Convergence in law to the multiple fractional integral

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Abstract

We study the convergence in law in $\mathcal{C}_0([0,1])$, as $\varepsilon \to 0$, of the family of continuous processes $\{I_{\eta_\varepsilon}(f)\}_{\varepsilon>0}$ defined by the multiple integrals

$$I_{\eta_\varepsilon}(f)_t = \int_0^t \cdots \int_0^t f(t_1,\dots,t_n) d\eta_\varepsilon(t_1)\cdots d\eta_\varepsilon(t_n); \quad t\in[0,1],$$

where f is a deterministic function and $\{\eta_\varepsilon\}_{\varepsilon>0}$ is a family of processes, with absolutely continuous paths, converging in law in $\mathcal{C}_0([0,1])$ to the fractional Brownian motion with Hurst parameter $H>\frac{1}{2}$. When f is given by a multimeasure and for any family $\{\eta_\varepsilon\}$ with trajectories absolutely continuous whose derivatives are in $L^2([0,1])$, we prove that $\{I_{\eta_\varepsilon}(f)\}$ converges in law to the multiple fractional integral of f. This last integral is a multiple Stratonovich-type integral defined by Dasgupta and Kallianpur (1999a) on the space $L^2(\tilde{\mu}_n)$, where $\tilde{\mu}_n$ is a measure on $[0,1]^n$.

Finally, we have shown that, for two natural families $\{\eta_\varepsilon\}$ converging in law in $\mathcal{C}_0([0,1])$ to the fractional Brownian motion, the family $\{I_{\eta_\varepsilon}(f)\}$ converges in law to the multiple fractional integral for any $f\in L^2(\tilde{\mu}_n)$.

In order to prove the convergence, we have shown that the integral introduced by Dasguta and Kallianpur (1999a) can be seen as an integral in the sense of Solé and Utzet (1990).

1 Introduction

In this work, we study the convergence in law, when $\varepsilon\to 0$, of the family of continuous processes $\{I_{\eta_\varepsilon}(f)\}_{\varepsilon>0}$ defined by the following multiple integrals

$$I_{\eta_\varepsilon}(f)_t = \int_0^t \cdots \int_0^t f(t_1,\dots,t_n) d\eta_\varepsilon(t_1)\cdots d\eta_\varepsilon(t_n); \quad t\in[0,1], \tag{1}$$

where f is a deterministic function and $\{\eta_\varepsilon\}_{\varepsilon>0}$ is a family of processes with absolutely continuous paths converging in law to the fractional Brownian motion with Hurst parameter $H>\frac{1}{2}$.

Avram (1988) studied a similar problem for $f(t_1,\dots,t_n)=I_{\{t_1<\cdots<t_n\}}$ and a family $\{\eta_\varepsilon\}$ of semimartingales in the Skorohod space $\mathcal{D}([0,1])$. He applied the obtained results to asymptotic distributions of some U-statistics. The author proved that a necessary and sufficient condition for the convergence of $\{I_{\eta_\varepsilon}(f)\}$ to the multiple iterated Itô integral of f with respect to η is the joint convergence of η_ε and its quadratic variation to η and its quadratic variation.

However, Avram's results do not cover the case when the limit process η has positive quadratic variation while the approximations η_ε have absolutely continuous paths. In that situation, the multiple Stratonovich integral rather than the multiple Itô integral should arise.

Bardina and Jolis (2000) studied this problem with a more general f and the η_ε being approximations in law of the Brownian motion in $\mathcal{C}_0([0,1])$, the space of continuous functions that are null at zero. In that paper, in the cases where the authors are able to prove its existence, the limit of $\{I_{\eta_\varepsilon}(f)\}$ is always the multiple Stratonovich integral of the function f. More precisely, in the case in which f is given by a multimeasure (see Section 3 for the definition of a multimeasure), $I_{\eta_\varepsilon}(f)$ is the evaluation of a continuous operator, and then, for any family $\{\eta_\varepsilon\}_{\varepsilon>0}$, converging in law to the Brownian motion, the convergence on $\mathcal{C}_0([0,1])$ of $\{I_{\eta_\varepsilon}(f)\}$ to the multiple Stratonovich integral of f is proved. For other types of functions only partial results are obtained. Concretely, when f is either continuous or $f(t_1,\ldots,t_n) = f_1(t_1)\cdots f_n(t_n)\, I_{\{t_1<\cdots<t_n\}}$, with $f_i \in L^2([0,1])$, the convergence is proved for certain classes of processes η_ε.

Recently, due to the great interest of the fractional Brownian motion as a model for many phenomena, several authors have raised the problem of the construction of a multiple integral with respect to this process. The fractional Brownian motion is not a semimartingale but, in the case of Hurst parameter $H > \frac{1}{2}$, Dasgupta and Kallianpur (1999a) defined a multiple integral of Stratonovich type on a reasonable space of functions. This space is $L^2(\tilde{\mu}_n)$, where $\tilde{\mu}_n$ is a measure on [0,1], and these authors called their integral the multiple fractional integral.

The above results lead us to study the convergence of $\{I_{\eta_\varepsilon}(f)\}$ when $\{\eta_\varepsilon\}$ are absolutely continuous processes that converge in law in $\mathcal{C}_0([0,1])$ to the fractional Brownian motion of parameter $H > \frac{1}{2}$.

In Section 4 we prove that for any function $f \in L^2(\tilde{\mu}_n)$ and two natural families of processes $\{\eta_\varepsilon\}$, defined in Section 2, the family of multiple integrals $\{I_{\eta_\varepsilon}(f)\}_{\varepsilon>0}$ converges in law (also in $\mathcal{C}_0([0,1])$) to the multiple fractional integral of f with respect to the fractional Brownian motion (see Theorem 4.2). This result is based on the inequality stated in Lemma 4.1 whose long proof is given in an appendix.

On the other hand, when f is given by a multimeasure, we prove in Theorem 3.1 a result that is analogous to Corollary 3.2 of Bardina and Jolis (2000).

We have organized the paper as follows. Section 2 of preliminaries is devoted to introduce the notations, to construct two families of processes converging in law to the fractional Brownian motion, and to define the multiple fractional integral. In this same section, we prove in Proposition 2.4 that the multiple fractional integral defined for $f \in L^2(\tilde{\mu}_n)$ has the form of the multiple Stratonovich-type integral defined by Solé and Utzet (1990) for the standard Brownian motion. In Section 3 we give the result for the functions defined by multimeasures and in Section 4 we prove the convergence in law of $\{I_{\eta_\varepsilon}(f)\}_\varepsilon$ to the multiple fractional integral of f, when $f \in L^2(\tilde{\mu}_n)$ and the processes η_ε are those constructed in Section 2. We have added an Appendix with the proof of the inequality given in Lemma 4.1, that is the basis of the result of Section 4.

Positive constants, denoted by C with subscripts indicating appropriate parameters, e.g. C_H or $C_{n,H}$, may vary from line to line.

2 Preliminaries

2.1 The fractional Brownian motion and some approximations in law

The fractional Brownian motion (fBm, for short) of parameter $H \in (0,1)$ is a centered Gaussian process $B^H = \{B^H_t,\ t \geq 0\}$, defined on some probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ with expectation denoted by $\bar{E}$, that has the following covariance

$$R(s,t) = \bar{E}[B_s^H B_t^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),$$

see for instance Mandelbrot and Van Ness (1968). When $H = \frac{1}{2}$ this process is the standard Brownian motion. It is well known that there exists a version of B^H with continuous trajectories.

The representation of B^H as the integral of a deterministic kernel with respect to a standard Brownian motion,

$$B_t^H = \int_0^t K_H(t,s) dW_s, \tag{2}$$

is a very useful result for the construction of a stochastic calculus with respect to the fBm. The kernel $K_H(t,s)$ is defined on the set $\{0 < s < t\}$ and given by

$$K_H(t,s) = d_H (t-s)^{H-\frac{1}{2}} + d_H(\frac{1}{2} - H) \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du, \tag{3}$$

where d_H is the following normalizing constant:

$$d_H = \left(\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)}\right)^{\frac{1}{2}}.$$

(See, for instance, Section 8 of Alòs et al., 2001 and also Decreusefond and Üstünel, 1999 and Norros et al., 1999).

Although there exist easier integral representations of the fBm (see Mandelbrot and Van Ness, 1968) they have the inconvenience that their domain of integration is unbounded.

When $H > \frac{1}{2}$ the kernel K_H has the simpler expression

$$K_H(t,s) = d_H(H - \frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du.$$

¿From representation (2), a natural way to obtain approximations in law for the fBm is to define

$$\eta_\varepsilon(t) = \int_0^t K_H(t,s) \theta_\varepsilon(s) ds,$$

where $\{\theta_\varepsilon\}_\varepsilon$ is a "weak approximation of the white noise". More precisely, $\{\theta_\varepsilon\}_\varepsilon$ is a family of processes, defined on some probability space $(\Omega, \mathcal{F}, P)$, with expectation denoted by E, such that, for

$$\rho_\varepsilon(t) := \int_0^t \theta_\varepsilon(s) ds,$$

$\{\rho_\varepsilon\}_{\varepsilon > 0}$ converges in law in $\mathcal{C}_0([0,1])$ to the standard Wiener process as $\varepsilon \to 0$.

We consider two examples of this situation. The first one is to take

$$\theta_\varepsilon(s) = \frac{1}{\varepsilon}(-1)^{N(\frac{s}{\varepsilon^2})},$$

where $\{N(s),\, s \geq 0\}$ is a standard Poisson process. The convergence in law in $\mathcal{C}_0([0,1])$ of

$$\rho_\varepsilon(t) = \int_0^t \theta_\varepsilon(s)ds$$

to the standard Brownian motion was proved by Stroock (1982). We will call functions θ_ε Stroock kernels.

To give the other example consider

$$\theta_\varepsilon(s) = \frac{1}{\varepsilon}\sum_{k=1}^{\infty} \xi_k I_{[k-1,k)}(\frac{s}{\varepsilon^2}),$$

where $\{\xi_k\}$ is a sequence of independent, identically distributed random variables satisfying $E(\xi_1) = 0$ and $\mathrm{Var}(\xi_1) = 1$. In this case, we will call θ_ε Donsker kernels because the convergence in law of $\rho_\varepsilon(t) = \int_0^t \theta_\varepsilon(s)ds$ to the Brownian motion is the well-known Donsker's Invariance Principle. In order to obtain easily the convergence of $\{\eta_\varepsilon\}_\varepsilon$ to the fBm, when $H < \frac{1}{2}$, we will assume that the random variables ξ_k have finite moments of order m, with $m \in \mathbb{N}$ and $m > \frac{1}{H}$.

Proposition 2.1 *Let θ_ε be either the Stroock or the Donsker kernels. Then, the family of laws in $\mathcal{C}_0([0,1])$ of the processes*

$$\eta_\varepsilon(t) = \int_0^t K_H(t,s)\theta_\varepsilon(s)ds$$

converges weakly to the law of a fractional Brownian motion of parameter $H \in (0,1)$, when $\varepsilon \to 0$.

Proof: First of all, by representation (2) and the local boundedness of the trajectories of θ_ε, it is easy to see that the η_ε are continuous processes. Indeed, if $s < t$,

$$\begin{aligned} |\eta_\varepsilon(t) - \eta_\varepsilon(s)| &= \left| \int_0^t [K_H(t,r) - K_H(s,r)I_{(0,s)}(r)]\theta_\varepsilon(r)dr \right| \\ &\leq \sup_{0\leq r\leq t} |\theta_\varepsilon(r)| \left(\int_0^t [K_H(t,r) - K_H(s,r)I_{(0,s)}(r)]^2 dr \right)^{\frac{1}{2}} \\ &= \sup_{0\leq r\leq t} |\theta_\varepsilon(r)| \left(\bar{E}(B_t^H - B_s^H)^2 \right)^{\frac{1}{2}} \\ &= \sup_{0\leq r\leq t} |\theta_\varepsilon(r)| (t-s)^H. \end{aligned}$$

The proposition for $H = \frac{1}{2}$ is already known. Suppose, then, that $H \neq \frac{1}{2}$.

