Random Obstacle Problems

Lorenzo Zambotti
(LPMA, Univ. Paris 6)

July 2015, Saint-Flour

A course in 8 chapters:

- Introduction
- The reflecting Brownian motion
- Bessel processes
- The stochastic heat equation
- Obstacle problems
- Integration by Parts Formulae
- The contact set
- Convergence and Dirichlet Forms

A course in 8 chapters:

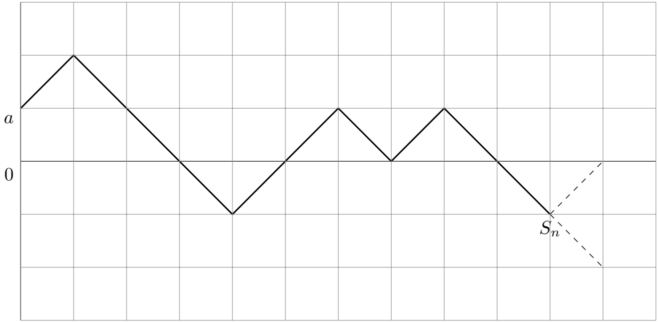
- Introduction
- The reflecting Brownian motion
- Bessel processes
- The stochastic heat equation
- Obstacle problems
- Integration by Parts Formulae
- The contact set
- Open problems

Symmetric simple random walk

$(Y_i)_{i\geq 1}$ i.i.d. sequence with $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = 1/2$

$$S_0 := a \in \mathbb{Z}, \qquad S_{n+1} = S_n + Y_{n+1},$$

$(S_n)_{n\geq 0}$ defines a Markov chain with valued in $\mathbb{Z}$.

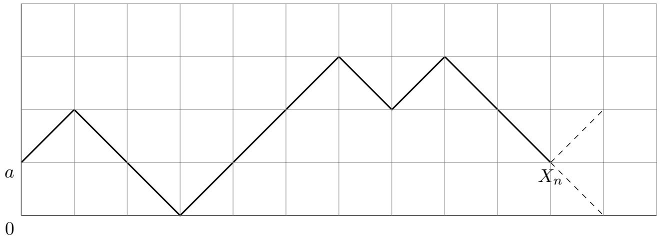


Symmetric simple random walk with reflection

$$X_{n+1} := \begin{cases} X_n + Y_{n+1} & \text{if } X_n > 0 \\ 1 & \text{if } X_n = 0. \end{cases}$$

This defines a Markov chain with values in $\mathbb{Z}_+ = \{0, 1, \ldots\}$. It is easy to prove by recurrence on n that

$$X_n = S_n + \sum_{i=1}^{n} \mathbb{1}_{(X_{i-1}=0)}(1 - Y_i), \qquad n \geq 0. \tag{1}$$



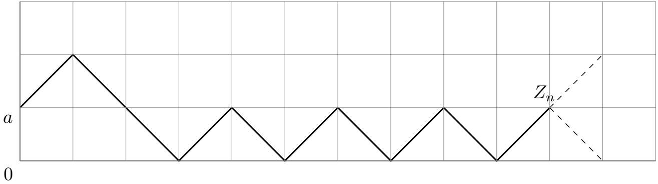
Absolute value of the SSRW

Let $Z_n := |S_n|$, with $S_0 = a \geq 0$. Although this is not obvious at first sight, $(Z_n)_{n\geq 0}$ is also a Markov chain:

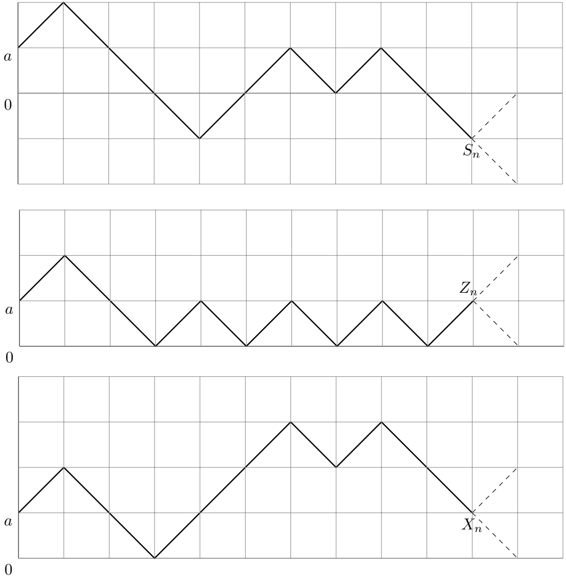
$$Z_{n+1} = \begin{cases} Z_n + W_{n+1} & \text{if } Z_n > 0 \\ 1 & \text{if } Z_n = 0, \end{cases}$$

where $W_{n+1} := Y_{n+1}(\mathbb{1}_{S_n \geq 0} - \mathbb{1}_{S_n < 0})$.

It is easy to see that $(W_i)_{i\geq 1}$ is a copy of $(Y_i)_{i\geq 1}$ and then $(Z_n)_{n\geq 0}$ and $(X_n)_{n\geq 0}$ (for the same fixed a) have the same law.



S, Z, X



Discrete interfaces

We define now a Markov chain with values in discrete paths. We fix $N \in \mathbb{N}$ and we define the state space

$$E_N := \{w \in \mathbb{Z}^{2N} : w(0) = w(2N) = 0, \\ |w(i) - w(i-1)| = 1, \ \forall\, i = 1, \ldots, 2N\}.$$

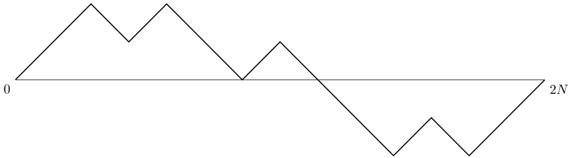
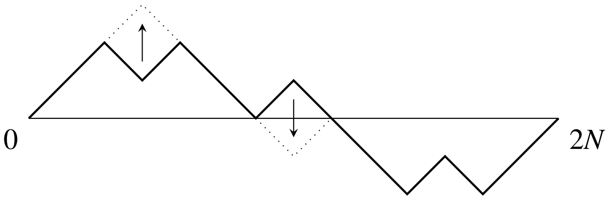


Figure : A typical path in E_N

The free evolution

Then we define a Markov chain with values in E_N as follows: we define a map $F : E_N \times \{1, \dots, 2N-1\} \to E_N$, $\quad F(w, j) = \hat{w} \in E_N$, where

$$\hat{w}_i = \begin{cases} w_i + 2 & \text{if } i = j \text{ and } w_{i-1} = w_{i+1} > w_i \\ w_i - 2 & \text{if } i = j \text{ and } w_{i-1} = w_{i+1} < w_i \\ w_i & \text{otherwise.} \end{cases}$$



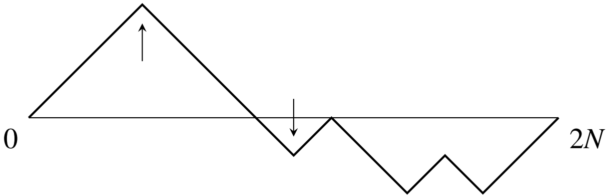
We let $(U_n)_{n \geq 1}$ be an i.i.d. sequence of uniform random variables on $\{1, \dots, 2N-1\}$; then

$$e_{n+1} := F(e_n, U_{n+1}), \qquad e_0 \in E_N.$$

The free evolution

Then we define a Markov chain with values in E_N as follows: we define a map $F : E_N \times \{1, \dots, 2N-1\} \to E_N$, $\quad F(w, j) = \hat{w} \in E_N$, where

$$\hat{w}_i = \begin{cases} w_i + 2 & \text{if } i = j \text{ and } w_{i-1} = w_{i+1} > w_i \\ w_i - 2 & \text{if } i = j \text{ and } w_{i-1} = w_{i+1} < w_i \\ w_i & \text{otherwise.} \end{cases}$$



We let $(U_n)_{n\geq 1}$ be an i.i.d. sequence of uniform random variables on $\{1, \dots, 2N-1\}$; then

$$e_{n+1} := F(e_n, U_{n+1}), \qquad e_0 \in E_N.$$

The free evolution

We denote the transition matrix of $(e_n)_{n\geq 0}$ by

$$P(x,y) = \mathbb{P}(F(x,U_1) = y).$$

It is easy to see that $P(x,y) = P(y,x)$ and therefore the uniform measure on E_N is invariant and reversible for P.

