as in the Hilbert expansion:

$$F_1(t, x, w) = -\beta(x, w) \cdot \nabla_x F(t, x)$$

$$F_2(t, x, w) = -(I - P)(v_i \partial_{x_j}(\beta_j \partial_{x_j} F_0(t, x)))$$
• Finally, consider the truncated Hilbert expansion

\[ \tilde{F}_\epsilon = F + \epsilon F_1 + \epsilon^2 F_2 \]

• One easily checks that \( \tilde{F}_\epsilon \) satisfies

\[
\begin{align*}
\epsilon^2 \partial_t \tilde{F}_\epsilon + \epsilon v(w) \cdot \nabla_x \tilde{F}_\epsilon + \mathcal{L} \tilde{F}_\epsilon &= \epsilon^4 \partial_t F_2 + \epsilon^3 \partial_t F_1 + \epsilon^3 v(w) \cdot \nabla_x F_2 \\
&= O(\epsilon^3) \text{ in } L^\infty_t (L^2 (dx d\mu(w)))
\end{align*}
\]

while

\[
\tilde{F}_\epsilon|_{t=0} = F^{\text{in}} + O(\epsilon) \text{ in } L^\infty_t (L^2 (dx d\mu(w)))
\]
By the uniform stability of the scaled, linear Boltzmann in $L^2(dx d\mu(w))$ we conclude that
\[
\|F_\epsilon(t) - \tilde{F}_\epsilon(t)\|_{L^2(dx d\mu(w))} = O(\epsilon) \text{ uniformly in } t \in [0, T]
\]
and since
\[
\tilde{F}_\epsilon = F + O(\epsilon) \text{ in } L^\infty_t(L^2(dx d\mu(w)))
\]
we eventually obtain the following statement

**Theorem.** Let $F^{in} \equiv F^{in}(x) \in L^2(\mathbb{R}^D)$, and let $F_\epsilon$ be for each $\epsilon > 0$ the solution of the scaled linear Boltzmann equation. Assume that the scattering kernel $k$ is such that $\beta \equiv \beta(x, w)$ is smooth in $x$, and let $F$ be the solution of the diffusion equation with initial data $F^{in}$. Then
\[
F_\epsilon(t, x, w) \to F(t, x) \text{ in } L^2(dx d\mu(w)) \text{ uniformly in } t \in [0, T].
\]
Comparing the three methods

• The relative entropy method is based on the regularity of the target (diffusion) limit, while the Hilbert expansion method is based on the uniform stability of the linear Boltzmann equation in the $\epsilon$.

• The moment method only uses the most fundamental “entropy estimate” on $F_\epsilon$, which is the only uniform a priori estimate known to be satisfied by the family $F_\epsilon$.

• Therefore, one can hope that the moment method can be extended to nonlinear models where little is known about either the strong stability of the scaled Boltzmann equation or the regularity of the limiting solution (for instance in the case of degenerate diffusions)
One can also hope to use the relative entropy method in cases where the strong stability of the scaled Boltzmann equation uniformly in the small parameter is not known, but where the target equation is known to possess smooth solutions.

Finally, the Hilbert expansion method requires knowing that the scaled Boltzmann equation is strongly stable uniformly in $\epsilon$; in that case, one can obtain not only the diffusion limit but also higher order corrections thereof.
Here are some instances where these methods have been successfully used:

- Hilbert expansion: Caflisch (CPAM 1980): compressible Euler limit of the Boltzmann equation; Bardos-Santos-Sentis (TAMS 1984): diffusion limit of the linear Boltzmann equation, with boundary layer analysis