The convergence in law of $\{\eta_\varepsilon\}$ to the fractional Brownian motion was proved for the case of the Stroock kernels in Delgado and Jolis (2000).

The case of Donsker kernels, with finite moments of order $m \in \mathbb{N}$ and $m > \frac{1}{H}$, can be seen by proving, in a similar way as in Lemma 4.2 of Bardina and Jolis (2000), that there exists a constant C_m, depending only on m, such that if $g \in L^2([0,1])$ then

$$E \left| \int_0^1 g(x)\theta_\varepsilon(x)dx \right|^m \leq C_m \left(\int_0^1 g^2(x)dx \right)^{m/2}.$$

¿From this inequality, the proof concludes as in Delgado and Jolis (2000), Theorem 1. □

¿From the expression of the kernel $K_H(t,s)$, it is easy to check that it is differentiable with respect to the variable t on the set $\{0 < s < t\}$ and that

$$\frac{\partial}{\partial t} K_H(t,s) = d_H(H - \frac{1}{2}) \left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}. \tag{4}$$

The properties of this function, when $H > \frac{1}{2}$, are fundamental in what follows. For instance, if $H > \frac{1}{2}$, the processes η_ε defined above are not only continuous but absolutely continuous. This is a consequence of the following deterministic lemma.

Lemma 2.2 *Let $T > 0$ and $\theta \in L^\infty([0,T])$. Define*

$$\eta(t) = \int_0^t K_H(t,r)\theta(r)dr, \quad t \in [0,T],$$

with $H > \frac{1}{2}$. Then, η is differentiable and

$$\eta'(t) = \int_0^t \frac{\partial}{\partial t} K_H(t,r)\theta(r)dr = d_H(H - \frac{1}{2}) t^{H-\frac{1}{2}} \int_0^t r^{\frac{1}{2}-H}(t-r)^{H-\frac{3}{2}}\theta(r)dr. \tag{5}$$

Moreover, η' is a continuous function and hence η is absolutely continuous.

Proof:
It follows from standard arguments. □

2.2 The multiple integral with respect to the fBm with parameter $H > \frac{1}{2}$

As for the standard Brownian motion, when $H > \frac{1}{2}$, two kinds of multiple integrals with respect to the fractional Brownian motion have been constructed. One of them is of Itô-Wiener type and the other one is of Stratonovich type. Unlike the case $H = \frac{1}{2}$, in which the Itô-Wiener integral can be interpreted (if $n > 1$) as a limit of Riemann sums with exclusion of the diagonals, when $H > \frac{1}{2}$ the only integral that can be always interpreted as a limit of Riemann sums is the Stratonovich integral (see, for instance, Proposition 2.4 and Remark 2.5). In the case $H \neq \frac{1}{2}$, the Itô-Wiener type integral can be defined in terms of tensor products of elements of the linear space generated by the fBm. We refer, for instance, to Huang and Cambanis (1978), Dasgupta and Kallianpur (1999a), Dasgupta and Kallianpur (1999b) and Perez-Abreu and Tudor (2001) for an account on these constructions.

Dasgupta and Kallianpur (1999a) defined a multiple integral (they only consider the case $n > 1$) of Stratonovich type with respect to the fBm of parameter $H > \frac{1}{2}$ for a function $f \in L^2(\tilde{\mu}_n)$, where the measure $\tilde{\mu}_n$ will be defined below. They call this kind of integral the multiple fractional integral. We explain their construction in what follows and we also consider the case $n = 1$.

Let $\pi = \{0 = t_0 < t_1 < \cdots < t_{m+1} = 1\}$ be a finite partition of the unit interval, and denote by $\Delta_i = [t_i, t_{i+1}),\ i = 1, \ldots, m$ the intervals defined by this partition. A function of the form

$$f(x_1, \ldots, x_n) = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} I_{\Delta_{i_1}}(x_1) \cdots I_{\Delta_{i_n}}(x_n),$$

with $a_{i_1,\ldots,i_n}$ some constants, will be called an elementary function.

The multiple fractional integral with respect to the fBm of such a function is defined as

$$I_n \circ (f) = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} B^H(\Delta_{i_1}) \cdots B^H(\Delta_{i_n}).$$

We consider a measure on $[0,1]^n$ given by

$$\mu_n(dx_1, ..., dx_n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \nu_n^k(dx_1, ..., dx_n),$$

where the measures ν_n^k are defined as follows:

$$\nu_n^0(dx_1, ..., dx_n) = H^n n! \left(\prod_{j=1}^n \left[x_j^{2H-1} + (1-x_j)^{2H-1} \right] \right) dx_1 \cdots dx_n,$$

and for $k = 1, \ldots, \left[\frac{n}{2}\right]$,

$$\begin{aligned} \nu_n^k(dx_1, ..., dx_n) &= H^{n-k}(2H-1)^k c_{n,k}^2 (n-2k)! \left(\prod_{j=1}^k |x_{2j-1} - x_{2j}|^{2H-2} \right) \\ &\quad \times \left(\prod_{j=2k+1}^n \left[x_j^{2H-1} + (1-x_j)^{2H-1} \right] \right) dx_1 \cdots dx_n, \end{aligned}$$

where

$$c_{n,k} = \frac{n!}{2^k k! \, (n-2k)!}.$$

In Section 5 of Dasgupta and Kallianpur (1999a) it is proved for $n > 1$ the following inequality

$$\bar{E}[I_n \circ (f)]^2 \le \|\tilde{f}\|^2_{L^2(\mu_n)} \le n! \|f\|^2_{L^2(\tilde{\mu}_n)},$$

where $\tilde{f}$ denotes the symmetrization of the elementary function f, and $\tilde{\mu}_n$ the symmetrization of the measure μ_n.

When $n = 1$ it is easy to check that if f is an elementary function then

$$\begin{aligned} \bar{E}\left[I_1 \circ (f)\right]^2 &= \bar{E}\left[\sum_{i=0}^m a_i B^H(\Delta_i) \right]^2 = H(2H-1) \int_0^1 \int_0^1 f(s) f(t) |s-t|^{2H-2} ds dt \\ &\le H(2H-1) \int_0^1 \int_0^1 |f(t)|^2 |s-t|^{2H-2} ds dt = \|f\|^2_{L^2(\mu_1)}. \end{aligned}$$

¿From these facts, the extension of the integral to all $L^2(\tilde{\mu}_n)$ is obtained (since the elementary functions are dense in $L^2(\tilde{\mu}_n)$), and also the continuity of the extended operator from $L^2(\tilde{\mu}_n)$ into $L^2(\bar{\Omega})$. We denote the extended operator in the same way.

Notice that the measures ν_n^0 can be bounded from above and from below by positive constants (depending only on H and n) multiplied by the Lebesgue measure.

On the other hand, when $n > 1$, we define

$$\nu_n(dx_1, \ldots, dx_n) = \left(\prod_{k=1}^{\left[\frac{n}{2}\right]} |x_{2k-1} - x_{2k}|^{2H-2} \right) dx_1 \cdots dx_n.$$

It is clear that there exist constants depending only on n, k and H, denoted by $C_{n,k,H}$, such that

$$\nu_n^k \le C_{n,k,H}\nu_n, \quad \text{for any } k = 0, \ldots, \left[\frac{n}{2}\right],$$

in particular, ν_n dominates the Lebesgue measure.

Moreover, there exists another constant $C_{n,H}$ such that $C_{n,H}\nu_n \le \mu_n$.

We will use these last facts in many places in the rest of the paper. Observe that they imply that $L^2(\mu_n) = L^2(\nu_n)$.

We finish this section with a proposition that states that the integral defined by Dasgupta and Kallianpur (1999a) coincides with the integral in the sense of Solé and Utzet (1990).

Proposition 2.3 *If $f \in L^2(\mu_n)$ is a symmetric function, then*

$$I_n \circ (f) = L^2(\bar\Omega) - \lim \sum_{i_1,\ldots,i_n} \frac{1}{|\Delta_{i_1}| \cdots |\Delta_{i_n}|} \Big(\int_{\Delta_{i_1} \times \cdots \times \Delta_{i_n}} f(t_1, \ldots, t_n) dt_1 \cdots dt_n \Big) B^H(\Delta_{i_1}) \cdots B^H(\Delta_{i_n}),$$

where the Δ_{i_j} are the intervals defined by a partition π of $[0,1]$, and the limit is taken when $|\pi|$ tends to zero.

Proof: Given a symmetric function $f \in L^2(\mu_n)$ and a partition π of $[0,1]$, we denote by f^π the following (symmetric) elementary function

$$f^\pi(s_1, \ldots, s_n) = \sum_{i_1,\ldots,i_n} \frac{1}{|\Delta_{i_1}| \cdots |\Delta_{i_n}|} \Big(\int_{\Delta_{i_1} \times \cdots \times \Delta_{i_n}} f(x_1, \ldots, x_n) dx_1 \cdots dx_n \Big) I_{\Delta_{i_1}}(s_1) \cdots I_{\Delta_{i_n}}(s_n).$$

In order to prove the proposition, it suffices to see that f^π converges in $L^2(\mu_n)$ to f, as $|\pi|$ tends to zero. This convergence is well-known when $n = 1$. For the case $n > 1$, we will prove that there exists a constant $C_{H,n}$ such that

$$\|f^\pi\|_{L^2(\mu_n)} \le C_{H,n} \|f\|_{L^2(\mu_n)}. \tag{6}$$

¿From this fact, the proof is standard since g^π clearly tends to g in $L^2(\mu_n)$ when g is a continuous function.

Inequality (6) is a consequence of the fact that there exists a constant C_H such that, if $i \le j$ then

$$\int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} |s-t|^{2H-2} ds dt \le C_H |x-y|^{2H-2} |\Delta_i||\Delta_j|, \tag{7}$$

for any $(x, y) \in \Delta_i \times \Delta_j$.

Indeed, we have that

$$\|f^\pi\|^2_{L^2(\mu_n)} = \sum_{i_1,\dots,i_n} \left(\frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|} \Big(\int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} f(x_1,\dots,x_n)dx_1\cdots dx_n \Big) \right)^2 \int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} \mu_n(ds_1,\dots,ds_n)$$

$$\leq C_{H,n} \sum_{i_1,\dots,i_n} \frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|} \Big(\int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} f^2(x_1,\dots,x_n)dx_1\cdots dx_n \Big) \int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} |s_1-s_2|^{2H-2}|s_3-s_4|^{2H-2}$$

$$\times\cdots\times|s_{2[\frac{n}{2}]-1}-s_{2[\frac{n}{2}]}|^{2H-2}ds_1\cdots ds_n = C_{H,n} \sum_{i_1,\dots,i_n} \int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} C_{i_1,\dots,i_n} f^2(x_1,\dots,x_n)dx_1\cdots dx_n,$$

where we have denoted by $C_{i_1,\dots,i_n}$ the following constants:

$$C_{i_1,\dots,i_n} = \frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|} \int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} |s_1-s_2|^{2H-2}|s_3-s_4|^{2H-2}\cdots|s_{2[\frac{n}{2}]-1}-s_{2[\frac{n}{2}]}|^{2H-2}ds_1\cdots ds_n.$$

By using inequality (7), we obtain that

$$C_{i_1,\dots,i_n} \leq C_{H,n} \prod_{j=1}^{[\frac{n}{2}]} |x_{2j-1}-x_{2j}|^{2H-2}$$

for any $(x_1,\dots,x_n)\in\Delta_{i_1}\times\cdots\times\Delta_{i_n}$, and then the norm $\|f^\pi\|^2_{L^2(\mu_n)}$ is bounded by

$$C_{H,n} \sum_{i_1\dots,i_n} \int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} f^2(x_1,\dots,x_n) \left(\prod_{j=1}^{[\frac{n}{2}]} |x_{2j-1}-x_{2j}|^{2H-2} \right) dx_1\cdots dx_n$$

$$\leq C_{H,n}\|f\|^2_{L^2(\mu_n)}.$$

Then, let us now prove inequality (7). Consider first the case $i=j$. We have that

$$\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} |s-t|^{2H-2}dsdt = \frac{1}{H(2H-1)}(t_{i+1}-t_i)^{2H} \leq \frac{1}{H(2H-1)}|x-y|^{2H-2}(t_{i+1}-t_i)^2,$$

for all $x,y\in\Delta_i$.

