CONVERTIBLE BONDS IN A DEFAULTABLE DIFFUSION MODEL

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Contents

1 Introduction

The goal of this work is a detailed and rigorous examination of *convertible securities* (CS) in a financial market model endowed with the following primary traded assets: a savings account, a stock underlying the CS and an associated *credit default swap* (CDS) contract or, alternatively to the latter, a *rolling CDS*. Let us stress that we deal here not only with the valuation, but also, even more crucially, with the issue of hedging convertible securities that are subject to credit risk. Special emphasis is put on the properties of convertible bonds (CB) with credit risk, which constitute an important class of actively traded convertible securities. It should be acknowledged that convertible bonds were already extensively studied in the past by, among others, Andersen and Buffum [1], Ayache et al. [2], Brennan and Schwartz [12, 13], Davis and Lischka [22], Kallsen and Kühn [31], Kwok and Lau [35], Lvov et al. [37], Sîrbu et al. [42], Takahashi et al. [43], Tsiveriotis and Fernandes [45], to mention just a few. Of course, it is not possible to give here even a brief overview of models, methods and results from the abovementioned papers (for a discussion of some of them and further references, we refer to [4]-[6]). Despite the existence of these papers, it was nevertheless our feeling that a rigorous, systematic and fully consistent approach to hedging-based valuation of convertible securities with credit risk (as opposed to a formal risk-neutral valuation approach) was not available in the literature, and thus we decided to make an attempt to fill this gap in a series of papers for which this work can be seen as the final episode. We strive to provide here the most explicit valuation and hedging techniques, including numerical analysis of specific features of convertible bonds with call protection and call notice periods.

The main original contributions of the present paper, in which we apply and make concrete several results of previous works, can be summarized as follows:

• we make a judicious choice of primary traded instruments used for hedging of a convertible security, specifically, the underlying stock and the rolling credit default swap written on the same credit name,

• the completeness of the model until the default time of the underlying name in terms of uniqueness of a martingale measure is studied,

• a detailed specification of the model assumptions that subsequently allow us to apply in the present framework our general results from the preceding papers [4]-[6] is provided,

• it is shown that super-hedging of the arbitrage value of a convertible security is feasible in the present set-up for both issuer and holder at the same initial cost,

• sufficient regularity conditions for the validity of the aggregation property for the value of a convertible bond at call time in the case of positive call notice period are given,

• numerical results for the decomposition of the value of a convertible bond into straight bond and embedded option components are provided,

• the precise definitions of the implied spread and implied volatility of a convertible bond are stated and some numerical analysis for both quantities is conducted.

Before commenting further on this work, let us first describe very briefly the results of our preceding papers. In [4], working in an abstract set-up, we characterized arbitrage prices of generic *convertible securities* (CS), such as *convertible bonds* (CB), and we provided a rigorous decomposition of a CB into a straight bond component and a game option component, in order to give a definite meaning to commonly used terms of 'CB spread' and 'CB implied volatility.' Subsequently, in [5], we showed that in the hazard process set-up, the theoretical problem of pricing and hedging CS can essentially be reduced to a problem of solving an associated doubly reflected Backward Stochastic Differential Equation (BSDE for short). Finally, in [6], we established a formal connection between this BSDE and the corresponding variational inequalities with double obstacles in a generic Markovian intensity model. The related mathematical issues are dealt with in companion papers by Crépey [18] and Crépey and Matoussi [19].

In the present paper, we focus on a detailed study of convertible securities in a specific market set-up with the following traded assets: a savings account, a stock underlying a convertible security, and an associated rolling credit default swap. In Section 2, the dynamics of these three securities are formally introduced in terms of Markovian diffusion set-up with default. We also study there the arbitrage-free property of this model, as well as its completeness. The model considered in this work appears as the simplest equity-to-credit reduced-form model, in which the connection between equity and credit is reflected by the fact that the *default intensity* γ depends on the stock level S. To the best of our knowledge, it is widely used by the financial industry for dealing with convertible bonds with credit risk. This specific model's choice was the first rationale for the present study. Our second motivation was to show that all assumptions that were postulated in our previous theoretical works [4]-[6] are indeed satisfied within this set-up; in this sense, the model can be seen as a practical implementation of the general theory of arbitrage pricing and hedging of convertible securities.

Section 3 is devoted to the study of convertible securities. We first provide a general result on the valuation of a convertible security within the present framework (see Proposition 3.1). Next, we address the issue of valuation and hedging through a study of the associated doubly reflected BSDE. Proposition 3.3 provides a set of explicit conditions, obtained by applying general results of Crépey [18], which ensure that the BSDE associated with a convertible security has a unique solution. This allows us to establish in Proposition 3.2 the form of the (super-)hedging strategy for a convertible security. Subsequently, we characterize in Proposition 3.4 the pricing function of a convertible security in terms of the viscosity solution to associated variational inequalities and we prove in Proposition 3.5 the convergence of suitable approximation schemes for the pricing function.

In Section 4, we further concretize these results in the special case of a convertible bond. In [4, 6] we worked under the postulate that the value U_t^{cb} of a convertible bond upon a call at time t yields, as a function of time, a well-defined process satisfying some natural conditions. In the specific framework considered here, using the uniqueness of arbitrage prices established in Propositions 2.1 and 3.1 and the *continuous aggregation* property for the value U_t^{cb} of a convertible bond upon a call at time t furnished by Proposition 4.7, we actually prove that this assumption is satisfied and we subsequently discuss in Propositions 4.6 and 4.8 the methods for computation of U_t^{cb}. We also examine in some detail the decomposition into straight bond and embedded game option components, which is both and practically relevant, since it provides a formal way of defining the implied volatility of a convertible bond. We conclude the paper by illustrating some results through numerical computations of relevant quantities in a simple example of an equity-to-credit model.

2 Markovian Equity-to-Credit Framework

We first introduce a generic Markovian default intensity set-up. More precisely, we consider a *defaultable diffusion model* with time- and stock-dependent *local default intensity* and *local volatility* (see [1, 2, 6, 14, 22, 24, 34]). We denote by $\int_0^t$ the integrals over $(0,t]$.

2.1 Default Time and Pre-Default Equity Dynamics

Let us be given a standard stochastic basis $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})$, over $[0,\Theta]$ for some fixed $\Theta \in \mathbb{R}_+$, endowed with a standard Brownian motion $(W_t)_{t\in[0,\Theta]}$. We assume that $\mathbb{F}$ is the filtration generated by W. The underlying probability measure $\mathbb{Q}$ is aimed to represent a risk-neutral probability measure (or 'pricing probability') on a financial market model that we are now going to construct.

In the first step, we define the *pre-default factor process* $(\widetilde{S}_t)_{t\in[0,\Theta]}$ (to be interpreted later as the *pre-default stock price* of the firm underlying a convertible security) as the diffusion process with the initial condition $\widetilde{S}_0$ and the dynamics over $[0,\Theta]$ given by the stochastic differential equation (SDE)

$$d\widetilde{S}_t = \widetilde{S}_t\Big(\big(r(t) - q(t) + \eta\gamma(t,\widetilde{S}_t)\big)\,dt + \sigma(t,\widetilde{S}_t)\,dW_t\Big) \tag{1}$$

with a strictly positive initial value $\widetilde{S}_0$. We denote by $\mathcal{L}$ the infinitesimal generator of $\widetilde{S}$, that is, the differential operator given by the formula

$$\mathcal{L} = \partial_t + \big(r(t) - q(t) + \eta\gamma(t,S)\big)S\,\partial_S + \frac{\sigma^2(t,S)S^2}{2}\,\partial^2_{S^2}. \tag{2}$$

Assumption 2.1 **(i)** The riskless short interest rate $r(t)$, the equity dividend yield $q(t)$, and the local default intensity $\gamma(t,S) \geq 0$ are bounded, Borel-measurable functions and $\eta \leq 1$ is a real constant, to be interpreted later as the *fractional loss upon default* on the stock price.

(ii) The local volatility $\sigma(t,S)$ is a positively bounded, Borel-measurable function, so, in particular, we have that $\sigma(t,S) \geq \underline{\sigma} > 0$ for some constant $\underline{\sigma}$.
(iii) The functions $\gamma(t,S)S$ and $\sigma(t,S)S$ are Lipschitz continuous in S, uniformly in t.

Note that we allow for negative values of r and q in order, for instance, to possibly account for *repo rates* in the model. Under Assumption 2.1, SDE (1) is known to admit a unique strong solution $\widetilde{S}$, which is non-negative over $[0,\Theta]$. Moreover, the following (standard) a priori estimate is available, for any $p \in [2,+\infty)$

$$\mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,\Theta]} |\widetilde{S}_t|^p\Big) \leq C\left(1 + |\widetilde{S}_0|^p\right). \tag{3}$$

In the next step, we define the $[0,\Theta]\cup\{+\infty\}$-valued *default time* τ_d, using the so-called *canonical construction* [8]. Specifically, we set (by convention, $\inf\emptyset = \infty$)

$$\tau_d = \inf\Big\{\, t \in [0,\Theta];\ \int_0^t \gamma(u,\widetilde{S}_u)\,du \geq \varepsilon\Big\}, \tag{4}$$

where ε is a random variable on $(\Omega,\mathcal{G},\mathbb{F},\mathbb{Q})$ with the unit exponential distribution and independent of $\mathbb{F}$. Because of our construction of τ_d, the process $G_t := \mathbb{Q}(\tau > t\,|\,\mathcal{F}_t)$ satisfies, for every $t \in [0,\Theta]$,

$$G_t = e^{-\int_0^t \gamma(u,\widetilde{S}_u)\,du}$$

and thus it has continuous and non-increasing sample paths. This also means that the process $\gamma(t,\widetilde{S}_t)$ is the $\mathbb{F}$-*hazard rate* of τ_d (see, e.g., [8, 30]). The fact that the hazard rate γ may depend on $\widetilde{S}$ is crucial, since this dependence actually conveys all the 'equity-to-credit' information in the model. A natural choice for γ is a decreasing (e.g., negative power) function of $\widetilde{S}$ capped when $\widetilde{S}$ is close to zero. A possible further refinement would be to put a positive floor on the function γ. The lower bound on γ would then reflect the perceived level the systemic default risk, as opposed to firm-specific default risk.

Let $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$ be the *default indicator process* and let the process $(M_t^d)_{t\in[0,\Theta]}$ be given by the formula

$$M_t^d = H_t - \int_0^t (1-H_u)\gamma(u,\widetilde{S}_u)\,du.$$

We denote by $\mathbb{H}$ the filtration generated by the process H and by $\mathbb{G}$ the *enlarged filtration* given as $\mathbb{F}\vee\mathbb{H}$. Then the process M^d is known to be a $\mathbb{G}$-martingale, called the *compensated jump martingale*. Moreover, the filtration $\mathbb{F}$ is *immersed* in $\mathbb{G}$, in the sense that all $\mathbb{F}$-martingales are $\mathbb{G}$-martingales; this property is also frequently referred to as Hypothesis (H). It implies, in particular, that the $\mathbb{F}$-Brownian motion W remains a Brownian motion with respect to the enlarged filtration $\mathbb{G}$ under $\mathbb{Q}$.

2.2 Market Model

We are now in a position to define the prices of primary traded assets in our market model. Assuming that τ_d is the default time of a reference entity (firm), we consider a continuous-time market on the time interval $[0,\Theta]$ composed of three primary assets:
• the savings account evolving according to the deterministic short-term interest rate r; we denote by β the *discount factor* (the inverse of the savings account), so that $\beta_t = e^{-\int_0^t r(u)\,du}$;
• the stock of the reference entity with the pre-default price process $\widetilde{S}$ given by (1) and the *fractional loss upon default* determined by a constant $\eta \leq 1$;
• a CDS contract written at time 0 on the reference entity, with maturity Θ, the *protection payment* given by a Borel-measurable, bounded function $\nu : [0,\Theta] \to \mathbb{R}$ and the fixed *CDS spread* $\bar{\nu}$.

Remarks 2.1 It is worth noting that the choice of a fixed-maturity CDS as a primary traded asset is only temporary and it is made here mainly for the sake of expositional simplicity. In Section 2.3 below, we will replace this asset by a more practical concept of a rolling CDS, which essentially is a self-financing trading strategy in market CDSs.

The *stock price* process $(S_t)_{t\in[0,\Theta]}$ is formally defined by setting

$$dS_t = S_{t-}\Big(\big(r(t) - q(t)\big)\,dt + \sigma(t, S_t)\,dW_t - \eta\,dM_t^d\Big), \quad S_0 = \widetilde{S}_0, \tag{5}$$

so that, as required, the equality $(1 - H_t)S_t = (1 - H_t)\widetilde{S}_t$ holds for every $t \in [0, \Theta]$. Note that estimate (3) enforces the following moment condition on the process S

$$\mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,\tau_d\wedge\Theta]} S_t\Big) < \infty, \quad \text{a.s.} \tag{6}$$

We define the *discounted cumulative stock price* $\beta\widehat{S}$ stopped at τ_d by setting, for every $t \in [0, \Theta]$,

$$\beta_t\widehat{S}_t = \beta_t(1 - H_t)\widetilde{S}_t + \int_0^{t\wedge\tau_d} \beta_u\big((1-\eta)\widetilde{S}_u\,dH_u + q(u)\widetilde{S}_u\,du\big)$$

or, equivalently, in terms of S

$$\beta_t\widehat{S}_t = \beta_{t\wedge\tau_d}S_{t\wedge\tau_d} + \int_0^{t\wedge\tau_d} \beta_u q(u) S_u\,du.$$

Note that we deliberately stopped $\beta\widehat{S}$ at default time τ_d, since we will not need to consider the behavior of the stock price strictly after default. Indeed, it will be enough to work under the assumption that all trading activities are stopped no later than at the random time $\tau_d \wedge \Theta$.

Let us now examine the valuation in the present model of a CDS written on the reference entity. We take the perspective of the credit protection buyer. Consistently with the no-arbitrage requirements (cf. [7]), we assume that the *pre-default CDS price* $(\widetilde{B}_t)_{t\in[0,\Theta]}$ is given as $\widetilde{B}_t = \widetilde{B}(t, \widetilde{S}_t)$, where the *pre-default CDS pricing function* $\widetilde{B}(t, S)$ is the unique (classical) solution on $[0, \Theta] \times \mathbb{R}_+$ to the following parabolic PDE

$$\mathcal{L}\widetilde{B}(t, S) + \delta(t, S) - \mu(t, S)\widetilde{B}(t, S) = 0, \quad \widetilde{B}(\Theta, S) = 0, \tag{7}$$

where
- the differential operator $\mathcal{L}$ is given by (2),
- $\delta(t, S) = \nu(t)\gamma(t, S) - \bar{\nu}$ is the *pre-default dividend function* of the CDS,
- $\mu(t, S) = r(t) + \gamma(t, S)$ is the *credit-risk adjusted interest rate*.

The *discounted cumulative CDS price* $\beta\widehat{B}$ equals, for every $t \in [0, \Theta]$,

$$\beta_t\widehat{B}_t = \beta_t(1 - H_t)\widetilde{B}_t + \int_0^{t\wedge\tau_d} \beta_u\big(\nu(u)\,dH_u - \bar{\nu}\,du\big). \tag{8}$$

Remarks 2.2 It is worth noting that as soon as the risk-neutral parameters in the dynamics of the stock price S are given by (5), the dynamics (8) of the CDS price is derived from the dynamics of S and our postulate that $\mathbb{Q}$ is the 'pricing probability' for a CDS. This procedure resembles the standard method of completing a stochastic volatility model by taking a particular option as an additional primary traded asset (see, e.g., Romano and Touzi [41]). We will sometimes refer to dynamics (5) as the model; it will be implicitly assumed that this model is actually completed either by trading a fixed-maturity CDS (as in Section 2.2.1) or by trading a rolling CDS (see Section 2.3).

Given the interest rate r, dividend yield q, the parameter η, and the covenants of a (rolling) CDS, the model calibration will then reduce to a specification of the local intensity γ and the local volatility σ only. We refer, in particular, Section 4.3.6 in which the concepts of the implied spread and the implied volatility of a convertible bond are examined.

2.2.1 Risk-Neutral Measures and Model Completeness

Since $\beta\widehat{S}$ and $\beta\widehat{B}$ are manifestly locally bounded processes, a *risk-neutral measure* for the market model is defined as any probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ such that the discounted cumulative prices $\beta\widehat{S}$ and $\beta\widehat{B}$ are $(\mathbb{G}, \widetilde{\mathbb{Q}})$-local martingales (see, for instance, Page 234 in Björk [9]). In particular, we note that the underlying probability measure $\mathbb{Q}$ is a risk-neutral measure for the market model. The following lemma can be easily proved using the Itô formula.

