The law of the total progeny of multitype branching processes.

Loïc Chaumont
Université d’Angers

Joint work with Rongli Liu,
University of Nanjing, Université d’Angers
Introduction

- $\mu$ distribution on $\mathbb{Z}_+$ such that $\sum_{k=0}^{\infty} k \mu(k) \leq 1$, $\mu(1) < 1$.

- $\tau$ branching tree with offspring distribution $\mu$.

- $O(\tau)$ total progeny of $\tau$.

- $u_0, \ldots, u_{O(\tau)-1}$ vertices of $\tau$ ranked in the breadth first search order.

- $k_u(\tau)$ number of children of $u \in \tau$. 
Introduction

The genealogy of any tree $\tau$ is encoded through:

$$X_0 = 0, \quad X_{n+1}(\tau) - X_n(\tau) = k_{u_n}(\tau) - 1, \quad 0 \leq n \leq O(\tau) - 1.$$  

$$(X_n)_{n \geq 0}$$ downward skip free random walk with step distribution:

$$\mathbb{P}(X_1 = i) = \mu(i + 1).$$
Introduction

The law of $O(\tau)$ follows from the identity :

$$O(\tau) = \inf\{n : X_n = -1\}$$

and the theorem :

**Theorem (Ballot theorem)**

Let $T_k = \inf\{n : X_n = -k\}$, then for any $k \geq 1$,

$$\mathbb{P}(T_k = n) = \frac{k}{n} \mathbb{P}(X_n = -k).$$

**Consequence (Dwass, 1969)** :

$$\mathbb{P}(O(\tau) = n) = \frac{1}{n} \mu^*(n-1).$$
Progeny of 2-type branching processes

\( \mu_1 \) and \( \mu_2 \) probabilities on \( \mathbb{Z}_+ \times \mathbb{Z}_+ \).

\( Z_n := (Z_n^{(1)}, Z_n^{(2)}), \ n \geq 0, \) 2-type branching process with progeny law \((\mu_1, \mu_2)\), such that \( Z_0 = (1, 0) \). Assume that

\[
T := \inf\{n : Z_n = 0\} < \infty, \text{ a.s.}
\]

What is the joint law of

\[
O_1 = \sum_{n=0}^{T} Z_n^{(1)} = \text{total number of individuals of type 1 at time } T
\]

\[
O_2 = \sum_{n=0}^{T} Z_n^{(2)} = \text{total number of individuals of type 2 at time } T?
\]
Progeny of 2-type branching processes

Define the mean matrix:

\[ m_{ij} = \sum_{\mathbf{z} \in \mathbb{Z}^2_+} z_j \mu_i(\mathbf{z}), \quad i, j \in \{1, 2\}. \]

- \( m_{12} > 0, 1 \geq m_{11} > 0 \) and \( m_{22} = m_{21} = 0 \), (Bertoin, 2010):

\[ \mathbb{P}(O_1 = n_1, O_2 = n_2) = \frac{1}{n_1} \mu^*_1(n_1-1, n_2), \quad n_1 \geq 1, n_2 \geq 0. \]

- \( m_{12} > 0, 1 \geq m_{11}, m_{22} > 0 \) but \( m_{21} = 0 \),

\[ \mathbb{P}(O_1 = n_1, O_2 = n_2) = \frac{1}{n_1 n_2} \sum_{j=0}^{n_2} j \mu^*_1(n_1-1, j) \mu^*_2(0, n_2-j). \]
Progeny of 2-type branching processes

In all the remaining cases the matrix \((m_{ij})_{i,j\in\{1,2\}}\) (or the process \(Z\)) is irreducible, i.e.

\[
m_{12} > 0 \text{ and } m_{21} > 0.
\]

Let \(\rho\) be the dominant eigenvalue (Perron-Frobenius).

Then,

\[
\rho \leq 1 \iff T := \inf\{n : Z_n = 0\} < \infty, \quad \text{a.s.}
\]

The process is said to be critical \((\rho = 1)\) or subcritical \((\rho < 1)\).
Progeny of 2-type branching processes

$O_1$: total number of individuals of type 1.