When $i<j$, it suffices to show that

$$\int_{t_j}^{t_{j+1}}\int_{t_i}^{t_{i+1}} |s-t|^{2H-2}dsdt \leq C_H|t_{j+1}-t_i|^{2H-2}|\Delta_i||\Delta_j|.$$

To see this last inequality, consider, for fixed $t_{i+1}\leq t_j$, the function

$$g(u,v) = (v-u)^{2H-2}(t_{i+1}-u)(v-t_j)$$

defined for $(u,v)\in[0,t_{i+1}]\times[t_j,\infty)$.

It is clear that

$$g(u,v) = -\int_{t_j}^{v}\int_{u}^{t_{i+1}} \frac{\partial^2}{\partial s\partial t} g(s,t)dsdt$$

and so, we will prove that there exists a constant $M_H > 0$ such that

$$-\frac{\partial^2}{\partial s \partial t} g(s,t) \geq M_H |s-t|^{2H-2}.$$

We have that

$$\begin{aligned} -\frac{\partial^2}{\partial s \partial t} g(s,t) &= (2H-2)(2H-3)(t-s)^{2H-4}(t_{i+1}-s)(t-t_j) + (2H-2)(t-s)^{2H-3}(t_{i+1}-s) \\ &\quad +(2H-2)(t-s)^{2H-3}(t-t_j) + (t-s)^{2H-2}. \end{aligned}$$

Taking into account that $2H-2<0$, $(2H-2)(2H-3)>0$ and that $t_{i+1}-t_j \leq 0$, the last expression can be bounded from below by $(2H-1)(t-s)^{2H-2}$.

This fact concludes the proof. □

Remark 2.4 *We point out that in Proposition 2.4, the Riemann sums can be taken excluding the diagonal terms. This is a consequence of inequality (6) and the fact that the measure μ_n does not charge the diagonals.*

3 The case of f given by a multimeasure

We recall that if $(X_1, \mathcal{B}_1), \ldots, (X_n, \mathcal{B}_n)$ are measurable spaces, a mapping $\mu: \mathcal{B}_1 \times \cdots \times \mathcal{B}_n \longrightarrow \mathbb{R}$ is said to be a multimeasure if for every $i \in \{1, \ldots, n\}$ and fixed $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ with $A_j \in \mathcal{B}_j$ for all $j \in \{1, \ldots, n\} \backslash \{i\}$, $\mu(A_1, \ldots, A_{i-1}, F, A_{i+1}, \ldots, A_n)$ is a finite signed measure in the variable $F \in \mathcal{B}_i$. Notice that we are denoting by $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$, the cartesian product of the $\mathcal{B}_i$ instead of their product σ-field. We refer to Nualart and Zakai (1990) for an account on the properties of multimeasures and the integration with respect to them.

We will say that a function $f:[0,1]^n \rightarrow \mathbb{R}$ is given by a multimeasure if it is Lebesgue measurable and there exists a multimeasure μ defined on $\mathcal{B}([0,1]) \times \cdots \times \mathcal{B}([0,1])$ such that

$$f(t_1, \ldots, t_n) = \mu((t_1, 1], \ldots, (t_n, 1]),$$

for all $(t_1, \ldots, t_n) \in [0,1]^n$ a.e.

There are many different extensions to the $d-$dimensional space of the notion of bounded variation function (see Clarkson and Adams, 1933). One of them is the concept of multimeasure and another one is the notion of function of bounded variation on each variable. The paper of Nualart and Zakai (1990) gives an example of function with bounded variation on each coordinate that is not given by a multimeasure.

Let us introduce the Cameron-Martin space

$$\mathcal{H} = \{\eta \in \mathcal{C}_0([0,1]) : \eta_t = \int_0^t \dot{\eta}_s ds,\ \dot{\eta} \in L^2([0,1])\}.$$

The main result of this section is the following theorem:

Theorem 3.1 *Suppose that f is a symmetric function given by a multimeasure μ. Suppose also that $\{\eta_\varepsilon\}_{\varepsilon>0}$ are stochastic processes with trajectories in $\mathcal{H}$ converging in law to a fractional Brownian motion with parameter $H > 1/2$ in the space $\mathcal{C}_0([0,1])$ as $\varepsilon \to 0$. Then the family $\{I_{\eta_\varepsilon}(f)\}_{\varepsilon>0}$ (defined by expression (1)) converges in law in $\mathcal{C}_0([0,1])$ to the multiple fractional integral process $\{I_n \circ (f I_{[0,t]^n}),\ t \in [0,1]\}$.*

Proof: In Bardina and Jolis (2000) the following family of multimeasures $\{\overline{\mu}_t\}_{t\in[0,1]}$ is introduced:

$$\overline{\mu}_t(A_1,\ldots,A_n)=\mu(A_1(t),\ldots,A_n(t)),$$

where for any $A\in\mathcal{B}([0,1])$ and $t\in[0,1]$ define $A(t)$ as $A\cap[0,t]$ if $t\notin A$ and $A\cup(t,1]$ if $t\in A$.

These multimeasures have the property

$$\overline{\mu}_t((x_1,t],\ldots,(x_n,t])=\mu((x_1,1],\ldots,(x_n,1])=f(x_1\ldots,x_n),$$

for all $x_i<t,\quad i\in\{1,\ldots,n\}$.

Since the η_ε are absolutely continuous functions, we have that

$$\int_{[0,t]^n}f(x_1,\ldots,x_n)d\eta_\varepsilon(x_1)\cdots d\eta_\varepsilon(x_n)=\int_{[0,t]^n}\overline{\mu}_t((x_1,t],\ldots,(x_n,t])d\eta_\varepsilon(x_1)\cdots d\eta_\varepsilon(x_n).$$

Using now integration by parts on each coordinate (η_ε are continuous), the fact that $\overline{\mu}_t((x_1,t],\ldots,(x_n,t])=0$ when some $x_i=t$ and that $\eta_\varepsilon(0)=0$, the last equality becomes

$$\int_{[0,t]^n}f(x_1,\ldots,x_n)d\eta_\varepsilon(x_1)\cdots d\eta_\varepsilon(x_n)=\int_{[0,t]^n}\eta_\varepsilon(x_1)\cdots\eta_\varepsilon(x_n)\overline{\mu}_t(dx_1,\ldots,dx_n),\tag{8}$$

Even more, if f is given by a multimeasure, the operator

$$\begin{array}{cccl}\varphi_f: & \mathcal{H} & \longrightarrow & \mathcal{C}_0([0,1])\\ & \eta & \longrightarrow & \varphi_f(\eta)_t=\int_0^t\cdots\int_0^t f(x_1,\ldots,x_n)d\eta(x_1)\cdots d\eta(x_n).\end{array}$$

possesses a continuous extension on $\mathcal{C}_0([0,1])$, denoted by $\overline{\varphi}_f$, given by

$$\overline{\varphi}_f(\eta)_t=\int_{[0,t]^n}\eta(x_1)\cdots\eta(x_n)\overline{\mu}_t(dx_1\ldots,dx_n),$$

for any $\eta\in\mathcal{C}_0([0,1])$ (see Theorem 3.1 of Bardina and Jolis, 2000).

Then $I_{\eta_\varepsilon}(f)=\overline{\varphi}_f(\eta_\varepsilon)$ will converge in law in $\mathcal{C}_0([0,1])$ to $\overline{\varphi}(B^H)$, that is

$$\int_{[0,t]^n}B^H_{x_1}\cdots B^H_{x_n}\overline{\mu}_t(dx_1,\ldots,dx_n).$$

For all fixed $t\in[0,1]$, $f(x_1,\ldots,x_n)I_{[0,t]^n}(x_1,\ldots,x_n)=\overline{\mu}_t((x_1,t],\ldots,(x_n,t])I_{[0,t]^n}(x_1,\ldots,x_n)$ a.e. is Stratonovich integrable with respect to B^H, because it is an essentially bounded function. Hence, in order to finish the proof, it suffices to see that (a.s)

$$I_n\circ(fI_{[0,t]^n})=\int_{[0,t]^n}B^H_{x_1}\cdots B^H_{x_n}\overline{\mu}_t(dx_1,\ldots,dx_n).\tag{9}$$

By using Proposition 2.4, the left hand side of (9) is the limit when $|\pi|\to0$ in $L^2(\bar{\Omega})$ of

$$\sum_{i_1,\ldots,i_n}\frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|}\Big(\int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}}f(t_1,\ldots,t_m)1_{[0,t]^n}dt_1\cdots dt_n\Big)B^H(\Delta_{i_1})\cdots B^H(\Delta_{i_n}).\tag{10}$$

But, if we define the following absolutely continuous processes

$$B^{\pi}(x)=\sum_{j}\left[B^H_{t_{j-1}}+\frac{x-t_{j-1}}{t_j-t_{j-1}}\Big(B^H_{t_j}-B^H_{t_{j-1}}\Big)\right]I_{(t_{j-1},t_j]}(x),$$

expression (10) coincides with

$$\int_{[0,t]^n}\overline{\mu}_t((x_1,t],\ldots,(x_n,t])dB^{\pi}(x_1)\cdots dB^{\pi}(x_n).$$

By using once more integration by parts on each coordinate this last expression turns to be equal to

$$\int_{[0,t]^n}B^{\pi}(x_1)\cdots B^{\pi}(x_n)\overline{\mu}_t(dx_1,\ldots,dx_n).$$

Since B^{π} tends, as $|\pi|\to 0$, to B^H uniformly in $[0,1]$ almost surely, by using the properties of the integrals with respect to multimeasures, the last expression converges almost surely to

$$\int_{[0,t]^n}B^H_{x_1}\cdots B^H_{x_n}\overline{\mu}_t(dx_1,\ldots,dx_n).$$

This fact concludes the proof. □

4 The case of $f\in L^2(\tilde{\mu}_n)$

In this section we consider the processes

$$\eta_\varepsilon(t)=\int_0^t K_H(s,t)\theta_\varepsilon(s)ds,\quad t\in[0,1],$$

where θ_ε are either Stroock or Donsker kernels (see Section 2.1). We will assume, in the case of Donsker's kernels, that the random variables ξ_k have finite moments of order $2n$.