Lemma 2.1 *Let us denote* $\widehat{X}_t = \begin{bmatrix} \widehat{S}_t \\ \widehat{B}_t \end{bmatrix}$. *We have, for every* $t \in [0, \Theta]$,

$$d(\beta_t \widehat{X}_t) = d \begin{bmatrix} \beta_t \widehat{S}_t \\ \beta_t \widehat{B}_t \end{bmatrix} = \mathbb{1}_{\{t \le \tau_d\}} \beta_t \Sigma_t \, d \begin{bmatrix} W_t \\ M_t^d \end{bmatrix}, \tag{9}$$

where the $\mathbb{F}$*-predictable, matrix-valued process* Σ *is given by the formula*

$$\Sigma_t = \begin{bmatrix} \sigma(t, \widetilde{S}_t)\widetilde{S}_t & -\eta \widetilde{S}_t \\ \sigma(t, \widetilde{S}_t)\widetilde{S}_t \, \partial_S \widetilde{B}(t, \widetilde{S}_t) & \nu(t) - \widetilde{B}(t, \widetilde{S}_t) \end{bmatrix}. \tag{10}$$

We work in the sequel under the following standing assumption.

Assumption 2.2 The matrix-valued process Σ is invertible on $[0, \Theta]$.

The next proposition suggests that, under Assumption 2.2, our market model is complete with respect to defaultable claims maturing at $\tau_d \wedge \Theta$.

Proposition 2.1 *For any risk-neutral measure* $\widetilde{\mathbb{Q}}$ *for the market model, we have that the Radon-Nikodym density* $Z_t := \mathbb{E}_{\mathbb{Q}} \left(\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} \,\middle|\, \mathcal{G}_t \right) = 1$ *on* $[0, \tau_d \wedge \Theta]$.

Proof. For any probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_\Theta)$, the Radon-Nikodym density process Z_t, $t \in [0, \Theta]$, is a strictly positive $(\mathbb{G}, \mathbb{Q})$-martingale. Therefore, by the predictable representation theorem due to Kusuoka [33], there exist two $\mathbb{G}$-predictable processes, φ and φ^d say, such that

$$dZ_t = Z_{t-} \left(\varphi_t \, dW_t + \varphi_t^d \, dM_t^d \right), \quad t \in [0, \Theta]. \tag{11}$$

A probability measure $\widetilde{\mathbb{Q}}$ is then a risk-neutral measure whenever the process $\beta \widehat{X}$ is a $(\mathbb{G}, \widetilde{\mathbb{Q}})$-local martingale or, equivalently, whenever the process $\beta \widehat{X} Z$ is a $(\mathbb{G}, \mathbb{Q})$-local martingale. The latter condition is satisfied if and only if

$$\Sigma_t \begin{bmatrix} \varphi_t \\ \gamma(t, \widetilde{S}_t) \varphi_t^d \end{bmatrix} = 0. \tag{12}$$

The unique solution to (12) on $[0, \tau_d \wedge \Theta]$ is $\varphi = \varphi^d = 0$ and thus $Z = 1$ on $[0, \tau_d \wedge \Theta]$. □

2.3 Modified Market Model

In market practice, traders would typically prefer to use for hedging purposes the *rolling CDS*, rather than a fixed-maturity CDS considered in Section 2.2. Formally, the rolling CDS is defined as the wealth process of a self-financing trading strategy that amounts to continuously rolling one unit of long CDS contracts indexed by their *inception date* $t \in [0, \Theta]$, with respective maturities $\theta(t)$, where $\theta : [0, \Theta] \to [0, \Theta]$ is an increasing and piecewise constant function satisfying $\theta(t) \ge t$ (in particular, $\theta(\Theta) = \Theta$). We shall denote such contracts as $CDS(t, \theta(t))$.

Intuitively, the above mentioned strategy amounts to holding at every time $t \in [0, \Theta]$ one unit of the $CDS(t, \theta(t))$ combined with the *margin account*, that is, either positive or negative positions in the savings account. At time $t + dt$ the unit position in the $CDS(t, \theta(t))$ is unwounded (or offset) and the net mark-to-market proceeds, which may be either positive or negative depending on the evolution of the CDS market spread between the dates t and $t + dt$, are reinvested in the savings account. Simultaneously, a freshly issued unit credit default swap $CDS(t + dt, \theta(t + dt))$ is entered into at no cost. This procedure is carried on in continuous time (in practice, on a daily basis) until the hedging horizon. In the case of the rolling CDS, the entry $\beta \widehat{B}$ in (9) is meant to represent the discounted cumulative wealth process of this trading strategy. The next results shows that the only modification with respect to the case of a fixed-maturity CDS is that the matrix-valued process Σ, which was given previously by (10), should now be adjusted to Σ given by (13).

Lemma 2.2 *Under the assumption that $\widehat{B}$ represents the rolling CDS, Lemma 2.1 holds with the $\mathbb{F}$-predictable, matrix-valued process Σ given by the expression*

$$\Sigma_t = \left[\begin{array}{cc} \sigma(t,\widetilde{S}_t)\widetilde{S}_t & -\eta\widetilde{S}_t \\ \sigma(t,\widetilde{S}_t)\widetilde{S}_t\,\partial_S\widetilde{P}_{\theta(t)}(t,\widetilde{S}_t) - \bar{\nu}(t,\widetilde{S}_t)\sigma(t,\widetilde{S}_t)\widetilde{S}_t\,\partial_S\widetilde{F}_{\theta(t)}(t,\widetilde{S}_t) & \nu(t) \end{array} \right] \tag{13}$$

where the functions $\widetilde{P}_{\theta(t)}$ and $\widetilde{F}_{\theta(t)}$ are the pre-default pricing functions of the protection leg and fee legs of the $CDS(t,\theta(t))$, respectively, and the quantity

$$\bar{\nu}(t,\widetilde{S}_t) = \frac{\widetilde{P}_{\theta(t)}(t,\widetilde{S}_t)}{\widetilde{F}_{\theta(t)}(t,\widetilde{S}_t)}$$

represents the related CDS spread.

Proof. Of course, it suffices to focus on the second row in matrix Σ. We start by noting that Lemma 2.4 in [7], when specified to the present set-up, yields the following dynamics for the discounted cumulative wealth $\beta\widehat{B}$ of the rolling CDS between the deterministic times representing the jump times of the function θ

$$d(\beta_t\widehat{B}_t) = (1-H_t)\beta_t\alpha_t^{-1}\big(dp_t - \bar{\nu}(t,\widetilde{S}_t)\,df_t\big) + \beta_t\nu(t)\,dM_t^d, \tag{14}$$

where we denote

$$p_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^{\theta(t)} \alpha_u\nu(u)\gamma(u,\widetilde{S}_u)\,du\,\Big|\,\mathcal{F}_t\Big), \quad f_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^{\theta(t)} \alpha_u\,du\,\Big|\,\mathcal{F}_t\Big),$$

and where in turn the process α is given by

$$\alpha_t = e^{-\int_0^t \mu(u,\widetilde{S}_u)\,du} = e^{-\int_0^t (r(u)+\gamma(u,\widetilde{S}_u))\,du}.$$

In addition, being a $(\mathbb{G},\mathbb{Q})$-local martingale, the process $\beta\widehat{B}$ is necessarily continuous prior to default time τ_d (this follows, for instance, from Kusuoka [33]). It is therefore justified to use (14) for the computation of a diffusion term in the dynamics of $\beta\widehat{B}$.

To establish (13), it remains to compute explicitly the diffusion term in (14). Since the function θ is piecewise constant, it suffices in fact to examine the stochastic differentials dp_t and df_t for a fixed value $\theta = \theta(t)$ over each interval of constancy of θ. By the standard valuation formulae in an intensity-based framework, the pre-default price of a protection payment ν with a fixed horizon θ is given by, for $t \in [0,\theta]$,

$$\widetilde{P}_\theta(t,\widetilde{S}_t) = \alpha_t^{-1}\,\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\theta \alpha_u\nu(u)\gamma(u,\widetilde{S}_u)\,du\,\Big|\,\mathcal{F}_t\Big).$$

Therefore, by the definition of p, we have that, for $t \in [0,\theta]$,

$$p_t = \int_0^t \alpha_u\nu(u)\gamma(u,\widetilde{S}_u)\,du + \alpha_t\widetilde{P}_\theta(t,\widetilde{S}_t). \tag{15}$$

Since p is manifestly a $(\mathbb{G},\mathbb{Q})$-martingale, an application of the Itô formula to (15) yields, in view of (1),

$$dp_t = \alpha_t\sigma(t,\widetilde{S}_t)\widetilde{S}_t\,\partial_S\widetilde{P}_\theta(t,\widetilde{S}_t)\,dW_t.$$

Likewise, the pre-default price of a unit rate fee payment with a fixed horizon θ is given by

$$\widetilde{F}_\theta(t,\widetilde{S}_t) = \alpha_t^{-1}\,\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\theta \alpha_u\,du\,\Big|\,\mathcal{F}_t\Big).$$

By the definition of f, we obtain, for $t \in [0, \theta]$,

$$f_t = \int_0^t \alpha_u \, du + \alpha_t \widetilde{F}_\theta(t, \widetilde{S}_t)$$

and thus, noting that f is a $(\mathbb{G}, \mathbb{Q})$-martingale, we conclude easily that

$$df_t = \alpha_t \sigma(t, \widetilde{S}_t) \widetilde{S}_t \, \partial_S \widetilde{F}_\theta(t, \widetilde{S}_t) \, dW_t.$$

By inserting dp_t and df_t into (14), we complete the derivation of (13). □

Remarks 2.3 It is worth noting that for a fixed u the pricing functions $\widetilde{P}_{\theta(u)}$ and $\widetilde{F}_{\theta(u)}$ can be characterized as solutions of the PDE of the form (7) on $[u, \theta(u)] \times \mathbb{R}_+$ with the function δ therein given by $\delta^1(t, S) = \nu(t)\gamma(t, S)$ and $\delta^2(t, S) = 1$, respectively. Hence the use of the Itô formula in the proof of Lemma 2.2 can indeed be justified. Note also that, under the standing Assumption 2.2, a suitable form of completeness of the modified market model will follow from Proposition 2.1.

3 Convertible Securities

In this section, we first recall the concept of a *convertible security* (CS). Subsequently, we establish, or specify to the present situation, the fundamental results related to its valuation and hedging.

We start by providing a formal specification in the present set-up of the notion of a convertible security. Let 0 (resp. $T \le \Theta$) stand for the *inception date* (resp. the *maturity date*) of a CS with the underlying asset S. For any $t \in [0, T]$, we write $\mathcal{F}_T^t$ (resp. $\mathcal{G}_T^t$) to denote the set of all $\mathbb{F}$-stopping times (resp. $\mathbb{G}$-stopping times) with values in $[t, T]$. Given the *time of lifting of a call protection of a CS*, which is modeled by a stopping time $\bar{\tau}$ belonging to $\mathcal{G}_T^0$, we denote by $\bar{\mathcal{G}}_T^t$ the following class of stopping times

$$\bar{\mathcal{G}}_T^t = \left\{ \vartheta \in \mathcal{G}_T^t \, ; \, \vartheta \wedge \tau_d \ge \bar{\tau} \wedge \tau_d \right\}.$$

We will frequently use τ as a shorthand notation for $\tau_p \wedge \tau_c$, for any choice of $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$. For the definition of the game option, we refer to Kallsen and Kühn [31] and Kiefer [32].

Definition 3.1 A *convertible security* with the underlying S is a *game option* with the *ex-dividend cumulative discounted cash flows* $\pi(t; \tau_p, \tau_c)$ given by the following expression, for any $t \in [0, T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$,

$$\beta_t \pi(t; \tau_p, \tau_c) = \int_t^\tau \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau_c < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau = T\}} \xi \Big),$$

where:

- the *dividend process* $D = (D_t)_{t \in [0,T]}$ equals

$$D_t = \int_{[0,t]} (1 - H_u) \, dC_u + \int_{[0,t]} R_u \, dH_u$$

for some *coupon process* $C = (C_t)_{t \in [0,T]}$, which is a $\mathbb{G}$-adapted, real-valued, càdlàg process with bounded variation, and a $\mathbb{G}$-adapted, real-valued, càdlàg *recovery process* $R = (R_t)_{t \in [0,T]}$,
- the *put/conversion payment* L is given as a $\mathbb{G}$-adapted, real-valued, càdlàg process on $[0, T]$,
- the *call payment* U is a $\mathbb{G}$-adapted, real-valued, càdlàg process on $[0, T]$, such that $L_t \le U_t$ on $[\tau_d \wedge \bar{\tau}, \tau_d \wedge T)$,
- the *payment at maturity* ξ is a $\mathcal{G}_T$-measurable, real-valued random variable,
- the processes R, L and the random variable ξ are assumed to satisfy the following inequalities, for a positive constant c,

$$\begin{aligned} -c \le R_t &\le c\,(1 \vee S_t)\,, \quad t \in [0, T], \\ -c \le L_t &\le c\,(1 \vee S_t)\,, \quad t \in [0, T], \\ -c \le \xi &\le c\,(1 \vee S_T)\,. \end{aligned} \tag{16}$$

3.1 Arbitrage Valuation of a Convertible Security

We are in a position to recall and specify to the present set-up a general valuation result for a convertible security. Let us mention that the notion of an arbitrage price of a convertible security, referred to in what follows, is a suitable extension to game options (see Definition 2.6 in Kallsen and Kühn [31]) of the *No Free Lunch with Vanishing Risk* (NFLVR) condition of Delbaen and Schachermayer [23]. We also use here the well known connection between Dynkin games and the valuation of game options (see Kiefer [32]).

Proposition 3.1 *If the Dynkin game related to a convertible security admits a value* Π*, in the sense that*

$$\begin{aligned}\operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) &= \Pi_t \\ = \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big), &\quad t \in [0,T],\end{aligned} \tag{17}$$

and Π *is a* $\mathbb{G}$*-semimartingale, then* Π *is the unique arbitrage (ex-dividend) price of the CS.*

Proof. Except for the uniqueness statement, this follows by applying the general results in [4]. To verify the uniqueness property, we first note that for any risk-neutral measure $\widetilde{\mathbb{Q}}$, we have that $Z_t = \mathbb{E}_{\mathbb{Q}}\left(\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} \,\middle|\, \mathcal{G}_t\right) = 1$ on $[0,\tau_d \wedge T]$, by Proposition 2.1. In view of the estimate (6) on $\sup_{t\in[0,T\wedge\tau_d]} S_t$, and since $\sup_{t\in[0,T\wedge\tau_d]} S_t$ is a $\mathcal{G}_{\tau_d\wedge T}$-measurable random variable, this implies that, for any risk-neutral measure $\widetilde{\mathbb{Q}}$,

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\sup_{t\in[0,T\wedge\tau_d]} S_t\Big) = \mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T\wedge\tau_d]} S_t\Big) < \infty. \tag{18}$$

Obviously, for the supremum over the set $\mathcal{M}$ of all risk-neutral measures $\widetilde{\mathbb{Q}}$ we thus have that

$$\sup_{\widetilde{\mathbb{Q}}\in\mathcal{M}} \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\sup_{t\in[0,T\wedge\tau_d]} S_t\Big) < \infty. \tag{19}$$

Under condition (19), any arbitrage price of a CS with underlying S is then given by the value of the related Dynkin game for some risk-neutral measure $\widetilde{\mathbb{Q}}$, by the general results of [4]. Furthermore, $\pi(t;\tau_p,\tau_c)$ is a $\mathcal{G}_{\tau_d\wedge T}$-measurable random variable. Therefore, for any $t\in[0,T], \tau_p \in \mathcal{G}_T^t, \tau_c \in \bar{\mathcal{G}}_T^t$,

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big). \tag{20}$$

We conclude that the $\widetilde{\mathbb{Q}}$-Dynkin game has the value Π for any risk-neutral measure $\widetilde{\mathbb{Q}}$. □

We now define two special cases of CSs that correspond to American- and European-style CSs.

Definition 3.2 A *puttable security* (as opposed to puttable *and* callable, in the case of a general convertible security) is a convertible security with $\bar{\tau} = T$. An *elementary security* is a puttable security with a *bounded variation* dividend process D over $[0,T]$, a *bounded* payment at maturity ξ, and such that

$$\int_{[0,t]} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > t\}} \beta_t L_t \;\le\; \int_{[0,T]} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi, \quad t \in [0,T). \tag{21}$$

3.2 Doubly Reflected BSDEs Approach

We will now apply to convertible securities the method proposed by El Karoui et al. [27] for American options and extended by Cvitanić and Karatzas [20] to the case of stochastic games. In order to effectively deal with the doubly reflected BSDE associated with a convertible security, which is introduced in Definition 3.3 below, we need to impose some technical assumptions. We refer the reader to Section 4 for concrete examples in which all these assumptions are indeed satisfied.

