Directed Polymers in Random Environment\textsuperscript{a}

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Abstract

Directed polymers in random environment can be thought of as a model of statistical mechanics in which paths of stochastic processes interact with a quenched disorder (impurities), depending on both time and space. We review a number of results which have been obtained during the last fifteen years. Our topics will include: 1- Discrete and continuous models, related models; 2- Weak disorder regime and strong disorder regime; 3- Localization phenomenon at strong disorder; 4- Diffusivity at weak disorder; 5- Large deviations for a Brownian model.

We will use a thermodynamics approach, a martingale approach, and we prove that the transition between delocalized and localized phases is characterized by equality or strict inequality between the quenched and the annealed free energy.
Contents

1 Simple random walk model for directed polymers 2

2 Thermodynamics 5
   2.1 Free energy ......................................................... 5
   2.2 Upper bounds ....................................................... 7
   2.3 Monotonicity ....................................................... 10

3 A martingale approach 11
   3.1 A martingale associated to the partition function .................... 11
   3.2 The second moment method and the $L^2$ region ........................ 12
   3.3 Semimartingale decomposition ..................................... 19
   3.4 The diffusive regime ............................................... 22
   3.5 Localization and delocalization .................................. 24

4 Low dimensions 25

5 A Brownian motion model for directed polymers 27
   5.1 Connection to the Kardar-Parisi-Zhang equation ....................... 30

6 Exponents 31
   6.1 Large deviation for the Brownian polymer .......................... 34
   6.2 Fluctuation results ................................................ 36

1 Simple random walk model for directed polymers

The model we consider here is defined as a random walk in a random potential. We first fix the notations for the random walk and the random environment, and then introduce the polymer measure.

- **The random walk:** $(\{\omega_n\}_{n \geq 0}, P)$ is a simple random walk on the $d$-dimensional integer lattice $\mathbb{Z}^d$. More precisely, we let $\Omega_\omega$ be the path space $\Omega_\omega = \{ \omega = (\omega_n)_{n \geq 0}; \omega_n \in \mathbb{Z}^d, n \geq 0 \}$, $\mathcal{F}$ be the cylindrical $\sigma$-field on $\Omega$, and, for all $n \geq 0$, $\omega_n : \omega \mapsto \omega_n$ be the projection map. We consider the unique probability measure $P$ on $(\Omega_\omega, \mathcal{F})$ such that $\omega_0 = 0$ and $P[\omega_{n-1} = 0] = (2d)^{-1}$, $j = 1, 2, \ldots, d$.

- **The random environment:** $\eta = \{\eta(n, x) : n \in \mathbb{N}, x \in \mathbb{Z}^d\}$ is a sequence of r.v.’s which are real valued, non-constant, and i.i.d.(independent identically distributed) r.v.’s defined on a probability space $(\Omega_\eta, \mathcal{G}, Q)$ such that
  
  $Q[\exp(\beta \eta(n, x))] < \infty$ for all $\beta \in \mathbb{R}$.

Here, and in the sequel, $Q[Y]$ denotes the $Q$-expectation of a r.v. $Y$ on $(\Omega_\eta, \mathcal{G}, Q)$. 

• **The polymer measure:** For any $n > 0$, define the probability measure $\mu_n$ on the path space $(\Omega, \mathcal{F})$ by

$$\mu_n(d\omega) = \frac{1}{Z_n} \exp \beta H_n(\omega) \, P(d\omega),$$

where $\beta > 0$ is a parameter (the inverse temperature), where

$$H_n(\omega) = H_n(\eta, \omega) = \sum_{1 \leq j \leq n} \eta(j, \omega_j)$$

and

$$Z_n = Z_n(\beta, \eta) = P \left[ \exp \left( \beta \sum_{1 \leq j \leq n} \eta(j, \omega_j) \right) \right]$$

is the normalizing constant (the partition function).

The polymer measure $\mu_n$ can be thought of as a Gibbs measure on the path space $(\Omega, \mathcal{F})$ with the Hamiltonian $H_n$. We stress that the random environment $\eta$ is contained in both $Z_n$ and $\mu_n$ without being integrated out, so that they are r.v.’s on the probability space $(\Omega, \mathcal{G}, Q)$. The polymer is attracted to sites where the random environment is positive, and repelled by sites where the environment is negative.

**Remark 1.1** This model was originally introduced in physics literature [28] to mimic the phase boundary of Ising model subject to random impurities. Later on, the model reached the mathematics community [29, 7], where it was reformulated as above.

- Other physical realizations: torn paper sheets [34] A rectangular sheet of paper is tightened by a machine. The strain is applied on two opposite sides, and it is slowly increased. A small notch is made on a third side to initiate the tear. The fracture is governed by the random
geometry of the fiber network. The fracture line is highly correlated to the weakest bonds in the sheet, as can be checked by microdensitometry. Directed polymers with $d = 1$ and its zero temperature counterpart, oriented first passage percolation, are natural model for the fracture line. We will come back to this example in section 6.

- Corresponding to $\beta = \infty$, the ground states, i.e. the paths $\omega$ maximizing $H_n$, are the geodesics of last passage percolation. Relations to percolation, last passage percolation, and all the related models (tandem queuing systems, totally asymmetric exclusion process, . . .)

- Model for one-dimensional directed chain interacting with a random environment.

- Model for normal growth with deposition, see Kardar-Parisi-Zhang equation (5.94).

- Interpretations in terms of walkers in deadly traps which regenerates with time ($\eta \leq 0, \beta \geq 0$): $Z_n = \text{probability of survival up to time } n \text{ of a walker which is killed at time } t \text{ with probability } 1 - e^{\beta \eta(t,x)}$, where $x$ denotes its position at this time.

- $(2d)^n Z_n = E S_n$, expected size $S_n$ at time $n$ of a population where each individual arriving at time $t$ in $x$ is killed with probability $1 - e^{\beta \eta(t,x)}$ independently of the others; if he survives, he sends one child at each of the $2d$ sites $x \pm e_i (i = 1, \ldots d)$ at time $t + 1$.

The binary case $\eta = -1, 0$, can be related to non-homogeneous oriented percolation: the probability for a site to be open is itself random (either 1 or $e^{-\beta}$). Perform percolation with these probabilities. Then, $Z_n$ is the expected number of ”wet points” at distance $n$ (with their multiplicity!!).

- The parabolic Anderson model is time-continuous version of our discrete model. It is given by the parabolic partial differential equation:

\[
\frac{\partial u}{\partial t} = k \Delta u + \eta_t(x) u , \quad u(0,x) \equiv 1
\]

where $\eta_t(x)$ is a random (time dependent) potential. The time $t$ is continuous ($t \in \mathbb{R}$), and $\Delta$ is discrete laplacian. The solution $u$ can be expressed by the Feynman-Kac formula time-continuous random walk on $\mathbb{Z}^d$, and is completely similar to $Z_n$. Asymptotic behaviour of integer moments of the solution are studied, together with their accordance or discrepancy which is a sign of the so-called intermittancy. Intermittency means that the solution of the system develops pronounced spatial structures on islands located far from each other. We will get a rather precise picture of this in our model, without averaging with respect to the medium. Also, the sharp asymptotics when the diffusion coefficient $k$ tends to 0 are of interest [17]. This model was introduced in [11] to check the predictions of Anderson’s theory of (time independent) random Schrödinger operators. It has motivated a number of papers, but the picture is much less clear understood than the time independent case, see [24] for a recent account.

In view of our assumptions, an important quantity for this model is the logarithmic moment generating function $\lambda$ of $\eta(n,x)$,

\[
\lambda(\beta) = \ln Q[\exp(\beta \eta(n,x))], \quad \beta \in \mathbb{R}.
\]  

The function $\lambda(\beta)$ can be explicitly computed for some typical choice of the distribution of $\eta(n,x)$.

For example, $\lambda(\beta) = \ln (p e^{-\beta} + (1-p) e^\beta)$ for the Bernoulli environment and $\lambda(\beta) = \frac{1}{2} \beta^2$ for the Gaussian environment.
2 Thermodynamics

2.1 Free energy

We want to study the free energy (pressure)
\[ p_n = p_n(\beta, \eta) = \frac{1}{n} \ln Z_n, \]
more precisely to know if it converges as \( n \to \infty \) and if this case, how the limit depends on the environment \( \eta \).

**Proposition 2.1** As \( n \to \infty \), \( p_n(\beta, \eta) \to p(\beta) \) \( Q \)-a.s., and the limit is deterministic.

The proof splits in two steps.

**Step 1:**

**Lemma 2.2**
\[
\lim_{\beta \to \infty} Q p_n(\beta) = \sup_{n \in \mathbb{N}} Q p_n(\beta) \in \mathbb{R}
\]

The variable
\[
Z_{n,m}^x = P^x \left[ \exp \left( \sum_{1 \leq t \leq m} (\beta \eta(\omega_t, t + n)) \right) \right], \quad n, m \geq 1,
\]
has the same law as \( Z_m \), and note that for \( m, n \geq 1 \),
\[
Z_{n+m} = Z_n \sum_x \mu_n(\omega_n = x) Z_{n,m}^x,
\]
using Markov property for \( P \). Now, \( Z_{n,m}^x \) has the same law as \( Z_m \), and we have by Jensen’s inequality
\[
\ln Z_{n+m} \geq \ln Z_n + \sum_x \mu_n(\omega_n = x) \ln Z_{n,m}^x.
\]

Let \( G_n \) be the \( \sigma \)-field generated by \( \eta(t, \cdot), t \leq n \). Taking expectation and using independence, we obtain
\[
Q[\ln Z_{n+m}] \geq Q[\ln Z_n] + Q \left[ \sum_x \mu_n(\omega_n = x) \ln Z_{n,m}^x | G_n \right]
\]
\[
\geq Q[\ln Z_n] + \sum_x \mu_n(\omega_n = x) Q[\ln Z_{n,m}^x | G_n] \quad (\mu_n G_n - \text{measurable})
\]
\[
= Q[\ln Z_n] + Q[\ln Z_m] \quad (Z_{n,m}^x \text{ law } Z_m)
\]
i.e., \( Q[\ln Z_n] \) is super-additive. From the super-additive Lemma [20, lemma 3.1.3] we see that
\[
\lim_{n \to \infty} \frac{1}{n} Q[\ln Z_n] = \sup_n \frac{1}{n} Q[\ln Z_n].
\]

Now, the finiteness of \( p \) follows from the annealed bound (2.6) below.

**Step 2:** We use a concentration inequality. To keep things self-contained, we use --and prove -- Azuma’s lemma, at the price of assuming the \( \eta \)'s bounded. For statements and proof of this step for general \( \eta \), see [12].
Lemma 2.3 (Concentration) Let \((X_k)_{k \leq n}\) a sequence of independent random variables in \(E\), and \(Y_n = f_n(X_1, \ldots, X_n)\) with \(f_n : E^n \to \mathbb{R}\) a measurable. Assume that for \(k = 1, \ldots, n\), all \(x_1, \ldots, x_n, y_k \in E\),
\[
|f_n(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) - f_n(x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n)| \leq 1
\]
Then, for \(r \geq 0\),
\[
P(Y_n - EY_n \geq nr) \leq e^{-nr^2/2}
\]
\(\Box\) Let \(\mathcal{H}_k = \sigma(X_1, \ldots, X_k)\), and \(M_k = E^{\mathcal{H}_k}(Y_n) - E(Y_n)\). Then, \((M_k, k \leq n)\) is a \((\mathcal{H}_k, k \leq n)\)-martingale, with \(M_n = Y_n - EY_n\). Moreover, by independence,
\[
\Delta M_{k-1} = E^{\mathcal{H}_k}(Y_n) - E^{\mathcal{H}_{k-1}}(Y_n)
\]
\[
= E^{\mathcal{H}_k}[f_n(X_1, \ldots, X_{k-1}, X_k, X_{k+1}, \ldots, X_n)
- f_n(X_1, \ldots, X_{k-1}, Y_k, X_{k+1}, \ldots, X_n)]
\]
with \(Y_k\) an independent copy of \(X_k\). We see that
\[
|\Delta M_k| \leq 1, \quad k = 0, \ldots, n - 1.
\]
From the barycentric relation
\[
\Delta M_k = \frac{1 + \Delta M_k}{2} \cdot 1 + \frac{1 - \Delta M_k}{2} \cdot (-1),
\]
it follows by convexity that
\[
e^{\lambda \Delta M_k} \leq \frac{1 + \Delta M_k}{2} e^\lambda + \frac{1 - \Delta M_k}{2} e^{-\lambda},
\]
and by the martingale property,
\[
E^{\mathcal{H}_k} e^{\lambda \Delta M_k} \leq \cosh \lambda \leq e^{\lambda^2/2}.
\]
Hence, for \(\lambda \in \mathbb{R}\),
\[
E(e^{\lambda|Y_n - EY_n|}) = E E^{\mathcal{H}_{n-1}} \prod_{k=0}^{n-1} e^{\lambda \Delta M_k}
\]
\[
= E \left( \prod_{k=0}^{n-2} e^{\lambda \Delta M_k} \right) E^{\mathcal{H}_{n-1}} e^{\lambda \Delta M_{n-1}}
\]
\[
\leq E \prod_{k=0}^{n-2} e^{\lambda \Delta M_k} e^{\lambda^2/2}
\]
\[
\leq \exp n \lambda^2 / 2
\]
by induction. From this and Markov inequality we obtain for \(\lambda \geq 0\),
\[
e^{-n\lambda r} P(Y_n - EY_n \geq nr) \leq E(e^{\lambda|Y_n - EY_n|}; \{Y_n - EY_n \geq nr\})
\]
\[
\leq E(e^{\lambda|Y_n - EY_n|})
\]
\[
\leq \exp n \lambda^2 / 2.
\]
Hence, 

\[ P(Y_n - EY_n \geq nr) \leq \exp \left( -n \sup_{\lambda \geq 0} \left( r\lambda - \lambda^2/2 \right) \right), \]

which yields the result by taking the optimal \( \lambda (\lambda = r \text{ is non negative when } r \geq 0) \).

Notes: a- Refe [21, cor. 2.4.14]: we have proved in fact \( P(\ln Z_n - EY_n \geq nr) \leq e^{-nI(x)} \), with 

\[ I(x) = \frac{1}{2} \ln(1 + x) + \frac{1 - x}{2} \ln(1 - x) = \sup_{\lambda \geq 0} \left( r\lambda - \ln \cosh \lambda \right). \]

b- We have obtained a subgaussian bound for the fluctuation of \( \ln Z_n \). All standard methods for concentration inequalities give similar bound (up to a constant factor in the exponent). Though it is believed that the bound is not of the right order, for no value of the parameters! This phenomenon also occurs for the fluctuations of the top eigenvalue of large random matrices [26].