$O_2$: total number of individuals of type 2.

$N_1$: total number of individuals of type 1 whose parent is of type 2.

$N_2$: total number of individuals of type 2 whose parent is of type 1.

**Theorem**

Assume that $Z$ is irreducible and critical or subcritical and $Z_0 = (1, 0)$. Then for all $n_1 \geq 1 \, n_2 \geq 0$, $0 \leq k_1 \leq n_1$ and $0 \leq k_2 \leq n_2$,

$$
P(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) = \frac{k_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2) \mu_2^{*n_2}(k_1, n_2 - k_2).$$
Progeny of 2-type branching processes

$O_1$ : total number of individuals of type 1.

$O_2$ : total number of individuals of type 2.

$N_1$ : total number of individuals of type 1 whose parent is of type 2.

$N_2$ : total number of individuals of type 2 whose parent is of type 1.

**Theorem**

Assume that $Z$ is irreducible and critical or subcritical and $Z_0 = (1, 0)$. Then for all $n_1 \geq 1$ $n_2 \geq 0$, $0 \leq k_1 \leq n_1$ and $0 \leq k_2 \leq n_2$,

$$
\mathbb{P}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) = \frac{k_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2) \mu_2^{*n_2}(k_1, n_2 - k_2).
$$
Encoding 2-type forests

Define a 2-type forest,

\[ \mathcal{F} = \{t_1, t_2, \ldots \}, \]

as an infinite sequence of independent 2-type rooted trees, with progeny law \((\mu_1, \mu_2)\).

- Each vertex \(u \in t_i\) is either of type 1 or type 2.
- The root of each tree is of type 1.
- Vertices of \( \mathcal{F} \) are ranked in the lexicographical order.
Type 1 = ⬤

Type 2 = ⬤
Encoding 2-type forests

Ordering vertices of type 1:

- Subtrees of type 1 are ranked according to the breadth first search order of their roots in the forest:

  \[ t_1^{(1)}, t_2^{(1)}, \ldots, t_n^{(1)}, \ldots \]

- Then vertices \( u_{i_1}^{(1)}, \ldots, u_j^{(1)} \) of \( t_n^{(1)} \) are ranked according to the 'local' breadth first search order of \( t_n^{(1)} \):

  \[ u_0^{(1)}, \ldots, u_{i_1-1}^{(1)}, u_{i_1}^{(1)}, \ldots, u_{i_1+i_2-1}^{(1)}, \ldots \]
Encoding 2-type forests

Let $k_i(u)$ be the number of children of type $i$ of the vertex $u$.

Then define the integer valued chains $X = (X^{(1)}, X^{(2)})$ and $Y = (Y^{(1)}, Y^{(2)})$ by:

$$X^{(1)}_{n+1} - X^{(1)}_n = k_1(u^{(1)}_n) - 1$$
$$X^{(2)}_{n+1} - X^{(2)}_n = k_2(u^{(1)}_n)$$
$$Y^{(1)}_{n+1} - Y^{(1)}_n = k_1(u^{(2)}_n)$$
$$Y^{(2)}_{n+1} - Y^{(2)}_n = k_2(u^{(2)}_n) - 1.$$

Proposition

The chains $X$ and $Y$ are independent random walks in $\mathbb{Z} \times \mathbb{Z}_+$ and $\mathbb{Z}_+ \times \mathbb{Z}$, respectively, with step distributions:

$$\mathbb{P}(X_1 = (i, j)) = \mu_1(i+1, j), \quad \mathbb{P}(Y_1 = (i, j)) = \mu_2(i, j+1).$$
Encoding 2-type forests

Define

\[ T_k = \inf \{ n : X_n^{(1)} = -k \} \quad S_k = \inf \{ n : Y_n^{(2)} = -k \} . \]

Then,

\[ X^{(2)}(T_k) \] is the number of subtrees of type 2 encountered when \( k \) subtrees of type 1 have been visited,

\[ Y^{(1)}(S_k) \] is the number of subtrees of type 1 encountered when \( k \) subtrees of type 2 have been visited.

