A simple path to asymptotics for the frontier of a branching Brownian motion

Matt Roberts
mattiroberts@gmail.com

McGill University, Montreal (from next Monday)

9th September, 2011
1 Introduction
   • Definition of branching Brownian motion
   • The maximal particle

2 A new proof of Bramson’s theorem
   • The expected number of particles
   • The lower bound
   • The upper bound

3 The almost sure behaviour of $M_t$
Definition of standard branching Brownian motion

- Start with one particle at the origin.
Definition of standard branching Brownian motion

- Start with one particle at the origin.
- This particle moves around in $\mathbb{R}$ like a Brownian motion.
Definition of standard branching Brownian motion

- Start with one particle at the origin.
- This particle moves around in $\mathbb{R}$ like a Brownian motion.
- At a random time, exponentially distributed with parameter 1, the particle dies and is replaced by two new particles.
Definition of standard branching Brownian motion

- Start with one particle at the origin.
- This particle moves around in $\mathbb{R}$ like a Brownian motion.
- At a random time, exponentially distributed with parameter 1, the particle dies and is replaced by two new particles.
- These new particles repeat the random behaviour of their parent, behaving independently given their birth time and position.
Definition of standard branching Brownian motion

- Start with one particle at the origin.
- This particle moves around in \( \mathbb{R} \) like a Brownian motion.
- At a random time, exponentially distributed with parameter 1, the particle dies and is replaced by two new particles.
- These new particles repeat the random behaviour of their parent, behaving independently given their birth time and position.
Definition of standard branching Brownian motion

- Start with one particle at the origin.
- This particle moves around in $\mathbb{R}$ like a Brownian motion.
- At a random time, exponentially distributed with parameter 1, the particle dies and is replaced by two new particles.
- These new particles repeat the random behaviour of their parent, behaving independently given their birth time and position.
Definition of standard branching Brownian motion

- Start with one particle at the origin.
- This particle moves around in $\mathbb{R}$ like a Brownian motion.
- At a random time, exponentially distributed with parameter 1, the particle dies and is replaced by two new particles.
- These new particles repeat the random behaviour of their parent, behaving independently given their birth time and position.
Definition of standard branching Brownian motion

- Start with one particle at the origin.
- This particle moves around in $\mathbb{R}$ like a Brownian motion.
- At a random time, exponentially distributed with parameter 1, the particle dies and is replaced by two new particles.
- These new particles repeat the random behaviour of their parent, behaving independently given their birth time and position.
A simulation of a branching Brownian motion
Some notation

- We want to know about the position of the maximal (top-most) particle in this picture.
Some notation

- We want to know about the position of the maximal (top-most) particle in this picture.

- Let $N(t)$ be the set of particles alive at time $t$. 

Matt Roberts  mattiroberts@gmail.com  (McGill)
Some notation

- We want to know about the position of the maximal (top-most) particle in this picture.

- Let $N(t)$ be the set of particles alive at time $t$.

- For $u \in N(t)$ and $s \leq t$, let $X_u(s)$ be the position of the unique ancestor of $u$ that was alive at time $s$. 
Some notation

- We want to know about the position of the maximal (top-most) particle in this picture.
- Let $N(t)$ be the set of particles alive at time $t$.
- For $u \in N(t)$ and $s \leq t$, let $X_u(s)$ be the position of the unique ancestor of $u$ that was alive at time $s$.
- Define $M_t = \max_{u \in N(t)} X_u(t)$.
Some notation

- We want to know about the position of the maximal (top-most) particle in this picture.
- Let $N(t)$ be the set of particles alive at time $t$.
- For $u \in N(t)$ and $s \leq t$, let $X_u(s)$ be the position of the unique ancestor of $u$ that was alive at time $s$.
- Define $M_t = \max_{u \in N(t)} X_u(t)$.
Two theorems about the distribution of $M_t$

"$M_t$ looks like $\sqrt{2t}$".