This is in fact the unique probability invariant measure of $(e_n)_{n\geq 0}$ by the following

Lemma

The Markov chain $(e_n)_{n\geq 0}$ is aperiodic and irreducible.

Proof.

Equivalence with a particle system:

$$E_N \ni w \longleftrightarrow (w_i - w_{i-1})_{i=1,\dots,2N} \in \{1,-1\}^{2N} \longleftrightarrow \{\bullet,\circ\}^{2N}$$

$$E_N \longleftrightarrow C_N := \{c \in \{\bullet,\circ\}^{2N} : \#\bullet = \#\circ = N\}.$$

$P(x,x') > 0$ iff

$$x \longleftrightarrow \cdots | \bullet | \circ | \cdots , \qquad x' \longleftrightarrow \cdots | \circ | \bullet | \cdots$$

or viceversa.

Let $c_0 := | \bullet | \cdots | \bullet | \circ | \cdots | \circ | \in C_N$. From any $c \in C_N$ we can reach c_0 with finitely many transitions having positive probability by moving all particles in c to the left starting with the leftmost one. By symmetry one can go from c_0 to any $c \in C_N$. Aperiodicity: $P(c_0,c_0) > 0$. □

The reflected interface

Let us now add reflection to our discrete interface. We set

$$E_N^+ := \{w \in E_N : w(i) \geq 0,\ \forall\, i = 0, \ldots, 2N\}.$$

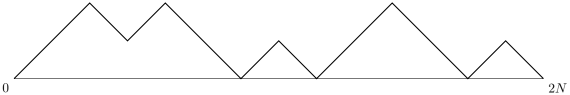
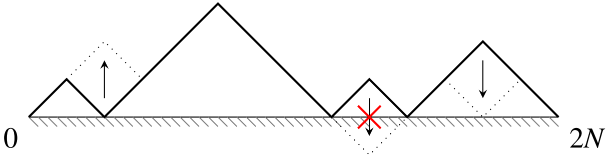


Figure : A typical path in E_N^+

Reflection means now suppression of transitions which would let e_n^+ take negative values:



Let us now add reflection to our discrete interface. We set

$$E_N^+ := \{w \in E_N : w(i) \geq 0,\ \forall\, i = 0, \dots, 2N\}.$$

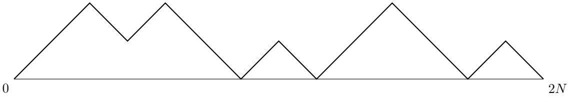
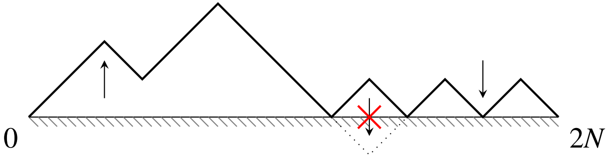


Figure : A typical path in E_N^+

Reflection means now suppression of transitions which would let e_n^+ take negative values:



The reflected interface

The Markov evolution in E_N^+ is defined as follows:
$F^+ : E_N^+ \times \{1, \ldots, 2N-1\} \to E_N^+$

$$F^+(w,j) = \begin{cases} F(w,j) & \text{if } F(w,j) \in E_N^+ \\ w & \text{otherwise.} \end{cases}$$

Then our E_N^+-valued Markov chain $(e_n^+)_{n\geq 0}$ is defined by

$$e_{n+1}^+ := F^+(e_n^+, U_{n+1}), \qquad e_0^+ \in E_N^+.$$

Lemma

The Markov chain $(e_n^+)_{n\geq 0}$ is aperiodic irreducible and has a unique invariant probability measure, given by the uniform probability measure on E_N^+. This probability measure is furthermore reversible for $(e_n^+)_{n\geq 0}$.

Proof.

We recall that the transition matrix P of $(e_n)_{n\geq 0}$ is symmetric. Now we have for $x, y \in E_N^+$:

$$P_+(x,y) = \mathbb{P}(f_+(x,U_1) = y) = P(x,y) + P(x, E_N \setminus E_N^+)\mathbb{1}_{(y=x)}.$$

Then P_+ is also symmetric and therefore the uniform measure on E_N^+ is invariant and reversible for P_+.

$E_N^+ \longleftrightarrow C_N^+$, set of $c \in C_N$ such that for all $k \in \{1, \ldots, 2N\}$ the first k sites contain at least $s(k)$ particles, where $s(2i) := i$, $s(2i+1) := i+1$.

Then $c_0 = |\bullet|\cdots|\bullet|\circ|\cdots|\circ| \in C_N^+$ and we can find a path in C_N^+ from any $c \in C_N^+$ to c_0 by moving particles leftwards starting with the leftmost one. Finally $P_+(c_0, c_0) > 0$, so that this Markov chain is aperiodic. □

An important remark

We see that the reflection for the dynamics is equivalent to a conditioning for the invariant measure.

Let $|e_n|$ be the absolute value of the free interface $(e_n)_{n\geq 0}$. If $(e_n)_{n\geq 0}$ is stationary then the distribution of $|e_0|$ is a probability measure on E_N^+

$$\mathbb{P}(|e_0| = w) \propto \#\{w' \in E_N : |w'| = w\} = 2^{L(w)},$$

$$L(w) := \sum_{i=1}^{2N} \mathbb{1}_{(w_i=0)}, \qquad w \in E_N^+.$$

In other words $L(w)$ is the number of excursions of w.

On the other hand, if $(e_n^+)_{n\geq 0}$ is stationary then the law of e_0^+ is uniform on E_N^+, so that e_0^+ and $|e_0|$ have different laws.

Moreover $(|e_n|)_{n\geq 0}$ is not Markovian.

Scaling limits

Let $(S_n)_{n\geq 0}$ be the SSRW with $S_0 := 0$ and

$$B_t^N := \frac{1}{\sqrt{N}} S_{\lfloor Nt \rfloor}, \qquad t \geq 0.$$

By Donsker's theorem $(B_t^N)_{t\geq 0} \Longrightarrow (B_t)_{t\geq 0}$ as $N \to +\infty$.

Under the same scaling the reflecting SSRW $(X_n)_{n\geq 0}$ converges to the reflecting Brownian motion $(\rho_t)_{t\geq 0}$ (see chapter 2).

This process is given by a stochastic differential equation

$$\mathrm{d}\rho_t = \mathrm{d}B_t + \mathrm{d}\ell_t, \qquad \rho_t \geq 0, \quad \mathrm{d}\ell_t \geq 0, \quad \int_0^\infty \rho_t \, \mathrm{d}\ell_t = 0,$$

- $t \mapsto (\rho_t, \ell_t)$ is continuous, ρ_t is non-negative
- $\ell_0 = 0$, $\ell_s \leq \ell_t$ for $s \leq t$
- $\mathrm{supp}(\mathrm{d}\ell_t) \subset \{t \geq 0 : \rho_t = 0\}$.

The measure $\mathrm{d}\ell_t$ is the reflection term: see (1).