Assumption 3.1 We postulate that:
• the coupon process C satisfies

$$C_t = C(t) := \int_0^t c(u)\,du + \sum_{0 \le T_i \le t} c^i, \tag{22}$$

for a bounded, Borel-measurable *continuous-time coupon rate function* $c(\cdot)$ and deterministic *discrete times* and *coupons* T_i and c^i, respectively; we take the tenor of the discrete coupons as $T_0 = 0 < T_1 < \cdots < T_{I-1} < T_I$ with $T_{I-1} < T \le T_I$;
• the recovery process $(R_t)_{t\in[0,T]}$ is of the form $R(t, S_{t-})$ for a Borel-measurable function R;
• $L_t = L(t, S_t)$, $U_t = U(t, S_t)$, $\xi = \xi(S_T)$ for some Borel-measurable functions L, U and ξ such that, for any t, S, we have

$$L(t,S) \le U(t,S), \quad L(T,S) \le \xi(S) \le U(T,S);$$

• the call protection time $\bar\tau \in \mathcal{F}^0_T$.

The *accrued interest* at time t is given by

$$A(t) = \frac{t - T_{i_t-1}}{T_{i_t} - T_{i_t-1}}\, c^{i_t}, \tag{23}$$

where i_t is the integer satisfying $T_{i_t-1} \le t < T_{i_t}$. On open intervals between the discrete coupon dates we thus have $dA(t) = a(t)\,dt$ with $a(t) = \frac{c^{i_t}}{T_{i_t} - T_{i_t-1}}$.

To a CS with data (functions) C, R, ξ, L, U and lifting time of call protection $\bar\tau$, we associate the Borel-measurable functions $f(t,S,x)$ (for x real), $g(S)$, $\ell(t,S)$ and $h(t,S)$ defined by

$$g(S) = \xi(S) - A(T), \quad \ell(t,S) = L(t,S) - A(t), \quad h(t,S) = U(t,S) - A(t), \tag{24}$$

and (recall that $\mu(t,S) = r(t) + \gamma(t,S)$)

$$f(t,S,x) = \gamma(t,S)R(t,S) + \Gamma(t,S) - \mu(t,S)x, \tag{25}$$

where we set

$$\Gamma(t,S) = c(t) + a(t) - \mu(t,S)A(t). \tag{26}$$

Remarks 3.1 In the case of a puttable security, the process U is not relevant and thus we may and do set $h(t,S) = +\infty$. Moreover, in the case of an elementary security, the process L plays no role either, and we redefine further $\ell(t,S) = -\infty$.

We define the quadruplet (f, g, ℓ, h) associated to a CS (parameterized by $x \in \mathbb{R}$, regarding f) as

$$f_t(x) = f(t, \widetilde S_t, x),\ g = g(\widetilde S_T),\ \ell_t = \ell(t, \widetilde S_t),\ h_t = \mathbb{1}_{\{t<\bar\tau\}}\infty + \mathbb{1}_{\{t\ge\bar\tau\}} h(t, \widetilde S_t) \tag{27}$$

with the convention that $0 \times \infty = 0$ in the last equality. Let us also write

$$\gamma_t = \gamma(t, \widetilde S_t)\,,\ \ \mu_t = \mu(t, \widetilde S_t)\,,\ \ \alpha_t = e^{-\int_0^t \mu_u\,du}. \tag{28}$$

It is well known that game options (in particular, convertible securities) can be studied by analyzing the corresponding doubly reflected Backward Stochastic Differential Equations (cf. [20]). In our set-up, this connection is formalized through the following definition.

Definition 3.3 Consider a convertible security with data $C, R, \xi, L, U, \bar\tau$ and the associated quadruplet (f, g, ℓ, h) given by (27). The associated *doubly reflected Backward Stochastic Differential Equation* has the form, for $t \in [0, T)$,

$$\begin{cases} -d\widehat\Pi_t = f_t(\widehat\Pi_t)\,dt + dK_t - Z_t\,dW_t, \\ \ell_t \le \widehat\Pi_t \le h_t, \\ (\widehat\Pi_t - \ell_t)\,dK_t^+ = (h_t - \widehat\Pi_t)\,dK_t^- = 0, \end{cases} \tag{$\mathcal{E}$}$$

with the terminal condition $\widehat\Pi_T = g$.

To define a solution of the doubly reflected BSDE ($\mathcal{E}$), we need to introduce the following spaces:

$\mathcal{H}^2$ – the set of real-valued, $\mathbb{F}$-predictable processes X such that $\mathbb{E}_{\mathbb{Q}}\Big(\int_0^T X_t^2\,dt\Big) < \infty$,

$\mathcal{S}^2$ – the set of real-valued, $\mathbb{F}$-adapted, continuous processes X such that $\mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T]} X_t^2\Big) < \infty$,

$\mathcal{A}^2$ – the space of continuous processes of finite variation K with (continuous and non decreasing) Jordan components $K^{\pm} \in \mathcal{S}^2$ null at time 0,

$\mathcal{A}_i^2$ – the space of non-decreasing, continuous processes null at 0 and belonging to $\mathcal{S}^2$.

For any $K \in \mathcal{A}^2$, we thus have that $K = K^+ - K^-$, where $K^{\pm} \in \mathcal{A}_i^2$ define mutually singular measures on $\mathbb{R}^+$.

Definition 3.4 By a *solution* to the doubly reflected BSDE ($\mathcal{E}$) with data (f, g, ℓ, h), we mean a triplet of processes $(\widehat{\Pi}, Z, K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{A}^2$ satisfying all conditions in ($\mathcal{E}$). In particular, the process K, and thus also the process $\widehat{\Pi}$, have to be continuous.

Remarks 3.2 **(i)** For a puttable security, we have that $\bar{\tau} = T$ and thus $K^- = 0$ in any solution $(\widehat{\Pi}, Z, K)$ to ($\mathcal{E}$). Therefore, the doubly reflected BSDE ($\mathcal{E}$) reduces to the reflected BSDE ($\mathcal{E}$.1) with data (f, g, ℓ) and $K = K^+ \in \mathcal{A}_i^2$ in any solution; specifically,

$$\begin{cases} -d\widehat{\Pi}_t = f_t(\widehat{\Pi}_t)\,dt + dK_t^+ - Z_t\,dW_t, \\ \ell_t \le \widehat{\Pi}_t, \\ (\widehat{\Pi}_t - \ell_t)\,dK_t^+ = 0, \end{cases} \tag{$\mathcal{E}$.1}$$

with the terminal condition $\widehat{\Pi}_T = g$.

(ii) For an elementary security, we have $K = 0$ in any solution $(\widehat{\Pi}, Z, K)$ to ($\mathcal{E}$). Consequently, the doubly reflected BSDE ($\mathcal{E}$) becomes the standard BSDE ($\mathcal{E}$.2) with data (f, g), that is,

$$-\,d\widehat{\Pi}_t = f_t(\widehat{\Pi}_t)\,dt - Z_t\,dW_t \tag{$\mathcal{E}$.2}$$

with the terminal condition $\widehat{\Pi}_T = g$.

In order to establish the well-posedness of the doubly reflected BSDE, as well as its connection with the related *obstacles problem* examined in the next section, we will work henceforth under the following additional assumption.

Assumption 3.2 The functions r, q, γ, σ, c, R, g, h, ℓ are continuous.

3.2.1 Super-Hedging Strategies for a Convertible Security

The following definition of a self-financing trading strategy is standard.

Definition 3.5 By a *self-financing strategy* over the time interval $[0, T]$, we mean a pair (V_0, ζ) such that:

- V_0 is a real number representing the *initial wealth*,
- $(\zeta_t)_{t\in[0,T]}$ is an $\mathbb{R}^{1\otimes 2}$-valued (bi-dimensional row vector), $\beta\widehat{X}$-integrable process (cf. (9)) representing holdings (number of units held) in primary risky assets.

The *wealth process* V of a self-financing strategy (V_0, ζ) is given by

$$\beta_t V_t = V_0 + \int_0^t \zeta_u\,d(\beta_u \widehat{X}_u), \quad t \in [0, T]. \tag{32}$$

Remarks 3.3 It should be emphasized that as $\beta\widehat{B}$ we can take in Definition 3.5 either the dynamics of the discounted wealth a fixed-maturity CDS, given by (8), or of a rolling CDS, given by (14). Consequently, in view of Lemmas 2.1 and 2.2, equality (32) becomes

$$\beta_t V_t = V_0 + \int_0^t [\zeta_u^1, \zeta_u^2]\,d\begin{bmatrix} \beta_u \widehat{S}_u \\ \beta_u \widehat{B}_u \end{bmatrix} = V_0 + \int_0^t \mathbb{1}_{\{u\le\tau_d\}} \beta_u [\zeta_u^1, \zeta_u^2]\,\Sigma_u\,d\begin{bmatrix} W_u \\ M_u^d \end{bmatrix}, \tag{33}$$

where the matrix-valued process Σ is given by (10) in the case of a fixed-maturity CDS, and it is given by (13) in the case of a rolling CDS. Formula (33) makes it clear that the wealth process V is stopped at time τ_d; this property reflects the fact that we are only interested in trading on the stochastic interval $[0, \tau_d \wedge T]$, where T is the maturity date of a considered convertible security.

In the set-up of this paper, the notions of the *issuer's* and *holder's (super-)hedges* take the following form. Recall that we denote $\tau = \tau_p \wedge \tau_c$.

Definition 3.6 (i) An *issuer's hedge* for a convertible security is represented by a triplet (V_0, ζ, τ_c) such that:
- (V_0, ζ) is a self-financing strategy with the wealth process V,
- the call time τ_c belongs to $\bar{\mathcal{G}}^0_T$,
- the following inequality is valid, for every put time $\tau_p \in \mathcal{G}^0_T$,

$$\beta_\tau V_\tau \geq \beta_0 \pi(0; \tau_p, \tau_c), \quad \text{a.s.} \tag{34}$$

(ii) A *holder's hedge* for a convertible security is a triplet (V_0, ζ, τ_p) such that:
- (V_0, ζ) is a self-financing strategy with the wealth process V,
- the put time τ_p belongs to $\mathcal{G}^0_T$,
- the following inequality is valid, for every call time $\tau_c \in \bar{\mathcal{G}}^0_T$,

$$\beta_\tau V_\tau \geq -\beta_0 \pi(0; \tau_p, \tau_c), \quad \text{a.s.} \tag{35}$$

Remarks 3.4 Definition 3.6 can be easily extended to hedges starting at any initial date $t \in [0, T]$, as well as specified to the particular cases of puttable and elementary securities (see [5, 6]).

By applying the general results of [5, 6], we obtain the following (super-)hedging result. Obviously, the conclusion of Proposition 3.2 hinges on the temporary assumption that the related BSDE $(\mathcal{E})$ has a solution. The issue of existence and uniqueness of a solution to $(\mathcal{E})$ will be addressed in the foregoing subsection. See also Remarks 3.9 for a more explicit representation of a hedging strategy.

Proposition 3.2 *Assume that a solution $(\widehat{\Pi}, Z, K)$ to the doubly reflected BSDE $(\mathcal{E})$ exists. Let Π_t denote $\mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t$ with $\widetilde{\Pi} := \widehat{\Pi} + A$. Then Π is the unique arbitrage price process of a convertible security.*
(i) *For any $t \in [0, T]$, an issuer's hedge with the initial wealth Π_t is furnished by*

$$\tau_c^* = \inf\left\{ u \in [\bar{\tau} \vee t, T];\, \widehat{\Pi}_u = h_u \right\} \wedge T$$

and

$$\zeta^*_u := \mathbb{1}_{\{u \leq \tau_d\}}\big[Z_u\,,\, R_u - \widetilde{\Pi}_{u-}\big]\Lambda_u, \quad u \in [t, T], \tag{36}$$

where $[Z_u, R_u - \widetilde{\Pi}_{u-}]$ denotes the concatenation of Z_u and $R_u - \widetilde{\Pi}_{u-}$ and where Λ denotes of the inverse of the matrix-valued process Σ over $[0, \tau_d \wedge T]$ (cf. Assumption 2.2). Moreover, Π_t is the smallest initial wealth of an issuer's hedge.
(ii) *For any $t \in [0, T]$, a holder's hedge with the initial wealth $-\Pi_t$ is furnished by*

$$\tau_p^* = \inf\left\{ u \in [t, T]\,;\, \widehat{\Pi}_u = \ell_u \right\} \wedge T$$

and $\zeta = -\zeta^*$ with ζ^* given by (36). Moreover, in case of a CS with bounded cash cash flows, $-\Pi_t$ is the smallest initial wealth of a holder's hedge.

Proof. In view of the general results of [5, 6], we see that the process Π defined in the statement of the proposition satisfies all the assumptions for the process Π introduced in Proposition 3.1. Hence it is the unique arbitrage price process of a CS. As for statements (i) and (ii), they are rather straightforward consequences of the general results of [5, 6]. □

Proposition 3.2 shows that in the present set-up a CS has a *bilateral hedging price*, in the sense that the price Π_t ensures super-hedging to both its issuer and holder, starting from the initial wealth Π_t for the former and $-\Pi_t$ for the latter, where process Π is also the unique arbitrage price. Note also that in the case of an elementary security, there are no stopping times involved and process K is equal to 0, so that (Π_t, ζ^*) in fact defines a replicating strategy.

Remarks 3.5 Let us recall that $\widehat{B}$ is aimed to represent either a fixed-maturity CDS or a rolling CDS. Since Assumption 2.2 was postulated for both cases then the underlying probability $\mathbb{Q}$ is the unique risk-neutral probability on $[0, \tau_d \wedge \Theta]$ no matter whether a fixed-maturity CDS or a rolling CDS is chosen to be a traded primary asset. Consequently, the hedging price of a CS does not depend on the choice of primary traded CDSs. By contrast, the super-hedging strategies of Proposition 3.2 are clearly dependent on the choice of traded CDSs through the matrix-valued process $\Lambda = \Sigma^{-1}$, where Σ is given either by (10) or by (13).

3.2.2 Solutions of the Doubly Reflected BSDE

As mentioned above, the existence of hedging strategies for a convertible security will be derived from the existence of a solution to the related doubly reflected BSDE. To establish the latter, we need to impose further technical assumptions on a convertible security under study.

Let then $\mathcal{P}$ stand for the class of functions Π of the real variable S bounded by $C(1 + |S|^p)$ for some real C and integer p that may depend on Π. By a slight abuse of terminology, we shall say that a function $\Pi(S, \dots)$ is of class $\mathcal{P}$ if it has polynomial growth in S, uniformly in other arguments. We postulate henceforth the following additional assumptions regarding the specification of a convertible security.

Assumption 3.3 The functions R, g, h, ℓ associated to a CS are of class $\mathcal{P}$ (or $h = +\infty$, in the case of a puttable security, and $\ell = -\infty$, in the case of an elementary security), and $\bar{\tau}$ is given as

$$\bar{\tau} = \inf\{\, t > 0\, ;\, \widetilde{S}_t \geq \bar{S}\} \wedge \bar{T} \tag{37}$$

for some constants $\bar{T} \in [0, T]$ and $\bar{S} \in \mathbb{R}_+ \cup \{+\infty\}$ (so, in particular, $\bar{\tau} = 0$ in case $\bar{S} = 0$, and $\bar{\tau} = \bar{T}$ in case $\bar{S} = +\infty$). As for ℓ, it satisfies, more specifically, the following *structure condition*: $\ell(t, S) = \lambda(t, S) \vee c$ for some constant $c \in \mathbb{R} \cup \{-\infty\}$, and a function λ of class $\mathcal{C}^{1,2}$ with

$$\lambda,\ \partial_t\lambda,\ S\partial_S\lambda,\ S^2\partial^2_{S^2}\lambda \in \mathcal{P} \tag{38}$$

(or $\ell = -\infty$, in the case of an elementary security).

Example 3.1 The standard example of the function $\lambda(t, S)$ satisfying (38) is $\lambda(t, S) = S$. In that case, ℓ corresponds to the payoff function of a call option (or, more precisely, to the lower payoff function of a convertible bond, see Section 4).

By an application of the general results of [6, 18], we then have the following proposition, which complements Proposition 3.2.