Final argument:
We prove the proposition for bounded \( \eta \), i.e., \( |\eta(t,x)| \leq K \). We apply Lemma 2.3 with 

\[ X_k = (\eta(k,x), x \in \mathbb{Z}^d), f_n = np_n/(2\beta K). \]

Indeed, if \( \eta(t,x) = \eta'(t,x) \) for all \( t \) but \( k \), it holds

\[ |p_n(\eta) - p_n(\eta')| \leq \beta \sup_x |\eta(k,x) - \eta'(k,x)| \leq 2\beta K \]

Together with Chebychev’s inequality, the lemma implies that \( p_n - Qp_n \to 0 \text{ a.s.} \), which, in addition to Lemma 2.2, proves the proposition.

2.2 Upper bounds

Annealed bound: By Jensen inequality,

\[ Q \ln Z_n \leq \ln QZ_n = n\lambda, \]

hence

\[ p(\beta) \leq \lambda(\beta) \tag{2.6} \]

For other upper bounds, we can use standard monotonicity properties from thermodynamics:

**Proposition 2.4**.

(i) \( p_n(\beta) \) is a smooth convex function, \( p_n(0) = 0 \); \( p \) is convex with \( p(0) = 0 \).

(ii) \( \beta \mapsto \beta^{-1} p_n(\beta) \) is increasing.

(iii) \( \beta \mapsto \beta^{-1} [p_n(\beta) + \ln(2d)] \) is decreasing.

\[ \square \]

By differentiation, one gets

\[ \frac{d}{d\beta} np_n = \mu_n[H_n], \quad \frac{d^2}{d\beta^2} np_n = \text{Var}_{\mu_n}[H_n] > 0, \tag{2.7} \]

and all properties in (i) follows easily. By convexity, \( \beta^{-1} p_n(\beta) = \beta^{-1} [p_n(\beta) - p_n(0)] \) is non-decreasing. Turning to (iii), we have the identity

\[ \frac{d}{d\beta} \frac{1}{\beta} [p_n(\beta) + \ln(2d)] = -\frac{1}{\beta^2} [p_n(\beta) + \ln(2d)] + \frac{1}{n\beta} \mu_n[H_n] = \frac{1}{n\beta^2} h(\mu_n), \]
where \( h(\nu) \) is the Boltzmann entropy of a probability measure on the path space,

\[
h(\nu) := \sum_{\omega} \nu(\omega) \ln \nu(\omega),
\]

(2.8)

which is non-positive, and negative if \( \nu \) is not a Dirac mass.

We derive an other upper bound,

\[
Q[p_n(\beta) + \ln(2d)] \leq^{(\text{iii})} \inf_{m \in [0,1]} \frac{1}{m} Q[p_n(m\beta) + \ln(2d)]
\]

\[
\leq^{(\text{Jensen})} \inf_{m \in [0,1]} \frac{1}{m} [\lambda(m\beta) + \ln(2d)]
\]

\[
= \beta \inf_{\beta' \in [0,\beta]} \frac{1}{\beta'} [\lambda(\beta') + \ln(2d)],
\]

which is depicted in figure 2. The optimal \( m \) is equal to 1 if \( \beta \lambda'(\beta) - \lambda(\beta) \leq \ln(2d) \); In this case we recover the annealed bound. Assume there exists \( \beta_1 \in (0, \infty) \) such that \( \beta \lambda'(\beta) = \ln(2d) + \lambda(\beta) \). When \( \beta > \beta_1 \), the optimal \( m \) is such that \( m\beta = \beta_1 \) and we find the bound

\[
Q_{p_n}(\beta) \leq \frac{\beta}{\beta_1} [\lambda(\beta_1) + \ln(2d)] - \ln(2d) < \lambda(\beta)
\]

Summarizing, we have:

**Proposition 2.5** If

\[
\beta \lambda'(\beta) - \lambda(\beta) > \ln(2d),
\]

(2.9)

then \( p(\beta) < \lambda(\beta) \). More precisely, \( p(\beta) \leq (\beta/\beta_1)[\lambda(\beta_1) + \ln(2d)] - \ln(2d) \), where \( \beta_1 \in (0, \beta) \) is the unique positive root of \( \beta \lambda'(\beta) = \ln(2d) + \lambda(\beta) \).

We now look for conditions ensuring \( p(\beta) < \lambda(\beta) \) for large \( \beta \), in terms of the the distribution \( q \) of \( \eta \).

**Corollary 2.6** Both following conditions implies that there exists \( \beta_1 \in (0, \infty) \) such that \( p(\beta) < \lambda(\beta) \) for \( \beta > \beta_1 \):

![Figure 2: An upper bound.](image-url)
1. \( \text{esssup } q = +\infty \), i.e. \( \eta \) is unbounded from above,

2. \( \text{esssup } q < +\infty \) and \( q\{\text{esssup } q\} < 1/2d \), i.e. \( \eta \) is bounded from above and has enough mass at its maximum value.

We check condition (2.9). Let \( \lambda^* \) the Legendre transform of \( \lambda \),

\[
\lambda^*(u) = \sup\{u\beta - \lambda(\beta)\}
\]

Then,

\[
\beta \lambda'(\beta) - \lambda(\beta) = \lambda^*(\lambda'(\beta)) = \bar{q}^\beta[\ln \frac{dq^\beta}{q}]
\]

where \( \bar{q}^\beta \) is the tilted probability measure on \( \mathbb{R} \) given by

\[
\bar{q}^\beta(d\eta) = e^{\beta \eta - \lambda(\beta)} q(d\eta)
\]

As illustrated in figure 2, \( \lambda'(\beta) \to \text{esssup } q \) when \( \beta \to +\infty \).

![Figure 3: The functions \( \lambda \) and \( \lambda^* \).](image)

1. If \( \text{esssup } q = +\infty \), then \( \lambda^* \to \infty \) at infinity, and (2.9) holds for large \( \beta \).
2. If \( \text{esssup } q < +\infty \),

\[
\lim_{\beta \to +\infty} \lambda^*(\lambda'(\beta)) = -\ln q\{\text{esssup } q\}
\]

and (2.9) holds for large \( \beta \) if we assume \( q\{\text{esssup } q\} < 1/2d \)

\[\blacksquare\]

**Remark 2.1** Lower bounds are less useful. We can use lemma 2.2, and the simplest application leads to

\[
p(\beta) \geq Q p_1(\beta) = Q \ln P[\exp\{\beta \eta(1, \omega_1)\}]
\]

which is already better than using Jensen inequality \( (p(\beta) \geq \ln P[\exp\{\beta Q \eta(1, \omega_1)\}] = Q \eta(1, \omega_1)) \). But all these bounds correspond to local optimization in comparison with the polymer measure which is in fact highly non local.

\[\blacksquare\]
2.3 Monotonicity

The function $p$ being convex, its left and right derivatives $p'_g(\beta), p'_d(\beta)$, are non-decreasing. The function $\lambda$ is increasing. Their difference has a nice monotonicity property.

**Proposition 2.7**. The function $\beta \mapsto \lambda(\beta) - p(\beta)$ is non-decreasing on $\mathbb{R}^+$.

With $\zeta_n = \exp \beta H_n$, it is straightforward to check

$$\frac{\partial}{\partial \beta}Q \ln Z_n = Q \frac{\partial}{\partial \beta} \ln Z_n$$

$$= Q[Z_n^{-1} \frac{\partial}{\partial \beta} Z_n]$$

$$= P[Q[Z_n^{-1} H_n \zeta_n]]$$

At this point, we will use the fact that independent variables are positively correlated.

**Proposition 2.8** (Fortuyn-Kasteleyn-Ginibre inequality for independent variables, due to Harris) Let $X = (X_i; 1 \leq i \leq k)$ be a family of independent real random variables. For any $f, g : \mathbb{R}^k \to \mathbb{R}$ bounded increasing functions (increasing means $f(x) \leq f(y)$ if $x_i \leq y_i \forall i \leq k$),

$$\mathbb{E}[f(X)g(X)] \geq [\mathbb{E}f(X)][\mathbb{E}g(X)]$$

We finish our proof, deferring the one of FKG. For any fixed path $\omega$, the probability measure $\zeta_n e^{-n\lambda(\beta)}dQ$ is product, and therefore the family $\eta$ satisfies the FKG inequality. Note that the function $H_n$ is increasing in $\eta$, while $Z_n^{-1}$ is a decreasing. We apply Proposition 2.8 for fixed $\omega$,

$$Q[Z_n^{-1} H_n \zeta_n] \leq e^{-n\lambda(\beta)}Q[Z_n^{-1} \zeta_n] \times Q[H_n \zeta_n]$$

$$= Q[Z_n^{-1} \zeta_n] \times n\lambda'(\beta)$$

using that $Q[\eta(t,x)e^{\beta \eta(t,x)}] = \lambda'(\beta)e^{\lambda(\beta)}$. Integrating with respect to $P$, we get

$$\frac{\partial}{\partial \beta}Q \ln Z_n \leq n\lambda'(\beta)P[Q[Z_n^{-1} \zeta_n]]$$

$$= n\lambda'(\beta)Q[Z_n^{-1} P[\zeta_n]]$$

$$= n\lambda'(\beta)$$

which is the desired result.

We prove Proposition 2.8 by induction on $k$. For $k = 1$, consider an independent copy $Y_1$ of $X_1$, and write

$$\text{Cov}(f(X_1), g(X_1)) = \mathbb{E}[f(X_1)g(X_1)] - [\mathbb{E}f(X_1)][\mathbb{E}g(X_1)]$$

$$= \frac{1}{2} \mathbb{E}[[f(X_1) - f(Y_1)][g(X_1) - g(Y_1)]]$$

where the integrand is a.s. non-negative by monotonicity of $f, g$. Let now $k > 1$, and write

$$\text{Cov}(f(X), g(X)) = \text{Cov}(\mathbb{E}[f(X)|X_1], \mathbb{E}[g(X)|X_1]) + \mathbb{E}\text{Cov}(f(X), g(X)|X_1),$$

with Cov$(U, V|X_1)$ the covariance of $U, V$ given $X_1$. For $f$ non-decreasing, $\mathbb{E}[f(X)|X_1]$ is itself non-decreasing, and the first term is non-negative according to the case $k = 1$. In the second term, the conditional covariance Cov$(f(X), g(X)|X_1)$ is a.s. non-negative by the induction assumption.
Theorem 2.9 There exists $\beta_c = \beta_c(Q, d) \in [0, \infty)$ such that

\[
\begin{align*}
 p(\beta) &= \lambda(\beta) & \text{if } \beta \leq \beta_c, \\
 p(\beta) &< \lambda(\beta) & \text{if } \beta > \beta_c
\end{align*}
\] (2.10)

□ This is a direct consequence of Proposition 2.7

Remark 2.2 (i) This result implies the absence of reentrant phase transition in the phase diagram of the model.  
(ii) We have seen sufficient conditions for $\beta_c < \infty$.  
(iii) For more information on FKG inequality, see [38, p.77-83]. (iv) It is not hard to see that $\mu_n(H_n)$ is not an increasing function of $\eta$. However we proved that $Q(\mu_n(H_n) \times Z_n) \geq Q(\mu_n(H_n))Q(Z_n)$.

3 A martingale approach

Martingale theory is a powerful tool to study random sequences. In this section, we dig in this direction. To start with, it is efficient for proving that equality can hold in (2.6).

3.1 A martingale associated to the partition function

Classical considerations from thermodynamics and common sense made us consider $\ln Z_n$, a rather difficult quantity to study directly since we cannot compute its expectation. It is easier to consider the partition function $Z_n$ itself, for which we easily see that $QZ_n = \exp(n\lambda)$. Define the normalized partition function by

\[
W_n = Z_n \exp(n\lambda(\beta)) , \quad n \geq 1.
\] (3.11)

Not only $W_n$ has mean 1, but also $(W_n)_{n \geq 1}$ is a positive $(\mathcal{G}_n)$-martingale on $(G, \mathcal{G}, Q)$. Indeed, for all fixed path $\omega$, the sequence $H_n(\omega, \eta)$ is a random walk defined on $(G, \mathcal{G}, Q)$, and

\[
\bar{\zeta}_n = \bar{\zeta}_n(\omega, \eta) = \exp(\beta H_n - n\lambda(\beta)),
\] (3.12)

is its exponential martingale. Hence, the average over $\omega$, $W_n = P[\bar{\zeta}_n]$, is itself a mean-one, positive $(\mathcal{G}_n)$-martingale on $(G, \mathcal{G}, Q)$. As we will see now, the martingale property makes this sequence much easier to study than $\ln Z_n$ itself, a fact which was used first by Bolthausen in [7].

By the martingale convergence theorem, the limit $W_\infty$ exists $Q$-a.s. It is clear that the event $\{W_\infty = 0\}$ is measurable with respect to the tail $\sigma$-field

\[
\bigcap_{n \geq 1} \sigma[\eta(j, x) ; j \geq n, x \in \mathbb{Z}^d].
\]

Indeed, we can write in the notations of (2.5),

\[
W_{n+m} = P[\bar{\zeta}_n e^{-m\lambda} Z_{n,m}] ,
\]

\[
W_\infty = \lim_{m \to \infty} W_{n+m} = P[\bar{\zeta}_n \times \lim_{m \to \infty} (e^{-m\lambda} Z_{n,m})].
\]
By positivity of $\zeta_n$ we see that
\[
\{W_\infty = 0\} = \{ \lim_{m \to \infty} e^{-m\lambda} Z_{n,m}^x = 0 \forall x : P[\omega_n = x] > 0 \}
\in \sigma[\eta(j, x) : j \geq n, x \in \mathbb{Z}^d]
\]
for all $n$. By Kolmogorov’s zero-one law [22], every event in the tail $\sigma$-field has probability 0 or 1. We then have

**Proposition 3.1** The limit
\[
W_\infty = \lim_{n \to \infty} W_n
\]
exists $Q$-a.s. Moreover, there are only two possibilities for the positivity of the limit;

\[
Q\{W_\infty > 0\} = 1,
\]

or
\[
Q\{W_\infty = 0\} = 1.
\]

The above contrasting situations (3.14) and (3.15) will be called the **weak disorder** phase and the **strong disorder** phase, respectively.

We will see later that the polymer is diffusive in the regime (3.14), as well as other consequences.

**Remark 3.1** (i) It is an interesting question to find a characterization of (3.14) (or (3.15)) in terms of the distribution of $\eta(n, x)$.

(ii) The question of whether the positive martingale $W_n$ vanishes or not as $n \to \infty$, has somewhat similar flavor to some other topics in the probability theory such as Kakutani’s dichotomy for infinite product measure (e.g., [22, page 244]), nontriviality of the limit of the normalized Galton-Watson process [2] and of multiplicative chaos [32].