Let $f\in L^2(\mu_n)$, with μ_n defined in Section 2.2, be a symmetric function. We want to study the weak convergence in $\mathcal{C}_0([0,1])$ of the family of laws of $\{I_{\eta_\varepsilon}(f)\}_{\varepsilon>0}$, with

$$I_{\eta_\varepsilon}(f)_t=\int_0^t\cdots\int_0^t f(t_1,\ldots,t_n)d\eta_\varepsilon(t_1)\cdots d\eta_\varepsilon(t_n);\quad t\in[0,1].$$

Due to identity (5), we have the following equivalent expression for I_{η_ε}:

$$I_{\eta_\varepsilon}(f)_t=\int_0^t\cdots\int_0^t f(x_1,\ldots,x_n)\tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_n)dx_1\cdots dx_n,$$

where, by definition

$$\tilde{\theta}_\varepsilon(x)=\int_0^x\frac{\partial}{\partial x}K_H(x,r)\theta_\varepsilon(r)dr.$$

Before giving the main result of this section, let us state the following lemma that provides the basic inequality for the proof of the convergence in law of $\{I_{\eta_\varepsilon}\}$. This lemma is proved in the Appendix.

Lemma 4.1 *With the previous definitions, let f be a symmetric function in the space $L^2(\mu_n)$. Then, for any $0 \le s \le t \le 1$,*

$$\begin{aligned} & E\left[\int_{[s,t]\times[0,t]^{(n-1)}} f(x_1,\ldots,x_n)\tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_n)dx_1\cdots dx_n\right]^2 \\ \le\ & C_{H,n}(t-s)^{2H-1}\left(\int_{[s,t]\times[0,t]^{n-1}} f^2(x_1,\ldots,x_n)\mu_n(dx_1,\ldots,dx_n)\right). \end{aligned}$$

We can now state the principal result of the section.

Theorem 4.2 *Let $f \in L^2(\mu_n)$ be a symmetric function and consider the approximations η_ε of the fBm corresponding to Donsker or Stroock kernels. Then, the family of processes $\{I_{\eta_\varepsilon}(f)\}_\varepsilon$ converges in law in $\mathcal{C}_0([0,1])$ to the multiple fractional integral process $\{I_n \circ (fI_{[0,t]^n}),\ t \in [0,1]\}$ as $\varepsilon \to 0$.*

Proof: Let us prove first the tightness of the family. We have that for all $s \le t$

$$\begin{aligned} & E(I_{\eta_\varepsilon}(f)_t - I_{\eta_\varepsilon}(f)_s)^2 \\ =\ & E\left(\int_{[0,t]^n\setminus[0,s]^n} f(x_1,\ldots,x_n)\tilde{\theta}_\varepsilon(x_1)\ldots\tilde{\theta}_\varepsilon(x_n)dx_1\cdots dx_n\right)^2 \\ \le\ & C_{H,n}(t-s)^{2H-1}\left(\int_{[s,t]\times[0,t]^{(n-1)}} f^2(x_1,\ldots,x_n)\mu_n(dx_1,\ldots,dx_n)\right), \end{aligned}$$

where we have applied Lemma 4.1 and the symmetry of f.

Since $c \cdot d \le \frac{1}{p}c^p + \frac{1}{q}d^q$ for all $c, d > 0$ and any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then the last expression can be bounded by

$$\begin{aligned} & C_{H,p,n}(t-s)^{(2H-1)p} + C_{H,p,n}\left(\int_{[s,t]\times[0,1]^{(n-1)}} f^2(x_1,\ldots,x_n)\mu_n(dx_1,\ldots,dx_n)\right)^q \\ \le\ & C(F(t)-F(s))^{[(2H-1)p]\wedge q}, \end{aligned}$$

with $F(x) = x + \int_{[0,x]\times[0,1]^{(n-1)}} f^2(x_1,\ldots,x_n)\mu_n(dx_1,\ldots,dx_n)$, and the constant C depending on H, p, n and f.

Take now $p > \frac{1}{2H-1}$, so that $[(2H-1)p]\wedge q > 1$. Then, the tightness is obtained by Billingsley's criterion (see Theorem 12.3 of Billingsley, 1968).

Let us now prove that the finite dimensional distributions of $I_{\eta_\varepsilon}(f)$ converge weakly to those of $I_n \circ (f)$.

Let h be a function defined on $\mathbb{R}^m$, with continuous and bounded first partial derivatives. We will check that, for all $t_1, \ldots, t_m \in [0,1]$,

$$|E[h(I_{\eta_\varepsilon}(f)_{t_1},\ldots,I_{\eta_\varepsilon}(f)_{t_n})] - \bar{E}[h(I_n \circ (fI_{[0,t_1]^n}),\ldots,I_n \circ (fI_{[0,t_m]^n}))]|$$

converges to zero when ε tends to zero.

For the sake of simplicity, we introduce the following notations:

$$\begin{aligned}
X_t &= I_n \circ (f I_{[0,t]^n}),\\
X_t^{\varepsilon} &= I_{\eta_\varepsilon}(f)_t = \int_{[0,t]^n} \tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_n) f(x_1,\ldots,x_n)dx_1\cdots dx_n,\\
X_t^{\varepsilon,\pi} &= \int_{[0,1]^n} \tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_n)\Big(\sum_{i_1,\ldots,i_n} \frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|}\\
&\quad\times\big(\int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} I_{[0,t]^n} f(y_1,\ldots,y_n)dy_1\cdots dy_n\big) I_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}}(x_1,\ldots,x_n)\Big)dx_1\cdots dx_n,\\
X_t^{\pi} &= \sum_{i_1,\ldots,i_n} \big(\frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|}\int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} I_{[0,t]^n} f(y_1,\ldots,y_n)dy_1\cdots dy_n\big) B^H(\Delta_{i_1})\cdots B^H(\Delta_{i_n}),
\end{aligned}$$

where Δ_{i_j} are the intervals of a partition π of $[0,1]$ containing the points $t_1,\ldots,t_m$. We have that

$$|E[h(X_{t_1}^{\varepsilon},\ldots,X_{t_m}^{\varepsilon})] - \bar{E}[h(X_{t_1},\ldots,X_{t_m})]| \le I_1 + I_2 + I_3,$$

with

$$\begin{aligned}
I_1 &= |E[h(X_{t_1}^{\varepsilon},\ldots,X_{t_m}^{\varepsilon}) - h(X_{t_1}^{\varepsilon,\pi},\ldots,X_{t_m}^{\varepsilon,\pi})]|\\
I_2 &= |E[h(X_{t_1}^{\varepsilon,\pi},\ldots,X_{t_m}^{\varepsilon,\pi})] - \bar{E}[h(X_{t_1}^{\pi},\ldots,X_{t_m}^{\pi})]|\\
I_3 &= |\bar{E}[h(X_{t_1}^{\pi},\ldots,X_{t_m}^{\pi})] - \bar{E}[h(X_{t_1},\ldots,X^{\pi})]|.
\end{aligned}$$

Observe that

$$\begin{aligned}
I_1 &\le K\max_j E|X_{t_j}^{\varepsilon} - X_{t_j}^{\varepsilon,\pi}|\\
&\le K\max_j (E(X_{t_j}^{\varepsilon} - X_{t_j}^{\varepsilon,\pi})^2)^{\frac{1}{2}},
\end{aligned}$$

and that

$$E(X_{t_j}^{\varepsilon} - X_{t_j}^{\varepsilon,\pi})^2 = E\big(\int_{[0,t_j]^n} \tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_n)(f-f^{\pi})(x_1,\ldots,x_n)dx_1\cdots dx_n\big)^2,$$

where

$$\begin{aligned}
f^{\pi}(x_1,\ldots,x_n) &= \sum_i \frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|}\int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} f(y_1,\ldots,y_n)dy_1\cdots dy_n\\
&\quad\times I_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}}(x_1,\ldots,x_n).
\end{aligned}$$

So, by Lemma 4.1, we obtain

$$E(X_{t_j}^{\varepsilon} - X_{t_j}^{\varepsilon,\pi})^2 \le C_{H,n}\|f-f^{\pi}\|^2_{L^2(\mu_n)}.$$

By the arguments of the proof of Proposition 2.4, this last norm tends to zero as $|\pi| \to 0$. Hence, given $\delta > 0$, we can take $|\pi|$ sufficiently small such that $I_1 < \frac{\delta}{3}$ for any $\varepsilon > 0$.

On the other hand,

$$\begin{aligned} I_3 &= |\bar{E}[h(X^\pi_{t_1}, \ldots, X^\pi_{t_m})] - \bar{E}[h(X_{t_1}, \ldots, X_{t_m})]| \\ &\leq K \max_j \bar{E}|X^\pi_{t_j} - X_{t_j}|. \end{aligned}$$

Then $I_3 < \frac{\delta}{3}$ when $|\pi|$ is small enough because, by Proposition 2.4, $X^\pi_t \overset{L^2(\bar{\Omega})}{\longrightarrow} X_t$ when $|\pi|$ tends to zero.

Finally, for a fixed partition π, with sufficiently small norm,

$$I_2 = |E[h(X^{\varepsilon,\pi}_{t_1}, \ldots, X^{\varepsilon,\pi}_{t_m})] - \bar{E}[h(X^\pi_{t_1}, \ldots, X^\pi_{t_m})]|$$

converges to zero when ε tends to zero because $\mathcal{L}(X^{\varepsilon,\pi}_{t_1}, \ldots, X^{\varepsilon,\pi}_{t_m}) \overset{w}{\longrightarrow} \mathcal{L}(X^\pi_{t_1}, \ldots, X^\pi_{t_m})$ since the processes η_ε converge weakly to the fractional Brownian motion. Then $I_2 < \frac{\delta}{3}$ if ε is small enough. This fact concludes the proof. □

Remark 4.3 *Observe that, in view of the proof of Theorem 4.2, the inequality given by Lemma 4.1 can be seen as a sufficient condition for a family* $\{\eta_\varepsilon\}$ *to have the convergence in law of* $\{I_{\eta_\varepsilon}(f)\}$ *to the process* $\{I_n \circ (f I_{[0,t]^n}),\, t \in [0,1]\}$.

Appendix

Prior to the proof of Lemma 4.1, we need a technical result. Its first statement is related to Stroock kernels and statements (b) and (c) are related to Donsker kernels.

Lemma A.1 *For all different* $x, y, z \in (0, \infty)$ *and* $\frac{1}{2} < H < 1$ *there exists a constant* C_H*, depending only on* H*, such that*

(*a*)

$$(xy)^{H-\frac{1}{2}} \int_0^y \int_0^x (uv)^{\frac{1}{2}-H} [(x-u)(y-v)]^{H-\frac{3}{2}} \frac{1}{\varepsilon^2} \exp(-2\frac{|v-u|}{\varepsilon^2}) dudv \leq C_H |y-x|^{2H-2},$$

(*b*)

$$(xy)^{H-\frac{1}{2}} \int_0^y \int_0^x (uv)^{\frac{1}{2}-H} [(x-u)(y-v)]^{H-\frac{3}{2}} \frac{1}{\varepsilon^2} I_{[0,\varepsilon^2)}(|v-u|) dudv \leq C_H |y-x|^{2H-2},$$

(*c*)

$$\begin{aligned} &(xyz)^{H-\frac{1}{2}} \int_0^z \int_0^y \int_0^x (uvw)^{\frac{1}{2}-H} [(x-u)(y-v)(z-w)]^{H-\frac{3}{2}} \\ &\times \frac{1}{\varepsilon^3} I_{[0,\varepsilon^2)}(u \vee v \vee w - u \wedge v \wedge w) dudvdw \leq C_H (|x-y||x-z||y-z|)^{H-1}. \end{aligned}$$

Notice that inequality (b) follows from (a), since for all $x \in \mathbb{R}$, $I_{[0,\varepsilon^2)}(x) \leq e^2 e^{-2\frac{x}{\varepsilon^2}}$.