Proposition 3.3 *The doubly reflected BSDE ($\mathcal{E}$) admits a unique solution* $(\widehat{\Pi}, Z, K)$. □

In the next section, we will study the variational inequalities approach to convertible securities in the present Markovian set-up, as well as the link between the variational inequalities and the doubly reflected BSDEs.

3.3 Variational Inequalities Approach

In Section 3.3, we will give analytical characterizations of the so-called *pre-default clean prices* (that is, the pre-default prices less accrued interest) in terms of viscosity solutions to the associated variational inequalities. In the context of convertible bonds, the variational inequalities approach was examined, though without formal proofs, in Ayache et al. [2].

Convention. ***Unless explicitly stated otherwise, by a 'price' of a convertible security we mean henceforth its 'pre-default clean price.'***

Note that the clean prices correspond to the state-process $\widehat{\Pi}$ of a solution to ($\mathcal{E}$); see Proposition 3.2 and [6]. To obtain the corresponding pre-default price, it suffices to add to the clean price process the related accrued interest given by (23), provided, of course, that there are any discrete coupons present in the product under consideration.

For any $\bar{\tau} \in \mathcal{F}_T^0$, the associated price coincides on $[\bar{\tau}, T]$ with the price corresponding to a lifting time of call protection given by $\bar{\tau}^0 := 0$. This observation follows from the general results in [5], using also the fact that, under the standing assumptions, the BSDEs related to the problems with lifting times of call protection $\bar{\tau}$ and $\bar{\tau}^0$ both have solutions.

The *no-protection prices* (i.e., prices obtained for the lifting time of call protection $\bar{\tau}^0 = 0$) can thus also be interpreted as post-protection prices for an arbitrary stopping time $\bar{\tau} \in \mathcal{F}_T^0$, where by the *post-protection price* we mean the price restricted to the random time interval $[\bar{\tau}, T]$. Likewise, we define the *protection prices* as prices restricted to the random time interval $[0, \bar{\tau}]$.

For a closed domain $\mathcal{D} \subseteq [0,T] \times \mathbb{R}$, let $\mathrm{Int}_p\mathcal{D}$ and $\partial_p\mathcal{D}$ stand for the *parabolic interior* and the *parabolic boundary* of $\mathcal{D}$, respectively. For instance, if $\mathcal{D} = [0,\bar{T}] \times (-\infty, \bar{S}] =: \mathcal{D}(\bar{T}, \bar{S})$ for some $\bar{T} \in [0,T]$ and $\bar{S} \in \mathbb{R}$ (cf. formula (37) in Assumption 3.3) then

$$\mathrm{Int}_p\mathcal{D} = [0,\bar{T}) \times (-\infty, \bar{S}), \ \ \partial_p\mathcal{D} = \big([0,\bar{T}] \times \{\bar{S}\}\big) \cup \big(\{\bar{T}\} \times (-\infty, \bar{S})\big).$$

If $\mathcal{D} = [0,\bar{T}] \times \mathbb{R} =: \mathcal{D}(\bar{T}, +\infty)$ for some $\bar{T} \in [0,T]$ then

$$\mathrm{Int}_p\mathcal{D} = [0,\bar{T}) \times \mathbb{R}, \ \ \partial_p\mathcal{D} = \{\bar{T}\} \times \mathbb{R}.$$

Definition 3.7 Assume that we are given a closed domain $\mathcal{D} \subseteq [0,T] \times \mathbb{R}$ and a continuous boundary condition b of class $\mathcal{P}$ on $\partial_p\mathcal{D}$. We then introduce the following *obstacles problem* (or *variational inequality*, $(\mathcal{VI})$ for short) on $\mathrm{Int}_p\mathcal{D}$

$$\max\Big(\min\big(-\mathcal{L}\Pi(t,S) - f(t,S,\Pi(t,S)),\, \Pi(t,S) - \ell(t,S)\big),\, \Pi(t,S) - h(t,S)\Big) = 0 \qquad (\mathcal{VI})$$

with the boundary condition $\Pi = b$ on $\partial_p\mathcal{D}$, where $\mathcal{L}, \ell, h, f$ are defined in (2), (24) and (25).

Remarks 3.6 Note that the problem $(\mathcal{VI})$ is defined over a domain in space variable S ranging to $-\infty$, although only the positive part of this domain is meaningful for the financial purposes. Had we decided instead to pose the problem $(\mathcal{VI})$ over bounded spatial domains then, in order to get a well-posed problem, we would need to impose some appropriate non-trivial boundary condition at the lower space boundary.

The foregoing remarks, in which we deal with special cases of convertible securities, corresponds to Remarks 3.2.

Remarks 3.7 **(i)** For a puttable security, we have that $h = +\infty$ and thus the associated problem $(\mathcal{VI})$ simplifies to

$$\min\big(-\mathcal{L}\Pi(t,S) - f(t,S,\Pi(t,S)),\, \Pi(t,S) - \ell(t,S)\big) = 0 \qquad (\mathcal{VI}.1)$$

with the boundary condition $\Pi = b$ on $\partial_p\mathcal{D}$.
(ii) For an elementary security, we also have that $\ell = -\infty$ and thus the corresponding problem $(\mathcal{VI})$ reduces to the linear parabolic PDE

$$-\mathcal{L}\Pi(t,S) - f(t,S,\Pi(t,S)) = 0 \qquad (\mathcal{VI}.2)$$

with the boundary condition $\Pi = b$ on $\partial_p\mathcal{D}$.

Let us state the definition of a viscosity solution to the problem $(\mathcal{VI})$, which is required to handle potential discontinuities in time of f at the T_is in case there are discrete coupons (cf. (25)). Given a closed domain $\mathcal{D} \subseteq [0,T] \times \mathbb{R}$, we denote, for $i = 1, 2, \ldots, I$,

$$\mathcal{D}^i = \mathcal{D} \cap \{T_{i-1} \le t \le T_i\}, \ \ \mathrm{Int}_p\mathcal{D}^i = \mathrm{Int}_p\mathcal{D} \cap \{T_{i-1} \le t < T_i\}.$$

Note that the sets $\mathrm{Int}_p\mathcal{D}^i$ provide a partition of $\mathrm{Int}_p\mathcal{D}$.

Definition 3.8 **(i)** A locally bounded upper semicontinuous function Π on $\mathcal{D}$ is called a *viscosity subsolution* of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ if and only if $\Pi \leq h$, and $\Pi(t,S) > \ell(t,S)$ implies

$$-\mathcal{L}\varphi(t,S) - f(t,S,\Pi(t,S)) \leq 0$$

for any $(t,S) \in \mathrm{Int}_p\mathcal{D}^i$ and $\varphi \in \mathcal{C}^{1,2}(\mathcal{D}^i)$ such that $\Pi - \varphi$ is maximal on $\mathcal{D}^i$ at (t,S), for some $i \in 1,2,\ldots,I$.
(ii) A locally bounded lower semicontinuous function Π on $\mathcal{D}$ is called a *viscosity supersolution* of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ if and only if $\Pi \geq \ell$, and $\Pi(t,S) < h(t,S)$ implies

$$-\mathcal{L}\varphi(t,S) - f(t,S,\Pi(t,S)) \geq 0$$

for any $(t,S) \in \mathrm{Int}_p\mathcal{D}^i$ and $\varphi \in \mathcal{C}^{1,2}(\mathcal{D}^i)$ such that $\Pi - \varphi$ is minimal on $\mathcal{D}^i$ at (t,S), for some $i \in 1,2,\ldots,I$.
(iii) A function Π is called a *viscosity solution* of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ if and only if it is both a viscosity subsolution and a viscosity supersolution of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ (in which case Π is a continuous function).

Remarks 3.8 **(i)** In the case of a CS with no discrete coupons, the previous definitions reduce to the standard definitions of viscosity (semi-)solutions for obstacles problems (see, for instance, [17, 28]).
(ii) A classical solution of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ is necessarily a viscosity solution of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$.
(iii) A viscosity subsolution (resp. supersolution) Π of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ does not need to verify $\Pi \geq \ell$ (resp. $\Pi \leq h$) on $\mathrm{Int}_p\mathcal{D}$. A viscosity solution Π of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ necessarily satisfies $\ell \leq \Pi \leq h$ on $\mathrm{Int}_p\mathcal{D}$.

Building upon Definition 3.8, we introduce the following definition of $\mathcal{P}$*-(semi-)solutions* to $(\mathcal{VI})$ on $\mathcal{D}$.

Definition 3.9 By a $\mathcal{P}$*-subsolution* (resp. $\mathcal{P}$*-supersolution*, resp. $\mathcal{P}$*-solution*) Π of $(\mathcal{VI})$ on $\mathcal{D}$ for the boundary condition b, we mean a function of class $\mathcal{P}$ on $\mathrm{Int}_p\mathcal{D}$, which is a viscosity subsolution (resp. supersolution, resp. solution) of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$, and such that $\Pi \leq b$ (resp. $\Pi \geq b$, resp. $\Pi = b$) pointwise on $\partial_p\mathcal{D}$.

3.3.1 Pricing and Hedging Through Variational Inequalities

In the following results, the process $\widehat{\Pi}$ represents the state-process of the solution to the doubly reflected BSDE $(\mathcal{E})$ in Proposition 3.3. It thus depends, in particular, on the stopping time $\bar{\tau}$ representing the end of call protection period.

Lemma 3.1 (No-protection price) *Assume that $\bar{\tau} := \bar{\tau}^0 = 0$. Then the solution to the doubly reflected BSDE $(\mathcal{E})$ can be represented as $\widehat{\Pi}^0_t = \widehat{\Pi}^0(t, \widetilde{S}_t)$, where the function $\widehat{\Pi}^0$ is a $\mathcal{P}$-solution of $(\mathcal{VI})$ on $[0,T] \times \mathbb{R}$, with the terminal condition $\widehat{\Pi}^0(T,S) = g(S)$, where g is given by (24).*

Proof. This follows by the application of the results from [18]. □

Proposition 3.4 *Let $\bar{\tau}$ be given by (37) for some constants $\bar{T} \in [0,T]$ and $\bar{S} \in \mathbb{R}_+ \cup \{+\infty\}$.*
(i) ***Post-protection price.*** *On $[\bar{\tau}, T]$, the solution to the doubly reflected BSDE $(\mathcal{E})$ can be represented as $\widehat{\Pi}^0_t = \widehat{\Pi}^0(t, \widetilde{S}_t)$, where $\widehat{\Pi}^0$ is the function defined in Lemma 3.1;*
(ii) ***Protection price.*** *On $[0, \bar{\tau}]$, the solution to the reflected BSDE $(\mathcal{E}.1)$ can be represented as $\widehat{\Pi}^1_t = \widehat{\Pi}^1(t, \widetilde{S}_t)$, where the function $\widehat{\Pi}^1$ is a $\mathcal{P}$-solution of the problem $(\mathcal{VI}.1)$ on $\mathcal{D} = \mathcal{D}(\bar{T}, \bar{S})$ and the boundary condition $\widehat{\Pi}^1 = \widehat{\Pi}^0$ on $\partial_p\mathcal{D}$.*

Proof. In view of the observations made above, Lemma 3.1 immediately implies (i). In particular, we then have that $\widehat{\Pi}^0_{\bar{\tau}} = \widehat{\Pi}^0(\bar{\tau}, \widetilde{S}_{\bar{\tau}})$, where the restriction of $\widehat{\Pi}^0$ to $\partial_p\mathcal{D}$ defines a continuous function of class $\mathcal{P}$ over $\partial_p\mathcal{D}$. Part (ii) then follows by the application of the results from [18]. □

We are in a position to state the following corollary to Propositions 3.2 and 3.4.

Corollary 3.1 **(i)** ***Post-protection optimal exercise policies.*** *The* post-protection optimal put and call times (τ_p^*, τ_c^*) *after time* $t \in [0,T]$ *for the CS are given by*

$$\tau_p^* = \inf\left\{u \in [t,T]\,;\,(u,\widetilde{S}_u) \in \mathcal{E}_p\right\} \wedge T,$$
$$\tau_c^* = \inf\left\{u \in [t,T]\,;\,(u,\widetilde{S}_u) \in \mathcal{E}_c\right\} \wedge T,$$

where

$$\mathcal{E}_p = \left\{(u,S) \in [0,T] \times \mathbb{R}\,;\,\widehat{\Pi}^0(u,S) = \ell(u,S)\right\},$$
$$\mathcal{E}_c = \left\{(u,S) \in [0,T] \times \mathbb{R}\,;\,\widehat{\Pi}^0(u,S) = h(u,S)\right\},$$

are the post-protection put region *and the* post-protection call region*, respectively.*
(ii) ***Protection optimal exercise policy.*** *The* protection optimal put time τ_p^* *after time* $t \in [0,T]$ *for the CS is given by*

$$\tau_p^* = \inf\left\{u \in [t,\bar{\tau}]\,;\,(u,\widetilde{S}_u) \in \bar{\mathcal{E}}_p\right\},$$

where

$$\bar{\mathcal{E}}_p = \left\{(u,S) \in [0,T] \times \mathbb{R}\,;\,\widehat{\Pi}^1(u,S) = \ell(u,S)\right\}$$

is the protection put region*.* □

Assume that the call protection has not been lifted yet ($t < \bar{\tau}$) and that the CS is still alive at time t. Then an optimal strategy for the holder of the CS is to put the CS as soon as $(u,\widetilde{S}_u)$ hits $\bar{\mathcal{E}}_p$ for the first time after t, if this event actually happens before $\tau_d \wedge \bar{\tau}$.

If we assume instead that the call protection has already been lifted ($t \geq \bar{\tau}$) and that the CS is still alive at time t then:
- an optimal call time for the issuer of the CS is given by the first hitting time of $\mathcal{E}_c$ by $(u,\widetilde{S}_u)$ after t, provided this hitting time is realized before $T \wedge \tau_d$;
- an optimal put policy for the holder of the CS consists in putting when $(u,\widetilde{S}_u)$ hits $\mathcal{E}_p$ for the first time after t, if this event occurs before $T \wedge \tau_d$.

Remarks 3.9 Let us set (see Proposition 3.4)

$$\widehat{\Pi}(t,\widetilde{S}_t) = \mathbb{1}_{\{t \leq \bar{\tau}\}}\widehat{\Pi}^1(t,\widetilde{S}_t) + \mathbb{1}_{\{t > \bar{\tau}\}}\widehat{\Pi}^0(t,\widetilde{S}_t)$$

and let $\Pi_t = \mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t$ with $\widetilde{\Pi} = \widehat{\Pi} + A$. It then follows from Proposition 3.2 that $(\Pi_t)_{t\in[0,T]}$ is the arbitrage price process of the CS and the issuer's hedge with the initial wealth $\Pi_0 = \widehat{\Pi}_0$ is furnished by

$$\tau_c^* = \inf\left\{\, t \in [\bar{\tau},T];\, \widehat{\Pi}_t = h_t\right\} \wedge T$$

and

$$\zeta_t^* = [\zeta_t^{*1}, \zeta_t^{*2}] = \mathbb{1}_{\{t \leq \tau_d\}}\left[Z_t\,,\, R(t,\widetilde{S}_t) - \widetilde{\Pi}_{t-}\right]\Lambda_t, \tag{42}$$

where, as usual, Λ denotes of the inverse of the matrix-valued process Σ and the process Z is given by the expression

$$Z_t = \sigma(t,\widetilde{S}_t)\widetilde{S}_t\big(\mathbb{1}_{\{t\leq\bar{\tau}\}}\partial_S\widehat{\Pi}^1(t,\widetilde{S}_t) + \mathbb{1}_{\{t>\bar{\tau}\}}\partial_S\widehat{\Pi}^0(t,\widetilde{S}_t)\big), \tag{43}$$

where the last equality holds provided that the pricing functions $\widehat{\Pi}^0$ and $\widehat{\Pi}^1$ are sufficiently regular for the Itô formula to be applicable. Recall that the process Σ is given either by (10) or by (13), depending on whether we choose a fixed-maturity CDS of Section 2.2 or a rolling CDS of Section 2.3 as a traded asset $\widehat{B}$.

3.3.2 Approximation Schemes for Variational Inequalities

We now come to the issues of uniqueness and approximation of solutions for $(\mathcal{VI})$. For this, we make the following additional standing

Assumption 3.4 The functions r, q, γ, σ are locally Lipschitz continuous.