(iii) When $W_\infty > 0$, we have clearly $p = \lambda$.

**Open problem:** Does (3.15) implies $p < \lambda$? We will see in Theorem 4.4 that the answer is yes when $d = 1$.

### 3.2 The second moment method and the $L^2$ region

The results we present in this subsection show that the impurities do not change too much the behavior of the polymer if $d \geq 3$ and $\beta$ is small enough. We first recall the following fact about the return probability $\pi_d$ for the simple random walk,

\[
\pi_d \overset{\text{def}}{=} P\{\omega_n = 0 \text{ for some } n \geq 1\} =
\begin{cases}
1 & \text{if } d \leq 2, \\
< 1 & \text{if } d \geq 3.
\end{cases}
\]

(3.16)

More precisely, it is known that $\pi_{d+1} < \pi_d$ for all $d \geq 3$ (e.g., [42, Lemma 1]) and that $\pi_3 = 0.3405...$ [48, page 103]. In particular, $\pi_d \leq 0.3405...$ for all $d \geq 3$.

**Theorem 3.2** [7] Suppose that $d \geq 3$ (hence $\pi_d < 1$) and that

\[
\gamma_1(\beta) \overset{\text{def}}{=} \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_d).
\]

Then, $W_\infty > 0$ a.s.
Note that \( \gamma_1(\beta) \) is increasing on \([0, \infty)\) and \( \gamma_1(0) = 0 \) so that the condition in (3.17) does hold if \( \beta \) is small, whatever the distribution of the environment is. This theorem shows that there always exists a weak disorder regime for \( d \geq 3 \) and small \( \beta \). In particular, \( p = \lambda \) holds under these assumptions.

**Example 3.1** Gaussian environment. If \( \eta \) is standard gaussian \( \mathcal{N}(0, 1) \), then \( \gamma_1(\beta) = \beta^2 \) and hence (3.17) holds if \( \beta < \sqrt{\ln(1/\pi d)} \).

**Example 3.2** Absence of strong disorder regime. Consider the case of Bernoulli environment, where \( \eta(t, x) = 1 \) or 0 with probability \( p \) and \( 1 - p \) respectively. Recalling that \( \lambda = \ln[pe^{\beta} + (1 - p)] \), we see from direct computations that

\[
\lim_{\beta \to \infty} \gamma_1(\beta) = -\ln(p).
\]

Hence, (3.17) holds for all \( \beta \geq 0 \) if \( p > \pi_d \). Theorem 3.2 shows that, in this case, weak disorder holds for all \( \beta \geq 0 \).

\[\Box\]

Proof of Theorem 3.2. We compute the \( L^2 \)-norm of the martingale \( W_n \). To do so, we consider on the product space \((\Omega^2, \mathcal{F}^\otimes 2)\), the probability measure \( P^\otimes 2 = P^\otimes 2(d\omega, d\bar{\omega}) \), that we will view as the distribution of the couple \((\omega, \bar{\omega})\) with \( \bar{\omega} = (\bar{\omega}_k)_{k \geq 0} \) an independent copy of \( \omega = (\omega_k)_{k \geq 0} \).

\[
Q[W_n^2] = Q \left[ P^\otimes 2 \prod_{t=1}^{n} e^{\beta[\eta(t, \omega_t) + \eta(t, \bar{\omega}_t)] - 2\lambda(\beta)} \right]
\]

\[
= P^\otimes 2 \left[ \prod_{t=1}^{n} (e^{\lambda(2\beta) - 2\lambda(\beta)}1_{\omega_t = \bar{\omega}_t} + 1_{\omega_t \neq \bar{\omega}_t}) \right]
\]

\[
= P^\otimes 2 \left[ e^{\gamma_1(\beta)N_n} \right],
\]

with \( N_n \) the number of intersections of the paths \( \omega, \bar{\omega} \) up to time \( n \),

\[
N_n = N_n(\omega, \bar{\omega}) = \sum_{t=1}^{n} 1_{\omega_t = \bar{\omega}_t}
\]

As \( n \to \infty \), \( N_n \not
\to N_\infty \), and by monotone convergence \( Q[W_n^2] \not
\to P^\otimes 2 \left[ e^{\gamma_1(\beta)N_\infty} \right] \). It is easy to see that \( N_\infty \) is the number of visit to 0 of the simple random walk starting from 0. Hence, \( N_\infty \) is geometrically distributed with success probability \( \pi_d \), and

\[
\sup_n Q[W_n^2] < \infty \iff \gamma_1 + \ln \pi_d < 0,
\]

i.e., iff (3.17) is fulfills. Then, the martingale \( W_n \) is bounded in \( L^2 \), and by a classical convergence result [52], it converges in \( L^2 \) to a limit, which is necessarily equal to \( M_\infty \). So \( QM_\infty = \lim_n QM_n = 1 \), which excludes the possibility that the limit vanishes in theorem 3.1.

\[\blacksquare\]

We call \( \text{L}^2 \text{ region} \), the set of parameters \( \beta \) such that (3.17) holds. In this region, the natural martingale is bounded in \( \text{L}^2 \), and a number of results are known via second moment computations.

\footnote{Indeed, we can compute \( \gamma_1(\beta) = 2[\lambda'(2\beta) - \lambda'(\beta)] \) and use the convexity of \( \lambda \) to see that \( \gamma_1(\beta) \) is increasing on \( \mathbb{R}^+ \) and decreasing on \( \mathbb{R}^- \).}
**Remark 3.2** An improved sufficient condition – weaker than (3.17) – for weak disorder was recently obtained in [5] using size-biasing.

The next theorem states that that the impurities do not change the transversal fluctuations of the polymer for large $d$ and small enough $\beta$.

**Theorem 3.3 (Diffusive behavior in $L^2$ region [29, 7, 46])** Under the assumptions of Theorem 3.2, we have

$$\lim_{n/\infty} \mu_n [\omega_n^2/n] = 1 \quad Q\text{-a.s.}, \quad (3.19)$$

and for all $f \in C(\mathbb{R}^d)$ with at most polynomial growth at infinity

$$\lim_{n/\infty} \mu_n \left[ f\left(\omega_n/\sqrt{n}\right) \right] = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f\left(x/\sqrt{d}\right) \exp(-|x|^2/2) dx, \quad Q\text{-a.s.} \quad (3.20)$$

In particular,

$$\mu_n \left( \frac{\omega_n}{n} \in \cdot \right) \xrightarrow{\text{law}} \mathcal{N}(0, d^{-1}I_d)$$

**Remark 3.3** The first rigorous proof of (3.19) was obtained by Imbrie and Spencer [29] in the case of Bernoulli environment. The fact that the polymer is diffusive in some regime was much of a surprise. Soon afterwards, a more transparent proof based on the martingale analysis was given by Bolthausen [7]. The martingale proof was then extended to general environment under condition (3.17) by Song and Zhou [46]. The diffusive behavior (3.19) follows from (3.20) by choosing $f(x) = |x|^2$. In [7], (3.20) is obtained for the Bernoulli environment only. However, with the help of the observation made in [46], it is not difficult to extend the central limit theorem to general environment under the assumption in Theorem 3.3. In [1] Albeverio and Zhou proved, under the assumptions of Theorem 3.2, that under the polymer measure $\mu_n$, the path $\omega$ satisfies the invariance principle for almost every realization of the environment.

We now start the proof of Theorem 3.3. The proofs are based on the $L^2$ analysis of certain martingales on $(G, \mathcal{G}, Q)$. This approach was introduced by Bolthausen [7] and later taken up by Song and Zhou [46]. We summarize the main step in their analysis as Proposition 3.4 below. The proposition deals with a process $(M_n)_{n \geq 1}$ on $(G, \mathcal{G}, Q)$ of the form;

$$M_n = P[\varphi(n, \omega_n) \tilde{c}_n]. \quad (3.21)$$

Here, $\tilde{c}_n$ has been introduced in (3.12) and $\varphi : \mathbb{N} \times \mathbb{Z}^d \to \mathbb{R}$ is a function for which we assume the following properties:

**(P1)** There are constants $C_i, p \in [0, \infty)$, $i = 0, 1, 2$ such that

$$|\varphi(n, x)| \leq C_0 + C_1 |x|^p + C_2 n^{p/2} \quad \text{for all } (n, x) \in \mathbb{N} \times \mathbb{Z}^d. \quad (3.22)$$

**(P2)** $\Phi_n \overset{\text{def}}{=} \varphi(n, \omega_n)$, $n \geq 1$ is a martingale on $(\Omega, \mathcal{F}, P)$ with respect to the filtration

$$\mathcal{F}_n = \sigma[\omega_j ; j \leq n]. \quad (3.23)$$
It is easy to see that \((M_n)_{n \geq 1}\) is a \((\mathcal{G}_n)\)-martingale on \((G, \mathcal{G}, Q)\): Denoting conditional expectations by \(Q^{\mathcal{G}_n}, P^{\mathcal{F}_n}\), we have

\[
Q^{\mathcal{G}_n} M_{n+1} = P[\varphi(n+1, \omega(n+1)) Q^{\mathcal{G}_n} e^{\beta n}] = P[\varphi(n+1, \omega(n+1)) e^{\beta n}] = M_n,
\]

by (P2). The following proposition generalizes [7, Lemma 4] and [46, Theorem 2].

**Proposition 3.4** Consider the martingale \((M_n)_{n \geq 1}\) defined by (3.21). Suppose that \(d \geq 3\) and that (3.17), (P1), (P2) are satisfied. Then, there exists \(\kappa \in [0, p/2)\) such that

\[
\max_{0 \leq j \leq n} |M_j| = \mathcal{O}(n^\kappa), \quad \text{as } n \nrightarrow \infty, \quad Q\text{-a.s.} \tag{3.25}
\]

If in addition, \(p < \frac{1}{2}d - 1\), then

\[
\lim_{n \rightarrow \infty} M_n \text{ exists } Q\text{-a.s. and in } L^2(Q). \tag{3.26}
\]

**Remark 3.4**

(i) Note that theorem 3.2 can be viewed as a consequence of (3.26) by choosing \(\varphi \equiv 1\), in which case \(M_n = W_n\).

(ii) As will be seen from the way (3.25) is used below, it is crucial that the divergence of the right-hand-side is strictly slower than \(n^{p/2}\), and this is where the property (P2) is relevant.

If we drop the property (P2) from the assumption of Proposition 3.4, we then have a larger bound:

\[
M_n = \mathcal{O}(n^{p/2}), \quad \text{as } n \nrightarrow \infty, \quad Q\text{-a.s.} \tag{3.27}
\]

This larger bound from the weaker assumption can be obtained via Proposition 3.4 as in the proof of (2.1) in [7].

Proof of Theorem 3.3. We will prove Proposition 3.4 later on, independently. With Proposition 3.4 in hand, we first derive the statements in Theorem 3.3, following the lines of [7].

- To prove (3.19), we take \(\varphi(n, x) = |x|^2 - n\) (hence \(p = 2\)). Then, by theorem 3.2 and Proposition 3.4, there exists \(\kappa \in [0, 1]\) such that

\[
\mu_n[|\omega|^2] - n = P[\varphi(n, \omega(n)) e^{\beta n}] = \mathcal{O}(n^\kappa) \quad Q\text{-a.s.} \tag{3.28}
\]

- we now explain the route to (3.20). We let \(a = (a_j)_{j=1}^d\) and \(b = (b_j)_{j=1}^d\) denote multi indices in what follows. We will use standard notation \(|a|_1 = a_1 + \ldots + a_d\), \(x^a = x_1^{a_1} \cdots x_d^{a_d}\) and \((\frac{\partial}{\partial x})^a = \left(\frac{\partial}{\partial x_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{a_d}\) for \(x \in \mathbb{R}^d\). It is enough to prove (3.20) for any monomial of the form \(f(x) = x^a\). We will do this by induction on \(|a|_1\). We introduce

\[
\varphi(n, x) = \left(\frac{\partial}{\partial \theta}\right)^a \exp(\theta \cdot x - n \rho(\theta)) \bigg|_{\theta = 0},
\]

\[
\psi(n, x) = \left(\frac{\partial}{\partial \theta}\right)^a \exp(\theta \cdot x - n \frac{|\theta|^2}{2d}) \bigg|_{\theta = 0},
\]
where \( \rho(\theta) = \ln \left( \frac{1}{d} \sum_{1 \leq j \leq d} \cosh(\theta_j) \right) \). Clearly, the function \( \varphi \) satisfies (P1) and (P2) with \( p = |a|_1 \). On the other hand, we see from the definition of \( \psi \) that

\[
(2\pi)^{-d/2} \int_{\mathbb{R}^d} \psi(1, x/\sqrt{d}) e^{-|x|^2/2} dx = 0. \tag{3.29}
\]

Moreover, it is not difficult to see [7, Lemma 3c] that \( \varphi(n, x) = x^a + \varphi_0(n, x) \) and \( \psi(n, x) = x^a + \psi_0(n, x) \) where

\[
\varphi_0(n, x) = \sum_{|b| + 2j \leq |a|_1} A_a(b, j)x^b n^j, \quad \psi_0(n, x) = \sum_{|b| + 2j = |a|_1} A_a(b, j)x^b n^j.
\]

for some \( A_a(b, j) \in \mathbb{R} \). In particular, \( \varphi_0 \) and \( \psi_0 \) have the same coefficients for \( x^b n^j \) with \( |b| + 2j = |a|_1 \). We now write \( \mu_n[(\omega_n/\sqrt{n})^a] \) as

\[
\mu_n[(\omega_n/\sqrt{n})^a] = \frac{1}{W_n} P[\varphi(n, \omega_n) \tilde{\zeta}_n] n^{-|a|_1/2} - \frac{1}{W_n} P[\psi_0(1, \omega_n/\sqrt{n}) \tilde{\zeta}_n] + \frac{1}{W_n} P[\psi_0(n, \omega_n) - \varphi_0(n, \omega_n)] \tilde{\zeta}_n] n^{-|a|_1/2}
\]

As \( n \nrightarrow \infty \), the second term converges to \( (2\pi)^{-d/2} \int_{\mathbb{R}^d} (x/\sqrt{d})^p e^{-|x|^2/2} dx \) by the induction hypothesis and (3.29). The first and the third terms on the right-hand side vanish as \( n \nrightarrow \infty \). In fact, we use theorem 3.2, Proposition 3.4 for the first term and theorem 3.2, (3.27) for the third term.

We now turn to the proof of Proposition 3.4. Here, we follow [46]. We present a key step in the proof as a lemma.