Scaling of the free interface

Let us first consider the stationary version of $(e_n)_{n\geq 0}$ and define

$$v_N(t,x) = \frac{1}{\sqrt{2N}}\, e_{\lfloor 4N^2 t\rfloor}(\lfloor 2Nx\rfloor), \qquad t \geq 0,\ x \in [0,1].$$

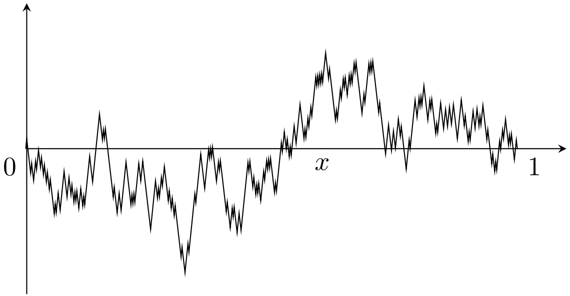


Figure : A typical path of $v_N(t,\cdot)$ for any $t \geq 0$ when N is large

As $N \to +\infty$, $v_N \Longrightarrow (v(t,x), t \geq 0, x \in [0,1])$, that we are going to study in detail in chapter 4 and is a stationary solution to a stochastic partial differential equation (SPDE)

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2} + W, \\ v(t,0) = v(t,1) = 0, \qquad t \geq 0, \\ v(0,x) = v_0(x), \qquad x \in [0,1]. \end{cases}$$

Here W is a space-time white noise, a two-parameter version of the derivative of a Brownian motion that we shall define properly later.

Scaling of the reflected interface

Let us now consider the stationary version of $(e_n^+)_{n\geq 0}$ and define

$$u_N(t,x) = \frac{1}{\sqrt{2N}}\, e^+_{\lfloor 4N^2 t\rfloor}(\lfloor 2Nx\rfloor), \qquad t\geq 0,\ x\in[0,1].$$

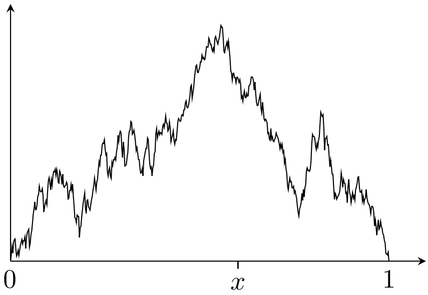


Figure : A typical path of $u_N(t,\cdot)$ for any $t\geq 0$ when N is large

A SPDE with reflection

As $N \to +\infty$, $u_N \Longrightarrow (u(t,x), t \geq 0, x \in [0,1])$, t, that we are going to study in detail from chapter 5 on and is a stationary solution to a SPDE with reflection

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + W + \eta \\ \\ u(0,x) = u_0(x), \quad u(t,0) = u(t,1) = 0 \\ \\ u \geq 0,\ \mathrm{d}\eta \geq 0,\ \int u\,\mathrm{d}\eta = 0. \end{cases}$$

Here (u, η) is a random pair that consists of

- a continuous non-negative functions $u(t,x) \geq 0$
- a Radon measure η on $]0,+\infty[\times]0,1[$,

such that the support of η is contained in $\{(t,x) : u(t,x) = 0\}$.

The contact set

For all $t \geq 0$ the typical profile of $u(t, \cdot)$ is positive on $]0, 1[$. Where does the reflection act?

The contact set

For all $t \geq 0$ the typical profile of $u(t,\cdot)$ is positive on $]0,1[$. Where does the reflection act?

This apparent paradox is solved if we formulate the sentence more precisely: the correct result is that for all $t \geq 0$, a.s. $u(t,\cdot) > 0$ on $]0,1[$:

$$\forall\, t > 0, \qquad \mathbb{P}(\exists\, x \in\,]0,1[\,:\ u(t,x) = 0) = 0.$$

However this does not exclude the existence, with positive probability, of exceptional times $t \geq 0$ and $x \in\,]0,1[$ such that $u(t,x) = 0$:

$$\mathbb{P}(\exists\, t > 0,\ x \in\,]0,1[\,:\ u(t,x) = 0) > 0.$$

The contact set

The next question is: what can be said about the contact set

$$\mathscr{Z} := \{(t,x) : t > 0,\ x \in]0,1[,\ u(t,x) = 0\}.$$

After proving that with positive probability u visits 0, one can ask:

- what is the typical behavior at exceptional times $t \geq 0$?
- That is, if $t > 0$ is such that there exists $x \in]0,1[$ so that $u(t,x) = 0$, then how many such points x exist?

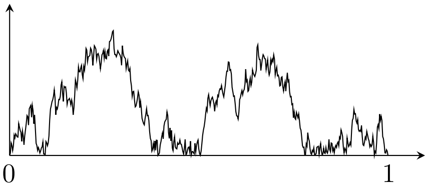


Figure : How many x such that $u(t,x) = 0$: infinitely many?

The contact set

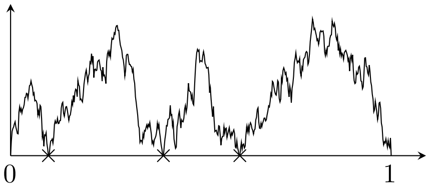


Figure : How many x such that $u(t,x) = 0$: finitely many?

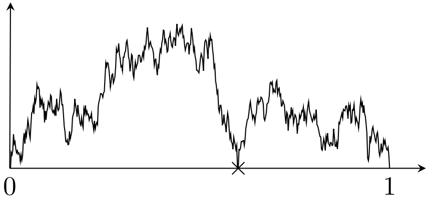


Figure : How many x such that $u(t,x) = 0$: just one? or two? or three?

Difficulties

- $(u(t,x), t \geq 0, x \in [0,1])$ has no standard semi-martingale structure
- neither has $(u(t,x), t \geq 0)$
- which is by the way not markovian for fixed $x \in [0,1]$
- therefore we must study S(P)DEs without stochastic calculus

Difficulties

- $(u(t,x), t \geq 0, x \in [0,1])$ has no standard semi-martingale structure
- neither has $(u(t,x), t \geq 0)$
- which is by the way not markovian for fixed $x \in [0,1]$
- therefore we must study S(P)DEs without stochastic calculus

This course is about some fascinating phenomena, in search of a theory.

Difficulties

- $(u(t,x), t \geq 0, x \in [0,1])$ has no standard semi-martingale structure
- neither has $(u(t,x), t \geq 0)$
- which is by the way not markovian for fixed $x \in [0,1]$
- therefore we must study S(P)DEs without stochastic calculus

This course is about some fascinating phenomena, in search of a theory.

The dream is a stochastic calculus for infinite-dimensional processes, following the model of the classical theory (e.g. Revuz-Yor).

Difficulties

- $(u(t,x), t \geq 0, x \in [0,1])$ has no standard semi-martingale structure
- neither has $(u(t,x), t \geq 0)$
- which is by the way not markovian for fixed $x \in [0,1]$
- therefore we must study S(P)DEs without stochastic calculus

This course is about some fascinating phenomena, in search of a theory.

The dream is a stochastic calculus for infinite-dimensional processes, following the model of the classical theory (e.g. Revuz-Yor).

We shall discuss some open problems waiting for someone to solve them.

Let us meet our first obstacle problem.

Lemma (Skorokhod)

Let $y : \mathbb{R}_+ \mapsto \mathbb{R}$ *be a continuous function such that* $y(0) \geq 0$*. Then there exists a unique couple* (z, ℓ) *of continuous functions on* $\mathbb{R}_+$ *s.t.*

1. $z = y + \ell$
2. $z \geq 0$
3. $t \mapsto \ell_t$ *is monotone non-decreasing,* $\ell_0 = 0$*, and the support of the associated measure* $\mathrm{d}\ell_t$ *is contained in* $\{t \geq 0 : z_t = 0\}$*.*

Moreover, ℓ *are* z *given explicitly by*

$$\ell_t = \sup_{s \leq t} \max\{-y_s, 0\}, \qquad z_t = y_t + \ell_t. \tag{2}$$

Let us meet our first obstacle problem.