We refer the reader to Barles and Souganidis [3] (see also Crépey [18]) for the definition of *stable, monotone* and *consistent* approximation schemes to $(\mathcal{VI})$ and for the related notion of *convergence* of the scheme, involved in the following

Proposition 3.5 (i) ***Post-protection price.*** *The function* $\widehat{\Pi}^0$ *introduced in Proposition 3.4(i) is the* unique $\mathcal{P}$*-solution, the* maximal $\mathcal{P}$-subsolution*, and the* minimal $\mathcal{P}$-supersolution *of the related problem* $(\mathcal{VI})$ *on* $\mathcal{D} = [0,T]\times\mathbb{R}$*. Let* $(\widehat{\Pi}^0_h)_{h>0}$ *denote a stable, monotone and consistent approximation scheme for the function* $\widehat{\Pi}^0$*. Then* $\widehat{\Pi}^0_h \to \widehat{\Pi}^0$ *locally uniformly on* $\mathcal{D}$ *as* $h \to 0^+$*.*
(ii) ***Protection price.*** *The function* $\widehat{\Pi}^1$ *introduced in Proposition 3.4(ii) is the* unique $\mathcal{P}$*-solution, the* maximal $\mathcal{P}$-subsolution*, and the* minimal $\mathcal{P}$-supersolution *of the related problem* $(\mathcal{VI}.1)$ *on* $\mathcal{D} = \mathcal{D}(\bar{T},\bar{S})$*. Let* $(\widehat{\Pi}^1_h)_{h>0}$ *denote a stable, monotone and consistent approximation scheme for the function* $\widehat{\Pi}^1$*. Then* $\widehat{\Pi}^1_h \to \widehat{\Pi}^1$ *locally uniformly on* $\mathcal{D}$ *as* $h \to 0^+$*, provided (in case* $\bar{S} < +\infty$*)* $\widehat{\Pi}^1_h \to \widehat{\Pi}^1 = \widehat{\Pi}^0$ *on* $[0,\bar{T}]\times\{\bar{S}\}$*.*

Proof. Note, in particular, that under our assumptions:
• the functions $(r(t) - q(t) + \eta\gamma(t,S))S$ and $\sigma(t,S)S$ are locally Lipschitz continuous;
• the function f admits a *modulus of continuity* in S, in the sense that for every constant $c > 0$ there exists a continuous function $\eta_c : \mathbb{R}_+ \to \mathbb{R}_+$ with $\eta_c(0) = 0$ and such that, for any $t \in [0,T]$ and $S, S', x \in \mathbb{R}$ with $|S| \vee |S'| \vee |x| \le c$,

$$|f(t,S,x) - f(t,S',x)| \le \eta_c(|S - S'|).$$

The assertions are then consequences of the results in [18]. □

Remarks 3.10 We refer, in particular, the reader to [18] in regard to the fact that the potential discontinuities of f at the T_is (which represent a non-standard feature from the point of view of the classic theory of viscosity solutions as presented, for instance, in Crandall et al. [17]) are not a real issue in the previous results, provided one works with Definition 3.8 of viscosity solutions to our problems.

4 Convertible Bonds

As was already pointed out, a convertible bond is a special case of a convertible security. To describe the covenants of a typical convertible bond (CB), we introduce the following additional notation (for a detailed description and discussion of typical covenants of a CB, see, e.g., [2, 4, 35]):

$\bar{N}$: the par (nominal) value,

η: the fractional loss on the underlying equity upon default,

C: the deterministic coupon process given by (22),

$\bar{R}$: the recovery process on the CB upon default of the issuer at time t, given by $\bar{R}_t = \bar{R}(t,S_{t-})$ for a continuous bounded function $\bar{R}$,

κ: the conversion factor,

$R^{cb}_t = R^{cb}(t,S_{t-}) = (1-\eta)\kappa S_{t-} \vee \bar{R}_t$: the effective recovery process,

$\xi^{cb} = \bar{N} \vee \kappa S_T + A(T)$: the effective payoff at maturity; with A given by (23),

$\bar{P} \le \bar{C}$: the put and call nominal payments, respectively, such that $\bar{P} \le \bar{N} \le \bar{C}$,

$\delta \ge 0$: the length of the call notice period (see below),

$t^\delta = (t+\delta) \wedge T$: the end date of the call notice period started at t.

Note that *putting* a convertible bond at τ_p effectively means either putting or converting the bond at τ_p, whichever is best for the bondholder. This implies that, accounting for the accrued interest, the effective payment to the bondholder who decides to put at time t is

$$P_t^{ef} := \bar{P} \vee \kappa S_t + A(t). \tag{44}$$

As for *calling*, convertible bonds typically stipulate a positive *call notice period* δ clause, so that if the bond issuer makes a call at time τ_c, then the bondholder has the right to either redeem the bond for $\bar{C}$ or convert it into κ shares of stock at any time $t \in [\tau_c, \tau_c^\delta]$, where $\tau_c^\delta = (\tau_c + \delta) \wedge T$.

If the bond has been called at time t then, accounting for the accrued interest, the effective payment to the bondholder in case of exercise at time $u \in [t, (t+\delta) \wedge T]$ equals

$$C_t^{ef} := \bar{C} \vee \kappa S_t + A(t). \tag{45}$$

4.1 Reduced Convertible Bonds

A CB with a positive call notice period is rather hard to price directly. To overcome this difficulty, it is natural to use a two-step valuation method for a CB with a positive call notice period. In the first step, one searches for the value of a CB upon call, by considering a suitable family of puttable bonds indexed by the time variable t (see Proposition 4.7 and 4.8). In the second step, the price process obtained in the first step is used as the payoff at a call time of a CB with no call notice period, that is, with $\delta = 0$. To formalize this procedure, we find it convenient to introduce the concept of a *reduced convertible bond*, i.e., a particular convertible bond with no call notice period. Essentially, a reduced convertible bond associated with a given convertible bond with a positive call notice period is an 'equivalent' convertible bond with no call notice period, but with the payoff process at call adjusted upwards in order to account for the additional value due to the option-like feature of the positive call period for the bondholder.

Definition 4.1 A *reduced convertible bond* (RB) is a convertible security with coupon process C, recovery process R^{cb} and terminal payoffs L^{cb}, U^{cb}, ξ^{cb} such that (cf. (44)–(45))

$$R_t^{cb} = (1-\eta)\kappa S_{t-} \vee \bar{R}_t, \quad L_t^{cb} = \bar{P} \vee \kappa S_t + A(t) = P_t^{ef}, \quad \xi^{cb} = \bar{N} \vee \kappa S_T + A(T),$$

and, for every $t \in [0,T]$,

$$U_t^{cb} = \mathbb{1}_{\{t<\tau_d\}} \widetilde{U}^{cb}(t, S_t) + \mathbb{1}_{\{t \geq \tau_d\}} C_t^{ef}, \tag{46}$$

for a function $\widetilde{U}^{cb}(t,S)$ jointly continuous in time and space variables, except for negative left jumps of $-c^i$ at the T_is, and such that $\widetilde{U}^{cb}(t,S_t) \geq C_t^{ef}$ on the event $\{t < \tau_d\}$ (so $U_t^{cb} \geq C_t^{ef}$ for every $t \in [0,T]$).

The discounted dividend process of an RB is thus given by, for every $t \in [0,T]$,

$$\int_{[0,t]} \beta_u \, dD_u^{cb} = \int_0^{t \wedge \tau_d} \beta_u c(u) \, du + \sum_{0 \leq T_i \leq t, \, T_i < \tau_d} \beta_{T_i} c^i + \mathbb{1}_{\{0 \leq \tau_d \leq t\}} \beta_{\tau_d} R_{\tau_d}^{cb}. \tag{47}$$

Clearly, a CB with no notice period (i.e., with $\delta = 0$) is an RB with the function $\widetilde{U}^{cb}(t,S)$ given by the formula $\widetilde{U}^{cb}(t,S) = \bar{C} \vee \kappa S + A(t)$. More generally, the financial interpretation of the process $\widetilde{U}^{cb}$ in an RB is that $\widetilde{U}^{cb}$ represents the value of the RB upon a call at time t. In Section 4.2, we shall formally prove that, under mild regularity assumptions in our model, any CB (no matter whether the call period is positive or not) can be interpreted and priced as an RB prior to call.

In order to perform a deeper analysis of the bond and option features of a reduced convertible bond, it is useful to decompose an RB into the straight bond component, referred to as the *embedded bond*, and the option component, called the *embedded game exchange option*.

4.1.1 Embedded Bond

For an RB with the dividend process D^{cb} given by (47), we consider an elementary security with the same coupon process as the RB and with the quantities R^b and ξ^b given as follows:

$$R_t^b = \bar{R}_t, \quad \xi^b = \bar{N} + A(T), \tag{48}$$

so that

$$R_t^{cb} - R_t^b = \big((1-\eta)\kappa S_t - \bar{R}_t\big)^+ \geq 0, \quad \xi^{cb} - \xi^b = (\kappa S_T - \bar{N})^+ \geq 0.$$

This elementary security corresponds to the defaultable bond with discounted cash flows given by the expression

$$\begin{aligned} \beta_t \phi(t) &= \int_t^T \beta_u \, dD_u^b + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi^b \\ &:= \int_t^{T\wedge\tau_d} \beta_u c(u)\, du + \sum_{t<T_i\leq T, T_i<\tau_d} \beta_{T_i} c^i + \mathbb{1}_{\{t<\tau_d\leq T\}} \beta_{\tau_d} R_{\tau_d}^b + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi^b \end{aligned} \tag{49}$$

and the associated functions (cf. (24)–(25))

$$f(t,S,x) = \gamma(t,S)\bar{R}(t,S) + \Gamma(t,S) - \mu(t,S)x, \quad g(S) = \bar{N}.$$

Definition 4.2 The RB with discounted cash flows given by (48)–(49) is called the *bond embedded into the RB*, or simply the *embedded bond*. It can be seen as the 'straight bond' component of the RB, that is, the RB stripped of its optional clauses.

In the sequel, in addition to the assumptions made so far, we work under the following reinforcement of Assumption 3.4.

Assumption 4.1 The functions $r(t)$, $q(t)$, $\gamma(t,S)S$, $\sigma(t,S)S$, $\gamma(t,S)\bar{R}(t,S)$ and $c(t)$ are continuously differentiable in time variable, and thrice continuously differentiable in space variable, with bounded related spatial partial derivatives.

Note that these assumptions cover typical financial applications. In particular, they are satisfied when $\bar{R}$ is constant and for well-chosen parameterizations of σ and γ, which can be enforced at the time of the calibration of the model.

Proposition 4.1 **(i)** *In the case of an RB, the BSDE ($\mathcal{E}$) (see part (ii) in Remarks 3.2) associated with the embedded bond admits a unique solution $(\widehat{\Phi}, Z, K=0)$. Denoting $\widetilde{\Phi} = \widehat{\Phi} + A$, the embedded bond admits the unique arbitrage price*

$$\Phi_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{\Phi}_t, \quad t \in [0,T]. \tag{50}$$

(ii) *Moreover, we have that $\widehat{\Phi}_t = \widehat{\Phi}(t, \widetilde{S}_t)$ where the function $\widehat{\Phi}(t,S)$ is bounded, jointly continuous in time and space variables, twice continuously differentiable in space variable, and of class $\mathcal{C}^{1,2}$ on every time interval $[T_{i-1}, T_i)$ (or $[T_{I-1}, T)$, in case $i = I$). The process $\widehat{\Phi}(t, \widetilde{S}_t)$ is an Itô process with true martingale component; specifically, we have*

$$d\widehat{\Phi}_t = \big(\mu_t \widehat{\Phi}_t - (\gamma_t R_t^b + \Gamma_t)\big)\, dt + \sigma(t, \widetilde{S}_t)\widetilde{S}_t \partial_S \widehat{\Phi}_t \, dW_t = u_t\, dt + v_t \, dW_t, \tag{51}$$

where the process v belongs to $\mathcal{H}^2$.

Proof. **(i)** By standard results (see, e.g., [25, 27]), the BSDE ($\mathcal{E}$) with data $(\gamma\bar{R} + \Gamma - \mu x, \bar{N})$ admits a unique solution $(\widehat{\Phi}, Z, K=0)$. Therefore, from Proposition 3.2 specified to the particular case of an elementary security, we deduce that the embedded bond admits a unique arbitrage price given by (50).

(ii) The BSDE yields, for every $t \in [0,T]$,

$$\widehat{\Phi}_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \big(\gamma_u R_u^b + \Gamma_u - \mu_u \widehat{\Phi}_u\big)\,du + \big(\xi^b - A(T)\big)\,\Big|\,\mathcal{F}_t\Big)$$

or, equivalently,

$$\alpha_t \widehat{\Phi}_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \alpha_u\big(\gamma_u R_u^b + \Gamma_u\big)\,du + \alpha_T\big(\xi^b - A(T)\big)\,\Big|\,\mathcal{F}_t\Big). \tag{52}$$

Note that we have (cf. (23) and (28) with, by convention $A(0-) = 0$)

$$\alpha_T A(T) = \int_{[0,T]} d(\alpha_u A(u)) = \int_0^T \alpha_u\big(a(u) - \mu_u A(u)\big)\,du - \sum_{0 \le T_i \le T} \alpha_{T_i} c^i.$$

By plugging this into (52) and using the equalities $\widetilde{\Phi} = \widehat{\Phi} + A$ and (26), we obtain

$$\alpha_t \widetilde{\Phi}_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \alpha_u\big(\gamma_u R_u^b + c(u)\big)\,du + \sum_{t < T_i \le T} \alpha_{T_i} c^i + \alpha_T \xi^b\,\Big|\,\mathcal{F}_t\Big).$$

Let us set

$$\alpha_t \widehat{\Phi}_t^0 = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \alpha_u\big(\gamma_u R_u^b + c(u)\big)\,du + \alpha_T(\bar{N} + A(T))\,\Big|\,\mathcal{F}_t\Big), \quad t \le T, \tag{53}$$

$$\alpha_t \widehat{\Phi}_t^i = \mathbb{E}_{\mathbb{Q}}\big(\,\alpha_{T_i} c^i\,\big|\,\mathcal{F}_t\big), \quad t \le T_i. \tag{54}$$

We have $\widetilde{\Phi}_T = \widehat{\Phi}_T^0$ and $\widetilde{\Phi}_t = \widehat{\Phi}_t^0 + \sum_{j; T_i \le T_j \le T} \widehat{\Phi}_t^j$ on $[T_{i-1}, T_i)$ (or on $[T_{I-1}, T)$ in case $i = I$). Let us denote generically T or T^i by $\mathcal{T}$, and $\widehat{\Phi}^0$ or $\widehat{\Phi}^i$ by $\widehat{\Theta}$, as appropriate according to the problem at hand. Note that $\widehat{\Theta}$ is bounded. In addition, given our regularity assumptions, we have $\widehat{\Theta}_t = \widehat{\Theta}(t, \widetilde{S}_t)$, where $\widehat{\Theta}$ belongs to $\mathcal{C}^{1,2}([0,\mathcal{T}) \times \mathbb{R}) \cap \mathcal{C}^0([0,\mathcal{T}] \times \mathbb{R})$ (see [27, 40]). Therefore, $\widehat{\Phi}_t = \widetilde{\Phi}_t - A(t)$ is given by $\widehat{\Phi}(t, \widetilde{S}_t)$ for a function $\widehat{\Phi}(t, S)$, which is jointly continuous in (t, S) on $[0,T] \times \mathbb{R}$ and twice continuously differentiable in S on $[0,T) \times \mathbb{R}$. Moreover, given (53)–(54) and the above $\mathcal{C}^{1,2}$ regularity results, we have

$$d\widehat{\Phi}_t^0 = \Big(\mu_t \widehat{\Phi}_t^0 - \big(\gamma_t R_t^b + c(t)\big)\Big)\,dt + \sigma(t, \widetilde{S}_t)\widetilde{S}_t \partial_S \widehat{\Phi}^0(t, \widetilde{S}_t)\,dW_t, \quad t < T,$$
$$d\widehat{\Phi}_t^i = \mu_t \widehat{\Phi}_t^i\,dt + \sigma(t, \widetilde{S}_t)\widetilde{S}_t \partial_S \widehat{\Phi}^i(t, \widetilde{S}_t)\,dW_t, \quad t < T_i \wedge T, \text{ for } i = 1, 2, \ldots, I,$$
$$dA(t) = \rho(t)\,dt, \quad t \notin \{T_i\}_{i=0,1,\ldots,I}\,.$$

This yields

$$d\widehat{\Phi}(t, \widetilde{S}_t) = \Big(\mu_t \widetilde{\Phi}_t - \big(\gamma_t R_t^b + c(t) + \rho(t)\big)\Big)\,dt + \sigma(t, \widetilde{S}_t)\widetilde{S}_t \partial_S \widetilde{\Phi}(t, \widetilde{S}_t)\,dW_t = u_t\,dt + v_t\,dW_t.$$

Moreover, since $\widehat{\Phi}$ and u in (51) are bounded, we conclude that $v \in \mathcal{H}^2$. □

4.1.2 Embedded Game Exchange Option

The option component of an RB is formally defined as an RB with the dividend process $D^{cb} - D^b$, payment at maturity $\xi^{cb} - \xi^b$, put payment $L_t^{cb} - \Phi_t$, call payment $U_t^{cb} - \Phi_t$ and call protection lifting time $\bar{\tau}$, where Φ is the embedded bond price in (50). This can be formalized by means of the following definition.