**Lemma 3.5** Suppose that \( d \geq 3 \) and that (3.17), (P1), (P2) are satisfied. Then,

\[
Q[M_n^2] = O(b_n), \text{ as } n \nrightarrow \infty, \text{ Q.a.s.} \tag{3.30}
\]

where \( b_n = 1 \) if \( p = \frac{d}{2} - 1 \), \( b_n = \ln n \) if \( p = \frac{d}{2} - 1 \), and \( b_n = n^{p-\frac{d}{4}+1} \) if \( p > \frac{d}{2} - 1 \).

**Remark 3.5** The choice of \( b_n \) is made in order to have \( \sum_{1 \leq j \leq n} j^{p-\frac{d}{2}} = O(b_n) \). See (3.35) below for the reason of the power \( p - \frac{d}{2} \).

Proof of Lemma 3.5: On the product space \( (\Omega^2, \mathcal{F}^\otimes 2) \), we consider the probability measure \( P^\otimes 2 = P^\otimes 2(d\omega, d\tilde{\omega}) \), that we will view as the distribution of the couple \( (\omega, \tilde{\omega}) \) with \( \tilde{\omega} = (\tilde{\omega}_k)_{k \geq 0} \) an independent copy of \( \omega = (\omega_k)_{k \geq 0} \). We write \( \Upsilon_{i_1 \ldots i_k} \) for the indicator function of the event

\[
\{\omega_{i_1} = \tilde{\omega}_{i_1}, \omega_{i_2} = \tilde{\omega}_{i_2}, \ldots, \omega_{i_k} = \tilde{\omega}_{i_k}\}.
\]

We first expand the second moment \( Q[M_n^2] \) as follows:

\[
Q[M_n^2] = \Phi_0^2 + \sum_{1 \leq k \leq n} (e^{n\beta} - 1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq n} P^\otimes 2[\Phi_{i_k}^2(\omega)^2 \Upsilon_{i_1 \ldots i_k}]. \tag{3.31}
\]
To see this, we write $M_n^2$ in terms of the independent copy:

$$
M_n^2 = P[\Phi_n \tilde{\zeta}_n]^2 = P^{\otimes 2}[\Phi_n(\omega)\Phi_n(\tilde{\omega})\tilde{\zeta}_n(\omega, \eta)\tilde{\zeta}_n(\tilde{\omega}, \eta)].
$$

(3.32)

It follows from (3.32) that

$$
Q[M_n^2] = P^{\otimes 2}[\Phi_n(\omega)\Phi_n(\tilde{\omega})Q[\tilde{\zeta}_n(\omega, \eta)\tilde{\zeta}_n(\tilde{\omega}, \eta)]].
$$

On the other hand,

$$
Q \left[ e^{\beta\eta(j,\omega_j) + \beta\eta(j,\tilde{\omega}_j) - 2\lambda_n} \right] = 1 + (e^{\gamma_1(\beta)} - 1)\mathcal{Y}_j,
$$

and hence that

$$
Q[\tilde{\zeta}_n(\omega, \eta)\tilde{\zeta}_n(\tilde{\omega}, \eta)] = \prod_{1 \leq j \leq n} (1 + (e^{\gamma_1(\beta)} - 1)\mathcal{Y}_j)
$$

$$
= 1 + \sum_{1 \leq k \leq n} (e^{\gamma_1(\beta)} - 1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq n} \mathcal{Y}_{i_1 \ldots i_k}.
$$

(3.34)

The expansion (3.31) is now obtained by inserting (3.34) into (3.33) and by the martingale property of $\Phi_n$.

Let us fix $i_1, \ldots, i_k$ for a moment. We then have by (3.22) that

$$
P^{\otimes 2}[\Phi_k(\omega) \mathcal{Y}_{i_1 \ldots i_k}] \leq 3C_1 A_{i_1 \ldots i_k} + 3(C_0^2 + C_2^2)B_{i_1 \ldots i_k},
$$

where

$$
A_{i_1 \ldots i_k} = P^{\otimes 2}|\omega_{i_k}|^{2p}\mathcal{Y}_{i_1 \ldots i_k}, \quad B_{i_1 \ldots i_k} = i_k^{p} P^{\otimes 2}[\mathcal{Y}_{i_1 \ldots i_k}].
$$

We now bound $A_{i_1 \ldots i_k}$ from above. As will be seen from the way it is done, the same bound (up to the multiplicative constant) is obtained for $B_{i_1 \ldots i_k}$. Observe that

$$
P^{\otimes 2}|\omega_n|^{2p}\mathcal{Y}_n = \sum_{x \in \mathbb{Z}^d} P[|\omega_n|^{2p} : \omega_n = x]P[\omega_n = x]
$$

$$
\leq Cn^{-\frac{d}{2}} P[|\omega_n|^{2p}]
$$

$$
\leq Cn^{p-\frac{d}{2}},
$$

(3.35)

where we have used on the second line, the following well known fact for the simple random walk:

$$
\max_{x \in \mathbb{Z}^d} P\{\omega_n = x\} = O(n^{-d/2}), \quad \text{as } n \to \infty.
$$

We write $j_\ell = i_\ell - i_{\ell-1}$, $\ell = 1, 2, \ldots, k$ with $i_0 = 0$. We then see from the Markov property and (3.35) that

$$
A_{i_1 \ldots i_k} \leq k^{2p-1} \sum_{1 \leq \ell \leq k} P^{\otimes 2}|\omega_{i_\ell} - \omega_{i_{\ell-1}}|^{2p}\mathcal{Y}_{i_1 \ldots i_k}
$$

$$
= k^{2p-1} \sum_{1 \leq \ell \leq k} \left( \prod_{1 \leq m < \ell} P^{\otimes 2}[\mathcal{Y}_{j_m}] \right) P^{\otimes 2}|\omega_{j_\ell}|^{2p}\mathcal{Y}_{j_\ell} \left( \prod_{\ell < m \leq k} P^{\otimes 2}[\mathcal{Y}_{j_m}] \right)
$$

$$
\leq Ck^{2p-1} \sum_{1 \leq \ell \leq k} j_\ell^{p-\frac{d}{2}} \prod_{1 \leq m \leq k \atop m \neq \ell} P^{\otimes 2}[\mathcal{Y}_{j_m}]
$$
Note that $\sum_{1 \leq j \leq n} j^{p-d} = \mathcal{O}(b_n)$ and that $\sum_{j \geq 1} P^{\otimes 2}[\gamma_j] = \frac{\pi_d}{1-\pi_d}$. Therefore, we obtain from what we have seen above that

$$\sum_{1 \leq i_1 < \ldots < i_k \leq n} P^{\otimes 2}[\Phi_{i_k}(\omega)^2 \gamma_{i_1,\ldots,i_k}] \leq C k^{2p-1} \sum_{1 \leq \ell \leq k} \sum_{1 \leq j_1 \leq n} \ldots \sum_{1 \leq j_k \leq n} j_\ell^{p-d} \prod_{m \neq \ell} P^{\otimes 2}[\gamma_{j_m}] \leq \mathcal{O}(b_n) k^{2p} \left( \frac{\pi_d}{1-\pi_d} \right)^{k-1}.$$  

By this and (3.31), we now arrive at

$$Q[M_n^2] \leq \Phi_0^2 + \mathcal{O}(b_n) \sum_{k \geq 1} k^{2p} (e^{\gamma_1(\delta)} - 1)^k \left( \frac{\pi_d}{1-\pi_d} \right)^{k-1}.$$  

The summation in $k$ converges, thanks to the assumptions $d \geq 3$ and (3.17). This finishes the proof of (3.30). \hfill \blacksquare 

It is now easy to complete the proof of Proposition 3.4. We set $M_n^* = \max_{0 \leq j \leq n} |M_j|$ to simplify the notation. For (3.30), it is sufficient to prove that for any $\delta > 0$,

$$M_n^* = \mathcal{O}(n^{\delta} \sqrt{b_n}) \quad \text{as } n \nearrow \infty, \text{ Q-a.s.,} \quad (3.36)$$

where $b_n$ is the $L^2$-bound in Lemma 3.5. Moreover, by the monotonicity of $M_n^*$ and the polynomial growth of $n^{\delta} \sqrt{b_n}$, it is enough to prove (3.36) along a subsequence $\{n^k : n \geq 1\}$ for some power $k \geq 2$. Now, take $k > 1/\delta$. We then have by Chebychev’s inequality, Doob’s inequality and Lemma 3.5 that

$$Q\{M_{n^k}^* > n^{k^2} \sqrt{b_{n^k}}\} \leq Q\{M_{n^k}^* > n\sqrt{b_{n^k}}\} \leq Q[(M_{n^k}^*)^2/(n^2 b_{n^k})] \leq 4Q[M_{n^k}^2]/(n^2 b_{n^k}) \leq Cn^{-2}.$$  

Then, it follows from the Borel-Cantelli lemma that

$$Q\{M_{n^k}^* \leq n^{k^2} \sqrt{b_{n^k}} \text{ for large enough } n's\} = 1.$$  

This ends the proof of (3.25).

The second statement (3.26) in Proposition 3.4 follows from Lemma 3.5 and the martingale convergence theorem. This completes the proof of Proposition 3.4. \hfill \blacksquare 

**Remark 3.6** The parameters – mean and covariance – in the CLT 3.2 do not depend on the environment. On the contrary, the density in the LLT does depend on the environment as seen from the final point. Further, the first order corrections do depend on the particular realization of the environment. Indeed, for $d \geq 7$ the statement in (3.28) can be refined into

$$\mu_n(|\omega_n|^2) - n \quad \text{converges a.s. and in } L^2$$
in view of (3.26) and of \( p < d/2 - 1 \) with \( p = 2 \). It is easy to see that the limit is not a constant, and therefore
\[ \mu_n(|\omega_n|^2) = n + V + o(1) \]
as \( n \to \infty \), where \( V = V(\eta) \) is a non trivial function of the environment. Similarly for the first moment, we can use (3.26) for \( d \geq 5 \) since \( p = 1 \) in this case, and we conclude that \( \mu_n(\omega_n) \) converges to a random vector as \( n \to \infty \).

The first order corrections are studied in details in [8], where they are given in a rather explicit form in terms of a perturbation expansion.

3.3 Semimartingale decomposition

The next step in our martingale analysis is to consider \( \ln W_n \) as a semimartingale and to write its Doob’s decomposition. Although this is quite natural, it was realized only recently. Viewed as a “conditional second moment” method, this step is most a natural continuation of the last section.

It is convenient to introduce some more notation. For a sequence \( (a_n)_{n \geq 0} \) (random or non-random), we set \( a_n = a_n - a_{n-1} \) for \( n \geq 1 \). Let us now recall Doob’s decomposition in this context [52]: any \( (\mathcal{G}_n) \)-adapted process \( X = \{X_n\}_{n \geq 0} \subset L^1(Q) \) can be decomposed in a unique way as
\[ X_n = M_n(X) + A_n(X), \quad n \geq 1, \]
where \( M(X) \) is an \( (\mathcal{G}_n) \)-martingale and
\[ A_0 = 0, \quad \Delta A_n = Q[\Delta X_n|\mathcal{G}_{n-1}], \quad n \geq 1. \]

\( M_n(X) \) and \( A_n(X) \) are called respectively, the martingale part and compensator of the process \( X \). If \( X \) is a square integrable martingale, then the compensator \( A_n(X^2) \) of the process \( X^2 = \{(X_n)^2\}_{n \geq 0} \subset L^1(Q) \) is denoted by \( \langle X \rangle_n \) and is given by the following formula:
\[ \Delta \langle X \rangle_n = Q[(\Delta X_n)^2|\mathcal{G}_{n-1}] \]

Here, we are interested in the Doob’s decomposition of \( X_n = -\ln W_n \), whose martingale part and the compensator will be denoted \( M_n \) and \( A_n \) respectively
\[ -\ln W_n = M_n + A_n. \tag{3.37} \]

To compute \( M_n \) and \( A_n \), we introduce
\[ U_n = \mu_{n-1}[e^{\beta \eta(n,\omega_n)} - \lambda(\beta)] - 1. \]

It is then clear that
\[ W_n/W_{n-1} = 1 + U_n \tag{3.38} \]

and hence that
\[ \Delta A_n = -Q[\ln(1 + U_n)|\mathcal{G}_{n-1}], \tag{3.39} \]
\[ \Delta M_n = -\ln(1 + U_n) + Q[\ln(1 + U_n)|\mathcal{G}_{n-1}] \tag{3.40} \]

A key role in the asymptotics of the model is played by the following random variables on \( (\Omega_0, \mathcal{G}, Q) \),
\[ I_n = \sum_{x \in \mathbb{Z}^d} \mu_{n-1}\{\omega_n = x\}^2. \tag{3.41} \]
We now mention to an interpretation of $I_n$. On the product space $(\Omega^2, \mathcal{F}^2)$, we consider the probability measure $\mu_n^\otimes^2 = \mu_n^\otimes^2(d\omega, d\tilde{\omega})$, that we will view as the distribution of the couple $(\omega, \tilde{\omega})$ with $\tilde{\omega} = (\tilde{\omega}_k)_{k \geq 0}$ an independent copy of $\omega = (\omega_k)_{k \geq 0}$ with law $\mu_n$. We then have that

$$I_n = \mu_n^\otimes^2(\omega_n = \tilde{\omega}_n).$$

(3.42)

Hence, the summation

$$\sum_{1 \leq k \leq n} I_k$$

is the expected amount of the overlap up to time $n$ of two independent polymers in the same (fixed) environment. This can be viewed as an analogue to the so-called replica overlap often discussed in the context of disordered systems, e.g. mean field spin glass, and also of directed polymers on trees [19].

The large time behavior of (3.43) and the normalized partition function $W_n$ are related as follows.

**Theorem 3.6** Let $\beta \neq 0$. Then,

$$\{W_\infty = 0\} = \left\{\sum_{n \geq 1} I_n = \infty\right\}, \quad Q\text{-a.s.}$$

(3.44)

Moreover, if $Q\{W_\infty = 0\} = 1$, there exist $c_1, c_2 \in (0, \infty)$ such that $Q\text{-a.s.},$

$$c_1 \sum_{1 \leq k \leq n} I_k \leq -\ln W_n \leq c_2 \sum_{1 \leq k \leq n} I_k \quad \text{for large enough } n\text{'s.}$$

(3.45)

Proof of Theorem 3.6: To conclude (3.44) and (3.45), it is enough to show the following (3.46) and (3.47):

$$\{W_\infty = 0\} \subset \left\{\sum_{n \geq 1} I_n = \infty\right\}, \quad Q\text{-a.s.}$$

(3.46)

There are $c_1, c_2 \in (0, \infty)$ such that

$$\left\{\sum_{n \geq 1} I_n = \infty\right\} \subset \{(3.45) \text{ holds}\}, \quad Q\text{-a.s.}$$

(3.47)

In view of the second line in (3.39), and since the variance is bounded by the second moment (conditionally on $\mathcal{G}_{n-1}$),

$$\Delta(\langle M \rangle)_n \leq Q[\ln^2(1 + U_n)|\mathcal{G}_{n-1}].$$

(3.48)

We now claim that there is a constant $c \in (0, \infty)$ such that

$$\frac{1}{c}I_n \leq \Delta A_n \leq cI_n, \quad \Delta(\langle M \rangle)_n \leq cI_n.$$

(3.49)

Indeed, both follow from (3.39), (3.48) and Lemma 3.7 below; $\{e_i\}, \{\alpha_i\}$ and $Q$ in the lemma play the roles of $\{e^{\beta_1(n,z)-\lambda}\}_{z|1| \leq n}, \{\mu_{n-1}(\omega_n = z)|z|1| \leq n$ and $Q[\cdot |\mathcal{G}_{n-1}]$. 