Theorem (Kolmogorov, Petrovski, Piscounov (1937))

There exist functions $m(t)$ and $w(y)$ such that

$$P(M_t > m(t) + y) \to w(y) \text{ as } t \to \infty,$$

and

$$m(t) = \sqrt{2t} + o(t).$$

Theorem (Bramson (1978))

$$m(t) = \sqrt{2t} - \frac{3}{2}\sqrt{2\log t} + O(1).$$
Two theorems about the distribution of $M_t$

“$M_t$ looks like $\sqrt{2}t$”.

Theorem (Kolmogorov, Petrovski, Piscounov (1937))

There exist functions $m(t)$ and $w(y)$ such that

$$\mathbb{P}(M_t > m(t) + y) \to w(y) \quad \text{as} \quad t \to \infty,$$

and

$$m(t) = \sqrt{2}t + o(t).$$
Two theorems about the distribution of $M_t$

"$M_t$ looks like $\sqrt{2}t$".

Theorem (Kolmogorov, Petrovski, Piscounov (1937))

There exist functions $m(t)$ and $w(y)$ such that

$$\mathbb{P}(M_t > m(t) + y) \to w(y) \quad \text{as} \quad t \to \infty,$$

and

$$m(t) = \sqrt{2}t + o(t).$$

"$M_t$ looks like $\sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$".
Two theorems about the distribution of $M_t$

"$M_t$ looks like $\sqrt{2}t$".

**Theorem (Kolmogorov, Petrovski, Piscounov (1937))**

There exist functions $m(t)$ and $w(y)$ such that

$$
P(M_t > m(t) + y) \to w(y) \quad \text{as} \quad t \to \infty,
$$

and

$$m(t) = \sqrt{2}t + o(t).$$

"$M_t$ looks like $\sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$".

**Theorem (Bramson (1978))**

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1).$$
The expected number of particles above $\beta t$

Let $\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} + \frac{y}{t}$.
The expected number of particles above $\beta t$

Let $\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} + \frac{v}{t}$.

$$\mathbb{E}[\#\{ u \in N(t) : X_u(t) > \beta t \}] = e^t \mathbb{P}(B_t > \beta t)$$
The expected number of particles above $\beta t$

Let $\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} + \frac{\sqrt{2}}{t}$.

$$\mathbb{E}[\# \{ u \in N(t) : X_u(t) > \beta t \}] = e^t \mathbb{P}(B_t > \beta t) \approx te^{-\sqrt{2}y}.$$
The expected number of particles above $\beta t$

Let $\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} + \frac{\sqrt{2}}{2} \log t$.

$$\mathbb{E}[\#\{u \in N(t) : X_u(t) > \beta t\}] = e^t \mathbb{P}(B_t > \beta t) \approx te^{-\sqrt{2}y}.$$ 

But this is bigger than we hoped...
Let $\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} + \frac{y}{t}$.

$$
\mathbb{E}[\#\{u \in N(t) : X_u(t) > \beta t\}] = e^t \mathbb{P}(B_t > \beta t) \approx te^{-\sqrt{2}y}.
$$

But this is bigger than we hoped... what's going wrong?
Let $\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} + \frac{y}{t}$.

$$\mathbb{E}[\#\{u \in N(t) : X_u(t) > \beta t\}] = e^t \mathbb{P}(B_t > \beta t) \approx te^{-\sqrt{2}y}.$$ But this is bigger than we hoped... what's going wrong?
The expected number of particles above $\beta t$

Let $\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \log t + \frac{y}{t}$.

$$\mathbb{E}[\# \{ u \in N(t) : X_u(t) > \beta t \}] = e^t \mathbb{P}(B_t > \beta t) \approx te^{-\sqrt{2}y}.$$ 

But this is bigger than we hoped... what's going wrong?
The expected number of particles above $\beta t$

Let $\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} + \frac{\gamma}{t}$.

$$\mathbb{E}[\#\{u \in N(t) : X_u(t) > \beta t\}] = e^t \mathbb{P}(B_t > \beta t) \approx te^{-\sqrt{2}\gamma}.$$  