Lemma (Skorokhod)

Let $y : \mathbb{R}_+ \mapsto \mathbb{R}$ *be a continuous function such that* $y(0) \geq 0$*. Then there exists a unique couple* (z, ℓ) *of continuous functions on* $\mathbb{R}_+$ *s.t.*

1. $z = y + \ell$
2. $z \geq 0$
3. $t \mapsto \ell_t$ *is monotone non-decreasing,* $\ell_0 = 0$*, and the support of the associated measure* $\mathrm{d}\ell_t$ *is contained in* $\{t \geq 0 : z_t = 0\}$.

Moreover, ℓ *are* z *given explicitly by*

$$\ell_t = \sup_{s \leq t} \max\{-y_s, 0\}, \qquad z_t = y_t + \ell_t. \tag{2}$$

Remark: $\mathrm{supp}(\mathrm{d}\ell) \subset \{t \geq 0 : z_t = 0\} \iff \int_0^\infty z_t \, \mathrm{d}\ell_t = 0.$

The Skorokhod Lemma

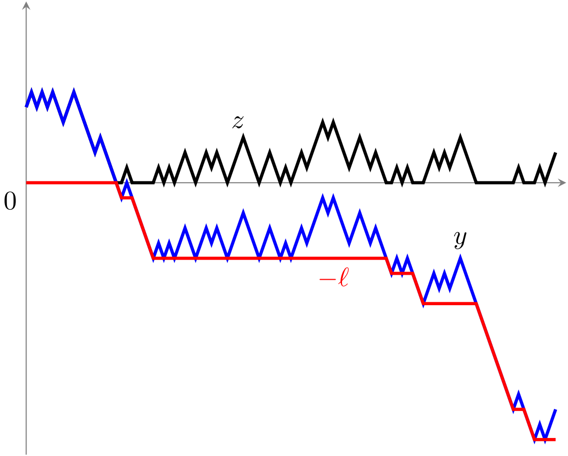


Figure : An example of a triple (y, z, ℓ) from the Skorokhod Lemma.

The Skorokhod Lemma

Proof.

To prove existence, one verifies that $\ell_t := \sup_{s \le t} \max\{-y_s, 0\}$ and $z := y + \ell$ satisfy the above properties.

If $(\bar{z}, \bar{\ell})$ is another pair with the same properties, then $z - \bar{z} = \ell - \bar{\ell}$ is a continuous function with bounded variation and by the chain rule

$$0 \le \frac{1}{2}(z_t - \bar{z}_t)^2 = \int_0^t (z_s - \bar{z}_s)\,\mathrm{d}(\ell_s - \bar{\ell}_s)$$

$$= \left[\int_0^t z_s\,\mathrm{d}\ell_s + \int_0^t \bar{z}_s\,\mathrm{d}\bar{\ell}_s - \int_0^t z_s\,\mathrm{d}\bar{\ell}_s - \int_0^t \bar{z}_s\,\mathrm{d}\ell_s\right].$$

Since $\mathrm{supp}(\mathrm{d}\ell_s) \subset \{s : z_s = 0\}$ and $\mathrm{supp}(\mathrm{d}\bar{\ell}_s) \subset \{s : \bar{z}_s = 0\}$, this is equal to

$$-\int_0^t z_s\,\mathrm{d}\bar{\ell}_s - \int_0^t \bar{z}_s\,\mathrm{d}\ell_s \le 0.$$

Then $(z_t - \bar{z}_t)^2 = (\ell_t - \bar{\ell}_t)^2 = 0$.

ODEs with reflection

Let $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$|f(x) - f(y)| \leq L|x - y|, \qquad x, y \in \mathbb{R}.$$

Proposition

Let $T \geq 0$ and $\mathrm{w} \in C([0,T])$ with $\mathrm{w}_0 \geq 0$. Then there exists a unique couple $(\rho_t, \ell_t)_{t\in[0,T]}$ of continuous real functions such that

$$\begin{cases} \rho_t = \mathrm{w}_t + \int_0^t f(\rho_s)\,\mathrm{d}s + \ell_t, \quad t \in [0,T] \\ \ell_0 = 0, \\ \rho_t \geq 0, \quad \mathrm{d}\ell_t \geq 0, \quad \int_0^T \rho_t\,\mathrm{d}\ell_t = 0. \end{cases} \tag{3}$$

Proof.

Let $E_T := C([0,T])$ with norm $\|g\|_L := \sup_{t\in[0,T]} e^{-3Lt}|g_t|$. We define $\Gamma : E_T \mapsto E_T$ by $\Gamma(g) := z$, where (z,ℓ) is the solution of the Skorokhod problem of Lemma 3 associated to y defined by

$$y_t := \mathrm{w}_t + \int_0^t f(g_s)\,\mathrm{d}s, \qquad t\in[0,T].$$

In particular, $z \geq 0$, $\mathrm{d}\ell \geq 0$, $\ell_0 = 0$, and for all $t \in [0,T]$

$$z_t := \mathrm{w}_t + \int_0^t f(g_s)\,\mathrm{d}s + \ell_t, \qquad \int_0^t z_s\,\mathrm{d}\ell_s = 0.$$

By the explicit representation (2) of (z,ℓ) in terms of y we obtain that Γ is a contraction in E_T and it has a unique fixed point $(\rho_t)_{t\in[0,T]}$. □

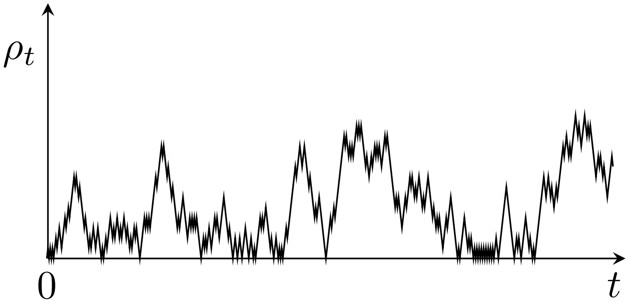
Corollary

Let $(B_t)_{t\geq 0}$ be a standard BM and $x \geq 0$. Then there exists a unique couple $(\rho_t, \ell_t)_{t\geq 0}$ of continuous real processes such that

$$\begin{cases} \rho_t = x + B_t + \int_0^t f(\rho_s)\,\mathrm{d}s + \ell_t, \quad t \geq 0 \\ \ell_0 = 0, \\ \rho_t \geq 0, \quad \mathrm{d}\ell_t \geq 0, \quad \int_0^\infty \rho_t\,\mathrm{d}\ell_t = 0. \end{cases} \tag{4}$$

If $f \equiv 0$ we call ρ the reflecting BM.

The reflecting BM



Reflection in $\mathbb{R}^d$

H. Tanaka (1979), *Stochastic differential equations with reflecting boundary condition in convex regions*. Hiroshima Math. J. 9, no. 1, 163–177.

Let $D \subset \mathbb{R}^d$ be a convex open bounded domain. We define:

- For all $x \in \partial D$ a exposing hyperplane $H \in \mathscr{H}_x(D)$ is a hyperplane such that D is contained in one of the two half-spaces whose boundary is H and $H \cap \partial D \neq \emptyset$.
- We call $\hat{n} \in \mathbb{R}^d$ an inward normal unit vector at $x \in \partial D$ if $x + \varepsilon\hat{n} \in D$ for some $\varepsilon > 0$, $\|\hat{n}\| = 1$ and $\hat{n}$ is perpendicular to some $H \in \mathscr{H}_x(D)$. We call $\mathscr{N}_x(D)$ the set of all inward unit normal vectors at $x \in \partial D$.