Therefore, if \( k_i, i = 1, 2 \) is the total number of subtrees of type \( i \) in the first tree \( t_1 \) of the 2-type forest \( \mathcal{F} \), then

\[
\begin{cases}
  k_2 = X^{(2)}(T_{k_1}) \\
  k_1 = 1 + Y^{(1)}(S_{k_2})
\end{cases}
\]
Encoding 2-type forests

Define

\[ T_k = \inf \{ n : X_n^{(1)} = -k \} \quad S_k = \inf \{ n : Y_n^{(2)} = -k \} . \]

Then,

- \( X^{(2)}(T_k) \) is the number of subtrees of type 2 encountered when \( k \) subtrees of type 1 have been visited,
- \( Y^{(1)}(S_k) \) is the number of subtrees of type 1 encountered when \( k \) subtrees of type 2 have been visited.

Therefore, if \( k_i, \ i = 1, 2 \) is the total number of subtrees of type \( i \) in the first tree \( t_1 \) of the 2-type forest \( \mathcal{F} \), then

\[
\begin{cases}
  k_2 = X^{(2)}(T_{k_1}) \\
  k_1 = 1 + Y^{(1)}(S_{k_2}) .
\end{cases}
\]
Encoding 2-type forests

Let \((k_1, k_2)\) be the smallest solution of

\[
\begin{cases}
  k_2 = X^{(2)}(T_{k_1}) \\
  k_1 = 1 + Y^{(1)}(S_{k_2})
\end{cases}
\]

Recall that \(t_1\) is the first tree of the 2-type forest \(\mathcal{F}\).

Proposition

- \(k_i, i = 1, 2\) is the total number of subtrees of type \(i\) in \(t_1\).
- \(T_{k_i}, i = 1, 2\) is the total number of individuals of type \(i\) in \(t_1\).
- \(t_1\) is encoded by the two 2-dimensional chains:

\[
\begin{align*}
  &\left[ (X_{n}^{(1)}, X_{n}^{(2)}), 0 \leq n \leq T_{k_1} \right] \\
  &\left[ (Y_{n}^{(1)}, Y_{n}^{(2)}), 0 \leq n \leq S_{k_2} \right].
\end{align*}
\]
The progeny law

Recall that:

- \( O_1 \): total number of individuals of type 1.
- \( O_2 \): total number of individuals of type 2.
- \( N_1 \): total number of individuals of type 1 whose parent is of type 2.
- \( N_2 \): total number of individuals of type 2 whose parent is of type 1.

\[
(S) \quad \left\{ \begin{array}{l}
k_2 = X^{(2)}(T_{k_1}) \\
k_1 = 1 + Y^{(1)}(S_{k_2})
\end{array} \right.
\]

Then,

\[
\mathbb{P}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) =
\mathbb{P}(T_{k_1} = n_1, S_{k_2} = n_2 \text{ and } (k_1, k_2) \text{ is the smallest solution of } (S).)
\]
The progeny law

\[
(S) \quad \left\{ \begin{array}{l}
k_2 = X^{(2)}(T_{k_1}) \\
k_1 = 1 + Y^{(1)}(S_{k_2}) .
\end{array} \right.
\]

Define

\[ U_k = X^{(2)}(T_{k_1}), \quad V_k = Y^{(1)}(S_{k_2}) \quad \text{and} \quad W_k = V(U_k). \]