We now conclude (3.46) from (3.49) as follows (the equalities and the inclusions here being understood as \( Q \)-a.s.):

\[
\left\{ \sum_{n \geq 1} I_n < \infty \right\} \subset \{ A_\infty < \infty, \langle M \rangle_\infty < \infty \}
\]
\[
\subset \{ A_\infty < \infty, \lim_{n/\infty} M_n \text{ exists and is finite} \}
\]
\[
\subset \{ W_\infty > 0 \}.
\]

Here, on the second line, we have used a well-known property for martingales, e.g. [22, page 255, (4.9)].

Finally we prove (3.47). By (3.49), it is enough to show that

\[
\{ A_\infty = \infty \} \subset \left\{ \lim_{n/\infty} - \ln W_n / A_n = 1 \right\}, \quad Q\text{-a.s.} \tag{3.50}
\]

Thus, let us suppose that \( A_\infty = \infty \). If \( \langle M \rangle_\infty < \infty \), then again by [22, page 255, (4.9)], \( \lim_{n/\infty} M_n \) exists and is finite and therefore (3.50) holds. If, on the contrary, \( \langle M \rangle_\infty = \infty \), then

\[
- \ln W_n / A_n = \langle M \rangle_n / A_n + 1 \to 1 \quad Q\text{-a.s.}
\]

by (3.49) and the law of large numbers for martingales, see [22, page 255, (4.10)]. This completes the proof of Theorem 3.6.

\[\square\]

**Lemma 3.7** Let \( e_i, 1 \leq i \leq m \) be positive, non-constant i.i.d. random variables on a probability space \((H, \mathcal{G}, Q)\) such that

\[
Q[e_1] = 1, \quad Q[e_1^3 + \ln^2 e_1] < \infty.
\]

For \( \{ \alpha_i \}_{1 \leq i \leq m} \subset [0, \infty) \) such that \( \sum_{1 \leq i \leq m} \alpha_i = 1 \), define a centered random variable \( U > -1 \) by \( U = \sum_{1 \leq i \leq m} \alpha_i e_i - 1 \). Then, there exists a constant \( c \in (0, \infty) \), independent of \( m \) and of \( \{ \alpha_i \}_{1 \leq i \leq m} \), such that

\[
\frac{1}{c} \sum_{1 \leq i \leq m} \alpha_i^2 \leq Q \left[ \frac{U^2}{2 + U} \right], \tag{3.51}
\]
\[
\frac{1}{c} \sum_{1 \leq i \leq m} \alpha_i^2 \leq -Q[\ln(1 + U)] \leq c \sum_{1 \leq i \leq m} \alpha_i^2, \tag{3.52}
\]
\[
Q[\ln^2(1 + U)] \leq c \sum_{1 \leq i \leq m} \alpha_i^2. \tag{3.53}
\]

The readers are invited to try the proof of this lemma as an interesting exercise. A solution can be found in [12]. Here, we prove it under the more restrictive assumption of bounded \( \eta \)'s.

\[\square\] Proof of lemma 3.52, when \( |\eta(t, x)| \leq K \) a.s. Then, for fixed \( \beta \), \( U_n \) stays in a fixed interval \( I \) which is bounded away from \(-1\) and \(+\infty\), and there exists constants \( C_\pm \in (0, \infty) \) such that

\[
u - C_- u^2 \leq \ln(1 + u) \leq u - C_+ u^2, \quad u \in I.
\]
Recalling the first line of (3.39), we have by in one direction:

\[
\Delta A_n = -Q_{\mathcal{G}^n-1}[\ln(1 + U_n)],
\]

(3.54)

\[
\leq -Q_{\mathcal{G}^n-1}[U_n] + C_-Q_{\mathcal{G}^n-1}[U_n^2],
\]

(3.55)

\[
= C_-\mu_n^{\otimes 2}Q_{\mathcal{G}^n-1}[(e^{\beta H(n,\omega_n)-\lambda(\beta)} - 1)(e^{\beta H(n,\tilde{\omega}_n)-\lambda(\beta)} - 1)]
\]

(3.56)

\[
= C_-\sum_x \mu_n^{\otimes 2}(\omega_n = \tilde{\omega}_n = x)Q[(e^{\beta H(n,x)-\lambda(\beta)} - 1)^2]
\]

(3.57)

\[
= C_-(e^{\gamma_1(\beta)} - 1)\sum_x \mu_n^{\otimes 2}(\omega_n = \tilde{\omega}_n)
\]

(3.58)

\[
= \text{Cst } I_n.
\]

(3.59)

Similarly, one gets the other direction \(\Delta A_n \geq \text{Cst } I_n\), which proves (3.52). It is clear that \(\ln^2(1 + u) \leq Cu^2\) for \(u \in \mathcal{I}\) with some constant finite \(C\), which is enough to get (3.53).

**Corollary 3.8** \(Q\)-a.s.,

\[
p(\beta) = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^n Q_{\mathcal{G}^t-1} \ln \mu_{t-1}[e^{\beta \eta(t,\omega)}]
\]

Since \(p_n = (1/n)\sum_{t=1}^n \ln(Z_t/Z_{t-1})\), and since \(p_n\) converge in \(L^1\) to a deterministic limit, the equalities

\[
p = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^n Q\left[\ln \mu_{t-1}[e^{\beta \eta(t,\omega)}]\right] = \text{a.s.} \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^n \ln \mu_{t-1}[e^{\beta \eta(t,\omega)}]
\]

were already known.

### 3.4 The diffusive regime

All through this section we assume \(d \geq 3\). We start by giving more results in the \(L^2\) region. The first one concerns the point-to-point partition function

\[
W_n^x(y) = P^x \left[\exp\{\beta H_n(\omega) - n\lambda\}\big|\omega_n = y\right]
\]

(3.60)

which has mean 1. As usual, we shorten \(W_n(y) = W_n^0(y)\).

**Theorem 3.9** *(Local limit theorem [47], [51]) Assume (3.17). Then,

\[
W_n(x) = W_\infty \times W_\infty \circ \theta_{n,x}^- + R_{n,x}
\]

(3.61)

where \(\theta_{n,x}^-\) is given by \(\theta_{n,x}^-(\eta(\cdot, \cdot)) : (u, y) \mapsto \eta(n - u, x + y)\), and the error term \(R_{n,x} \to 0\) in \(L^1\) uniformly in \(x : |x| \leq An^{1/2}\).

This result means that the polymer, at time \(n\) large, is distributed as the gaussian measure up to a factor which describe the environment seen from the final point. Essentially, it only
feels the environment in the neighborhood of the starting and ending points, and behave like
gaussian in between. It is quite instructive to make a heuristic computation:

\[
I_n = \sum_x \mu_n(\omega_n = x)^2 \\
\approx \sum_x [W_\infty \circ \theta_n^{-1}]^2 P[\omega_n = x]^2 \\
\approx Q[W_n^2] \times \sum_x P[\omega_n = x]^2 \\
= \mathcal{O}(n^{-d/2})
\] (3.62)

by arguing successively (3.61), and the ergodic theorem. At a rigorous level, only a slower
polynomial decay has been so far achieved for \(I_n ((1.17) in [12]) \). In view of this heuristic
computation, a natural question is whether \(\lim_n QW_n^2 = 1 \) implies that \(\sum_n I_n = \infty \) (and
therefore \(W_\infty = 0 \)) ? The answer is no, as we have seen in Remark 3.2.

In the \(L^2\) region, we can also consider the fluctuations of extensive thermodynamic quantities
other than the partition function: we show that these are typically of order 1, like \(\ln Z_n\) itself.

**Theorem 3.10** Assume \(d \geq 3\) and (3.17), and consider the two following variables: the energy
averaged over the path and the entropy of \(\mu_n\) with respect to \(P\), both shifted by a “centering”,

\[
\mu_n[H_n] - n\lambda'(\beta), \quad \mu_n^\beta[\ln(d\mu_n^\beta/dP)] - n[\beta\lambda'(\beta) - \lambda(\beta)].
\]

Both variables converges \(Q\)-a.s. to a finite random variable as \(n \to \infty\).

The proof [16] uses analytic functions arguments. The martingales \(W_n(\beta)\) are analytic in \(\beta\),
and the a.s. convergence as \(n \to \infty\) is shown, in the \(L^2\)-region, to be a convergence in the
sense of analytic functions. The crucial estimate is a bound on the second moment of some
complex random variable, this explains why we do assume (3.17).

It is natural to expect that diffusive behavior takes place in the whole weak disorder region,
not only under the stronger assumption (3.17).

**Theorem 3.11** [16] Assume \(d \geq 3\) and weak disorder (3.14). Then, for all bounded continu-
ous function \(F\) on the path space,

\[
\lim_n \mu_n[F(\omega(n))] = \mathbf{E}F(B)
\]

in probability, where \(\omega(n)\) is the rescaled path defined by \(\omega(n) = (\omega_{nt}/\sqrt{n})\) \(t \geq 0\) and \(B\) is the
Brownian motion with diffusion matrix \(d^{-1/2}I_d\). In particular, this holds for all \(\beta \in [0, \beta_c]\).

Incidently, the statement shows that the scaling relation between exponents does hold in the
full weak disorder region, with \(\xi = 1/2\) and \(\chi = 0\).

In the proof of theorem 3.11 convergence of the series \(\sum I_n\) is used as a main technical quan-
titative ingredient.


3.5 Localization and delocalization

We want to characterize the following phenomenon which can be observed experimentally or numerically: For large $\beta$ the polymer concentrates around the $n$-geodesics, i.e. the maximizers of $H_n$. For instance we could try to study $\mu_n(\omega \in G_n)$ for $G_n$ a neighborhood of the set of the $n$-geodesics. A difficulty is that little is known on the geodesics. For the case $d = 1$ the reader can refer to Part ?? of [40]. We will also reduce our ambition in considering only the ending point of the path. A simpler quantity is the probability of the favourite site for the polymer at time $n$,

$$J_n = \max_{x \in \mathbb{Z}^d} \mu_{n-1} \{ \omega_n = x \} .$$

Indeed, $J_n$ is small when the measure is spread out – for instance if $\beta = 0$, $J_n = \mathcal{O}(n^{-d/2})$ –, but $J_n$ should be much larger when $\mu_n$ concentrates on a small number of paths ($J_n \leq 1$). The advantage is that we don’t need to know where is (are) located the favourite point(s)!


The shift in the time index is harmless up to a constant factor, we could have taken in the definition on $I_n$ the maximum of $\mu_{n-1} \{ \omega_{n-1} = x \}$ without changing its essence, but the present one is more natural.

Hence the following definition from [9], [12], is quite natural:

**Definition 3.1** We say that the polymer is **localized** if

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} J_t > 0, \quad Q\text{-a.s.}$$

and that the polymer is **delocalized** if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} J_t = 0 \quad Q\text{-a.s.}$$

Roughly, delocalization and localization correspond to $J_n$ vanishing or not as $n \to \infty$. The following result shows that necessarily one of the two cases happens. But it is all the more a criterion for localization and delocalization.

**Theorem 3.12** Let $\beta \neq 0$. The polymer is

- localized if and only if $p < \lambda$,
- delocalized if and only if $p = \lambda$.

\[ \square \]

Observe first that, since $I_n = \sum_{x} \mu_{n-1} \{ \omega_n = x \}^2$,

$$J_n^2 \leq I_n \leq J_n.$$  

(3.66)

Theorem 3.12 clearly follows from Theorem 3.2, (3.44) and (3.66).

At this point we recall well known facts [48] for the simple random walk, i.e. the behavior of $I_n$ and $J_n$ in the case $\beta = 0$:

$$\max_{x \in \mathbb{Z}^d} P\{ \omega_n = x \} = \mathcal{O}(n^{-d/2}) ,$$

$$P^{\otimes 2}\{ \omega_n = \bar{\omega} \} = \mathcal{O}(n^{-d/2}) ,$$

(3.67)  

(3.68)
as $n \not\to \infty$. The decay rate $n^{-d/2}$ in (3.67) can be understood as the position of $\omega_n$ being roughly uniformly distributed over the euclidean ball in $\mathbb{Z}^d$ with radius const. $\times \sqrt{n}$.

For $\beta \not= 0$ but in the weak disorder region, we first note from the convergence of $\sum I_n$ that $J_n \to 0$. In this region, can still prove (3.67) in some specific models – see e.g. (5.91) – or at a heuristic level – see e.g.(3.62)., but, in general, only in a weaker form with a smaller exponent – see e.g. (1.17) in [12]. Anyway the picture remains similar, with the position $\omega_n$ of the polymer being widely spread out, or “delocalized”.

We summarize this in a table ($\alpha$ is some constant in $(0, d/2)$).

<table>
<thead>
<tr>
<th>order of magnitude</th>
<th>weak disorder</th>
<th>strong disorder</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_n$</td>
<td>$n^{-d/2}$</td>
<td>$n^{-\alpha}$</td>
</tr>
<tr>
<td>$J_n$</td>
<td>$n^{-d/2}$</td>
<td>$n^{-\alpha/2}$</td>
</tr>
</tbody>
</table>

4 Low dimensions

Dimensions $d = 1$ and 2 are special, due to recurrence of the simple random walk – more precisely, due to recurrence of the difference $\omega_n - \tilde{\omega}_m$ under the the product measure $P^\otimes 2$. We start with a simple computation which shows that, in dimension $d = 1$, strong disorder holds for all nonzero $\beta$. Observe that for all $z \in \mathbb{Z}$,

$$
\mu_{t-1}^{\otimes 2}(\omega_t = \tilde{\omega}_t + z) = \sum_x \mu_{t-1}(\omega_t = x)\mu_{t-1}(\omega_t = x + z)
\leq \left( \sum_x \mu_{t-1}(\omega_t = x)^2 \times \sum_x \mu_{t-1}(\omega_t = x + z)^2 \right)^{1/2}
= \mu_{t-1}^{\otimes 2}(\omega_t = \tilde{\omega}_t) = I_t
$$

where the inequality is from Cauchy-Schwarz. Now,

$$
1 = \sum_{z:z=0[\text{mod}2],|z|\leq 2n} \mu_{t-1}^{\otimes 2}(\omega_t = \tilde{\omega}_t + z) \leq (2t + 1)I_t
$$

Hence, $I_t \geq 1/(2t + 1)$ and $\sum_t I_t = \infty$, which shows that $W_\infty = 0$ when $d = 1$.