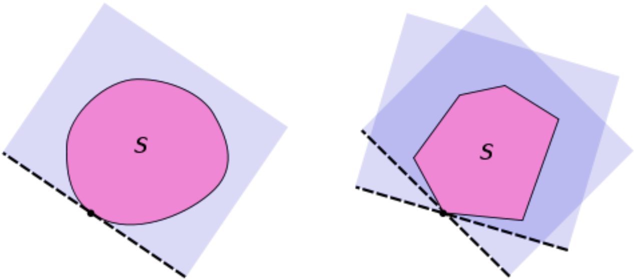


Figure : Exposing hyperplanes

(Special thanks to wikipedia)

Theorem (Tanaka)

For all $\mathrm{w} \in C([0,T];\mathbb{R}^d)$ *with* $\mathrm{w}_0 \in \overline{D}$ *there exists a unique pair* (ξ, φ) *such that*

- $\xi \in C([0,T];\overline{D})$, $\varphi \in C([0,T];\mathbb{R}^d)$, $\varphi(0) = 0$, $\xi = \mathrm{w} + \varphi$.
- *there exists a monotone non-decreasing* $\ell \in C([0,T])$ *such that*

$$\varphi_t = \int_0^t \mathbb{1}_{(\xi_s \in \partial D)} \, \mathbf{n}_s \, \mathrm{d}\ell_s, \quad t \in [0,T]$$

where $\mathbf{n}_s \in \mathscr{N}_{\xi_s}(D)$ *for* $\mathrm{d}\ell$*-a.e.* s.

Moreover the map $\mathrm{w} \mapsto \xi$ *is continuous in the sup-topology.*

One can write the equation as a differential inclusion

$$\mathrm{d}(\xi_t - \mathrm{w}_t) \in \mathbb{1}_{(\xi_t \in \partial D)} \, \mathscr{N}_{\xi_t}(D) \, \mathrm{d}\ell_t, \qquad \int_0^{\cdot} \mathbb{1}_{(\xi_s \in D)} \, \mathrm{d}\ell_s = 0.$$

Proof of Uniqueness

Let (ξ, φ) and $(\overline{\xi}, \overline{\varphi})$ be two solutions. Then

$$d\|\xi_t - \overline{\xi}_t\|^2 = (\xi_t - \overline{\xi}_t) \cdot (d\varphi_t - d\overline{\varphi}_t).$$

Now for $d\ell$-a.e. t

$$(\xi_t - \overline{\xi}_t) \cdot d\varphi_t = \mathbb{1}_{(\xi_s \in \partial D)} \, (\xi_t - \overline{\xi}_t) \cdot \mathbf{n}_t \, d\ell_t.$$

Since $\overline{\xi}_t \in \overline{D}$ and $\mathbf{n}_t \in \mathscr{N}_{\xi_t}(D)$, then

$$(\xi_t - \overline{\xi}_t) \cdot \mathbf{n}_t = -(\overline{\xi}_t - \xi_t) \cdot \mathbf{n}_t \leq 0.$$

By symmetry we conclude that $\|\xi_t - \overline{\xi}_t\|^2 \equiv 0$. □

Another approach to reflection: Penalisation

Let $n \geq 1$, $\mathrm{w} \in C([0,T])$ with $\mathrm{w}_0 \geq 0$ and

$$\rho^n_t = \mathrm{w}_t + n \int_0^t (\rho^n_s)^- \, \mathrm{d}s + \int_0^t f(\rho^n_s) \, \mathrm{d}s, \qquad t \in [0,T],$$

$$\text{where} \qquad r^- = (r)^- := \max\{-r, 0\}, \qquad r \in \mathbb{R}.$$

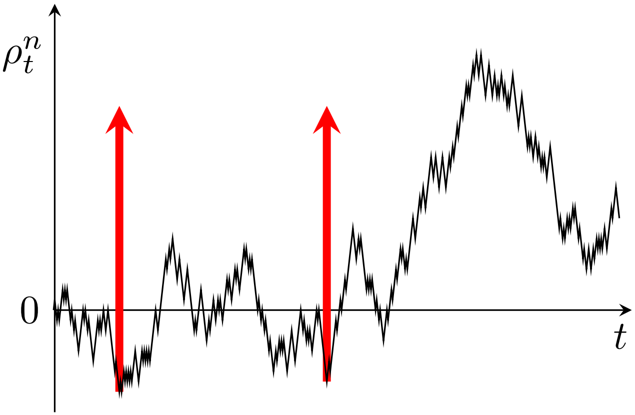
Additive (deterministic!) noise and Lipschitz drift, so clearly pathwise uniqueness and existence of solutions by a Picard iteration.

Proposition

1. *if $n \leq m$ then $\rho^n_t \leq \rho^m_t$ for all $t \in [0,T]$.*
2. *$\rho^n \uparrow \rho$ uniformly on $[0,T]$ as $n \uparrow +\infty$, where $(\rho_t, \ell_t)_{t \geq 0}$ is the unique solution to the equation with reflection* (3). *Moreover*

$$\lim_{n \uparrow +\infty} n \int_0^t (\rho^n_s)^- \, \mathrm{d}s = \ell_t, \qquad t \in [0,T].$$

Penalisation



Proof

We set $f_n : \mathbb{R} \mapsto \mathbb{R}$,

$$f_n(x) := n\,x^- + f(x).$$

Note that f_n is Lipschitz-continuous on $\mathbb{R}$ and satisfies the one-sided estimate:

$$f_n(x) - f_n(y) \le L(x-y), \qquad \forall\, n \ge 1,\ x > y.$$

For simplicity we assume that f is monotone non-increasing, so that in particular

$$f_n(x) - f_n(y) \le 0, \qquad \forall\, n \ge 1,\ x > y.$$

In the lecture notes you find the argument for the general case.

We call $S(n, \mathrm{w}, T) = \rho^n$ the unique solution of

$$\rho^n_t = \mathrm{w}_t + \int_0^t f_n(\rho^n_s)\,\mathrm{d}s, \qquad t \in [0,T].$$

Proof

Step 1 (Monotonicity). We show first that $\rho^n = S(n, \mathrm{w}, T) \le \rho^m = S(m, \mathrm{w}, T)$ if $n \le m$. Since $f_n(\cdot) \le f_m(\cdot)$

$$\frac{d}{dt}\left((\rho^n - \rho^m)^+\right)^2 \; = \; 2(\rho^n - \rho^m)^+ \left(f_n(\rho^n) - f_m(\rho^m)\right)$$
$$\le 2(\rho^n - \rho^m)^+ \left(f_m(\rho^n) - f_m(\rho^m)\right) \; \le 0.$$

Since $\rho^n_0 = \rho^m_0$, we conclude that $\rho^n \le \rho^m$. Then

$$S(\mathrm{w}, T) := \sup_n S(n, \mathrm{w}, T) = \lim_{n \to +\infty} S(n, \mathrm{w}, T)$$

where the supremum and the limit are taken pointwise, i.e. for all $t \in [0, T]$. It is not clear at this point whether $\rho := S(\mathrm{w}, T)$ is a continuous function. This will be proved in the next steps.

Proof

Step 2 (The case of a smooth obstacle). Let $\mathrm{w} \in C^\infty([0,T])$ with $\mathrm{w}_0 \geq 0$. Let us fix n. Writing for simplicity $x := \rho^n$

$$\dot{x}_t = f_n(x_t) + \dot{\mathrm{w}}_t,$$

$$\begin{aligned}\frac{d}{dt}|x_t - x_{t+h}| &= (f_n(x_t) - f_n(x_{t+h}) + \dot{\mathrm{w}}_t - \dot{\mathrm{w}}_{t+h}) \operatorname{sign}(x_t - x_{t+h}) \\ &\leq |\dot{\mathrm{w}}_t - \dot{\mathrm{w}}_{t+h}|\end{aligned}$$

so that

$$|x_t - x_{t+h}| \leq |x_0 - x_h| + \int_0^t |\dot{\mathrm{w}}_s - \dot{\mathrm{w}}_{s+h}| \, \mathrm{d}s.$$

If we divide by $h > 0$ and let $h \downarrow 0$ we obtain

$$\begin{aligned}|\dot{x}_t| &\leq |\dot{x}_0| + \int_0^t |\ddot{\mathrm{w}}_s| \, \mathrm{d}s \leq |f_n(\mathrm{w}_0)| + |\dot{\mathrm{w}}_0| + \int_0^t |\ddot{\mathrm{w}}_s| \, \mathrm{d}s \\ &= |f(x_0)| + |\dot{\mathrm{w}}_0| + \int_0^t |\ddot{\mathrm{w}}_s| \, \mathrm{d}s.\end{aligned}$$

Proof

Therefore

$$\sup_{n\geq 1}\sup_{t\in[0,T]}|\dot{x}_t|\leq C=C(T,\mathrm{w}).$$

By the Arzelà-Ascoli theorem, the sequence $x=\rho^n=S(n,\mathrm{w},T)$ in $n\geq 1$ is compact in $C([0,T])$. Therefore ρ^n converges uniformly,

$$\ell_t^n := n\int_0^t(\rho_s^n)^-\,\mathrm{d}s=\rho_t^n-\mathrm{w}_t-\int_0^t f(\rho_s^n)\,\mathrm{d}s\longrightarrow \ell_t$$

and one shows that (ρ,ℓ) is a solution to the SDE with reflection.