Then \((W_k - k, k \geq 0)\) is a downward skip free random walk and the smallest solution of \((S)\) is given by:

\[ k_1 = \inf\{k : W_k - k = -1\}. \]
The progeny law

So that

\[ P(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) \]

\[ = P(T_{k_1} = n_1, S_{k_2} = n_2, k_1 = \inf\{k : W_k - k = -1\}, X^{(2)}(T_{k_1}) = k_2) \]

\[ = \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, W_{k_1} = k_1 - 1, X^{(2)}(T_{k_1}) = k_2) \]

\[ = \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, Y^{(1)}(S_{k_2}) = k_1 - 1, X^{(2)}(T_{k_1}) = k_2) \]

\[ = \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, Y^{(1)}(n_2) = k_1 - 1, X^{(2)}(n_1) = k_2) \]

\[ = \frac{1}{k_1} P(T_{k_1} = n_1, X^{(2)}_{n_1} = k_2) P(S_{k_2} = n_2, Y^{(1)}_{n_2} = k_1 - 1) \]

\[ = \frac{k_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2) \mu_2^{*n_2}(k_1, n_2 - k_2). \]
The progeny law

So that

\[
\mathbb{P}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) \\
= \mathbb{P}(T_{k_1} = n_1, S_{k_2} = n_2, k_1 = \inf \{k : W_k - k = -1\}, X^{(2)}(T_{k_1}) = k_2) \\
= \frac{1}{k_1} \mathbb{P}(T_{k_1} = n_1, S_{k_2} = n_2, W_{k_1} = k_1 - 1, X^{(2)}(T_{k_1}) = k_2) \\
= \frac{1}{k_1} \mathbb{P}(T_{k_1} = n_1, S_{k_2} = n_2, Y^{(1)}(S_{k_2}) = k_1 - 1, X^{(2)}(T_{k_1}) = k_2) \\
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= \frac{1}{k_1} \mathbb{P}(T_{k_1} = n_1, X^{(2)}_{n_1} = k_2) \mathbb{P}(S_{k_2} = n_2, Y^{(1)}_{n_2} = k_1 - 1) \\
= \frac{k_2}{n_1 n_2} \mu^{*n_1}_{1}(n_1 - k_1, k_2) \mu^{*n_2}_{2}(k_1, n_2 - k_2).
\]
The progeny law

So that

\[
\mathbb{P}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2)
\]

\[
= \mathbb{P}(T_{k_1} = n_1, S_{k_2} = n_2, k_1 = \inf\{k : W_k - k = -1\}, X^{(2)}(T_{k_1}) = k_2)
\]

\[
= \frac{1}{k_1} \mathbb{P}(T_{k_1} = n_1, S_{k_2} = n_2, W_{k_1} = k_1 - 1, X^{(2)}(T_{k_1}) = k_2)
\]

\[
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\]

\[
= \frac{1}{k_1} \mathbb{P}(T_{k_1} = n_1, X_{n_1}^{(2)} = k_2) \mathbb{P}(S_{k_2} = n_2, Y_{n_2}^{(1)} = k_1 - 1)
\]

\[
= \frac{k_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2) \mu_2^{*n_2}(k_1, n_2 - k_2).
\]
The progeny law

So that

\[ \mathbb{P}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) \]

\[ = \mathbb{P}(T_{k_1} = n_1, S_{k_2} = n_2, k_1 = \inf\{k : W_k - k = -1\}, X^{(2)}(T_{k_1}) = k_2) \]

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\[ = \frac{1}{k_1} \mathbb{P}(T_{k_1} = n_1, Y^{(1)}(n_2) = k_1 - 1, X^{(2)}(n_1) = k_2) \]

\[ = \frac{1}{k_1} \mathbb{P}(T_{k_1} = n_1, X^{(2)}_{n_1} = k_2) \mathbb{P}(S_{k_2} = n_2, Y^{(1)}_{n_2} = k_1 - 1) \]

\[ = \frac{k_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2) \mu_2^{*n_2}(k_1, n_2 - k_2). \]
The progeny law