Definition 4.3 The *embedded game exchange option* is a zero-coupon convertible security with discounted cash flows, for any $t \in [0,T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$:

$$\beta_t \psi(t; \tau_p, \tau_c) = \mathbb{1}_{\{t < \tau_d \le \tau\}}\,\beta_{\tau_d}(R_{\tau_d}^{cb} - R_{\tau_d}^b) \tag{55}$$
$$+ \mathbb{1}_{\{\tau_d > \tau\}}\,\beta_\tau\Big(\mathbb{1}_{\{\tau = \tau_p < T\}}\big(L_{\tau_p}^{cb} - \Phi_{\tau_p}\big) + \mathbb{1}_{\{\tau = \tau_c < \tau_p\}}\,\big(U_{\tau_c}^{cb} - \Phi_{\tau_c}\big) + \mathbb{1}_{\{\tau = T\}}(\xi^{cb} - \xi^b)\Big).$$

Note that from the point of view of the financial interpretation (see [4] for more comments), the game exchange option corresponds to an option to exchange the embedded bond for either L^{cb}, U^{cb} or ξ^{cb} (as seen from the perspective of the holder), according to which player decides first to stop this game prior to or at T.

Also note that in the case of the game exchange option, there are clearly no coupons involved and thus the clean price and the price coincide.

4.1.3 Solutions of the Doubly Reflected BSDEs

The following auxiliary result can be easily proved by inspection.

Lemma 4.1 *Given an RB, the associated functions $f(t,S,x)$, $g=g(S)$, $\ell=\ell(t,S)$ and $h=h(t,S)$ are:*
- $f=\gamma R^{cb}+\Gamma-\mu x$, $g=\bar{N}\vee\kappa S$, $\ell=\bar{P}\vee\kappa S$ *and* $h=\widetilde{U}^{cb}-A$ *for the RB;*
- $f=\gamma(R^{cb}-R^{b})-\mu x$, $g=(\kappa S-\bar{N})^{+}$, $\ell=\bar{P}\vee\kappa S-\widehat{\Phi}$ *and* $h=\widetilde{U}^{cb}-A-\widehat{\Phi}$ *for the embedded game exchange option.* □

We will now show how our results can be applied to both an RB and an embedded game exchange option.

Proposition 4.2 **(i)** *The data f,g,ℓ,h (and $\bar{\tau}$ given, as usual, by (37)) associated to an RB satisfy all the assumptions of Propositions 3.4–3.5*
(ii) *The BSDEs $(\mathcal{E})$ related to an RB or to the embedded game exchange option have unique solutions.*

Proof. **(i)** This can be verified directly by inspection of the related data in Lemma 4.1 (we are in fact in the situation of Example 3.1).
(ii) Given part (i), the BSDE $(\mathcal{E})$ related to an RB has a unique solution $(\widehat{\Pi},V,K)$, by a direct application of Proposition 3.3. Now, $(\widehat{\Phi},Z,0)$ denoting the solution to the BSDE $(\mathcal{E})$ exhibited in Proposition 4.1(i), it is immediate to check that $(\widehat{\Psi},Y,K)$ solves the game exchange option-related problem $(\mathcal{E})$ iff $(\widehat{\Phi}+\widehat{\Psi},Z+Y,K)$ solves the RB-related problem $(\mathcal{E})$. Hence the result for the game exchange option follows from that for the RB. □

Given an RB and the embedded game exchange option, we denote by $\widehat{\Pi}$ and $\widehat{\Psi}$ the state-processes (i.e., the first components) of solutions to the related BSDEs. The following result summarizes the valuation of an RB and the embedded game exchange option.

Proposition 4.3 **(i)** *The process Ψ_t defined as $\mathbb{1}_{\{t<\tau_d\}}\widehat{\Psi}_t$ is the unique arbitrage price of the embedded game exchange option and $(\Psi_t,\zeta^*,\tau_c^*)$ (resp. $(-\Psi_t,-\zeta^*,\tau_p^*)$) as defined in Proposition 3.2 is an issuer's hedge with initial value Ψ_t (resp. holder's hedge with initial value $-\Psi_t$) starting from time t for the embedded game exchange option.*
(ii) *The process Π_t defined as $\mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t$, with $\widetilde{\Pi}:=\widehat{\Pi}+A$, is the unique arbitrage price of the RB, and (Π_t,ζ^*,τ_c^*) (resp. $(-\Pi_t,-\zeta^*,\tau_p^*)$) as defined in Proposition 3.2 is an issuer's hedge with initial value Π_t (resp. holder's hedge with initial value $-\Pi_t$) starting from time t for the RB.*
(iii) *With $\widehat{\Phi}$ and Φ defined as in Proposition 4.1, we have that $\Pi=\Phi+\Psi$ and $\widehat{\Pi}=\widehat{\Phi}+\widehat{\Psi}$.*

Proof. Given Proposition 4.2, statements (i) and (ii) follow by an application of Proposition 3.2., Part (iii) is then a consequence of the general results of [4]. □

4.1.4 Variational Inequalities for Post-Protection Prices

We consider the following problems $(\mathcal{VI})$ (for a game exchange option or an RB) or $(\mathcal{VI}.2)$ (for the defaultable bond) on $\mathcal{D}=[0,T]\times\mathbb{R}$:
- for a defaultable bond

$$
\begin{aligned}
-\mathcal{L}\widehat{\Phi}+\mu\widehat{\Phi}-(\gamma R^{b}+\Gamma)=0,&\quad t<T,\\
\widehat{\Phi}(T,S)=\bar{N},&
\end{aligned}
\tag{56}
$$

• for a game exchange option

$$\max\Big(\min\Big(-\mathcal{L}\widehat{\Psi}+\mu\widehat{\Psi}-\gamma\big(R^{cb}-R^{b}\big),\,\widehat{\Psi}-\big(\bar{P}\vee\kappa S-\widehat{\Phi}\big)\Big)\,,\,\widehat{\Psi}-\big(\widetilde{U}^{cb}-A-\widehat{\Phi}\big)\Big)=0,\quad t<T,$$
$$\widehat{\Psi}(T,S)=(\kappa S-\bar{N})^{+}, \tag{57}$$

• for an RB

$$\max\Big(\min\Big(-\mathcal{L}\widehat{\Pi}+\mu\widehat{\Pi}-\big(\gamma R^{cb}+\Gamma\big),\,\widehat{\Pi}-\bar{P}\vee\kappa S\Big)\,,\,\widehat{\Pi}-\big(\widetilde{U}^{cb}-A\big)\Big)=0,\quad t<T,$$
$$\widehat{\Pi}(T,S)=\bar{N}\vee\kappa S. \tag{58}$$

Convention. ***In the sequel we denote generically by $\widehat{\Theta}$ the state-process (i.e., the first component) of the solution to the BSDE related to an RB, the embedded game exchange option or the embedded bond, as is appropriate for the problem at hand.***

Proposition 4.4 (Post-Protection Prices) *For any of problems (56)-(58) there exists a $\mathcal{P}$-solution on $\mathcal{D}$, denoted generically as $\widehat{\Theta}(t,S)$, which determines the corresponding* post-protection price*, in the sense that*

$$\widehat{\Theta}_t=\widehat{\Theta}(t,\widetilde{S}_t),\quad t\in[\bar{\tau},T]. \tag{59}$$

Moreover, we have uniqueness of the $\mathcal{P}$-solution and any stable, monotone and consistent approximation scheme for $\widehat{\Theta}$ converges locally uniformly to $\widehat{\Theta}$ on $\mathcal{D}$ as $h\to 0^{+}$. In the case of the RB and the embedded game exchange option, the post-protection put/conversion region and the post-protection call/conversion region are given as

$$\mathcal{E}_p=\big\{(u,S)\in[0,T]\times\mathbb{R}\,;\,\widehat{\Pi}(u,S)=\bar{P}\vee\kappa S\big\},$$
$$\mathcal{E}_c=\big\{(u,S)\in[0,T]\times\mathbb{R}\,;\,\widehat{\Pi}(u,S)=\widetilde{U}^{cb}(u,S)-A(u)\big\}.$$

Proof. In the case of the RB or of the embedded bond, the results follow by direct application of Propositions 4.2, 3.4(i), 3.5(i) and Corollary 3.1(i). Now, given that $\widehat{\Pi}$ and $\widehat{\Phi}$ are $\mathcal{P}$-solutions to (58) and (56), respectively, and in view of the regularity properties of $\widehat{\Phi}$ stated in Proposition 4.1(ii), therefore $\widehat{\Psi}:=\widehat{\Pi}-\widehat{\Phi}$ is a $\mathcal{P}$-solution to (57). Since $\widehat{\Pi}$ and $\widehat{\Phi}$ satisfy the related identities (59), then so does $\widehat{\Psi}$, in view of Proposition 4.3(iii). Finally, given the last statement in Proposition 3.5, the game exchange option also satisfies the claimed uniqueness and convergence results. □

4.1.5 Variational Inequalities for Protection Prices

We now deal with the following problems ($\mathcal{VI}$.1) on $\mathcal{D}=\mathcal{D}(\bar{T},\bar{S})$, where the functions $\widehat{\Phi},\widehat{\Psi},\widehat{\Pi}$ are solutions to (60)–(61):

• for a game exchange option

$$\min\Big(-\mathcal{L}\bar{\Psi}+\mu\bar{\Psi}-\gamma\big(R^{cb}-R^{b}\big),\,\bar{\Psi}-\big(\bar{P}\vee\kappa S-\widehat{\Phi}\big)\Big)=0\text{ on }\mathrm{Int}_p\mathcal{D},$$
$$\bar{\Psi}=\widehat{\Psi}\text{ on }\partial_p\mathcal{D}, \tag{60}$$

• for an RB

$$\min\Big(-\mathcal{L}\bar{\Pi}+\mu\bar{\Pi}-\big(\gamma R^{cb}+\Gamma\big),\,\bar{\Pi}-\bar{P}\vee\kappa S\Big)=0\text{ on }\mathrm{Int}_p\mathcal{D},$$
$$\bar{\Pi}=\widehat{\Pi}\text{ on }\partial_p\mathcal{D}. \tag{61}$$

Proposition 4.5 (Protection Prices) *For any of the problems (60)–(61) there exists a $\mathcal{P}$-solution on $\mathcal{D}$, denoted generically as $\bar{\Theta}$, that determines the corresponding protection price, in the sense that*

$$\widehat{\Theta}_t=\bar{\Theta}(t,\widetilde{S}_t),\quad t\in[0,\bar{\tau}].$$

Moreover, we have uniqueness of the $\mathcal{P}$-solution and any stable, monotone and consistent approximation scheme for $\bar{\Theta}$ converges locally uniformly to $\bar{\Theta}$ on $\mathcal{D}$ as $h \to 0^+$, provided (in case $\bar{S} < +\infty$) it converges to $\bar{\Theta}(=\widehat{\Theta})$ at $\bar{S}$. In the case of the RB and the embedded game exchange option, the protection put/conversion region is given as

$$\bar{\mathcal{E}}_p = \big\{(u,S) \in [0,T] \times \mathbb{R}\,;\, \bar{\Pi}(u,S) = \bar{P} \vee \kappa S\big\}.$$

Proof. In the case of the RB, the results follow by direct application of Propositions 4.2, 3.4(ii), 3.5(ii) and Corollary 3.1(ii). In the case of the game exchange option, we proceed by taking the difference, as in the proof of Proposition 4.4 ($\widehat{\Phi}$ denoting the same function as before). □

4.2 Convertible Bonds with a Positive Call Notice Period

We now consider the case of a convertible bond with a positive call notice period. Note that between the call time t and the end of the notice period $t^\delta = (t+\delta) \wedge T$, a CB actually becomes a CB with no call clause (or *puttable bond*) over the time interval $[t, t^\delta]$, which is a special case of a puttable security (cf. Definition 3.2; formally, we set $\bar{\tau} = t^\delta$ in the related BSDE). For a fixed t, this puttable bond, denoted henceforth as the *t-PB*, has the effective payment given by the process C^{ef}_u, $u \in [t, t^\delta]$ for the original CB (see (45)).

Lemma 4.2 *In the case of the t-PB, the associated functions $f(u,S,x)$, $g = g(S)$ and $\ell = \ell(u,S)$ are ($h = +\infty$ in all three cases below):*
• ***embedded bond*** *(called the **t-bond**, in the sequel): $f(u,S,x) = \gamma(u,S)R^b(u,S) + \Gamma(u,S) - \mu(u,S)x$, $g(S) = \bar{C}$ and $\ell(u,S) = -\infty$;*
• ***embedded game exchange option*** *(called the **t-game exchange option**, in the sequel): $f(u,S,x) = \gamma(u,S)(R^{cb} - R^b)(u,S) - \mu(u,S)x$, $g(S) = \bar{C} \vee \kappa S - \widehat{\Phi}^t(t^\delta, S)$ and $\ell(u,S) = \bar{C} \vee \kappa S - \widehat{\Phi}^t(u,S)$, where $\widehat{\Phi}^t$ is the pricing function of the t-bond (obtained by an application of Proposition 4.4, see also (62) below);*
• ***t-PB****: $f(u,S,x) = \gamma(u,S)R^{cb}(u,S) + \Gamma(u,S) - \mu(u,S)x$, $g(S) = \bar{C} \vee \kappa S$ and $\ell(u,S) = \bar{C} \vee \kappa S$.*

Note that in view of the proof of Proposition 4.7 below, it is convenient to define the related pricing problems on $[0, t^\delta] \times \mathbb{R}$, rather than merely on $[t, t^\delta] \times \mathbb{R}$. Specifically, given $t \in [0,T]$, we define the following problems $(\mathcal{VI})$ on $[0, t^\delta] \times \mathbb{R}$:
• for the t-bond

$$\begin{aligned} -\mathcal{L}\widehat{\Phi}^t + \mu\widehat{\Phi}^t - \big(\gamma R^b + \Gamma\big) = 0, \quad u < t^\delta, \\ \widehat{\Phi}^t(t^\delta, S) = \bar{C}, \end{aligned} \tag{62}$$

• for the t-game exchange option

$$\begin{aligned} \min\Big(-\mathcal{L}\widehat{\Psi}^t + \mu\widehat{\Psi}^t - \gamma\big(R^{cb} - R^b\big),\, \widehat{\Psi}^t - \big(\bar{C} \vee \kappa S - \widehat{\Phi}^t\big)\Big) = 0, \quad u < t^\delta, \\ \widehat{\Psi}^t(t^\delta, S) = \bar{C} \vee \kappa S - \widehat{\Phi}^t(t^\delta, S), \end{aligned} \tag{63}$$

• for the t-PB

$$\begin{aligned} \min\Big(-\mathcal{L}\widehat{\Pi}^t + \mu\widehat{\Pi}^t - \big(\gamma R^{cb} + \Gamma\big),\, \widehat{\Pi}^t - \bar{C} \vee \kappa S\Big) = 0, \quad u < t^\delta, \\ \widehat{\Pi}^t(t^\delta, S) = \bar{C} \vee \kappa S. \end{aligned} \tag{64}$$

Proposition 4.6 *For any of problems (62)-(64), the corresponding BSDE ($\mathcal{E}$) has a solution, and the related t-price process $\widehat{\Theta}^t_u$ can be represented as $\widehat{\Theta}^t(u, \widetilde{S}_u)$, where the function $\widehat{\Theta}^t$ is a $\mathcal{P}$-solution of the related problem ($\mathcal{VI}$) on $[0, t^\delta] \times \mathbb{R}$. Moreover, the uniqueness of the $\mathcal{P}$-solution holds and any stable, monotone and consistent approximation scheme for $\widehat{\Theta}^t$ converges locally uniformly to $\widehat{\Theta}^t$ on $[0, t^\delta] \times \mathbb{R}$ as $h \to 0^+$. In the case of the t-PB and the t-game exchange option, the protection put/conversion region is given as*

$$\mathcal{E}^t_p = \big\{(u,S) \in [t, t^\delta] \times \mathbb{R}\,;\, \widehat{\Pi}^t(u,S) = \bar{C} \vee \kappa S\big\}.$$