But this is bigger than we hoped... what's going wrong?
Let $\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} + \frac{y}{t}$.

$$\mathbb{E}[\# \{u \in N(t) : X_u(t) > \beta t\}] = e^t \mathbb{P}(B_t > \beta t) \approx te^{-\sqrt{2}y}.$$ 

But this is bigger than we hoped... what's going wrong?
The expected number of particles near $\beta t$ that never go above $\alpha + \beta s$

- Define $H = \# \{ u \in N(t) : X_u(t) \approx \beta t, \ X_u(s) < \alpha + \beta s \ \forall s \leq t \}$. 
The expected number of particles near $\beta t$ that never go above $\alpha + \beta s$

- Define $H = \#\{u \in N(t) : X_u(t) \approx \beta t, \ X_u(s) < \alpha + \beta s \ \forall s \leq t\}$.

- $\mathbb{E}[H] = e^{t} \mathbb{P}(B_t \approx \beta t, \ B_s < \alpha + \beta s \ \forall s \leq t)$. 
The expected number of particles near $\beta t$ that never go above $\alpha + \beta s$

- Define $H = \#\{u \in N(t) : X_u(t) \approx \beta t, \ X_u(s) < \alpha + \beta s \ \forall s \leq t\}$.
- $\mathbb{E}[H] = e^t \mathbb{P}(B_t \approx \beta t, \ B_s < \alpha + \beta s \ \forall s \leq t)$.
- Let $\zeta(s) = \frac{\alpha + \beta s - B_s}{\alpha} e^{\beta B_s - \frac{1}{2} \beta^2 s} \mathbb{1}\{B_r < \alpha + \beta r \ \forall r \leq s\}$. 
The expected number of particles near $\beta t$ that never go above $\alpha + \beta s$

- Define $H = \#\{u \in N(t) : X_u(t) \approx \beta t, \; X_u(s) < \alpha + \beta s \; \forall s \leq t\}$.

- $E[H] = e^t P(B_t \approx \beta t, \; B_s < \alpha + \beta s \; \forall s \leq t)$.

- Let $\zeta(s) = \frac{\alpha + \beta s - B_s}{\alpha} e^{\beta B_s - \frac{1}{2} \beta^2 s} \mathbb{1}_{\{B_r < \alpha + \beta r \; \forall r \leq s\}}$.

- This is a martingale. Define $Q$ by setting

$$\left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = \zeta(t).$$
The expected number of particles near $\beta t$ that never go above $\alpha + \beta s$

- Define $H = \# \{ u \in N(t) : X_u(t) \approx \beta t, \ X_u(s) < \alpha + \beta s \ \forall s \leq t \}$. 

- $\mathbb{E}[H] = e^t \mathbb{P}(B_t \approx \beta t, \ B_s < \alpha + \beta s \ \forall s \leq t)$. 

- Let $\zeta(s) = \frac{\alpha + \beta s - B_s}{\alpha} e^{\beta B_s - \frac{1}{2} \beta^2 s} 1 \{ B_r < \alpha + \beta r \ \forall r \leq s \}$. 

- This is a martingale. Define $Q$ by setting 

$$\frac{dQ}{d\mathbb{P}} \bigg| \mathcal{F}_t = \zeta(t).$$

- Then under $Q$, $(\alpha + \beta s - B_s)_{s \geq 0}$ is a Bessel-3 process.
Using $Q$ to calculate $\mathbb{E}[H]$

$$\mathbb{E}[H] = e^t \mathbb{P}(B_t \approx \beta t, \ B_s < \alpha + \beta s \ \forall s \leq t)$$
Using $Q$ to calculate $\mathbb{E}[H]$

\[
\mathbb{E}[H] = e^t \mathbb{P}(B_t \approx \beta t, \ B_s < \alpha + \beta s \ \forall s \leq t)
\]

\[
= e^t Q \left[ \frac{1}{\zeta(t)} 1 \{B_t \approx \beta t, \ B_s \leq \alpha + \beta s \ \forall s \leq t\} \right]
\]
Using $Q$ to calculate $\mathbb{E}[H]$