So that

\[ P(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) \]

\[ = P(T_{k_1} = n_1, S_{k_2} = n_2, k_1 = \text{inf}\{k : W_k - k = -1\}, X^{(2)}(T_{k_1}) = k_2) \]

\[ = \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, W_{k_1} = k_1 - 1, X^{(2)}(T_{k_1}) = k_2) \]

\[ = \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, Y^{(1)}(S_{k_2}) = k_1 - 1, X^{(2)}(T_{k_1}) = k_2) \]

\[ = \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, Y^{(1)}(n_2) = k_1 - 1, X^{(2)}(n_1) = k_2) \]

\[ = \frac{1}{k_1} P(T_{k_1} = n_1, X^{(2)}_{n_1} = k_2) P(S_{k_2} = n_2, Y^{(1)}_{n_2} = k_1 - 1) \]

\[ = \frac{k_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2) \mu_2^{*n_2}(k_1, n_2 - k_2). \]
The progeny law

So that

\[ P(O_1 = n_1, O_2 = n_2, N_1 = k_1 - 1, N_2 = k_2) \]
\[ = P(T_{k_1} = n_1, S_{k_2} = n_2, k_1 = \inf\{k : W_k - k = -1\}, X^{(2)}(T_{k_1}) = k_2) \]
\[ = \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, W_{k_1} = k_1 - 1, X^{(2)}(T_{k_1}) = k_2) \]
\[ = \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, Y^{(1)}(S_{k_2}) = k_1 - 1, X^{(2)}(T_{k_1}) = k_2) \]
\[ = \frac{1}{k_1} P(T_{k_1} = n_1, S_{k_2} = n_2, Y^{(1)}(n_2) = k_1 - 1, X^{(2)}(n_1) = k_2) \]
\[ = \frac{1}{k_1} P(T_{k_1} = n_1, X^{(2)}_{n_1} = k_2) P(S_{k_2} = n_2, Y^{(1)}_{n_2} = k_1 - 1) \]
\[ = \frac{k_2}{n_1 n_2} \mu_1^{*n_1}(n_1 - k_1, k_2) \mu_2^{*n_2}(k_1, n_2 - k_2). \]
The progeny law

When \( Z_0 = (r_1, r_2) \):

**Theorem**

Assume that \( Z \) is irreducible and critical or subcritical and \( Z_0 = (r_1, r_2) \). Then for all \( n_1 \geq 1 \), \( n_2 \geq 0 \), \( 0 \leq k_1 \leq n_1 \) and \( 0 \leq k_2 \leq n_2 \),

\[
\mathbb{P}(O_1 = n_1, O_2 = n_2, N_1 = k_1 - r_1, N_2 = k_2 - r_2) = \frac{r_1 k_2 + r_2 k_1 - r_1 r_2}{n_1 n_2} \mu^*_{n_1}(n_1 - k_1, k_2) \mu^*_{n_2}(k_1, n_2 - k_2).
\]
The progeny law

Three types:

▶ \( A_{ij} \) = number of individuals of type \( j \) whose parent is of type \( i \).

**Theorem**

Assume that \( Z \) is irreducible and critical or subcritical and \( Z_0 = (r_1, r_2, r_3) \). Then for all \( n_j \geq 1 \) and \( 0 \leq k_{ij} \leq n_j, j = 1, 2, 3, \)

\[
\mathbb{P}(O_1 = n_1, O_2 = n_2, O_3 = n_3, A_{ij} = k_{ij}, i = 1, 2, 3, i \neq j) = \\
(n_1 n_2 n_3)^{-1} \{ r_1[(r_3 + k_{12})k_{23} + (k_{23} + r_2 + k_{12})(r_3 + k_{13})] + \\
k_{21}[r_3 k_{32} + (k_{23} + r_3 + k_{13})r_2] + k_{31}[r_2 k_{23} + (k_{32} + r_2 + k_{12})r_3] \} \\
\times \mu_1^{*n_1}(n_1 - k_{21} - k_{31} - r_1, k_{12}, k_{13}) \\
\times \mu_2^{*n_2}(k_{21}, n_2 - k_{12} - k_{32} - r_2, k_{23}) \\
\times \mu_3^{*n_3}(k_{31}, k_{32}, n_3 - k_{13} - k_{23} - r_3).
\]