Proof

Step 3 (Continuity of the map $\mathrm{w} \mapsto S(n, \mathrm{w}, T)$ uniformly in n). Let now $\mathrm{w} \in C([0, T])$ with $\mathrm{w}_0 \geq 0$. We still write $x := \rho^n$ for fixed n. Set

$$y_t := x_t - \mathrm{w}_t \implies \dot{y}_t = f_n(x_t). \tag{5}$$

Let $\hat{\mathrm{w}} \in C([0, T])$ with $\hat{\mathrm{w}}_0 \geq 0$, and set $x = S(n, \mathrm{w}, T)$, $\hat{x} = S(n, \hat{\mathrm{w}}, T)$. We denote

$$\|g\|_\infty := \sup_{[0,T]} |g_t|, \qquad g : [0, T] \mapsto \mathbb{R}.$$

We set y as in (5) and $\hat{y}$ analogously after replacing (x, w) by $(\hat{x}, \hat{\mathrm{w}})$.

Let $\kappa := \|\mathrm{w} - \hat{\mathrm{w}}\|_\infty$. Now if $y_t - \hat{y}_t - \kappa > 0$ then

$$\begin{aligned} x_t - \hat{x}_t &> \mathrm{w}_t - \hat{\mathrm{w}}_t + \kappa \\ &\geq -\|\mathrm{w} - \hat{\mathrm{w}}\|_\infty + \kappa = 0. \end{aligned}$$

Proof

Then since $r \mapsto f_n(r)$ is monotone non-increasing,

$$\frac{d}{dt}\left((y_t - \hat{y}_t - \kappa)^+\right)^2 = 2\,(y_t - \hat{y}_t - \kappa)^+\,(f_n(x_t) - f_n(\hat{x}_t)) \leq 0.$$

Since $y_0 = \hat{y}_0$, we obtain that $y_t - \hat{y}_t \leq \kappa$ for all $t \in [0, T]$, and by symmetry

$$\|y - \hat{y}\|_\infty \leq \kappa = \|\mathrm{w} - \hat{\mathrm{w}}\|_\infty.$$

By definition $y = x - \mathrm{w}$, so that

$$\|S(n, \mathrm{w}, T) - S(n, \hat{\mathrm{w}}, T)\|_\infty \leq 2\|\mathrm{w} - \hat{\mathrm{w}}\|_\infty.$$

By letting $n \to +\infty$:

$$\|S(\mathrm{w}, T) - S(\hat{\mathrm{w}}, T)\|_\infty \leq 2\|\mathrm{w} - \hat{\mathrm{w}}\|_\infty.$$

Proof

Step 4 (Conclusion).

- Let $w \in C([0,T])$, $w^m \in C^\infty([0,T])$ such that

$$w_0, w_0^m \geq 0, \qquad \lim_{m\to+\infty} \|w - w^m\|_\infty = 0.$$

- By Step 2, $S(w^m, T) \in C([0,T])$.
- By Step 3 $S(w^m, T)$ converges uniformly on $[0,T]$ to $\rho = S(w,T)$, so that $S(w,T) \in C([0,T])$.
- We can conclude the proof by Dini's Theorem. □

The penalised SDE

For $x \in \mathbb{R}$ and B a standard BM we set

$$\rho_t^n(x) = x + B_t + n \int_0^t (\rho_s^n(x))^- \, \mathrm{d}s + \int_0^t f(\rho_s^n(x)) \, \mathrm{d}s, \qquad t \geq 0,$$

The infinitesimal generator of ρ^n is for $\varphi \in C_c^2(\mathbb{R})$

$$L^n \varphi(x) := \frac{1}{2} \varphi''(x) + (nx^- + f(x)) \, \varphi'(x), \qquad x \in \mathbb{R}.$$

Moreover ρ^n admits as reversible invariant measure

$$\mu_n(\mathrm{d}x) = e^{-n\,(x^-)^2 + 2F(x)} \, \mathrm{d}x$$

where $F : \mathbb{R} \mapsto \mathbb{R}$ is any function such that

$$F'(x) = f(x), \qquad x \in \mathbb{R}.$$

Note that $\mu_n(]-\infty, 0]) < +\infty$ for n large, but $\mu_n([0, +\infty[) \leq +\infty$ in general.

The penalised SDE

Lemma

The measure μ is invariant and reversible for $(\rho_t)_{t\geq 0}$ and

$$\lim_{n\to+\infty}\int_{\mathbb{R}}\varphi\,\mathrm{d}\mu_n=\int_{\mathbb{R}}\varphi\,\mathrm{d}\mu,\qquad \forall\,\varphi\in C_c(\mathbb{R}),$$

where

$$\mu(\mathrm{d}x):=\mathbb{1}_{(x\geq 0)}\,e^{2F(x)}\,\mathrm{d}x.$$

Here is an important message, that we have already noticed for discrete interfaces:

Remark

A reflection for the dynamics means a conditioning for the invariant measure.

Proof

Since μ_n is reversible for ρ^n, for all $\varphi, \psi \in C_c(\mathbb{R})$

$$\int_{\mathbb{R}} \varphi(x)\, \mathbb{E}(\psi(\rho_t^n(x)))\, \mu_n(\mathrm{d}x) = \int_{\mathbb{R}} \psi(x)\, \mathbb{E}(\varphi(\rho_t^n(x)))\, \mu_n(\mathrm{d}x).$$

We see that $\mathbb{1}_{(x\geq 0)}\, \mu_n(\mathrm{d}x) = \mathbb{1}_{(x\geq 0)}\, \mu(\mathrm{d}x)$, while

$$\mu_n(]-\infty,0[) = \int_{]-\infty,0[} e^{-n\,x^2+2F(x)}\, \mathrm{d}x \to 0$$

as $n \to +\infty$. We conclude by dominated convergence $\square$

A useful formula

Lemma

Let $(\rho_t, \ell_t)_{t\geq 0}$ *be the solution to the SDE with reflection* (4). *Then*

$$\int_{\mathbb{R}_+} e^{2F(x)}\,\mathbb{E}(\ell_t(x))\,\mathrm{d}x = \frac{t}{2}\,e^{2F(0)}, \qquad t \geq 0. \tag{6}$$

Proof. We use the notation

$$\ell^n_t(x) := \int_0^t n\,(\rho^n_s(x))^-\,\mathrm{d}s.$$

We are going to prove two formulae:

$$\lim_{n\to+\infty}\int_{\mathbb{R}} \mu_n(\mathrm{d}x)\,\mathbb{E}(\ell^n_t(x)) = \int_{\mathbb{R}_+} \mu(\mathrm{d}x)\,\mathbb{E}(\ell_t(x)), \tag{7}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}} \mu_n(\mathrm{d}x)\,\mathbb{E}(\ell^n_t(x)) = \frac{t}{2}\,e^{2F(0)}. \tag{8}$$

Again, for simplicity we assume that f is monotone non-increasing. In the lecture notes you find the argument for the general case.