On the other hand, if we do the change of variables $u = u'\varepsilon^2$, $v = v'\varepsilon^2$, $w = w'\varepsilon^2$ and we replace $\frac{x}{\varepsilon^2}$, $\frac{y}{\varepsilon^2}$, $\frac{z}{\varepsilon^2}$ by x', y', z' respectively, statements (a) and (c) are respectively equivalent to

$$(x'y')^{H-\frac{1}{2}} \int_0^{y'} \int_0^{x'} (u'v')^{\frac{1}{2}-H} [(x'-u')(y'-v')]^{H-\frac{3}{2}} \exp(-2|v'-u'|) du' dv' \leq C_H |y'-x'|^{2H-2},$$

and

$$\begin{aligned} &(x'y'z')^{H-\frac{1}{2}} \int_0^{z'} \int_0^{y'} \int_0^{x'} (u'v'w')^{\frac{1}{2}-H} [(x'-u')(y'-v')(z'-w')]^{H-\frac{3}{2}} \\ &\times I_{[0,1)}(u' \vee v' \vee w' - u' \wedge v' \wedge w') du' dv' dw' \leq C_H (|x'-y'||x'-z'||y'-z'|)^{H-1}, \end{aligned}$$

where x', y', z' are three real numbers, different and positive.

If we perform now the change of variables $u = u'x$, $v = v'y$ and $w = w'z$, we see that Lemma A.1 is equivalent to the following result:

Lemma A.2 *Consider U, V and W three independent random variables with beta distribution of parameters $\alpha = \frac{3}{2} - H$ and $\beta = H - \frac{1}{2}$, where $H \in (\frac{1}{2}, 1)$. Then, for all different $x, y, z \in (0, \infty)$ there exists a constant C_H, depending only on H, such that*

(i) $E\left(\exp(-2|Ux - Vy|)\right) \leq C_H (xy)^{\frac{1}{2}-H} |x-y|^{2H-2}$

(ii) $E\left(I_{[0,1)}\left(\max(Ux, Vy, Wz) - \min(Ux, Vy, Wz)\right)\right) \leq C_H (xyz)^{\frac{1}{2}-H} \left(|x-y||x-z||y-z|\right)^{H-1}$.

Proof:

- Proof of statement (i).

Throughout this proof we will assume, without loosing the generality, that $y > x$. Observe that

$$\begin{aligned} & E\left(\exp(-2|Ux - Vy|)\right) \\ = \;& C_H \int_0^1 \int_0^1 (uv)^{\frac{1}{2}-H} \left[(1-u)(1-v)\right]^{H-\frac{3}{2}} e^{-2|ux-vy|} du dv \\ = \;& C_H \int_0^y \int_0^x (uv)^{\frac{1}{2}-H} \left[(x-u)(y-v)\right]^{H-\frac{3}{2}} e^{-2|u-v|} du dv, \end{aligned}$$

making a change of variables.

We have to prove that the last integral is bounded by $C_H (xy)^{\frac{1}{2}-H} |x-y|^{2H-2}$. We will divide the domain of integration in seven parts such as it is shown in Figure 1.

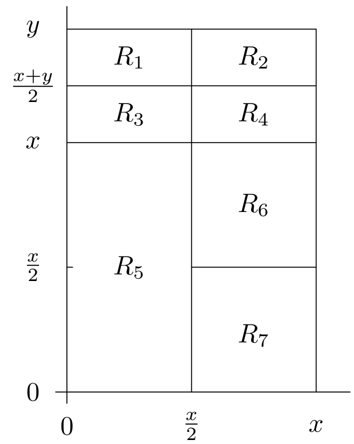

Figure 1: Partition in seven parts of the domain of integration.

Integral over R_1

$$I_1 = \int_{\frac{x+y}{2}}^{y} \int_{0}^{\frac{x}{2}} (uv)^{\frac{1}{2}-H} [(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2(v-u)} dudv.$$

Since over the region where we integrate $v \geq \frac{x+y}{2} \geq \frac{y}{2}$ and $x-u \geq \frac{x}{2}$, we obtain that

$$I_1 \leq C_H y^{\frac{1}{2}-H} x^{H-\frac{3}{2}} \int_{\frac{x+y}{2}}^{y} \int_{0}^{\frac{x}{2}} u^{\frac{1}{2}-H} (y-v)^{H-\frac{3}{2}} e^{-2(v-u)} dudv.$$

We have also that

$$v-u \geq \frac{x+y}{2} - \frac{x}{2} = \frac{y}{2}.$$

Then, $e^{-2(v-u)} \leq e^{-y} = e^{-x}e^{-(y-x)}$ and

$$\begin{aligned} I_1 &\leq C_H y^{\frac{1}{2}-H} e^{-x} e^{-(y-x)} (y-x)^{H-\frac{1}{2}} \\ &\leq C_H x^{\frac{1}{2}-H} y^{\frac{1}{2}-H} (y-x)^{2H-2}, \end{aligned}$$

since $e^{-x} \leq C_\alpha x^{-\alpha}$ for all $x > 0$ and $\alpha > 0$.

Integral over R_2

$$I_2 = \int_{\frac{x+y}{2}}^{y} \int_{\frac{x}{2}}^{x} (uv)^{\frac{1}{2}-H} [(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2(v-u)} dudv.$$

Since $v \geq \frac{x+y}{2} \geq \frac{y}{2}$ and $u \geq \frac{x}{2}$, we have that

$$I_2 \leq C_H x^{\frac{1}{2}-H} y^{\frac{1}{2}-H} \int_{\frac{x+y}{2}}^{y} \int_{\frac{x}{2}}^{x} [(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2(v-u)} du dv.$$

Moreover,

$$v - u \geq \frac{x+y}{2} - u = \frac{y-x}{2} + x - u.$$

Then, $e^{-2(v-u)} \leq e^{-(y-x)} e^{-2(x-u)}$ and

$$\begin{aligned} I_2 &\leq C_H x^{\frac{1}{2}-H} y^{\frac{1}{2}-H} e^{-(y-x)} \int_{\frac{x+y}{2}}^{y} \int_{\frac{x}{2}}^{x} [(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2(x-u)} du dv \\ &\leq C_H x^{\frac{1}{2}-H} y^{\frac{1}{2}-H} e^{-(y-x)} (y-x)^{H-\frac{1}{2}} \int_{\frac{x}{2}}^{x} (x-u)^{H-\frac{3}{2}} e^{-2(x-u)} du. \end{aligned}$$

Making the change of variables $x - u = u'$ and using that $\int_0^\infty u^{H-\frac{3}{2}} e^{-2u} du < \infty$, we have that

$$I_2 \leq C_H x^{\frac{1}{2}-H} y^{\frac{1}{2}-H} (y-x)^{2H-2}.$$

Integral over R_7

$$I_7 = \int_0^{\frac{x}{2}} \int_{\frac{x}{2}}^{x} (uv)^{\frac{1}{2}-H} [(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2(u-v)} du dv.$$

Writing $(y-v)^{H-\frac{3}{2}} = (y-v)^{2H-2}(y-v)^{\frac{1}{2}-H}$ and using that over the region where we integrate $u - v \geq u - \frac{x}{2}$, we can majorize I_7 as follows

$$\begin{aligned} I_7 &\leq C_H (y-x)^{2H-2} x^{\frac{1}{2}-H} \int_0^{\frac{x}{2}} \int_{\frac{x}{2}}^{x} (v)^{\frac{1}{2}-H} (x-u)^{H-\frac{3}{2}} (y-v)^{\frac{1}{2}-H} e^{-2\left(u-\frac{x}{2}\right)} du dv \\ &= C_H (y-x)^{2H-2} x^{\frac{1}{2}-H} \int_0^{\frac{x}{2}} \int_0^{\frac{x}{2}} (v)^{\frac{1}{2}-H} \left(\frac{x}{2} - u\right)^{H-\frac{3}{2}} (y-v)^{\frac{1}{2}-H} e^{-2u} du dv. \end{aligned}$$

It is easy to see that for any $a > 0$

$$\int_0^a (a-u)^{H-\frac{3}{2}} e^{-2u} du \leq C_H a^{H-\frac{3}{2}}.$$

¿From this fact, we can majorize the last expression by

$$\begin{aligned} & C_H (y-x)^{2H-2} x^{\frac{1}{2}-H} x^{H-\frac{3}{2}} \int_0^{\frac{x}{2}} [v(y-v)]^{\frac{1}{2}-H} dv \\ \leq\ & C_H (y-x)^{2H-2} x^{\frac{1}{2}-H} x^{H-\frac{3}{2}} \left(\frac{y}{2}\right)^{\frac{1}{2}-H} \int_0^{\frac{x}{2}} v^{\frac{1}{2}-H} dv \\ \leq\ & C_H (y-x)^{2H-2} x^{\frac{1}{2}-H} y^{\frac{1}{2}-H}. \end{aligned}$$

Integrals over regions from R_3 to R_6

We will consider two cases:

- Suppose first that $y < 2x$, and so, $y - x < x$.

Since over the integration area $(y-v)^{H-\frac{3}{2}} = (y-v)^{2H-2}(y-v)^{\frac{1}{2}-H} \le \left(\frac{y-x}{2}\right)^{2H-2}(y-v)^{\frac{1}{2}-H}$ the term

$$I_3 = \int_x^{\frac{x+y}{2}} \int_0^{\frac{x}{2}} (uv)^{\frac{1}{2}-H}[(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2(v-u)} du dv$$

can be majorized as follows:

$$\begin{aligned}
I_3 &\le C_H x^{\frac{1}{2}-H} x^{H-\frac{3}{2}} (y-x)^{2H-2} \int_x^{\frac{x+y}{2}} \int_0^{\frac{x}{2}} u^{\frac{1}{2}-H}(y-v)^{\frac{1}{2}-H} e^{-2(v-u)} du dv \\
&\le C_H \frac{1}{x}(y-x)^{2H-2} \int_x^{\frac{x+y}{2}} \int_0^{\frac{x}{2}} u^{1-2H} e^{-2(v-u)} du dv \\
&\quad + C_H \frac{1}{x}(y-x)^{2H-2} \int_x^{\frac{x+y}{2}} \int_0^{\frac{x}{2}} (y-v)^{1-2H} e^{-2(v-u)} du dv \\
&\le C_H \frac{1}{x}(y-x)^{2H-2}[x^{2-2H} + (y-x)^{2-2H}] \\
&\le C_H \frac{1}{x}(y-x)^{2H-2} x^{2-2H} \\
&\le C_H (y-x)^{2H-2} x^{\frac{1}{2}-H} y^{\frac{1}{2}-H},
\end{aligned}$$

because $y < 2x$.

Consider now the integral

$$I_4 = \int_x^{\frac{x+y}{2}} \int_{\frac{x}{2}}^{x} (uv)^{\frac{1}{2}-H}[(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2(v-u)} du dv.$$

Over the region R_4 we have that $(y-v)^{H-\frac{3}{2}} \le C_H(y-x)^{2H-2}(y-v)^{\frac{1}{2}-H} \le C_H(y-x)^{2H-2}(v-x)^{\frac{1}{2}-H}$. We have also that $\frac{x+y}{2} \le \frac{3x}{2}$. Then,

$$\begin{aligned}
I_4 &\le C_H x^{1-2H}(y-x)^{2H-2} \\
&\quad \times \int_x^{\frac{3}{2}x} \int_{\frac{x}{2}}^{x} (x-u)^{H-\frac{3}{2}}(v-x)^{\frac{1}{2}-H} e^{-2(x-u)} e^{-2(v-x)} du dv.
\end{aligned}$$

By the change of variables, $v - x = v'$ and $x - u = u'$, we have that the last expression equals to

$$\begin{aligned}
&C_H x^{1-2H}(y-x)^{2H-2} \int_0^{\frac{x}{2}} (v')^{\frac{1}{2}-H} e^{-2v'} dv' \int_0^{\frac{x}{2}} (u')^{H-\frac{3}{2}} e^{-2u'} du' \\
\le\; &C_H x^{\frac{1}{2}-H} y^{\frac{1}{2}-H} (y-x)^{2H-2},
\end{aligned}$$

since $y < 2x$.