The next theorem 1.1 is due to Carmona and Hu [9] for gaussian environment, and to [12] in the general case.

Theorem 4.1 Assume $d = 1$ or $d = 2$. For all $\beta \not= 0$, $W_\infty = 0$.

□ We follow the proof of [12], which yields additional information on the decay of $W_n$. It is carried out by estimating fractional moment, and by using the following

Lemma 4.2 Suppose that there exist constants $c \in (0, \infty)$, $\theta \in (0, 1)$ and a sequence $a_n \not\to \infty$ such that

$$
Q[W_n^\theta] \leq c \exp(-a_n), \quad n \geq 1.
$$

Then $Q\{W_\infty = 0\} = 1$. If moreover

$$
\sum_{n \geq 1} \exp(-\delta a_n) < \infty \quad \text{for some } \delta \in (0, 1),
$$

then $Q\{W_\infty = 0\} = 1$. If moreover
then there exists $c > 0$ such that
\[
Q \left\{ \lim_{n \to \infty} - \frac{1}{a_n} \ln Z_n \geq c \right\} = 1. \tag{4.70}
\]
(The first statement follows from Fatou’s lemma and the second from the Borel-Cantelli lemma.)

We will check (4.69) with
\[
a_n = \begin{cases} 
c_1 n^{1/3} & \text{if } d = 1 \\
c_2 \ln n & \text{if } d = 2
\end{cases}
\]  \tag{4.71}

where $c_1, c_2 \in (0, \infty)$ are some constants. In this respect, we first prove an auxiliary lemma.

**Lemma 4.3** For $\theta \in [0, 1]$ and $\Lambda \subset \mathbb{Z}^d$,
\[
Q [W_{n-1}^\theta I_n] \geq \frac{1}{|\Lambda|} Q [Z_{n-1}^\theta] - \frac{2}{|\Lambda|} P(\omega_n \not\in \Lambda)^\theta. \tag{4.72}
\]

**Proof:** Repeating the argument in [38, page 453], we see that
\[
I_n \geq \sum_{z \in \Lambda} \mu_{n-1}(\omega_n = z)^2 \\
\geq \frac{1}{|\Lambda|} \mu_{n-1}(\omega_n \in \Lambda)^2 \\
= \frac{1}{|\Lambda|} (1 - \mu_{n-1}(\omega_n \not\in \Lambda))^2 \\
\geq \frac{1}{|\Lambda|} (1 - 2\mu_{n-1}(\omega_n \not\in \Lambda)) \\
\geq \frac{1}{|\Lambda|} (1 - 2\mu_{n-1}(\omega_n \not\in \Lambda)^\theta).
\]

Note also that
\[
Q [W_{n-1}^\theta \mu_{n-1}(\omega_n \not\in \Lambda)^\theta] \leq Q [W_{n-1}^\theta \mu_{n-1}(\omega_n \not\in \Lambda)^\theta]^	heta \\
= P(\omega_n \not\in \Lambda)^\theta.
\]

We therefore see that
\[
Q [W_{n-1}^\theta I_n] \geq \frac{1}{|\Lambda|} Q [W_{n-1}^\theta] - \frac{2}{|\Lambda|} Q [W_{n-1}^\theta \mu_{n-1}(\omega_n \not\in \Lambda)^\theta] \\
\geq \frac{1}{|\Lambda|} Q [W_{n-1}^\theta] - \frac{2}{|\Lambda|} P(\omega_n \not\in \Lambda)^\theta.
\]

\( \square \)

Assume now that $\theta \in (0, 1)$, and define a function $f : (-1, \infty) \to [0, \infty)$ by
\[
f(u) = 1 + \theta u - (1 + u)^\theta.
\]

It is then clear that there are constants $c_1, c_2 \in (0, \infty)$ such that
\[
\frac{c_1 u^2}{2 + u} \leq f(u) \leq c_2 u^2 \quad \text{for all } u \in (-1, \infty). \tag{4.73}
\]
We see from (3.38), (4.73) and (3.51) that
\[
Q^{\mathcal{G}_{n-1}} \Delta W_n^\theta = W_{n-1}^\theta Q^{\mathcal{G}_{n-1}} ((1 + U_n)^\theta - 1) \\
= -W_{n-1}^\theta Q^{\mathcal{G}_{n-1}} f(U_n) \\
\leq -c_3 W_{n-1}^\theta I_n.
\]

We therefore have by (4.72) that
\[
Q W_n^\theta \leq \left(1 - \frac{c_3}{|\Lambda|}\right) Q [W_{n-1}^\theta] + \frac{2c_3}{|\Lambda|} P(\omega_n \not\in \Lambda)^\theta.
\]

(4.74)

For \(d = 1\), set \(\Lambda = (-n^{2/3}, n^{2/3}]\). Then,
\[
P(\omega_n \not\in \Lambda) = P\left(\frac{\omega_n}{n^{1/2}} \geq n^{1/6}\right) \leq 2 \exp\left(-\frac{n^{1/3}}{2}\right),
\]
so that (4.74) reads,
\[
Q W_n^\theta \leq \left(1 - \frac{c_3}{2n^{2/3}}\right) Q [W_{n-1}^\theta] + 4c_3 \exp\left(-\frac{n^{1/3}}{2}\right).
\]

It is not difficult to conclude (4.69) with \(a_n = c_1 n^{1/3}\) from the above.

For \(d = 2\), we set
\[
\Lambda = (-n^{1/2} \ln^{1/4} n, n^{1/2} \ln^{1/4} n]^2
\]
to get (4.69) with \(a_n = c_2 \sqrt{\ln n}\) in a similar way as above.

Untill recently, the following was only a conjecture:

**Theorem 4.4** [13] Assume the dimension is \(d = 1\).

\[\beta_c = 0\]

Equivalently, for all non degenerate \(Q\) and all \(\beta \neq 0\), \(p(\beta) < \lambda(\beta)\).

What is known in dimension 2 is that for all \(\beta \neq 0\), there is a constant \(c \in (0, \infty)\) such that
\[
\lim_{n \to \infty} I_n \geq c, \quad Q\text{-a.s.}
\]

(4.75)

The original proof is from Carmona and Hu [9] in the gaussian case with computations similar to those in the cavity method in spin-glass models. For a general proof, see [12, Proposition 1.4 (b)].

**5 A Brownian motion model for directed polymers**

The Brownian directed polymer discussed in this section can be thought of as a natural transposition of simple random walk model into continuum setting. It is defined in terms of Brownian motion for the free path and of a Poisson random measure for the environment. In what follows, \(\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)\) denotes the class of Borel sets in \(\mathbb{R}_+ \times \mathbb{R}^d\).
• **The Brownian motion:** Let \( \{\omega_t\}_{t \geq 0}, \{P^x\}_{x \in \mathbb{R}^d} \) denote a \( d \)-dimensional standard Brownian motion. Specifically, we let the measurable space \((\Omega_\omega, \mathcal{F})\) be the path space \(C(\mathbb{R}_+ \to \mathbb{R}^d)\) with the cylindrical \(\sigma\)-field, and \(P^x\) be the Wiener measure on \((\Omega_\omega, \mathcal{F})\) such that \(P^x\{\omega_0 = x\} = 1\).

• **The space-time Poisson random measure:** Let \(\eta\) denote the Poisson random measure on \(\mathbb{R}_+ \times \mathbb{R}^d\) with unit intensity, defined on a probability space \((\mathcal{M}, \mathcal{G}, Q)\). Then, \(\eta\) is an integer valued random measure characterized by the following property: If \(A_1, \ldots, A_n \in B(\mathbb{R}_+ \times \mathbb{R}^d)\) are disjoint and bounded, then

\[
Q\left(\bigcap_{j=1}^n \{\eta(A_j) = k_j\}\right) = \prod_{j=1}^n \exp(-|A_j|) \frac{|A_j|^{k_j}}{k_j!} \quad \text{for } k_1, \ldots, k_n \in \mathbb{N}.
\]

Here, \(|\cdot|\) denotes the Lebesgue measure in \(\mathbb{R}^{1+d}\). For \(t > 0\), it is natural and convenient to introduce

\[
\eta_t(A) = \eta(A \cap ((0,t] \times \mathbb{R}^d)), \quad A \in B(\mathbb{R}_+ \times \mathbb{R}^d)
\]

and the sub \(\sigma\)-field

\[
\mathcal{G}_t = \sigma[\eta_t(A) ; A \in B(\mathbb{R}_+ \times \mathbb{R}^d)].
\]

• **The polymer measure:** We let \(V_t\) denote a “tube” around the graph \(\{(s, \omega_s)\}_{0 \leq s \leq t}\) of the Brownian path,

\[
V_t = V_t(\omega) = \{(s, x) ; s \in (0,t], \ x \in U(\omega_s)\},
\]

where \(U(x) \subset \mathbb{R}^d\) is the closed ball with the unit volume, centered at \(x \in \mathbb{R}^d\). For any \(t > 0\) and \(x \in \mathbb{R}^d\), define a probability measure \(\mu_t^x\) on the path space \((\Omega, \mathcal{F})\)

\[
\mu_t^x(d\omega) = \frac{\exp(\beta \eta(V_t))}{Z_t^x} P^x(d\omega),
\]

where \(\beta \in \mathbb{R}\) is a parameter and

\[
Z_t^x = P^x[\exp(\beta \eta(V_t))].
\]

is the normalizing constant (the partition function). Note that \(\mu_t^x\) and \(Z_t^x\) contain \(\eta \in \mathcal{M}\) as a parameter and hence that they are random objects on the probability space \((\mathcal{M}, \mathcal{G}, Q)\). We will denote by \(P, \mu_t, Z_t, \ldots\), the quantities \(P^x, \mu_t^x, Z_t^x, \ldots\) with \(x = 0\).

Under the measure \(\mu_t^x\), the graph \(\{(s, \omega_s)\}_{0 \leq s \leq t}\) may be interpreted as a polymer chain living in the \((1+d)\)-dimensional space, constrained to stretch in the direction of the first coordinate \((t\text{-axis})\). At the heuristic level, the polymer measure is governed by the formal Hamiltonian

\[
\beta \mathcal{H}_t^x(\omega) = \frac{1}{2} \int_0^t |\dot{\omega}_s|^2 ds - \beta \#\{\text{Poisson points seen by } \omega \text{ on } (0,T)\}
\]

on the path space. We use the obvious definition: \((t, x)\) is seen by \(\omega\) if \(x \in U(\omega_t)\). The path \(\omega\) is attracted to Poisson points when \(\beta > 0\), and repelled by them when \(\beta < 0\). The sets \(\{s\} \times U(x)\) with \((s, x)\) a point of the Poisson field \(\eta\), appear as “rewards” in the first case, and “soft obstacles” in the second one. Note that the obstacles stretches in the transverse direction \((x\text{-hyperplane})\): This is a key technical point, allowing a simple use of stochastic calculus with respect to the Poisson field.

\[\text{\footnotesize{[2] it will be convenient to take for } } \Omega_\eta = \mathcal{M} \text{ the set of point measure on } \mathbb{R}_+ \times \mathbb{R}^d\]
Let us finish the definition of the model with some remarks on the notation we use. An important parameter is
\[ \lambda = \lambda(\beta) = e^\beta - 1 \in (-1, \infty), \] (5.81)
which is in fact the logarithmic moment generating function of a mean-one Poisson distribution.

When we want to stress the dependence of \( \lambda(\beta) \) on \( \beta \in \mathbb{R} \), we will use the notation \( \lambda(\beta) \). But otherwise, we will simply write \( \lambda \). The following short hand notation will frequently be used in the sequel:

\[ \zeta_t = \zeta_t(\omega, \eta) = \exp \left( \beta \eta(V_t) \right), \]
\[ \chi_{t,x} = \chi_{t,x}(\omega) = 1 \{ x \in U(\omega_t) \}. \] (5.82)

It is useful to note that
\[ \int_{\mathbb{R}^d} \chi_{t,x} \, dx = 1, \quad t \geq 0. \] (5.84)

Many arguments and results seen in the discrete setting have a counterpart here, see [14, 15].

For any fixed path \( \omega \), the process \( \{ \eta(V_t) \}_{t \geq 0} \) has independent, Poissonian increments, hence it is itself a standard Poisson process on the half-line, and \( \{ \exp(\beta \eta(V_t) - \lambda t) \}_{t \geq 0} \) is its exponential martingale. Therefore, the normalized partition function
\[ W_t = e^{-\lambda t} Z_t, \quad t \geq 0 \] (5.85)
is itself a mean-one, right-continuous and left-limited, positive martingale on \( (\mathcal{M}, \mathcal{G}, \mathbb{Q}) \), with respect to the filtration \( (\mathcal{G}_t)_{t \geq 0} \) defined by (5.77). In particular, the limit
\[ W_\infty \overset{\text{def}}{=} \lim_{t \to \infty} W_t. \] (5.86)
exists \( \mathbb{Q} \)-a.s., and as in the discrete case, we only have the two contrasting situations
\[ Q\{ W_\infty = 0 \} = 1, \]
or
\[ Q\{ W_\infty > 0 \} = 1, \]
as in (3.15) and (3.14). We define the former case (3.15) as the strong disorder phase, and the latter case (3.14) as the weak disorder phase.

**The replica overlap:** On the product space \( (\Omega^2, \mathcal{F}^\otimes 2) \), we consider the probability measure \( \mu_t^{\otimes 2} = \mu_t^{\otimes 2}(d\omega, d\bar{\omega}) \), that we will view as the distribution of the couple \( (\omega, \bar{\omega}) \) with \( \bar{\omega} \) an independent copy of \( \omega \) with law \( \mu_t \). We introduce a random variable \( I_t, t \geq 0 \), given by
\[ I_t = \mu_t^{\otimes 2} [U(\omega_t) \cap U(\bar{\omega}_t)]. \] (5.87)
Here we have used the notation \( | \cdot | \) for the Lebesgue measure on \( \mathbb{R}^d \). Note that for some constant \( c_1 = c_1(d) \in (0, 1) \),
\[ c_1 \sup_{y \in \mathbb{R}^d} \mu_t [\omega_t \in U(y)]^2 \leq I_t \leq \sup_{y \in \mathbb{R}^d} \mu_t [\omega_t \in U(y)]. \] (5.88)
The maximum appearing in the above bounds should be viewed as the probability of the favorite “location” for \( \omega_t \), under the polymer measure \( \mu_t \). It is shown in [14, Theorem 2.3.2] that, for \( \beta \neq 0 \),
\[ \{ W_\infty > 0 \} = \left\{ \int_0^\infty I_s \, ds < \infty \right\}, \quad \mathbb{Q}\text{-a.s.}, \] (5.89)
and that, if this set has probability 0, there exist \( c_1, c_2 \in (0, \infty) \) such that

\[
 c_1 \int_0^t I_s ds \leq \lambda t - \ln Z_t \leq c_2 \int_0^t I_s ds \quad \text{for large } t, \ Q\text{-a.s.} \tag{5.90}
\]

Many tools are available in this model, leading to nicer formulas. For instance, we obtain in [15, Theorem 2.1.1 (b)] for \( d \geq 3 \) and for \( \beta \) small enough so that \( \sup_t QW_t^2 < \infty \),

\[
 I_t = \mathcal{O}(t^{-d/2}) \quad \text{in } Q - \text{probability} \tag{5.91}
\]
as \( t \to \infty \), satisfactory a result since it is the order of magnitude in the case \( \beta = 0 \).