Proof

First, since μ_n is invariant for ρ^n,

$$\int_{\mathbb{R}} \mu_n(\mathrm{d}x)\, \mathbb{E}\left(\int_0^t n\,(\rho_s^n(x))^- \,\mathrm{d}s\right) = t\int_{\mathbb{R}} \mu_n(\mathrm{d}x)\, n\, x^-$$

and by a simple change of variable this is equal to

$$t\int_{\mathbb{R}} e^{-x^2+2F\left(n^{-1/2}x\right)}\, x^- \,\mathrm{d}x \to t\, e^{2F(0)}\int_{\mathbb{R}} e^{-x^2}\, x^- \,\mathrm{d}x = \frac{t}{2}\, e^{2F(0)} \qquad (9)$$

as $n \to +\infty$. In the above limit we can use dominated convergence since for $x \leq 0$

$$\begin{aligned} F(x) &= -\int_x^0 f(y)\,\mathrm{d}y + F(0) \leq -\int_x^0 f(0)\,\mathrm{d}y + F(0) \\ &= -f(0)\,x + F(0) \leq C(|x|+1) \end{aligned}$$

for some $C > 0$. We have therefore proved (8). We must now prove (7).

Proof

Since f is monotone non-increasing

$$\ell_t^n = \int_0^t n\,(\rho_s^n(x))^- \,\mathrm{d}s = \rho_t^n - B_t - x - \int_0^t f(\rho_s^n)\,\mathrm{d}s$$

becomes monotone non-decreasing in n.

This result is not so trivial, since $n \mapsto (\rho_s^n)^-$ is monotone non-increasing.

Let us now use the notation

$$\begin{aligned} G_n(x) &:= \mathbb{E}(\ell_t^n(x)) = \int_0^t \mathbb{E}\left(n\,(\rho_s^n(x))^-\right)\,\mathrm{d}s \\ &= \mathbb{E}\,(\rho_t^n(x)) - x - \int_0^t \mathbb{E}\,(f(\rho_s^n(x)))\,\mathrm{d}s, \\ G(x) &:= \mathbb{E}\,(\ell_t(x)). \end{aligned}$$

Proof

By the convergence of the penalisation procedure, by monotone convergence and since $\mathbb{1}_{(x\geq 0)}\mu_n(\mathrm{d}x) = \mu(\mathrm{d}x)$, as $n \uparrow +\infty$

$$\int_{\mathbb{R}_+} G_n(x)\,\mu_n(\mathrm{d}x) = \int_{\mathbb{R}_+} G_n(x)\,\mu(\mathrm{d}x) \uparrow \int_{\mathbb{R}_+} \mu(\mathrm{d}x)\,G(x).$$

For all $x \leq 0$ we have $\rho_t^n(x) \leq \rho_t^n(0)$ and $\rho_t^n(0) \leq \rho_t(0)$. Then

$$G_n(x) \leq \mathbb{E}\left(\rho_t(0)\right) - x - \int_0^t \mathbb{E}\left(f(\rho_s(0))\right) \mathrm{d}s =: C_t - x,$$

for all $x \leq 0$ and therefore as $n \uparrow +\infty$

$$\begin{aligned} 0 \leq \int_{\mathbb{R}_-} G_n(x)\,\mu_n(\mathrm{d}x) &= \int_{\mathbb{R}_-} G_n(x)\,e^{-n\,x^2+2F(x)}\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}_-} (C_t - x)\,e^{-n\,x^2+2C(x^2+1)}\,\mathrm{d}x \downarrow 0. \end{aligned}$$

We have thus proved (7).

Reflection and local time

Theorem

Let $(X_t)_{t\geq 0}$ be a real-valued continuous semimartingale with quadratic variation $(\langle X\rangle_t)_{t\geq 0}$. There exists a measurable process $(L^a_t)_{t\geq 0, a\in\mathbb{R}}$ such that a.s. for all bounded Borel $\varphi : \mathbb{R} \mapsto \mathbb{R}$

$$\int_0^t \varphi(X_s)\, \mathrm{d}\langle X\rangle_s = \int_{\mathbb{R}} \varphi(a)\, L^a_t\, \mathrm{d}a, \qquad t \geq 0. \tag{10}$$

The process $(L^a_t)_{t\geq 0, a\in\mathbb{R}}$ has a modification such that a.s. $(t,a) \mapsto L^a_t$ is continuous in t and cadlag in a.

The process $(L^a_t)_{t\geq 0}$ is called the local time of X at $a \in \mathbb{R}$ and (10) is known as the occupation time formula.

For all $a \in \mathbb{R}$ the process $(L^a_t)_{t\geq 0}$ is monotone non-decreasing and the measure $\mathrm{d}L^a_t$ is supported by $\{t \geq 0 : X_t = a\}$.

Reflection and local time

Proposition (Tanaka's formula)

For all $a \in \mathbb{R}$ *and* $t \geq 0$

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)\, \mathrm{d}X_s + L_t^a$$

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t \mathbb{1}_{(X_s > a)}\, \mathrm{d}X_s + \frac{1}{2} L_t^a$$

where $\operatorname{sgn}(x) := \mathbb{1}_{(x>0)} - \mathbb{1}_{(x\leq 0)}$.

The solution $(\rho_t)_{t\geq 0}$ to the reflected SDE (4) is a continuous semimartingale. By Tanaka's formula applied to $(\rho_t)^+ = \rho_t$

$$\ell_t = \frac{1}{2} L_t^0 \qquad t \geq 0, \text{ a.s.}$$

Reflection and local time

In particular, a.s. ℓ_t is adapted to the filtration of ρ.

All this is not true for generic w in the Skorokhod Lemma. Let us give two instructive and simple examples for $f \equiv 0$:

1. if $\mathrm{w} \equiv 0$, then $\rho \equiv 0$ and $\ell \equiv 0$.
2. if $\hat{\mathrm{w}}_t = -t$, then $\hat{\ell}_t = t$ and $\hat{\rho}_t = 0$, $t \in [0, T]$.

Then

$$\rho = \hat{\rho} = 0, \qquad \ell = 0, \qquad \hat{\ell}_t = t.$$

The same amount of time spent at 0 is associated with different values of the reflection process.

Moreover ℓ, though always an explicit function of w by the Skorokhod Lemma 3, may not be a function of ρ: indeed $\rho = \hat{\rho}$ but $\ell \neq \hat{\ell}$.

Finally neither ρ nor $\hat{\rho}$ have local times.

Proposition

Let $(\rho_t)_{t\geq 0}$ *be a reflecting Brownian motion started at* $x \geq 0$ *and* $(B_t)_{t\geq 0}$ *a standard Brownian motion in* $\mathbb{R}$*. Then* $(\rho_t)_{t\geq 0}$ *and* $(|x+B_t|)_{t\geq 0}$ *have the same law.*

Proof.

By Tanaka's formula (see Proposition 12)

$$|x+B_t| = x + \int_0^t \mathrm{sign}(x+B_s)\,\mathrm{d}B_s + L_t^{-x}, \qquad t \geq 0,$$

The process $\beta_t := \int_0^t \mathrm{sign}(x+B_s)\,\mathrm{d}B_s$ is a standard BM by Lévy's characterisation theorem. By the Skorokhod lemma

$$L_t^{-x} = \sup_{s\leq t} \max\{-x-\beta_s, 0\}, \qquad |x+B| = x + \beta + L^{-x},$$

and therefore $(\rho_t)_{t\geq 0} = \Gamma(B)$ and $(|x+B_t|)_{t\geq 0} = \Gamma(\beta)$.

Results on the contact set

In particular $\{t \geq 0 : \rho_t = 0\}$ and $\mathscr{Z} := \{t \geq 0 : \; B_t = 0\}$ have the same law (for simplicity $x = 0$ here).