Proof. In view of Lemma 4.2, the assertion follows by an application of Proposition 4.4. □

Proposition 4.7 (Continuous Aggregation Property) *The function $\widehat{U}(t,S) := \widehat{\Pi}^t(t,S)$ is jointly continuous in time and space variables. Hence the function $\widetilde{U}(t,S) = \widehat{U}(t,S) + A(t)$ is also continuous with respect to (t,S), except for left jumps of size $-c^i$ at the T_is.*

Proof. Let $(t_n, S_n) \to (t,S)$ as $n \to \infty$. We decompose

$$\widehat{\Pi}^{t_n}(t_n,S_n) = \widehat{\Pi}^t(t_n,S_n) + (\widehat{\Pi}^{t_n}(t_n,S_n) - \widehat{\Pi}^t(t_n,S_n)).$$

By Proposition 4.6, $\widehat{\Pi}^t(t_n,S_n) \to \widehat{\Pi}^t(t,S)$ as $n \to \infty$. Moreover, denoting $\widehat{C}_t = \bar{C} \vee \kappa \widetilde{S}_t$ and $F = \gamma R^{cb} + \Gamma$, we have that

$$\alpha_u \widehat{\Pi}^t_u = \operatorname{esssup}_{\tau_p \in \mathcal{F}^u_{t^\delta}} \mathbb{E}_{\mathbb{Q}}\Big(\int_u^{\tau_p} \alpha_v F_v \, dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_u \Big), \quad u \le t^\delta.$$

So, assuming t_n sufficiently close to the left of t, and in view of the Markov property of the process $\widetilde{S}$, we obtain, on the event $\{\widetilde{S}_{t_n} = S_n\}$,

$$\begin{aligned} \alpha_{t_n} \widehat{\Pi}^{t_n}(t_n,S_n) &= \operatorname{esssup}_{\tau_p \in \mathcal{F}^{t_n}_{t^\delta_n}} \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p} \alpha_v F_v \, dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_{t_n} \Big) \\ &\le \operatorname{esssup}_{\tau_p \in \mathcal{F}^{t_n}_{t^\delta}} \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p} \alpha_v F_v \, dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_{t_n} \Big) = \alpha_{t_n} \widehat{\Pi}^t(t_n,S_n). \end{aligned}$$

Conversely, for any $\tau_p \in \mathcal{F}^{t_n}_{t^\delta}$, we have $\tau^\delta_p := \tau_p \wedge t^\delta_n \in \mathcal{F}^{t_n}_{t^\delta_n}$, $0 \le \tau_p - \tau^\delta_p \le t - t_n$ and

$$\begin{aligned} &\Big| \int_{t_n}^{\tau_p} \alpha_v F_v \, dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} - \int_{t_n}^{\tau^\delta_p} \alpha_v F_v \, dv - \alpha_{\tau^\delta_p} \widehat{C}_{\tau^\delta_p} \Big| \\ &\qquad \le \int_{\tau^\delta_p}^{\tau_p} \alpha_v |F_v| \, dv + \big| \alpha_{\tau_p} \widehat{C}_{\tau_p} - \alpha_{\tau^\delta_p} \widehat{C}_{\tau^\delta_p} \big|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\Big| \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p} \alpha_v F_v \, dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_{t_n} \Big) - \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau^\delta_p} \alpha_v F_v \, dv + \alpha_{\tau^\delta_p} \widehat{C}_{\tau^\delta_p} \,\Big|\, \mathcal{F}_{t_n} \Big) \Big| \\ &\qquad \le \mathbb{E}_{\mathbb{Q}}\Big(\int_{\tau^\delta_p}^{\tau_p} \alpha_v |F_v| \, dv \,\Big|\, \mathcal{F}_{t_n} \Big) + \mathbb{E}_{\mathbb{Q}}\Big(\big| \alpha_{\tau_p} \widehat{C}_{\tau_p} - \alpha_{\tau^\delta_p} \widehat{C}_{\tau^\delta_p} \big| \,\Big|\, \mathcal{F}_{t_n} \Big) \\ &\qquad \le c\sqrt{t - t_n}\, \|F\|_{\mathcal{H}^2} + \mathbb{E}_{\mathbb{Q}}\Big(\big| \alpha_{\tau_p} \widehat{C}_{\tau_p} - \alpha_{\tau^\delta_p} \widehat{C}_{\tau^\delta_p} \big| \,\Big|\, \mathcal{F}_{t_n} \Big) \end{aligned}$$

for some constant c. We conclude that $\widehat{\Pi}^{t_n}(t_n,S_n) - \widehat{\Pi}^t(t_n,S_n) \to 0$ as $t_n \to t^-$. But this is also true, with the same proof, as $t_n \to t^+$. Hence $\widehat{\Pi}^{t_n}(t_n,S_n) - \widehat{\Pi}^t(t_n,S_n) \to 0$ as $t_n \to t$. Finally, $\widehat{\Pi}^{t_n}(t_n,S_n) \to \widehat{\Pi}^t(t,S)$ as $t_n \to t$, as desired. □

The next result shows that a CB can be formally reduced to the corresponding RB.

Proposition 4.8 *A CB with a positive notice period $\delta > 0$ can be interpreted as an RB with $\widetilde{U}^{cb}(t,S) = \widetilde{U}(t,S)$, where $\widetilde{U}(t,S)$ is the function defined in Proposition 4.7, so that (cf. (46))*

$$U^{cb}_t = \mathbb{1}_{\{\tau_d > t\}} \widetilde{U}(t,S_t) + \mathbb{1}_{\{\tau_d \le t\}} (\bar{C} \vee \kappa S_t + A(t)). \tag{65}$$

Proof. The t-PB related reflected BSDE $(\mathcal{E})$ has a solution, and thus, by Proposition 3.2, the t-PB has a unique arbitrage price process $\Pi^t_u = \mathbb{1}_{\{u < \tau_d\}} \widetilde{\Pi}^t_u$ with $\widetilde{\Pi}^t_u = \widehat{\Pi}^t_u + A(u)$. Hence the arbitrage price of the CB upon call time t (assuming the CB still alive at time t) is well defined, as $\Pi^t_t = \widetilde{\Pi}^t_t = \widetilde{U}(t,\widetilde{S}_t)$ (cf. Proposition 4.7).

Moreover, by Proposition 4.7, the function $\widetilde{U}(t,S)$ is jointly continuous in time and space, except for negative left jumps of $-c^i$ at the T_is, and we also have that $\Pi^t_t \geq \bar{C} \vee \kappa S_t + A(t)$ on the event $\{\tau_d > t\}$. Hence U^{cb} defined as (65) satisfies all the requirements in (46). □

Conclusion. ***An important conclusion from Proposition 4.8 is that the results of Section 4.1 are applicable to a CB also in the case of a positive call notice period, since, in the Markovian model of the present paper, a CB may always be interpreted as an RB.***

4.3 Numerical Analysis of a Convertible Bond

The remaining part of the paper is devoted to a numerical study of a convertible bond within the present Markovian framework. In particular, we examine the price decomposition into embedded bond and game exchange option, the behavior hedge ratios, the impact of the call protection period, as well as the values of implied spread and implied volatility of a convertible bond.

4.3.1 Numerical Issues

Let us first comment briefly on numerical issues related to the valuation of a convertible bond and of its bond and option components.

Assume that $\bar{\tau} = \bar{\tau}^0 := 0$ (no call protection) and that we have already specified all the parameters for one of the problems (56), (57) or (58), including, in the case of (57) or (58), the function $\widetilde{U}^{cb}$.

By Proposition 3.5 and the results of Sections 4.1, standard deterministic approximation schemes (see, e.g., [2, 35]) converge towards the $\mathcal{P}$-solution of the problem as the discretization step goes to 0. Solving the PDEs related to the embedded bond is complete routine and thus we shall not comment on this issue.

To have a fully endogenous specification of the problem in the case of the convertible bond and/or its option component, one can take $\widetilde{U}^{cb}(t,S) = \widetilde{U}(t,S)$ as defined in Proposition 4.7 in (57) or (58), where $\widetilde{U}(t,S)$ is numerically computed by solving the related obstacles problems, using Proposition 4.6. We provide below a practical algorithm for solving, say (58), with $\widetilde{U}^{cb}(t,S) = \widetilde{U}(t,S)$, using, for example, a fully implicit finite difference scheme (see, for instance, [39]) to discretize $\mathcal{L}$:

1. Localize problems (64) for the embedded t-PBs and problem (58) for the CB. A natural choice, for the t-PBs and the CB, is to localize the problems on the spatial domain $(-\infty, \frac{\bar{C}}{\kappa}]$, with a Dirichlet boundary condition equal to κS (or a Neumann boundary condition equal to κ) at level $\frac{\bar{C}}{\kappa}$;

2. Discretize the localized domain $\mathcal{D}^{lo} = [0,T] \times (-\infty, \frac{\bar{C}}{\kappa}]$, using, say, one time step per day between 0 and T;

3. Discretize problems (64) for the embedded t-PBs on the subdomain $[t, t^\delta]$ of $\mathcal{D}^{lo}$ for t in the time grid *(one problem per value of t in the time grid)*;

4. Solve for $\widehat{\Pi}^t$ the discretized problems (64) corresponding to the embedded t-PBs for t in the time grid *(one problem per value of t in the time grid)*;

5. Discretize problem (58) for the CB on $\mathcal{D}^{lo}$ and solve the discretized problem, using the numerical approximation of $\widetilde{U}(t,S) := \widehat{\Pi}^t(t,S) + A(t)$ as an input for $\widetilde{U}^{cb}(t,S)$ in (58).

Note that the problem for the t-PB only has to be solved on the time-strip $[t, t^\delta]$ of $\mathcal{D}^{lo}$. Hence the overall computational cost for solving a CB problem (58) with a positive call notice period is roughly the same as that required for solving one CB problem without call notice period, plus the cost of solving n PB problems that would be defined on the whole grid, where n is the number of time mesh points in the call notice period. For instance, for a call notice period δ equal to one month and a time step of one day, we have $n = 30$.

Finally, if a call protection is in force then we proceed along essentially the same lines, using the results of Section 4.1.5.

4.3.2 Embedded Bond and Game Exchange Option

On Figure 1,[1] we plot the price $\widehat{\Pi}^0(0, S)$ of a convertible bond with no call protection, its bond component and its game exchange option component, obtained through the method described in Section 4.3.1, as a function of the stock level S at time $t = 0$, in the simple case where there are no dividends on the convertible bond (neither coupons nor recovery), and for the remaining parameters as given in Table 1. Note that we take here $\eta = 0$; this corresponds to no jump of the stock price at default.

For each of the three products, we plot the curves corresponding to default intensities of the form $\gamma(t, S) = \gamma_0(\frac{S_0}{S})^{\gamma_1}$ where $\gamma_0 = 0.02$, and γ_1 equals either 1.2 (strong interaction between equity and credit), or 0 (no interaction between equity and credit). The corresponding curves are labeled `local` (red/solid curves) and `implied` (blue/dashed curves), respectively.

Note that in case $\gamma_1 = 1.2$, consistently with typical market data, the price of the CB as a function of S exhibits the so-called *ski–jump behavior*, namely, it is convex for high values of S and collapsing at the low values. This behavior at low levels of S comes from the collapse of the bond component of a CB ('collapse of the bond floor' at low S's in case $\gamma_1 > 0$ cf. the red/solid `local` embedded bond curve on Figure 1).

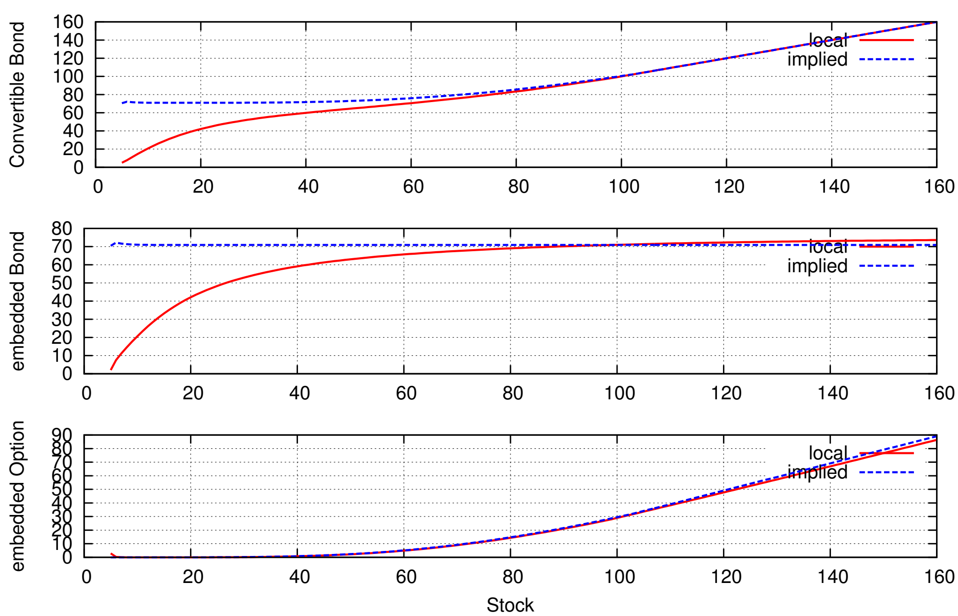

Figure 1: *The ski-jump diagram and its decomposition.*

r	q	η	σ	T	$\bar{P}$	$\bar{N}$	$\bar{C}$	κ
5%	0	0	20%	5y	0	100	103	1

Table 1: *Parameter values (all dividends set to 0).*

[1]We thank Abdallah Rahal from the Mathematics Departments at University of Evry, France, and Lebanese University, Lebanon, for numerical implementation of the model.

4.3.3 Hedge Ratios

Figure 2 shows the *no-protection delta* $\partial_S \widehat{\Pi}^0(0,S)$ as a function of S for the data of the previous subsection. Note that in the present context we have that $\bar{\tau} = 0$ and thus formula (43) reduces to $Z_t = \sigma \widetilde{S}_t \partial_S \widehat{\Pi}^0(t, \widetilde{S}_t)$. As explained in Remarks 3.9, the process Z is the essential ingredient in the computation of the hedge ratios $[\zeta_t^{*1}, \zeta_t^{*2}]$ with respect to traded primary assets (see, in particular, formula (42)). Under the present assumptions that $\eta = 0$ and $A \equiv 0$ (so that the process $\widetilde{\Pi}^0 = \widehat{\Pi}^0$ is continuous), after straightforward computations we obtain

$$\zeta_t^{*1} = \mathbb{1}_{\{t \le \tau_d\}} \Big(\partial_S \widetilde{\Pi}^0(t, \widetilde{S}_t) + \frac{\widetilde{\Pi}^0(t, \widetilde{S}_t) - R(t, \widetilde{S}_t)}{\nu(t) - \widetilde{B}(t, \widetilde{S}_t)} \, \partial_S \widetilde{B}(t, \widetilde{S}_t) \Big) \tag{66}$$

and

$$\zeta_t^{*2} = \mathbb{1}_{\{t \le \tau_d\}} \, \frac{R(t, \widetilde{S}_t) - \widetilde{\Pi}^0(t, \widetilde{S}_t)}{\nu(t) - \widetilde{B}(t, \widetilde{S}_t)}. \tag{67}$$

Equality (67) is rather clear since for $\eta = 0$ the stock price does not jump at default and thus jump risk should be hedged using a fixed-maturity CDS only. Formula (66) shows that in order to find the hedge ratio with respect to the stock price, one needs to adjust the usual delta $\partial_S \widetilde{\Pi}^0(t, \widetilde{S}_t)$ using the specific features of the CDS used for hedging.

Since the computation of the partial derivative $\partial_S \widetilde{\Pi}^0(0,S)$ by the finite difference approximation on the grid requires more accuracy than that of the prices, we only represented the values of deltas for S 'not too close' from the boundaries of the computational domain (specifically, on the subset $10 \le S \le 140$ of the computational domain $5 \le S \le 160$), where numerical boundary effects are not too important.

Observe that in the case of the convertible bond and of the embedded option, the corresponding deltas $\partial_S \widetilde{\Pi}^0(0,S)$ and $\partial_S \Psi(0,S)$ exhibit a rather irregular behavior (varies faster) in a small zone around $S = S_c \approx 99$ (in the case $\gamma_1 = 1.2$) or $S = S_c \approx 103$ (in the case $\gamma_1 = 0$). For greater values of S both deltas are equal to one; this behavior is consistent, in particular, with the fact (already present, but less easy to see, on Figure 1) that, in view of the values of the parameters in Table 1 and since there are no coupons involved, the price of the convertible bond is equal to

$$\bar{P} \vee \kappa S = \bar{C} \vee \kappa S = S$$

for any $S \ge \bar{C} = 103$.