Now, consider

$$I_5 = \int_0^x \int_0^{\frac{x}{2}} (uv)^{\frac{1}{2}-H}[(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2|u-v|} dudv.$$

We have

$$I_5 \le (y-x)^{2H-2}\left(\frac{x}{2}\right)^{2H-2} \int_0^x \int_0^{\frac{x}{2}} (uv)^{\frac{1}{2}-H}[(x-u)(y-v)]^{\frac{1}{2}-H} e^{-2|v-u|} dudv.$$

But, over R_5,

$$\begin{aligned} 2(uv)^{\frac{1}{2}-H}[(x-u)(y-v)]^{\frac{1}{2}-H} &\le 2(uv)^{\frac{1}{2}-H}[(x-u)(x-v)]^{\frac{1}{2}-H} \\ &\le [u(x-u)]^{1-2H} + [v(x-v)]^{1-2H}. \end{aligned}$$

Then,

$$\begin{aligned} I_5 &\le C_H (y-x)^{2H-2} x^{2H-2} \int_0^x \int_0^x [u(x-u)]^{1-2H} e^{-2|u-v|} dudv \\ &\le C_H (y-x)^{2H-2} x^{2H-2} \int_0^x [u(x-u)]^{1-2H} du \\ &= C_H (y-x)^{2H-2} x^{2H-2} x^{3-4H} \int_0^1 [u(1-u)]^{1-2H} du \\ &\le C_H (y-x)^{2H-2} x^{1-2H} \\ &\le C_H (y-x)^{2H-2} x^{\frac{1}{2}-H} y^{\frac{1}{2}-H}. \end{aligned}$$

On the other hand, for

$$I_6 = \int_{\frac{x}{2}}^x \int_{\frac{x}{2}}^x (uv)^{\frac{1}{2}-H}[(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2|u-v|} dudv,$$

we can write

$$\begin{aligned} I_6 &\le \left(\frac{x}{2}\right)^{2(\frac{1}{2}-H)} \int_{\frac{x}{2}}^x \int_{\frac{x}{2}}^x [(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2|v-u|} dudv \\ &= C_H x^{1-2H} \int_{\frac{x}{2}}^x \int_{\frac{x}{2}}^v [(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2(v-u)} dudv \\ &\quad + C_H x^{1-2H} \int_{\frac{x}{2}}^x \int_{\frac{x}{2}}^u [(x-u)(y-v)]^{H-\frac{3}{2}} e^{-2(u-v)} dvdu \\ &\le C_H x^{1-2H} \int_{\frac{x}{2}}^x [(x-v)(y-v)]^{H-\frac{3}{2}} \left(\int_{\frac{x}{2}}^v e^{-2(v-u)} du\right) dv \\ &\quad + C_H x^{1-2H} \int_{\frac{x}{2}}^x [(x-u)(y-u)]^{H-\frac{3}{2}} \left(\int_{\frac{x}{2}}^u e^{-2(u-v)} dv\right) du \\ &\le C_H x^{1-2H} \int_{\frac{x}{2}}^x [(x-v')(y-v')]^{H-\frac{3}{2}} dv'. \end{aligned}$$

Making the change of variables $\frac{x-v'}{y-x} = w$ the last expression is bounded by

$$\begin{aligned} & C_H x^{1-2H}(y-x)^{2H-2} \int_0^{\frac{x}{2(y-x)}} (1+w)^{H-\frac{3}{2}} w^{H-\frac{3}{2}} dv \\ \leq \quad & C_H x^{\frac{1}{2}-H} y^{\frac{1}{2}-H} (y-x)^{2H-2}. \end{aligned}$$

- Suppose now that $y > 2x$, i.e. $y < 2(y-x)$. This condition implies that $x^{\frac{1}{2}-H}(y-x)^{H-\frac{3}{2}} \leq C_H x^{\frac{1}{2}-H} y^{\frac{1}{2}-H}(y-x)^{2H-2}$. So, it is enough to bound the integrals $I_3, \ldots, I_6$ by $C_H x^{\frac{1}{2}-H}(y-x)^{H-\frac{3}{2}}$.

First,

$$\begin{aligned} I_3 \quad & \leq \quad C_H x^{\frac{1}{2}-H}(y-x)^{H-\frac{3}{2}} \int_x^{\frac{x+y}{2}} \int_0^{\frac{x}{2}} u^{\frac{1}{2}-H}(x-u)^{H-\frac{3}{2}} e^{-2(v-u)} du dv \\ & \leq \quad C_H x^{\frac{1}{2}-H}(y-x)^{H-\frac{3}{2}} \int_0^x u^{\frac{1}{2}-H}(x-u)^{H-\frac{3}{2}} du \\ & = \quad C_H x^{\frac{1}{2}-H}(y-x)^{H-\frac{3}{2}}. \end{aligned}$$

Consider now I_4,

$$\begin{aligned} I_4 \quad & \leq \quad C_H x^{2(\frac{1}{2}-H)}(y-x)^{H-\frac{3}{2}} \int_x^{\frac{x+y}{2}} \int_{\frac{x}{2}}^x (x-u)^{H-\frac{3}{2}} e^{-2(v-u)} du dv \\ & \leq \quad C_H x^{1-2H}(y-x)^{H-\frac{3}{2}} \int_{\frac{x}{2}}^x (x-u)^{H-\frac{3}{2}} du \\ & \leq \quad C_H (y-x)^{H-\frac{3}{2}} x^{\frac{1}{2}-H}. \end{aligned}$$

On the other hand,

$$I_5 \leq (y-x)^{H-\frac{3}{2}} \left(\frac{x}{2}\right)^{H-\frac{3}{2}} \int_0^x \int_0^{\frac{x}{2}} (uv)^{\frac{1}{2}-H} e^{-2|v-u|} du dv.$$

But, using that

$$2(uv)^{\frac{1}{2}-H} \leq u^{1-2H} + v^{1-2H},$$

we have that

$$\begin{aligned} I_5 \quad & \leq \quad C_H (y-x)^{H-\frac{3}{2}} x^{H-\frac{3}{2}} \int_0^x \int_0^x u^{1-2H} e^{-2|v-u|} du dv \\ & \leq \quad C_H (y-x)^{H-\frac{3}{2}} x^{H-\frac{3}{2}} \int_0^x u^{1-2H} du \\ & \leq \quad C_H (y-x)^{H-\frac{3}{2}} x^{\frac{1}{2}-H}. \end{aligned}$$

Finally,

$$
\begin{aligned}
I_6 &\le (y-x)^{H-\frac{3}{2}} \left(\frac{x}{2}\right)^{2(\frac{1}{2}-H)} \int_{\frac{x}{2}}^{x}\int_{\frac{x}{2}}^{x} (x-u)^{H-\frac{3}{2}} e^{-2|v-u|} du dv \\
&\le C_H (y-x)^{H-\frac{3}{2}} x^{1-2H} \int_{\frac{x}{2}}^{x} (x-u)^{H-\frac{3}{2}} du \\
&= C_H (y-x)^{H-\frac{3}{2}} x^{\frac{1}{2}-H}.
\end{aligned}
$$

- Proof of the statement (ii).

Notice that for all x, $I_{[0,1)}(x) \le e^2 e^{-2x}$. So, statement (i) implies that,

$$
E\left(I_{[0,1)}(|Ux - Vy|)\right) \le C_H (xy)^{\frac{1}{2}-H} |x-y|^{2H-2}. \tag{11}
$$

On the other hand,

$$
\begin{aligned}
&E\left(I_{[0,1)}\left(\max(Ux,Vy,Wz) - \min(Ux,Vy,Wz)\right)\right) \\
=&\int_0^1\int_0^1 I_{[0,1)}(|Vy-Wz|)\left(\int_0^1 I_{[0,1)}\left(\max(Ux,Vy.Wz) - \min(Ux,Vy,Wz)\right) f(u)du\right) f(v)f(w) dv dw,
\end{aligned}
$$

where f is the density function of a beta random variable of parameters $\alpha = \frac{3}{2} - H$ and $\beta = H - \frac{1}{2}$. Using Schwartz inequality, the last expression is bounded by

$$
\begin{aligned}
&\left(E\left(I_{[0,1)}|Vy - Wz|\right)\right)^{\frac{1}{2}} \\
\times&\left(E\left[I_{[0,1)}(\max(Ux,Vy,Wz) - \min(Ux,Vy,Wz)) I_{[0,1)}(\max(U'x,Vy,Wz) - \min(U'x,Vy,Wz))\right]\right)^{\frac{1}{2}},
\end{aligned}
$$

where U' is another beta random variable with the same parameters and independent of the variables U, V and W.

Now, we have the following inequalities:

$$
\begin{aligned}
I_{[0,1)}\left(\max(Ux,Vy,Wz) - \min(Ux,Vy,Wz)\right) &\le I_{[0,1)}(|Ux - Vy|) \quad \text{and} \\
I_{[0,1)}\left(\max(U'x,Vy,Wz) - \min(U'x,Vy,Wz)\right) &\le I_{[0,1)}(|U'x - Wz|).
\end{aligned}
$$

Finally, using these bounds, that U, U', V and W are independent random variables and inequality (11) we have that

$$
\begin{aligned}
&E\left(I_{[0,1)}\left(\max(Ux,Vy,Wz) - \min(Ux,Vy,Wz)\right)\right) \\
\le\ & C_H \left((yz)^{\frac{1}{2}-H}|y-z|^{2H-2}\right)^{\frac{1}{2}} \left((xy)^{\frac{1}{2}-H}||x-y|^{2H-2}(xz)^{\frac{1}{2}-H}|x-z|^{2H-2}\right)^{\frac{1}{2}} \\
\le\ & C_H (xyz)^{\frac{1}{2}-H} \left(|x-y||x-z||y-z|\right)^{H-1}.
\end{aligned}
$$

This completes the proof of Lemma A.2. □

Proof of Lemma 4.1 for the case of Stroock kernels and $n > 1$

We have that

$$\begin{aligned} & E\left[\int_{[s,t]\times[0,t]^{(n-1)}} f(x_1,\ldots,x_n)\tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_n)dx_1\cdots dx_n\right]^2 \\ = & \int_{([s,t]\times[0,t]^{(n-1)})^2} f(x_1,\ldots,x_n)f(x_{n+1},\ldots,x_{2n})E\left[\tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_{2n})\right]dx_1\cdots dx_{2n}. \end{aligned} \tag{12}$$

We will first obtain a bound for the expectation appearing in the last integral. By equality (4), we have

$$\begin{aligned} & E\left[\tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_{2n})\right] \\ = & E\left[\int_0^{x_1}\cdots\int_0^{x_{2n}}\frac{\partial}{\partial x_1}K(x_1,y_1)\theta_\varepsilon(y_1)\cdots\frac{\partial}{\partial x_{2n}}K(x_{2n},y_{2n})\theta_\varepsilon(y_{2n})dy_{2n}\cdots dy_1\right] \\ = & C_{H,n}\int_0^{x_1}\cdots\int_0^{x_{2n}}(x_1\cdots x_{2n})^{H-\frac{1}{2}}(y_1\cdots y_{2n})^{\frac{1}{2}-H}\left[(x_1-y_1)\cdots(x_{2n}-y_{2n})\right]^{H-\frac{3}{2}} \\ & \times|E\left[\theta_\varepsilon(y_1)\cdots\theta_\varepsilon(y_{2n})\right]|dy_{2n}\cdots dy_1. \end{aligned}$$

Since the Poisson process has independent increments and if $Z\sim\text{Poiss}(\lambda)$ then $E[(-1)^Z]) = \exp(-2\lambda)$, we obtain that the last expression equals to

$$\begin{aligned} & C_{H,n}\int_0^{x_1}\cdots\int_0^{x_{2n}}(x_1\cdots x_{2n})^{H-\frac{1}{2}}(y_1\cdots y_{2n})^{\frac{1}{2}-H}\left[(x_1-y_1)\cdots(x_{2n}-y_{2n})\right]^{H-\frac{3}{2}} \\ & \times\frac{1}{\varepsilon^2}\exp\left(-2\frac{y_{(2n)}-y_{(2n-1)}}{\varepsilon^2}\right)\cdots\frac{1}{\varepsilon^2}\exp\left(-2\frac{y_{(2)}-y_{(1)}}{\varepsilon^2}\right)dy_{2n}\cdots dy_1, \end{aligned}$$

where $y_{(1)}, y_{(2)},\ldots,y_{(2n)}$ are the variables $y_1, y_2,\cdots,y_{2n}$ in increasing order.