### 5.1 Connection to the Kardar-Parisi-Zhang equation

Another strong motivation for the present model is its relation to some stochastic partial differential equations. To describe the connection, it is necessary to relativize the partition function, by specifying the ending point of the Brownian motion at time \( t \). For \( 0 \leq s < t \), let \( P^{x-t}_{s-t} \) be the distribution of the Brownian bridge starting at point \( x \) at time \( s \) and ending at \( y \) at time \( t \). Define

\[
 Z^x_t(y) = g_t(y - x) P^{x-y}_{0-t} \left[ \exp(\beta \eta(V_t)) \right],
\]
with \( g_t(x) = (2\pi t)^{-d/2} \exp\{-|x|^2/2t\} \) the Gaussian density. Then, by definition of the Brownian bridge,

\[
 Z_t^x = \int_{\mathbb{R}^d} Z_t^x(y) dy.
\]

Similar to the Feynman-Kac formula, we will show the following stochastic heat equation (SHE) with multiplicative noise in a weak sense,

\[
dZ^x_t(y) = \frac{1}{2} \Delta_y Z^x_t(y) dt + \lambda Z^x_t(y) \eta(dt \times U(y)), \quad t \geq 0, x, y \in \mathbb{R}^d, \tag{5.93}
\]
where \( dZ^x_t(y) \) denotes the time differential and \( \Delta_y = (\frac{\partial}{\partial y_1})^2 + \cdots + (\frac{\partial}{\partial y_d})^2 \) the Laplacian operator. (SHE) will be properly formulated and be proved below. In the literature, this equation has been extensively considered in the case of a Gaussian driving noise, instead of the Poisson process \( \eta \) here. Although we are able to prove (5.93) only in the weak sense, let us now pretend that (5.93) is true for all \( y \in \mathbb{R}^d \). We would then see from Itô’s formula that the function \( h_t(y) = \ln Z^x_t(y) \) solves the Kardar-Parisi-Zhang equation (KPZ):

\[
dh_t(y) = \frac{1}{2} \left( \Delta h_t(y) + |\nabla h_t(y)|^2 \right) dt + \beta \eta(dt \times U(y)). \tag{5.94}
\]

We comment on this equation. The function \( h_t(y) \) can be interpreted as the height at time \( t \) and location \( y \) of a \( d \)-dimensional growing surface. Smoothing by diffusivity occurs from the first summand in the right-hand side. From the second term, the largest gradient the largest increase of the surface. From the last term a brick is added to the surface at random locations: for each point \((t, x)\) of the Poisson process, a brick is dropped vertically at time \( t \), with horizontal shape the ball \( U(x) \) centered at \( x \) and height \( \beta \). KPZ equation is a phenomenological for phenomena with normal growth and ballistic deposition.
We observe that, since \( h \) has jumps in the space variable \( y \), the non-linearity makes the precise meaning of this equation somewhat knotty. We will not address this equation in the present paper, but we make a few comments. To describe the large scale behavior of growing interface models this equation was introduced in [33], but driven by a space-time Gaussian white noise, making the equation ill posed. In fact, it is also used to describe scaling limit of large class of growth models – the so-called KPZ universality class. The scaling relation between exponents (6.96) is obtained from this equation (5.94) with Gaussian white noise.

See [36] for a detailed review of kinetic roughening of growth models within the physics literature, in particular to Section 5 for the status of this equation. In dimension \( d = 1 \), Bertini and Giacomin [4] proved that the KPZ equation comes as the limit of renormalized fluctuations for two microscopic models: the weakly asymmetric exclusion process, and the related Solid-On-Solid interface model.

\[ \square \] We prove that \( Z^x_t(y) \) is a weak solution of (5.93) in the following sense. For every compactly supported test function \( \Psi \in C^2(\mathbb{R}^d) \), \( Q \)-a.s.,

\[
\int_{\mathbb{R}^d} Z^x_t(y)\Psi(y)dy = \Psi(x) + \frac{1}{2} \int_0^t ds \int_{\mathbb{R}^d} Z^x_s(y)\Delta \Psi(y)dy + \lambda \int_{\mathbb{R}^d} \Psi(y)dy \int_{(0,t]} Z^x_s'(y)\eta(ds \times U(y))
\]

for all \( t \geq 0, x, y \in \mathbb{R}^d \). Indeed, observe that \( \int_{\mathbb{R}^d} Z^x_t(y)\Psi(y)dy = P^{x}[\zeta_t\Psi(\omega_t)] \). We obtain by applying first Itô’s formula to \( \zeta_t\Psi(\omega_t) \), and then by taking \( P^x \)-expectation that \( Q \)-a.s.,

\[
\int_{\mathbb{R}^d} Z^x_t(y)\Psi(y)dy = \Psi(x) + \frac{1}{2} \int_0^t ds \int_{\mathbb{R}^d} P^{x}[\zeta_s\Delta \Psi(\omega_s)]ds + \lambda \int_{(0,t] \times \mathbb{R}^d} \eta(dsdz) P^{x}[\zeta_{s-}\chi_{s,z}\Psi(\omega_s)]
\]

\[
= \Psi(x) + \frac{1}{2} \int_0^t ds \int_{\mathbb{R}^d} Z^x_s(y)\Delta \Psi(y)dy + \lambda \int_{(0,t]} \Psi(y)dy \int_{(0,t]} \eta(ds \times U(y)) Z^x_{s-}(y),
\]

which proves our claim. Here, we have used Fubini’s theorem on the last line. But this can easily be justified. For example,

\[
Q \int_{(0,t] \times \mathbb{R}^d} \eta(dsdz) P^{x}[\zeta_{s-}\chi_{s,z}\Psi(\omega_s)] = \int_0^t Q[\zeta_{s-}] P^{x}[\Psi(\omega_s)]ds
\]

\[
= \int_0^t e^{\lambda s} P^{x}[\Psi(\omega_s)]ds < \infty.
\]

\[ \blacksquare \]

6 Exponents

We consider a polymer model, which may be discrete or continuous. The \textbf{wandering exponent} \( \xi(d) \) describes the long-time behavior of the transversal fluctuations of the path, and the \textbf{roughness exponent} \( \chi(d) \) describes the (“longitudinal”) fluctuations of the free energy. Their definitions are roughly

\[
|\omega_t| \approx t^{\xi(d)} \quad \text{and} \quad \ln Z_t - Q[\ln Z_t] \approx t^{\chi(d)} \quad \text{as} \ t \to \infty.
\]  
(6.95)
Note that they depend, in principle, on the model, and we should write $\xi(d, \beta, Q, P)$. There are various ways to define rigorously these exponents, e.g. (0.6) and (0.10-11) in [53], (2.4) and (2.6-7-8) in [44], and the equivalence between these specific definitions are often non trivial. Here, we do not go into such subtleties and take (6.95) as “definitions”. The polymer is said to be diffusive if $\xi(d) = 1/2$ and super-diffusive if $\xi(d) > 1/2$.

These exponents are investigated in the context of various other models and in a large number of papers. In particular, it is conjectured in physics literature that the scaling identity holds in any dimension,
\begin{equation}
\chi(d) = 2\xi(d) - 1, \quad d \geq 1,
\end{equation}
and that the polymer is super-diffusive in dimension one;
\begin{equation}
\chi(1) = 1/3, \quad \xi(1) = 2/3 \quad \text{for } \beta \neq 0
\end{equation}

Also, universality is expected, i.e., their value should not be sensitive to the model but only on the dimension $d$ and on the fact that the disorder is weak or strong. See, e.g., [28],[23, (3.4),(5.11),(5.12)], [36, (5.19),(5.28)].

There is a wonderful story indicating the validity of the theory. We recall from section 1 that directed polymers in random environment for $d = 1$ can be physically realized as fracture lines of paper sheets. The strain is applied by a machine on opposite sides of a rectangular sheet. It is slowly increased, and the tear will initiate at a small notch made on a third side of the sheet. We explained that we can view the line of tear as a realization of the polymer measure. The experiment was realized by physicists [34], with many runs on different types of paper. Measuring the transverse fluctuations they ended with an estimation $\xi = 0.67 \pm 0.05$ for the exponent in agreement with the conjectured value $\xi(1) = 2/3 = 0.666\ldots$. The estimation did not change much between different types of paper, as predicted by the universality conjecture.

For dimension $d \geq 2$, a number of estimates and conjectures have been given for the value of the exponents. For instance Kim and Kosterlitz conjectured that $\xi(d) = \frac{d+3}{2(d+2)}$ and hence $\chi(d) = \frac{1}{d+2}$. A “proof” of this conjecture was even given by other physicists, but it was later disproved by the following numerics. Numerical studies yield estimates
\begin{align}
\xi(2) & = 0.624 \pm 0.001, \\
\xi(3) & = 0.585, \quad \chi(3) = 0.180 \pm 0.002
\end{align}

which is more or less in accordance with the scaling relation (6.96).

On the other hand, other rigorous results prove (or suggest) for example that
\begin{align}
\chi(d) & \leq 1/2 \text{ for all } d \geq 1, \\
\chi(d) & \geq 2\xi(d) - 1 \text{ for all } d \geq 1, \\
\xi(d) & \leq 3/4 \text{ for all } d \geq 1, \\
\xi(1) & > 1/2 \text{ if } \beta \neq 0, \\
\chi(1) & > 0 \text{ if } \beta \neq 0,
\end{align}

cf. Remark 6.1 below.

We start illustrating them through the results in the present paper. Our central limit Theorem 3.11 in the discrete case at weak disorder (hence, $d \geq 3$ necessarily) shows that (6.96) is trivially
satisfied with $\xi(d) = 1/2, \chi(d) = 0$. The interesting point is that it covers the full weak disorder region. For the Brownian directed polymer model, the central limit theorem so far is proved only in the region where the martingale $W_t$ is bounded in $L^2$ [15]. On the other hand, (6.118) or Lemma 2.3 in the discrete case implies (6.100) and Theorem 6.4 implies (6.104) with a lower bound $\chi(1) \geq 1/8$ for $\beta \neq 0$. If we insert $\chi(1) \geq 1/8$ in (6.96), we get the supersensitivity (6.103) with a lower bound $\xi(1) \geq 9/16$ for $\beta \neq 0$. Finally, the scaling inequality (6.101) follows from theorem 6.1.

Remark 6.1 We discuss the litterature.

- **Specific models with exact computations:** Recently Johansson, together with Baik and Deift, computed the exact values of exponents for oriented last passage percolation in dimension $d = 1$ and for specific distributions for $\eta$. In such cases, the distribution of passage times relates to that of the largest eigenvalue of some random matrix, which asymptotics can be studied using orthogonal polynomials. Specifically, in dimension $d = 1$ and for exponential and geometric distributions, it is proven in [30] that $\chi(1) = 1/3$, together with the Tracy-Widom law for limit fluctuations. Also, for a one-dimensional Poissonized of the longest increasing subsequence in a random permutation, $\chi(1, \infty) = 1/3$ is obtained in [3] together with the Tracy-Widom limit, though $\xi(1, \infty) = 2/3$ is proved in [31].

- **Last passage percolation with general passage time distribution:** For non oriented first passage percolation in 2 dimension – corresponding to $d = 1$ in our picture – , Newman [40] proved the existence of an infinite uni-geodesics with a given direction, and proved other results supporting the non-existence of doubly-infinite geodesics. In [37], soft versions of $\chi(1) \geq 1/5, \xi(1) \geq 3/5$ are obtained in this framework, under the so-called curvature assumption: the limit shape of rescaled set of points reachable in time smaller than $n$ (for first passage percolation) is a convex set, which is assumed to have a positive curvature. In Euclidean percolation is a continuous version of first passage percolation [27], where we are given the realisation $\Gamma$ of a standard Poisson field in $\mathbb{R}^{d+1}$ and a parameter $\alpha > 1$. The passage time between 0 and $x \in \mathbb{R}^{d+1}$ is the smallest value of $\sum_{k=1}^{n+1} |q_k - q_{k-1}|^\alpha$ over all sequences $q$ with $q_0 = 0, q_{n+1} = x$, and $q_i \in \Gamma$ for $i = 1, \ldots, n$. By rotational symmetry, the limit shape is a Euclidean ball, so it has a curvature! For oriented last passage percolation in 2 dimension ($d = 1$) with general passage time distribution, the passage time from the origin for being at location $-n + n^a$ at time $n$ ($a > 0$ small enough) has limiting fluctuations given by the Tracy-Widom law [6].

- **General directed polymers:** Piza [44] studies critical exponents for the simple random walk model for directed polymers. He proves various relations between $\chi(d)$ and $\xi(d)$ including an analogue of (6.123). Under the natural counterpart of the curvature assumption (page 589, “Definition” in that paper), he obtains the bounds $\xi(d) \leq 3/4$ and $\chi(1) \geq 1/8$. It is difficult

\[ \chi(d) := \frac{1}{2} \limsup_t \ln \text{Var}_Q(\ln Z_t)/\ln t = \frac{1}{2} \limsup_t \ln Q[\ln Z_t - Q \ln Z_t]^2)/\ln t \]

but not

\[ \chi(d) := \frac{1}{2} \limsup_t \ln Q[\ln Z_t - pt]^2)/\ln t \]
to check this assumption for last passage percolation since the shape function is not explicit in
general. For directed polymers, I don’t know any example where when the pressure function
can be computed explicitly. However, the large deviation principle (6.107)-(6.108) below
with \( \xi = 1 \), means that the curvature assumption is satisfied in our particular model: More
precisely, the minimizer \( \theta = 0 \) of the (quadratic) rate function \( I \) in (6.107) –corresponding
to the direction of the diagonal in Piza’s framework–, is a “direction of curvature” in the sense
of [44]. See remark 6.3 for a conjecture. Moreover, it is shown that for \( d = 1 \) and \( \beta \neq 0 \),
\[
\text{Var}_Q(\ln Z_n) \geq c \ln n, \quad n = 1, 2, \ldots \tag{6.105}
\]
which is much weaker than (6.125) for our Brownian polymer.
Petermann [43] proves for the Gaussian random walk model of directed polymers (in a Gaus-
sian medium) that \( \xi(1) = 5 \), while O. Mejane [39] proves \( \xi(d) = 3/4 \) for all \( d \geq 1 \).