$$\mathbb{E}[H] = e^t \mathbb{P}(B_t \approx \beta t, \ B_s < \alpha + \beta s \ \forall s \leq t)$$

$$= e^t Q \left[ \frac{1}{\zeta(t)} \mathbb{1}_{\{B_t \approx \beta t, \ B_s \leq \alpha + \beta s \ \forall s \leq t\}} \right]$$

$$\approx t^{3/2} e^{-\sqrt{2}y} Q(B_t \approx \beta t)$$
Using $Q$ to calculate $\mathbb{E}[H]$

\[ \mathbb{E}[H] = e^t \mathbb{P}(B_t \approx \beta t,\ B_s < \alpha + \beta s \ \forall s \leq t) \]

\[ = e^t Q \left[ \frac{1}{\zeta(t)} 1\{B_t \approx \beta t,\ B_s \leq \alpha + \beta s \ \forall s \leq t\} \right] \]

\[ \approx t^{3/2} e^{-\sqrt{2} y} Q(B_t \approx \beta t) \]

\[ \approx \alpha^2 e^{-\sqrt{2} y}. \]
The second moment method

- We want to use a second moment method for $H$. 

\[ \mathbb{E}[H] = \text{functional of a Bessel process}. \]

The many-to-two lemma says
\[ \mathbb{E}[H^2] = \text{functional of two dependent Bessel processes}. \]

We get
\[ \mathbb{E}[H^2] \leq C \mathbb{E}[H]. \]

Then
\[ \mathbb{P}(H \geq 1) \geq \frac{\mathbb{E}[H]}{\mathbb{E}[H^2]}, \]

which gives us our lower bound.
The second moment method

- We want to use a second moment method for $H$.
- We had $\mathbb{E}[H] = \mathcal{Q}[\text{functional of a Bessel process}]$. 
The second moment method

- We want to use a second moment method for $H$.
- We had $\mathbb{E}[H] = \mathbb{Q}[\text{functional of a Bessel process}]$.
- The many-to-two lemma says
  \[ \mathbb{E}[H^2] = \mathbb{Q}[\text{functional of two dependent Bessel processes}] . \]
The second moment method

We want to use a second moment method for $H$.

We had $\mathbb{E}[H] = \mathbb{Q}[\text{functional of a Bessel process}]$.

The many-to-two lemma says

$$\mathbb{E}[H^2] = \mathbb{Q}[\text{functional of two dependent Bessel processes}]$$

We get $\mathbb{E}[H^2] \leq C\mathbb{E}[H]$.
We want to use a second moment method for $H$.

We had $\mathbb{E}[H] = \mathbb{Q}[\text{functional of a Bessel process}]$.

The many-to-two lemma says

$$\mathbb{E}[H^2] = \mathbb{Q}[\text{functional of two dependent Bessel processes}].$$

We get $\mathbb{E}[H^2] \leq C\mathbb{E}[H]$.

Then $\mathbb{P}(H \geq 1) \geq \frac{\mathbb{E}[H]^2}{\mathbb{E}[H^2]}$, which gives us our lower bound.
The upper bound: a simple trick

Define

\[ B := \{ \exists u \in N(t), \ s \leq t : X_u(s) > \beta s + y \} \]
The upper bound: a simple trick

Define

- \( B := \{ \exists u \in N(t), \ s \leq t : X_u(s) > \beta s + y \} \)

- \( \Gamma := \#\{ u \in N(t) : X_u(t) \geq \beta t, X_u(s) \leq \beta s + y + 1 \ \forall s \leq t \} \).
Define

- \( B := \{ \exists u \in N(t), \ s \leq t : X_u(s) > \beta s + y \} \)

- \( \Gamma := \# \{ u \in N(t) : X_u(t) \geq \beta t, X_u(s) \leq \beta s + y + 1 \ \forall s \leq t \}. \)

\[
P(B) \leq \frac{\mathbb{E}[\Gamma]P(B)}{\mathbb{E}[\Gamma 1_B]}
\]
The upper bound: a simple trick