The above expression is bounded by

$$\begin{aligned} & C_{H,n}\int_0^{x_1}\cdots\int_0^{x_{2n}}(x_1\cdots x_{2n})^{H-\frac{1}{2}}(y_1\cdots y_{2n})^{\frac{1}{2}-H}\left[(x_1-y_1)\cdots(x_{2n}-y_{2n})\right]^{H-\frac{3}{2}} \\ & \times\sum_{\sigma\in\mathcal{P}(2n)}\frac{1}{\varepsilon^2}\exp\left(-2\frac{|y_{\sigma(1)}-y_{\sigma(2)}|}{\varepsilon^2}\right)\cdots\frac{1}{\varepsilon^2}\exp\left(-2\frac{|y_{\sigma(2n-1)}-y_{\sigma(2n)}|}{\varepsilon^2}\right)dy_{2n}\cdots dy_1, \end{aligned}$$

where $\mathcal{P}(2n)$ is the set of all the possible permutations of $\{1,\ldots,2n\}$.

Using statement (a) of Lemma A.1, the last expression is bounded by

$$C_{H,n}\sum_{\sigma\in\mathcal{P}(2n)}|x_{\sigma(1)}-x_{\sigma(2)}|^{2H-2}\cdots|x_{\sigma(2n-1)}-x_{\sigma(2n)}|^{2H-2}.$$

By substituting this last bound in expression (12), we have that

$$\begin{aligned}
& E\left[\int_{[s,t]\times[0,t]^{(n-1)}} f(x_1,\ldots,x_n)\tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_n)dx_1\cdots dx_n\right]^2 \\
\leq\ & C_{H,n}\sum_{\sigma\in\mathcal{P}(2n)}\int_{([s,t]\times[0,t]^{(n-1)})^2}|f(x_1,\ldots,x_n)||f(x_{n+1},\ldots,x_{2n})| \\
& \times|x_{\sigma(1)}-x_{\sigma(2)}|^{2H-2}\cdots|x_{\sigma(2n-1)}-x_{\sigma(2n)}|^{2H-2}dx_1\cdots dx_{2n} \\
\leq\ & C_{H,n}\sum_{\sigma\in\mathcal{P}(2n)}\int_{([s,t]\times[0,t]^{(n-1)})^2} f^2(x_1,\ldots,x_n) \\
& \times|x_{\sigma(1)}-x_{\sigma(2)}|^{2H-2}\cdots|x_{\sigma(2n-1)}-x_{\sigma(2n)}|^{2H-2}dx_1\cdots dx_{2n},
\end{aligned}$$

where we have made a change of variables in the second integral.

Fix now a permutation σ, we will prove that

$$\begin{aligned}
& \int_{([s,t]\times[0,t]^{(n-1)})^2} f^2(x_1,\ldots,x_n) \\
& \times|x_{\sigma(1)}-x_{\sigma(2)}|^{2H-2}\cdots|x_{\sigma(2n-1)}-x_{\sigma(2n)}|^{2H-2}dx_1\cdots dx_{2n} \\
\leq\ & C_{H,n}(t-s)^{2H-1}\left(\int_{[s,t]\times[0,t]^{n-1}} f^2(x_1,\ldots,x_n)\mu_n(dx_1,\ldots,dx_n)\right),
\end{aligned} \tag{13}$$

and this inequality will conclude the proof.

Indeed, in the left hand side of expression (13) we can find the variable x_{n+1} in three different situations depending on the permutation σ (notice that variables x_{n+1} and x_1 are the only two that we integrate over $[s,t]$):

- If we have $|x_{n+1}-x_j|^{2H-2}$ with $n+2\leq j\leq 2n$, then

$$\begin{aligned}
& \int_s^t\int_0^t|x_{n+1}-x_j|^{2H-2}dx_jdx_{n+1} \\
=\ & \int_s^t\int_0^t(x_{n+1}-x_j)^{2H-2}I_{\{x_j\leq x_{n+1}\}}dx_jdx_{n+1}+\int_s^t\int_0^t(x_j-x_{n+1})^{2H-2}I_{\{x_j>x_{n+1}\}}dx_jdx_{n+1} \\
\leq\ & C_{H,n}\left((t^{2H}-s^{2H})+(t-s)^{2H}\right) \\
\leq\ & C_{H,n}\left((t-s)+(t-s)^{2H}\right) \\
\leq\ & C_{H,n}(t-s)^{2H-1}.
\end{aligned}$$

  And, with this bound, inequality (13) follows by integrating with respect to the variables x_i for $i\neq j$ and $i>n+1$. We need also to use the symmetry of f and that the measure μ_n dominates all the measures ν_n^k.

- If we have $|x_{n+1}-x_j|^{2H-2}$ where $2 \le j \le n$, then

$$\begin{aligned} & \int_s^t \int_0^t f^2(x_1,\dots,x_j,\dots,x_n)|x_{n+1}-x_j|^{2H-2}dx_j dx_{n+1} \\ = & \int_0^t \int_s^t f^2(x_1,\dots,x_j,\dots,x_n)(x_{n+1}-x_j)^{2H-2}I_{\{x_{n+1}>x_j\}}dx_{n+1}dx_j \\ & + \int_0^t \int_s^t f^2(x_1,\dots,x_j,\dots,x_n)(x_j-x_{n+1})^{2H-2}I_{\{x_{n+1}<x_j\}}dx_{n+1}dx_j. \end{aligned}$$

The first integral is equal to

$$\begin{aligned} & C_{H,n}\int_0^t f^2(x_1,\dots,x_j,\dots,x_n)\left[(t-x_j)^{2H-1}-(x_j\vee s-x_j)^{2H-1}\right]dx_j \\ \le & C_{H,n}\int_0^t f^2(x_1,\dots,x_j,\dots,x_n)\left(t-x_j\vee s\right)^{2H-1}dx_j \\ \le & C_{H,n}(t-s)^{2H-1}\int_0^t f^2(x_1,\dots,x_j,\dots,x_n)dx_j, \end{aligned}$$

and the second one is equal to

$$\begin{aligned} & C_{H,n}\int_0^t f^2(x_1,\dots,x_j,\dots,x_n)(x_j-s)^{2H-1}dx_j \\ \le & C_{H,n}(t-s)^{2H-1}\int_0^t f^2(x_1,\dots,x_j,\dots,x_n)dx_j. \end{aligned}$$

And with these majorizations we obtain easily inequality (13).

- Finally, if for the permutation σ the term $|x_{n+1}-x_1|^{2H-2}$ appears, then one of the iterated integrals in the left hand side of (13) is

$$\int_s^t \int_s^t f^2(x_1,x_2,\dots,x_n)|x_{n+1}-x_1|^{2H-2}dx_1 dx_{n+1},$$

and by the same arguments of the previous situation the last expression is bounded by

$$C_{H,n}(t-s)^{2H-1}\int_s^t f^2(x_1,\dots,x_n)dx_1.$$

Proof of Lemma 4.1 for the case of Donsker kernels and $n > 1$

When θ_ε are the Donsker kernels, by the same arguments of Lemma 4.2 of Bardina and Jolis (2000) we have that

$$\begin{aligned}
& |E\left[\theta_\varepsilon(y_1)\cdots\theta_\varepsilon(y_{2n})\right]| \\
= \; & \frac{1}{\varepsilon^{2n}}|E\prod_{j=1}^{2n}\sum_{k=1}^{\infty}\zeta_k I_{[k-1,k)}\left(\frac{y_j}{\varepsilon^2}\right)| \\
\leq \; & \frac{1}{\varepsilon^{2n}}\sum_{\sigma\in\mathcal{P}(2n)}\sum_{j=0}^{\left[\frac{n}{3}\right]}\prod_{k=1}^{2j} I_{[0,\varepsilon^2)}\left(y_{\sigma(3k-2)}\vee y_{\sigma(3k-1)}\vee y_{\sigma(3k)} - y_{\sigma(3k-2)}\wedge y_{\sigma(3k-1)}\wedge y_{\sigma(3k)}\right) \\
& \times\prod_{k=3j+1}^{n} I_{[0,\varepsilon^2)}(|y_{\sigma(2k)} - y_{\sigma(2k-1)}|),
\end{aligned}$$

where if some product does not multiply any element we understand that it is equal to 1.

Then, following the first part of the proof for the case of Stroock kernels, we have that

$$\begin{aligned}
& |E\left[\tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_{2n})\right]| \\
\leq \; & C_{H,n}\sum_{\sigma\in\mathcal{P}(2n)}\sum_{j=0}^{\left[\frac{n}{3}\right]}\int_0^{x_1}\cdots\int_0^{x_{2n}}(x_1\cdots x_{2n})^{H-\frac{1}{2}}(y_1\cdots y_{2n})^{\frac{1}{2}-H}\left[(x_1-y_1)\cdots(x_{2n}-y_{2n})\right]^{H-\frac{3}{2}} \\
& \times\prod_{k=1}^{2j}\left(\frac{1}{\varepsilon^3}I_{[0,\varepsilon^2)}\left(y_{\sigma(3k-2)}\vee y_{\sigma(3k-1)}\vee y_{\sigma(3k)} - y_{\sigma(3k-2)}\wedge y_{\sigma(3k-1)}\wedge y_{\sigma(3k)}\right)\right) \\
& \times\prod_{k=3j+1}^{n}\left(\frac{1}{\varepsilon^2}I_{[0,\varepsilon^2)}(|y_{\sigma(2k)} - y_{\sigma(2k-1)}|)\right)dy_{2n}\cdots dy_1.
\end{aligned}$$

Using now statements (b) and (c) of Lemma A.1 we have that

$$\begin{aligned}
& E\left[\int_{[s,t]\times[0,t]^{(n-1)}} f(x_1,\ldots,x_n)\tilde{\theta}_\varepsilon(x_1)\cdots\tilde{\theta}_\varepsilon(x_n)dx_1\cdots dx_n\right]^2 \\
\leq \; & C_{H,n}\sum_{\sigma\in\mathcal{P}(2n)}\sum_{j=0}^{\left[\frac{n}{3}\right]}\int_{\left([s,t]\times[0,t]^{(n-1)}\right)^2}|f(x_1,\ldots,x_n)||f(x_{n+1},\ldots,x_{2n})| \\
& \times\prod_{k=1}^{2j}\left[|x_{\sigma(3k-2)}-x_{\sigma(3k-1)}||x_{\sigma(3k-2)}-x_{\sigma(3k)}||x_{\sigma(3k-1)}-x_{\sigma(3k)}|\right]^{H-1} \\
& \times\prod_{k=3j+1}^{n}|x_{\sigma(2k)}-x_{\sigma(2k-1)}|^{2H-2}dx_1\cdots dx_{2n}. \qquad (14)
\end{aligned}$$