• Wüthrich studies critical exponents for crossing Brownian motion in a soft Poissonian po-
tential in \( \mathbb{R}^{d+1} \) [53, 54, 55]. There he obtains \( \xi(d) \leq 3/4 \), \( \xi(1) \geq 3/5 \), \( \chi(1) \geq 1/8 \) and various
other relations between \( \chi(d) \) and \( \xi(d) \) including an analogue of (6.123). His techniques depend
quite heavily on the spatial invariance under rotation, which makes very precise information
on the quenched Lyapunov exponent available [49, Chapter5].

6.1 Large deviation for the Brownian polymer

We have the following large deviation principle for the transversal fluctuation of the Brownian
polymer.

**Theorem 6.1** Let \( t_n \) be a positive sequence tending to infinity as \( n \to \infty \), let \( \chi \geq 0 \) be such that\(^4\)
\[
\sum_{n \geq 1} Q (|\ln Z_{t_n} - Q[\ln Z_{t_n}]| > t_n^\chi) < \infty \tag{6.106}
\]
and let \( \xi \) be a number with
\[
\xi > (1 + \chi)/2 .
\]
Then,

(a) The large deviation principle for \( \mu_{t_n} \{ t_n^{-\xi} \omega_{t_n} \in \cdot \} \), \( n / \sim \infty \) holds Q-a.s., with the rate
function
\[
I(u) = |u|^2/2, \quad u \in \mathbb{R}^d
\]
and the speed \( t_n^{2\xi-1} \): There exists an event \( \mathcal{M}_\xi \) with \( Q(\mathcal{M}_\xi) = 1 \) such that, for any
\( \eta \in \mathcal{M}_\xi \) and for any Borel set \( B \subset \mathbb{R}^d \),
\[
- \inf_{\overline{B}} I-o(1) \leq t_n^{(2\xi-1)} \ln \mu_{t_n} \{ t_n^{-\xi} \omega_{t_n} \in B \} \leq - \inf_{\overline{B}} I+o(1) \tag{6.107}, \quad \text{as } n \not\to \infty.
\]
As a consequence, for any \( \varepsilon > 0 \),
\[
\lim_{n \not\to \infty} -t_n^{(2\xi-1)} \ln \mu_{t_n} \{ |\omega_{t_n}| \geq \varepsilon t_n^\xi \} = \varepsilon^2/2, \quad Q\text{-a.s.} \tag{6.108}
\]

\(^4\)Warning: \( \chi \) is not a short notation for the exponent \( \chi(d) \), but a real number. Same remark for \( \xi \).
(b) Assume that \( \lim_{n \to \infty} (t_n^\chi + t_n^{2\chi-1}) / \ln n = \infty \). Then, for \( d \geq 1 \) and \( \beta \in \mathbb{R} \), (6.106) holds true with any \( 1/2 < \chi \) and hence (6.107) and (6.108) hold for all \( \xi > 3/4 \).

(c) In the particular case \( \xi = 1 \), then (6.107) and (6.108) hold true without taking the subsequence, i.e., replacing \( t_n \) and \( n \to \infty \) in these statements by \( t \) and \( t \to \infty \), respectively.

Remark 6.2 In terms of critical exponents, Theorem 6.1 suggests (and shows, with an appropriate definition of the exponents) that

\[ \chi(d) \geq 2\xi(d) - 1 \, , \quad \xi(d) \leq 3/4 \, , \]

i.e. (6.102) and (6.101). If we combine the statements (6.123) and (6.102), we then get \( \chi(1) \geq 1/8 \). If we now insert this into the conjecture (6.96), we then obtain \( \xi(1) \geq 9/16 > 1/2 \), i.e., the polymer is super-diffusive in dimension \( d = 1 \), for all non-zero \( \beta \).

Remark 6.3 For the simple random walk model, Carmona and Hu proved in [10] the a.s. large deviation principle for the \( \omega_n \) under \( \mu_n \), if the environment is gaussian. In fact, they prove that \( \mu_n(\omega_n = x_n) = \exp\{nI(\theta) + o(n)\} \) for all sequence of attainable \( x_n \) with \( x_n/n \to \theta \). The argument is based on subadditivity. They show that \( I \) is deterministic, convex, symmetric. Unfortunately no information is obtained on the behavior at \( 0 \) of the rate function. If for some constant \( c > 0 \)

\[ I(u) \geq c|u|^2 \, , \quad |u| \leq 1 \, , \quad (6.109) \]

then both (6.101) and (6.102) would follows similarly to here. We conjecture that the curvature assumption (6.109) holds under the assumptions for the discrete model. Viewing polymers as interpolation between percolation and simple random walk supports strongly this conjecture.

\( \square \) of Thorem 6.107. We define \( \Lambda_t(\theta) = \ln \mu_t[\exp(\theta \cdot \omega_t)] \, , \quad \theta \in \mathbb{R}^d \). We will prove that, for each \( \xi > (1 + \chi)/2 \), there is an event \( \mathcal{M}_\xi \in \mathcal{G} \) with \( Q(\mathcal{M}_\xi) = 1 \) such that

\[ \lim_{n \to \infty} t_n^{-2(2\chi-1)} \Lambda_t(t_n^{\xi-1} \theta) = |\theta|^2/2 \, , \quad (6.110) \]

for all \( \eta \in \mathcal{M}_\xi \) and \( \theta \in \mathbb{R}^d \) and that

\[ \lim_{n \to \infty} t^{-1} \Lambda_t(\theta) = |\theta|^2/2 \, , \quad (6.111) \]

for all \( \eta \in \mathcal{M}_1 \) and \( \theta \in \mathbb{R}^d \). Then the theorem follows from the Gärtner-Ellis-Baldi theorem [18, page 44, Theorem 2.3.6]. The set \( \mathcal{M}_\xi \) should be independent of the choice of \( \theta \in \mathbb{R}^d \) for the Gärtner-Ellis-Baldi theorem to be applied. However, we first fix \( \theta \in \mathbb{R}^d \) and prove (6.110) and (6.111) on an event \( \mathcal{M}_{\xi,\theta} \) of \( Q \)-measure one, depending on \( \theta \). To do so, we define a transformation \( T_\theta \) on \( \mathbb{R}_+ \times \mathbb{R}^d \) by

\[ T_\theta(t, x) = (t, x + t\theta) \, , \]

and, for \( t \geq 0 \), the function \( \Theta_t : s \mapsto (s \wedge t)\theta \). We will abuse the notation slightly and write \( T_\theta \) also for the induced transformation \( \eta \mapsto \eta \circ (T_\theta)^{-1} \) on \( \mathcal{M} \). By Girsanov’s formula, the process
\[ \overline{\omega} = \omega - \Theta_t \] is a Brownian motion under the probability \( \overline{\mathcal{P}} \), \( \overline{\mathcal{P}}(d\omega) = \exp(\theta \cdot \omega_t - t|\theta|^2/2)P(d\omega) \). We therefore have that

\[
P[\exp\{\beta \eta(V_t(\omega))\} \exp(\theta \cdot \omega_t - t|\theta|^2/2)] = \overline{\mathcal{P}}[\exp(\beta \eta[(\overline{\omega} + \Theta_t)])]
\]

\[
= P[\exp(\beta \eta[T_\theta(V_t)])] = Z_t \circ T_\theta ,
\]

i.e.,

\[
\Lambda_t(\theta) = t|\theta|^2/2 + \ln Z_t \circ T_\theta - \ln Z_t .
\] (6.112)

Observe that \( Z_t \circ T_{-t\theta}^{-1} \) has the same distribution as \( Z_t \) for each fixed \( t > 0 \). Therefore, we see from (6.106) and the Borel-Cantelli lemma that

\[
\lim_{n \to \infty} \frac{1}{t_{\infty}} \left( \ln Z_{tn} \circ T_{-t_{\infty}}^{-1} - Q[\ln Z_{tn}] \right) = 0 \quad Q\text{-a.s.} \] (6.113)

Observe also that the process \((Z_t \circ T_\theta)_{t \geq 0}\) has the same distribution as \((Z_t)_{t \geq 0}\), and hence by (6.120),

\[
\lim_{t \to \infty} \frac{1}{t} (\ln Z_t \circ T_\theta - Q[\ln Z_t]) = 0 \quad Q\text{-a.s.} \] (6.114)

Combining (6.112) and (6.113), we see that for each \( \theta \in \mathbb{R}^d \), there is a set \( \mathcal{M}_{\xi, \theta} \subseteq \mathcal{G} \) with \( Q(\mathcal{M}_{\xi, \theta}) = 1 \) such that (6.110) holds for \( \eta \in \mathcal{M}_{\xi, \theta} \). We now define \( \mathcal{M}_\xi = \cap_{\theta \in \mathbb{Q}^d} \mathcal{M}_{\xi, \theta} \) and prove that the set \( \mathcal{M}_\xi \) has the desired property. This can be done with the help of convex analysis as follows. For any \( \eta \in \mathcal{M}_\xi \), \( \{\theta \mapsto t_n^{-2(\xi-1)} \Lambda_{tn}(t_n^{\xi-1}\theta)\}_{n \geq 1} \) is a sequence of convex (and hence continuous) functions which converges to \( |\theta|^2/2 \) for all \( \theta \in \mathbb{Q}^d \). Then, by [45, page 90, Theorem 10.8], the sequence converges for all \( \theta \in \mathbb{R}^d \) and the convergence is locally uniform, which imply that (6.110) holds for all \( \theta \in \mathbb{R}^d \). This ends the proof of (a).

A similar argument, based on (6.119), (6.112) and (6.114) proves that (6.111) holds for all \( \eta \in \mathcal{M}_1 \) and \( \theta \in \mathbb{R}^d \). We have proved the statement (c).

Finally, (b) follows from (6.112) and (6.119).

### 6.2 Fluctuation results

It not difficult to write the fluctuation of \( \ln Z_t \) as a stochastic integral with respect to the compensated Poisson field \( \tilde{\eta}(ds, dx) = \eta(ds, dx) - dsdx \), cf. [14],

\[
\ln Z_t - Q \ln Z_t = \int_{[0,t] \times \mathbb{R}^d} \tilde{\eta}(ds, dx) Q[\ln \{1 + \lambda \mu_t[\omega_s \in U(x)]\}]|\mathcal{G}_{s-} .
\]

This formula implies the following expression for the variance, yielding estimate for the longitudinal fluctuation.

**Proposition 6.2** Let \( d \geq 1 \) and \( \beta \in \mathbb{R} \) be arbitrary.

(a) With \( Q^{\mathcal{G}_s} \) the conditional expectation under \( Q \) given \( \mathcal{G}_s \), we have

\[
\text{Var}_Q(\ln Z_t) = Q \int_{[0,t] \times \mathbb{R}^d} dsdx \left[ Q^{\mathcal{G}_s} \ln(1 + \lambda \mu_t[\omega_s \ni U(x)]) \right]^2 .
\] (6.115)
As a consequence, the following inequalities hold:

\[
\text{Var}_Q(\ln Z_t) \geq c_2^2 Q \int_{[0,t] \times \mathbb{R}^d} dsdx \left( Q^{U_s} \mu_t \{ U(\omega_s) \ni x \} \right)^2,
\]

(6.116)

\[
\text{Var}_Q(\ln Z_t) \leq c_2^2 Q \int_{[0,t] \times \mathbb{R}^d} dsdx \left( Q^{U_s} \mu_t \{ U(\omega_s) \ni x \} \right)^2,
\]

(6.117)

and

\[
\text{Var}_Q(\ln Z_t) \leq c_+^2 t
\]

(6.118)

where \( c_- = 1 - e^{-|\beta|} \) and \( c_+ = e^{|\beta|} - 1 \).

(b) With \( c = c_+^2 \exp(c_+) \),

\[
Q \{ |\ln Z_t - Q[\ln Z_t]| > u \} \leq 2 \exp\left(-\frac{1}{2}(u \land \frac{u^2}{ct})\right), \quad u \geq 0.
\]

(6.119)

(c) Let \( d, \beta, \varepsilon > 0 \) be arbitrary. Then, as \( t \to \infty \),

\[
\ln Z_t - Q[\ln Z_t] = \mathcal{O}(t^{\frac{1+\varepsilon}{2}}), \quad Q\text{-a.s.}
\]

(6.120)

The formula (6.115) is analogous to (3.2) in [41]. Here, it is obtained rather easily, thanks to the power of stochastic calculus. It indicates that the variance is big when \( \mu_t \) concentrates on small regions.

In [14], the following is derived

**Corollary 6.3** Let \( \beta \neq 0 \). For \( \xi > 0 \) and \( C > 0 \), there exists \( c_1 = c_1(d, C) \in (0, \infty) \) such that

\[
\lim_{t \to \infty} t^{-(1-d\xi)} \text{Var}_Q(\ln Z_t) \geq c_1 \lim_{t \to \infty} \inf_{0 \leq s \leq t} (Q\mu_t \{ |\omega_s| \leq C + Ct^\xi \})^2
\]

(6.121)

\[
\geq c_1 \lim_{t \to \infty} \left( Q\mu_t \{ \sup_{0 \leq s \leq t} |\omega_s| \leq C + Ct^\xi \} \right)^2
\]

(6.122)

Corollary 6.3 suggests that

\[
\chi(d) \geq (1 - d\xi(d))/2,
\]

(6.123)

for non-zero \( \beta \). (Note that this inequality fails to hold for \( \beta = 0 \) and \( d = 1 \).) This inequality is useless for \( d \geq 2 \) since \( \xi(d) \) should be at least 1/2. We show how to use it in dimension \( d = 1 \). Together with \( \xi(d) \leq 3/4 \), it would mean that

\[
\chi(1) \geq 1/8
\]

(6.124)

with an appropriate definition of the exponent. The scaling equality (that we do not know) would imply superdiffusivity.

In [15], the same power divergence of the free energy fluctuation in \( d = 1 \) was obtained:

**Theorem 6.4** If \( d = 1 \) and \( \beta \neq 0 \), then for any \( \varepsilon > 0 \),

\[
\text{Var}_Q(\ln Z_t) \geq ct^{\frac{1}{1-\varepsilon}}, \quad t \geq 0.
\]

(6.125)

where the positive constant \( c \) depends only on \( \beta \) and \( \varepsilon \).
References