Define
- \( B := \{\exists u \in N(t), \ s \leq t : X_u(s) > \beta s + y\} \)
- \( \Gamma := \#\{u \in N(t) : X_u(t) \geq \beta t, X_u(s) \leq \beta s + y + 1 \ \forall s \leq t\} \).

\[
\mathbb{P}(B) \leq \frac{\mathbb{E}[\Gamma \mathbb{P}(B)]}{\mathbb{E}[\Gamma 1_B]} = \frac{\mathbb{E}[\Gamma]}{\mathbb{E}[\Gamma | B]}.
\]
The upper bound: a simple trick

Define

- \( B := \{ \exists u \in N(t), \ s \leq t : X_u(s) > \beta s + y \} \)

- \( \Gamma := \#\{ u \in N(t) : X_u(t) \geq \beta t, X_u(s) \leq \beta s + y + 1 \ \forall s \leq t \} \).

\[
P(B) \leq \frac{\mathbb{E}[\Gamma] \mathbb{P}(B)}{\mathbb{E}[\Gamma 1_B]} = \frac{\mathbb{E}[\Gamma]}{\mathbb{E}[\Gamma | B]}.
\]

- \( \mathbb{E}[\Gamma] \approx (y + 1)^4 e^{-\sqrt{2}y} \) (just like \( \mathbb{E}[H] \)).
Estimating $\mathbb{E}[\Gamma | B]$
Estimating $\mathbb{E}[\Gamma | B]$
Estimating $E[\Gamma | B]$
Estimating $\mathbb{E}[\Gamma|B]$ 

- $\mathbb{E}[\Gamma|B] \gtrsim C$
Estimating $\mathbb{E}[\Gamma | B]$

$\mathbb{E}[\Gamma | B] \gtrsim C$

$\mathbb{P}(\exists \text{ someone above } \beta t) \leq \mathbb{P}(\Gamma \geq 1) + \mathbb{P}(B) \leq C'(y + 1)^4 e^{-\sqrt{2}y}$. 
Estimating $\mathbb{E}[\Gamma|B]$

- $\mathbb{E}[\Gamma|B] \gtrsim C$

- $\mathbb{P}(\exists \text{ someone above } \beta t) \leq \mathbb{P}(\Gamma \geq 1) + \mathbb{P}(B) \leq C'(y + 1)^4 e^{-\sqrt{2}y}$.

- This gives us our upper bound.
Hu and Shi’s theorem (for branching Brownian motion)

Theorem (Hu and Shi (2009))

\[
\liminf_{t \to \infty} \frac{M_t - \sqrt{2}t}{\log t} = \frac{-3}{2\sqrt{2}} \quad \text{almost surely,}
\]
Hu and Shi’s theorem (for branching Brownian motion)

**Theorem (Hu and Shi (2009))**

\[
\liminf_{t \to \infty} \frac{M_t - \sqrt{2}t}{\log t} = \frac{-3}{2\sqrt{2}} \quad \text{almost surely, but}
\]

\[
\limsup_{t \to \infty} \frac{M_t - \sqrt{2}t}{\log t} = \frac{-1}{2\sqrt{2}} \quad \text{almost surely.}
\]
The almost sure behaviour of $M_t$

Hu and Shi’s theorem (for branching Brownian motion)

\[ \int_{0}^{t} 1 \{ \exists \text{ someone near upper line at time } s \} \, ds. \]

\[ \sqrt{2t} - \frac{1}{2\sqrt{2}} \log t \]

\[ \sqrt{2t} - \frac{3}{2\sqrt{2}} \log t \]

Never particles here
Sometimes particles here
Not always particles here
Always particles here
The almost sure behaviour of $M_t$

Hu and Shi’s theorem (for branching Brownian motion)

$$I_n = \int_n^{2n} \mathbb{1}\{\exists \text{ someone near upper line at time } s\} \, ds.$$
The end

Thanks for listening!