4.3.4 Separation of Credit and Volatility Risks

At the intuitive level, it seems clear that the embedded bond should concentrate most of the interest rate and credit risks of a convertible bond, whereas the value of the embedded game exchange option should explain most of the volatility risk (note in this respect that the embedded game exchange option always has a null coupon process). In the specific context of the model of this paper, this intuition can be assessed numerically. In what follows, we consider the case when $\eta = 1$, which corresponds to the total default of the stock.

Figure 3 shows the prices of a convertible bond and of its bond and option components as functions of the volatility parameter σ for a local default intensity $\gamma(t,S) = \gamma_0 (\frac{S_0}{S})^{\gamma_1}$ with $\gamma_0 = 0.02$ and $\gamma_1 = 1.2$. The embedded bond price varies between 69.19 and 72.98, whereas the embedded option price grows from 9.52E-05 to 8.20, as σ increases from 3% to 61%. Moreover, the embedded option price increases with σ, whereas the prices of the convertible bond and of the embedded bond exhibit a more complex behavior.

Figure 4 shows the same prices as functions of the parameter $\gamma = \gamma_0$ (constant default intensity) for the volatility $\sigma = 20\%$. The embedded bond price decreases from 97.72 to 60.89, whereas the embedded option price only rises from 0.0031 to 2.59, as γ_0 increases from 0.01 to 0.29.

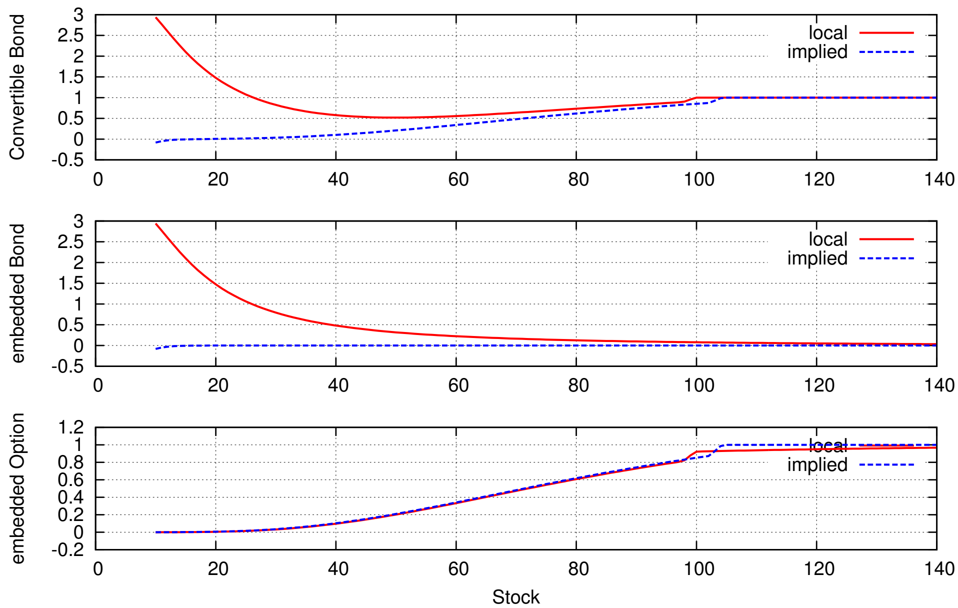

Figure 2: *Deltas corresponding to the prices of Figure 1.*

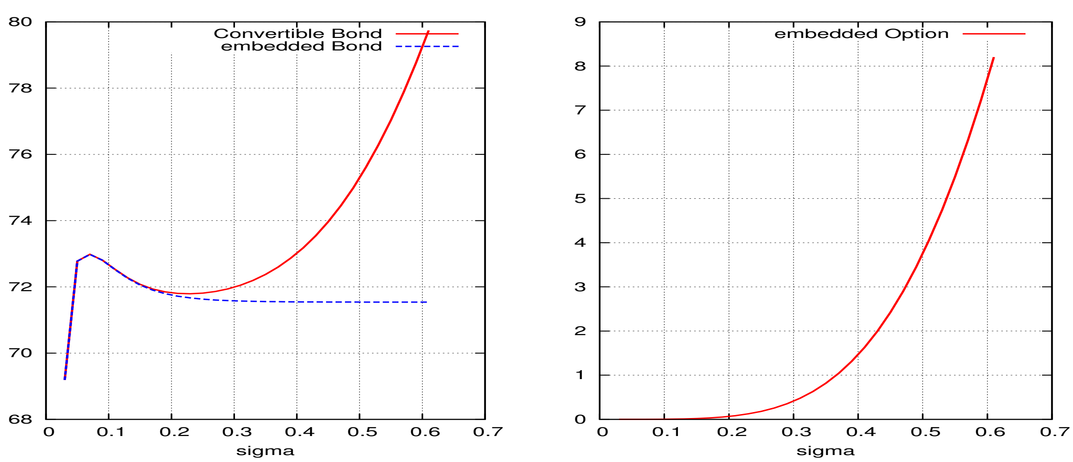

Figure 3: *Price of a convertible bond and of its option and bond components as a function of a constant volatility parameter* σ $(S = 100, \eta = 1, r = 5\%, q = 0, \gamma_0 = 0.02, \gamma_1 = 1.2)$.

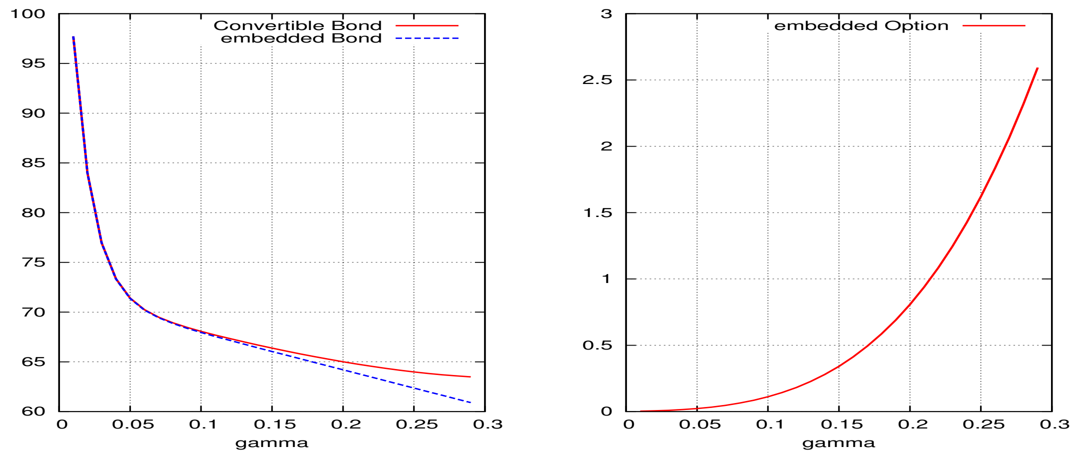

Figure 4: *Price of a convertible bond and of its option and bond components as a function of a constant default intensity parameter* $\gamma = \gamma_0$ *(*$S = 100, \eta = 1, r = 5\%, q = 0, \gamma_1 = 0, \sigma_0 = 20\%$*).*

4.3.5 Call Protection Period

The diversity and combination of all the possible clauses can give rise to a virtually infinite variation of situations. We shall thus not push much further the numerical analysis of convertible bonds in this paper, referring the interested reader to, e.g., the studies by Ayache et al. [2] and Kwok and Lau [35].

As for the impact of the call protection, in particular, we shall only present here few preliminary results in Table 3, which shows protection CB prices $\widehat{\Pi}^1(0, S)$ for $\bar{\tau}$ of the form (37), with $\bar{T} = T$ therein, for various pairs $(S, \bar{S})$. These prices were computed by the deterministic numerical scheme of Section 4.1.5, using the general data of Table 2 (in particular, a discrete coupon tenor is treated numerically by the transformation into clean prices). The protection prices are increasing with respect to $\bar{S}$, which means that a longer call protection period makes a convertible bond more expensive to the investor, as expected.

T	$\overline{P}$	$\overline{N}$	$\overline{C}$	$\overline{R}$	c^i	c	r	q	η	γ_0	γ_1	σ
6m	0	100	103	0	1.2/month	0	5%	0	1	0.02	1.2	20%

Table 2: *General data for the numerical results of Table 3.*

Note that real-life lifting times of call protections are typically highly path-dependent, e.g. (given a trigger level $\bar{S}$),

$$\bar{\tau} = \inf\{\, t > 0\,,\ S_{t_i} \geq \bar{S} \text{ at } d \text{ among } l \text{ consecutive monitoring times } t_i\text{s}\,\}.$$

$\bar{S} = 80$	S	78.55	79.55	80.55	81.55
	$\widehat{\Pi}^1(0, S)$	103.18	103.11	103	103
$\bar{S} = 103$	S	100.55	101.55	102.55	103.55
	$\widehat{\Pi}^1(0, S)$	103.87	103.78	103.69	103.55
$\bar{S} = 120$	S	100.55	101.55	102.55	103.55
	$\widehat{\Pi}^1(0, S)$	110.32	110.89	111.48	112.09

Table 3: *Time-0 protection price of the convertible bond of Table 2 for various pairs* $(S, \bar{S})$.

For such call protections, or, more generally, for contracts with highly path-dependent features, deterministic numerical schemes as above are ruled out by the curse of dimensionality, and simulation methods appear to be the only viable alternative.

The related simulation methods, which can be viewed as extensions to game problems of simulation methods for American options [36, 37, 44], correspond more generally to simulation numerical schemes for reflected BSDEs [10, 15, 38]. The possibility to effectively deal with various kinds of call protection through simulation pricing schemes is currently under study in [16].

It is interesting to note that besides computational tractability in high-dimension, simulation methods also present the theoretical interest of being amenable to a more complete convergence analysis than deterministic schemes. For instance, one can deal reasonably well with the issue of the numerical approximation by time discretization and simulation of the component Z of the solution to a BSDE, whereas in a PDE viscosity solution set-up underlying the convergence analysis of a deterministic scheme, this is more difficult. This is because the representation (cf. (43))

$$Z_t = \sigma(t, \widetilde{S}_t)\widetilde{S}_t\big(\mathbb{1}_{\{t\le\bar{\tau}\}}\partial_S\widehat{\Pi}^1(t,\widetilde{S}_t) + \mathbb{1}_{\{t>\bar{\tau}\}}\partial_S\widehat{\Pi}^0(t,\widetilde{S}_t)\big)$$

is only tentative for a continuous pricing function $\widehat{\Pi}^0$, unless further regularity estimates for the solution $\widehat{\Pi}^0$ may be established, or other notions of weak solutions to the pricing PDEs may be considered.

4.3.6 Implied Credit Spread and Implied Volatility

As argued in [4], in the context of hybrid equity-and-credit derivatives like convertible bonds, the standard notion of a (default-free) Black–Scholes implied volatility is not adequate. One should use instead a suitable notion of implied *credit spread* and *volatility* (two implied numbers rather than a single one), defined relatively to the prices of two financial contracts (rather than one, as for the Black–Scholes implied volatility), and in reference to a suitable benchmark model of equity-to-credit derivatives (rather than the Black–Scholes model).

In this regard, the decomposition of a convertible bond into the embedded bond and the game exchange option, which was established in [4] and is further studied in this paper in the context of model (1), allows one to give a rigorous definition of implied spread and volatility of a convertible bond, in terms of a special case of model (1) with a constant default intensity $\gamma(t,S)\equiv\gamma_0$ and a constant equity volatility $\sigma(t,S)\equiv\sigma_0$, that is,

$$d\widetilde{S}_t = \widetilde{S}_t\Big(\big(r(t)-q(t)+\eta\gamma_0\big)\,dt + \sigma_0\,dW_t\Big). \tag{68}$$

Let us thus recall from [4] the following definition of the implied spread and the implied volatility of a convertible bond at time 0, say, and for already known deterministic short-term risk-free interest-rate curve $r(t)$, the dividend yield curve $q(t)$, and the fractional loss parameter η upon default.

Definition 4.4 Given time-0 arbitrage prices Φ_0 of the bond component and Ψ_0 of the game exchange option component of a convertible bond, the *implied credit spread* and *implied volatility* at time 0 of the convertible bond are defined as any pair of constant positive parameters (γ_0,σ_0) that are consistent with the prices Φ_0 and Ψ_0, in the sense that the price at time 0 of the bond component (resp. the game exchange option component) in model (68) is equal to Φ_0 (resp. Ψ_0).

Figure 5 shows the implied credit spread and the implied volatility of the convertible bond as a function of S_0 for the data of Table 1 (except that we now set $\eta=1$). This means that the 'observed' prices Φ_0 and Ψ_0 are in fact computed using model (1) with the local default intensity $\gamma(t,S)=\gamma_0(\frac{S_0}{S})^{\gamma_1}$ where $\gamma_0=0.02$ and $\gamma_1=1.2$ and a constant volatility $\sigma=20\%$.

Remarks 4.1 In view of the no-arbitrage relationship $\Phi_0+\Psi_0=\Pi_0$, a formally equivalent definition would be that of a pair (γ_0,σ_0) consistent with the prices Π_0 and Φ_0 of the convertible bond and its embedded bond in a model (1) with a constant default intensity γ_0 and a constant equity volatility σ_0. We have argued in Section 4.3.2, however, that the embedded bond concentrates most of the

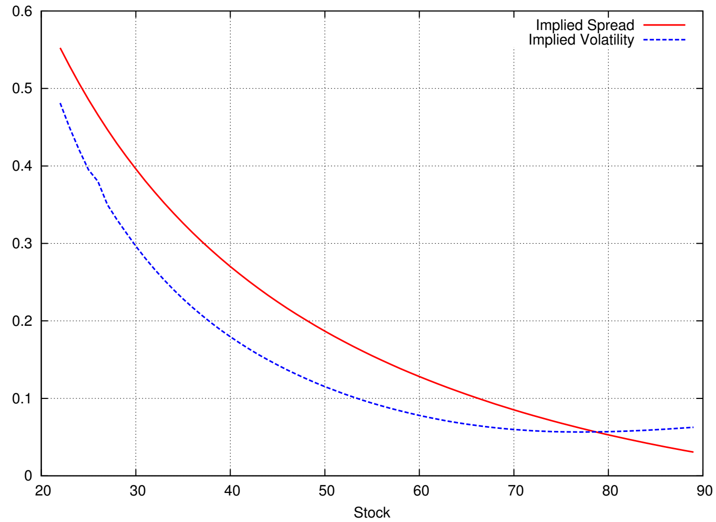

Figure 5: *Implied credit spread and implied volatility of a convertible bond at time 0 as a function of* S_0 *(*$\gamma_0 = 0.02, \gamma_1 = 1.2$*).*

interest rate and credit risks (cf. Figure 4), whereas the value of the embedded game exchange option explains most of the volatility risk (cf. Figure 3). It is thus more natural to use the prices of the embedded bond and game exchange option to infer the implied credit spread and the implied volatility.

4.3.7 Calibration Issues

A further numerical issue is the *calibration* of the model, which consists in fitting some specific parameters of the model, such as the local volatility σ and the local intensity γ in our case, to market quotes of related liquidly traded assets. Various sets of input instruments can be used in this calibration process, including traded options on the underlying equity and/or CDSs related to bond issues of the reference name.

The specification of a pertaining calibration procedure typically depends on the specification of the model at hand (non-parametric model with arbitrary local volatility and/or default intensity functions σ and γ or versus parameterized sub-classes of models with specific parametric forms for σ and γ), as well as on the nature of the calibration input instruments.

We shall not dwell more on calibration issues in the present paper, referring instead the reader to Andersen and Buffum [1] for interesting developments and numerical algorithms in this regard. Let us only observe that, as it can be seen in Figure 1, the price of the embedded game exchange option enjoys much better properties than the price of the CB in terms of convexity with respect to the stock price, and thus in turn (see Figure 3) in terms of monotonicity with respect to the volatility. Moreover, as we have argued above, the embedded bond concentrates most of the interest rate and credit risks of a convertible bond, whereas the value of the embedded game exchange option explains most of the volatility risk. These features suggest that for the calibration purposes it could be advantageous to use the (synthetic) prices of the embedded option and of the embedded bond, rather than those of the CB and of the embedded bond (in a situation where the related market prices would be available or could be reconstructed 'synthetically' in terms of prices of liquidly traded bonds and/or other traded instruments).

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