Now we can finish the proof by seeing that the last expression can be bounded by

$$C_{H,n}(t-s)^{2H-1}\left(\int_{[s,t]\times[0,t]^{n-1}} f^2(x_1,\ldots,x_n)\mu_n(dx_1,\ldots,dx_n)\right).$$

Indeed, fix a permutation $\sigma \in \mathcal{P}_{2n}$ and, for any k, denote by $I_k^\sigma = \{\sigma(3k-2), \sigma(3k-1), \sigma(3k)\}$, $a_k^\sigma = \max(I_k^\sigma)$, $c_k^\sigma = \min(I_k^\sigma)$ and $b_k^\sigma = \max\{I_k^\sigma - \{a_k^\sigma\}\}$. Then $c_k^\sigma < b_k^\sigma < a_k^\sigma$ and $I_k^\sigma = \{a_k^\sigma, b_k^\sigma, c_k^\sigma\}$. With these notations, we will have

$$\begin{aligned} & \sum_{j=0}^{[\frac{n}{3}]} \int_{\left([s,t]\times[0,t]^{(n-1)}\right)^2} |f(x_1,\dots,x_n)||f(x_{n+1},\dots,x_{2n})| \\ & \times \prod_{k=1}^{2j} \left[|x_{\sigma(3k-2)} - x_{\sigma(3k-1)}||x_{\sigma(3k-2)} - x_{\sigma(3k)}||x_{\sigma(3k-1)} - x_{\sigma(3k)}|\right]^{H-1} \\ & \times \prod_{k=3j+1}^{n} |x_{\sigma(2k)} - x_{\sigma(2k-1)}|^{2H-2} dx_1 \cdots dx_{2n} \\ = & \sum_{j=0}^{[\frac{n}{3}]} \int_{\left([s,t]\times[0,t]^{(n-1)}\right)^2} |f(x_1,\dots,x_n)||f(x_{n+1},\dots,x_{2n})| \\ & \times \prod_{k=1}^{2j} \left[|x_{a_k^\sigma} - x_{b_k^\sigma}||x_{a_k^\sigma} - x_{c_k^\sigma}||x_{c_k^\sigma} - x_{b_k^\sigma}|\right]^{H-1} \\ & \times \prod_{k=3j+1}^{n} |x_{\sigma(2k)} - x_{\sigma(2k-1)}|^{2H-2} dx_1 \cdots dx_{2n}. \end{aligned}$$

Observe that the first term of the previous sum ($j = 0$) is given by

$$\begin{aligned} & \int_{\left([s,t]\times[0,t]^{(n-1)}\right)^2} |f(x_1,\dots,x_n)||f(x_{n+1},\dots,x_{2n})| \\ & \times \prod_{k=1}^{n} |x_{\sigma(2k)} - x_{\sigma(2k-1)}|^{2H-2} dx_1 \cdots dx_{2n}, \end{aligned}$$

and this integral was studied before (see (13)). Let us regard now the terms with $j \geq 1$.

For any k, we will proceed in a different way with the term

$$\left[|x_{a_k^\sigma} - x_{b_k^\sigma}||x_{a_k^\sigma} - x_{c_k^\sigma}||x_{c_k^\sigma} - x_{b_k^\sigma}|\right]^{H-1}$$

depending on if $b_k^\sigma \leq n$ or $b_k^\sigma > n$.

We have the following majorization

$$\begin{aligned} & \sum_{j=1}^{[\frac{n}{3}]} \int_{\left([s,t]\times[0,t]^{(n-1)}\right)^2} |f(x_1,\dots,x_n)||f(x_{n+1},\dots,x_{2n})| \\ & \times \prod_{k=1}^{2j} \left[|x_{a_k^\sigma} - x_{b_k^\sigma}||x_{a_k^\sigma} - x_{c_k^\sigma}||x_{c_k^\sigma} - x_{b_k^\sigma}|\right]^{H-1} \\ & \times \prod_{k=3j+1}^{n} |x_{\sigma(2k)} - x_{\sigma(2k-1)}|^{2H-2} dx_1 \cdots dx_{2n} \\ \leq & \; S_1 + S_2, \end{aligned}$$

with

$$
\begin{aligned}
S_1 &= \frac{1}{2}\sum_{j=1}^{\left[\frac{n}{3}\right]}\int_{\left([s,t]\times[0,t]^{(n-1)}\right)^2} |f^2(x_1,\dots,x_n)| \prod_{k=1}^{2j}\Big(|x_{a_k^\sigma}-x_{b_k^\sigma}|^{2H-2} I_{\{b_k^\sigma\le n\}} \\
&+\left[|x_{a_k^\sigma}-x_{b_k^\sigma}||x_{b_k^\sigma}-x_{c_k^\sigma}|\right]^{2H-2} I_{\{b_k^\sigma>n\}}\Big) \\
&\times \prod_{k=3j+1}^{n} |x_{\sigma(2k)}-x_{\sigma(2k-1)}|^{2H-2} dx_1\cdots dx_{2n}, \qquad \text{and} \\
S_2 &= \frac{1}{2}\sum_{j=1}^{\left[\frac{n}{3}\right]}\int_{\left([s,t]\times[0,t]^{(n-1)}\right)^2} f^2(x_{n+1},\dots,x_{2n}) \prod_{k=1}^{2j}\Big(\left[|x_{a_k^\sigma}-x_{c_k^\sigma}||x_{b_k^\sigma}-x_{c_k^\sigma}|\right]^{2H-2} I_{\{b_k^\sigma\le n\}} \\
&+|x_{a_k^\sigma}-x_{c_k^\sigma}|^{2H-2} I_{\{b_k^\sigma>n\}}\Big) \\
&\times \prod_{k=3j+1}^{n} |x_{\sigma(2k)}-x_{\sigma(2k-1)}|^{2H-2} dx_1\cdots dx_{2n}.
\end{aligned}
$$

Notice that for the indexes k such that $b_k^\sigma > n$ in the summand S_1 we can integrate with respect to $x_{a_k^\sigma}$ and

$$\int_0^t |x_{a_k^\sigma}-x_{b_k^\sigma}|^{2H-2} dx_{a_k^\sigma} \le C_H.$$

On the other hand, in the summand S_2, and for the indexes k such that $b_k^\sigma \le n$ we can integrate with respect to $x_{b_k^\sigma}$ obtaining

$$\int_0^t |x_{b_k^\sigma}-x_{c_k^\sigma}|^{2H-2} dx_{b_k^\sigma} \le C_H.$$

Proceeding in this way for all permutation $\sigma \in \mathcal{P}(2n)$ we can bound the integral that appears in (14) by the sum of terms that are of one of the two following types:

- The integral of $f^2(x_1,\dots,x_n)$ multiplied by factors of the form $|y-z|^{2H-2}$ where y and z are variables that belong to $\{x_1,\dots,x_{2n}\}$, without repetition of variables in different factors. Moreover, we can assure that the variable x_{n+1} appears in one of these factors.

- The integral of $f^2(x_{n+1},\dots,x_{2n})$ multiplied by factors of the same form that before without repetition of variables in different factors and, moreover, we can assure that the variable x_1 appears in one of these factors.

In this situation, by using the same arguments that in the final part of the proof for the Stroock kernels, we can bound these integrals by

$$C_{H,n}(t-s)^{2H-1}\left(\int_{[s,t]\times[0,t]^{n-1}} f^2(x_1,\dots,x_n)\mu_n(dx_1,\dots,dx_n)\right).$$

This completes the proof of the lemma in the case $n>1$. □

Proof of Lemma 4.1 for the case $n = 1$

Proof: Observe that in both cases, for Donsker and for Stroock kernels, we have that

$$\begin{aligned} E\left(\tilde{\theta}_\varepsilon(x)\tilde{\theta}_\varepsilon(y)\right) &= C_H(xy)^{\frac{1}{2}-H}\int_0^x\int_0^y (uv)^{H-\frac{1}{2}}(x-u)^{H-\frac{3}{2}}(y-v)^{H-\frac{3}{2}}E\left[\theta_\varepsilon(u)\theta_\varepsilon(v)\right]dvdu \\ &\leq C_H(xy)^{\frac{1}{2}-H}\int_0^x\int_0^y (uv)^{H-\frac{1}{2}}(x-u)^{H-\frac{3}{2}}(y-v)^{H-\frac{3}{2}}\frac{1}{\varepsilon^2}e^{-2|u-v|}dvdu. \end{aligned}$$

And applying Lemma A.1, the above expression can be bounded by

$$C_H|x-y|^{2H-2}.$$

By using this fact, for $s < t$,

$$\begin{aligned} E\left[\int_s^t f(x)\tilde{\theta}_\varepsilon(x)dx\right]^2 &= \int_s^t\int_s^t f(x)f(y)E\left[\tilde{\theta}_\varepsilon(x)\tilde{\theta}_\varepsilon(y)\right]dxdy \\ &\leq C_H\int_s^t\int_s^t |f(x)f(y)||x-y|^{2H-2}dxdy \\ &\leq C_H\int_s^t\int_s^t f^2(x)|x-y|^{2H-2}dxdy \\ &\leq C_H\int_s^t f^2(x)\left((x-s)^{2H-1}+(t-x)^{2H-1}\right)dx \\ &\leq C_H(t-s)^{2H-1}\int_s^t f^2(x)dx \\ &\leq C_H(t-s)^{2H-1}\int_s^t f^2(x)\mu_1(dx). \end{aligned}$$

The proof of Lemma 4.1 is now complete. □

References

E. Alòs, O.Mazet and D. Nualart, Stochastic calculus with respect to Gaussian processes. Ann. Probab., 29(2) (2001) 766-801

F. Avram, Weak convergence of the variations, iterated integrals and Doléans-Dade exponentials of sequences of semimartingales. Ann. Probab., 16(1) (1988) 246-250

X. Bardina and M. Jolis, Weak convergence to the multiple Stratonovich integrals. Stochastic Process. Appl., 90(2) (2000) 277-300

P. Billinsgley, Convergence of probability measures. John Wiley and Sons (1968)

J.A. Clarkson and C.R. Adams, On definitions of bounded variation for functions of two variables. Trans. Amer. Math. Soc., 35(4) (1933) 824-854

A. Dasgupta and G. Kallianpur, Multiple fractional integrals. Probab. Theory Relat. Fields, 115 (1999) 505-525

A. Dasgupta and G. Kallianpur, Chaos decomposition of multiple fractional integrals and applications. Probab. Theory Relat. Fields, 115 (1999) 527-548

L. Decreusefond and A.S. Üstünel, Stochastic analysis of the fractional Brownian motion. Potential Anal., 10 (1999) 177-214

R. Delgado and M. Jolis, Weak approximation for a class of Gaussian processes. J. Appl. Probab., 37(2) (2000) 400-407

S.T. Huang and S. Cambanis, Stochastic and multiple Wiener integrals for Gaussian processes. Ann. Probab., 6(4) (1978) 585-614

B.B. Mandelbrot and J.W. Van Ness, Fractional Brownian motion, fractional noises and application. SIAM Rev., 10(4) (1968) 422-437

I. Norros, E. Valkeila and J. Virtamo, An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli, 5 (1999) 571-587

D. Nualart and M. Zakai, Multiple Wiener-Itô integrals possesing a continuous extension. Probab. Theory Relat. Fields, 85 (1990) 131-145

V. Perez-Abreu and C. Tudor, A transfer principle for multiple stochastic fractional integrals. Preprint (2001)

J.L. Solé and F. Utzet, Stratonovich integral and trace. Stochastics Stochastics Rep., 29(2) (1990) 203-220

D. Stroock, Topics in Stochastic Differential Equations. (Tata Institute of Fundamental Research, Bombay). Springer Verlag (1982)