- a.s. $\mathscr{Z}$ is closed, unbounded, without isolated points and has zero Lebesgue measure.
- Let $(L_t)_{t \geq 0}$ be the local time at 0 of B and

  $$\tau_t := \inf\{s > 0 : L_s > t\}, \qquad t \geq 0.$$

  Then $(\tau_t)_{t \geq 0}$ is a stable subordinator of index $1/2$.
- The set $\mathscr{Z}$ is a.s. the closure of the image of $(\tau_t)_{t \geq 0}$: a regenerative set.
- The Hausdorff dimension of $\mathscr{Z}$ is a.s. equal to $1/2$.

δ-Bessel processes are solutions $(\rho_t)_{t\geq 0}$ to the SDE

$$\rho_t = x + \frac{\delta - 1}{2}\int_0^t \rho_s^{-1}\,\mathrm{d}s + B_t, \qquad \rho_t \geq 0, \quad t \geq 0, \tag{11}$$

where $(B_t)_{t\geq 0}$ is a BM, $x \geq 0$, and $\delta > 1$. For $\delta = 1$ the equation is

$$\rho_t = x + \ell_t + B_t, \quad \ell_0 = 0, \quad \mathrm{d}\ell \geq 0, \quad \int_0^t \rho_s\,\mathrm{d}\ell_s = 0. \tag{12}$$

This is the reflecting Brownian motion. For all $\delta \geq 1$, the law of the δ-Bessel process is $\mathbf{P}_x^\delta$.

- $X_t : C([0, +\infty[) \mapsto \mathbb{R}$ is the canonical process, $X_t(w) := w_t$, $t \geq 0$
- $\mathscr{F}_t^0 := \sigma(X_s, s \in [0, t])$ is the canonical filtration
- let $\mathbf{W}_x$ be the law of $(x + B_t)_{t\geq 0}$.

Existence and uniqueness

Let $g :]0, +\infty[\mapsto \mathbb{R}$ be continuous and monotone non-increasing such that g is Lipschitz-continuous on $[\varepsilon, +\infty[$ for all $\varepsilon > 0$.

We allow $g(x)$ to blow up as $x \downarrow 0$. The main example we have in mind is $g(x) = Cx^{-\alpha}$, $\alpha > 0$, $C > 0$.

Proposition

Let $\mathrm{w} \in C([0, T])$ *with* $\mathrm{w}_0 \geq 0$ *and* $g :]0, +\infty[\mapsto \mathbb{R}$ *as above. Then there exists a unique couple* $(\rho, \ell) \in (C([0, T]))^2$ *such that*

$$\rho \geq 0, \quad g \circ \rho \in L^1(0, T), \quad \ell_0 = 0, \quad \mathrm{d}\ell \geq 0, \quad \rho \, \mathrm{d}\ell = 0$$

$$\rho_t = \mathrm{w}_t + \int_0^t g(\rho_s) \, \mathrm{d}s + \ell_t, \qquad t \in [0, T].$$

Proof

Let us start with uniqueness. If (ρ, ℓ) and $(\hat{\rho}, \hat{\ell})$ are two solutions, then

$$\rho_t - \hat{\rho}_t = \int_0^t (g(\rho_s) - g(\hat{\rho}_s))\, \mathrm{d}s + \ell_t - \hat{\ell}_t,$$

i.e. $\rho - \hat{\rho}$ is continuous with bounded variation on $[0, T]$. Therefore by the chain rule

$$\mathrm{d}\,(\rho_t - \hat{\rho}_t)^2 = 2\,(\rho_t - \hat{\rho}_t)\left[(g(\rho_t) - g(\hat{\rho}_t))\ \mathrm{d}t + \ \mathrm{d}\ell_t - \ \mathrm{d}\hat{\ell}_t\right] \leq 0$$

and this proves uniqueness.

Proof

Let us show now existence. Let us set

$$A_\varepsilon(x) := g(\varepsilon + x^+), \qquad x \in \mathbb{R},$$

where $\varepsilon > 0$ is fixed and $x^+ := \max\{x, 0\}$. Note that A_ε is Lipschitz-continuous and monotone non-increasing, i.e.

$$A_\varepsilon(x) - A_\varepsilon(y) \le 0, \qquad \forall\, \varepsilon > 0,\ x > y.$$

There is a unique continuous solution $(\rho^\varepsilon, \ell^\varepsilon)$ to

$$\rho^\varepsilon_t = \mathrm{w}_t + \int_0^t A_\varepsilon(\rho^\varepsilon_s)\, \mathrm{d}s + \ell^\varepsilon_t, \qquad t \in [0, T],$$

with the usual conditions $\rho^\varepsilon \ge 0,\ \mathrm{d}\ell^\varepsilon \ge 0,\ \rho^\varepsilon\, \mathrm{d}\ell^\varepsilon = 0$.

Proof

Now, $\varepsilon \mapsto A_\varepsilon(x)$ is monotone non-increasing since $g(\cdot)$ is, and we expect $\varepsilon \mapsto \rho^\varepsilon$ to be also monotone non-increasing. Indeed, let $\varepsilon > \varepsilon' > 0$. Then

$$\mathrm{d}\left(\left(\rho_t^\varepsilon - \rho_t^{\varepsilon'}\right)^+\right)^2$$
$$= 2\left(\rho_t^\varepsilon - \rho_t^{\varepsilon'}\right)^+ \left(\left(A_\varepsilon(\rho_t^\varepsilon) - A_{\varepsilon'}(\rho_t^{\varepsilon'})\right)\mathrm{d}t + \mathrm{d}\ell_t^\varepsilon - \mathrm{d}\ell_t^{\varepsilon'}\right)$$
$$\leq 2\left(\rho_t^\varepsilon - \rho_t^{\varepsilon'}\right)^+ \left(\left(A_{\varepsilon'}(\rho_t^\varepsilon) - A_{\varepsilon'}(\rho_t^{\varepsilon'})\right)\mathrm{d}t + \mathrm{d}\ell_t^\varepsilon - \mathrm{d}\ell_t^{\varepsilon'}\right) \leq 0$$

- the first inequality: monotonicity of $\varepsilon \mapsto A_\varepsilon$
- the second: monotonicity of $A_\varepsilon(\cdot)$ and

$$\rho_t^\varepsilon > \rho_t^{\varepsilon'} \geq 0 \Longrightarrow \mathrm{d}\ell_t^\varepsilon = 0 \ \text{ and } \ -\mathrm{d}\ell_t^{\varepsilon'} \leq 0.$$

Therefore

$$\rho^\varepsilon \leq \rho^{\varepsilon'}, \qquad \forall\, \varepsilon > \varepsilon' > 0,$$

i.e. $\varepsilon \mapsto \rho^\varepsilon$ is monotone non-increasing.

Proof

Note now that $\gamma^\varepsilon := \varepsilon + \rho^\varepsilon$ satisfies

$$\gamma_t^\varepsilon = \varepsilon + \mathrm{w}_t + \int_0^t g(\gamma_s^\varepsilon)\,\mathrm{d}s + \ell_t^\varepsilon, \qquad t \in [0,T],$$

with $\gamma^\varepsilon \geq \varepsilon,\ \mathrm{d}\ell^\varepsilon \geq 0, (\gamma^\varepsilon - \varepsilon)\,\mathrm{d}\ell^\varepsilon = 0$. In other words, γ^ε is the solution to a SDE with reflection at level ε, a Lipschitz non-linearity $g(\cdot \vee \varepsilon)$ and driving function $\varepsilon + \mathrm{w}$; and we expect $\varepsilon \mapsto \gamma^\varepsilon$ to be monotone non-decreasing. Indeed, let $\varepsilon > \varepsilon' > 0$. Then

$$\mathrm{d}\left(\left(\gamma_t^{\varepsilon'} - \gamma_t^\varepsilon\right)^+\right)^2 =$$
$$= 2\left(\gamma_t^{\varepsilon'} - \gamma_t^\varepsilon\right)^+ \left((g(\gamma_t^{\varepsilon'}) - g(\gamma_t^\varepsilon))\,\mathrm{d}t + \mathrm{d}\ell_t^{\varepsilon'} - \mathrm{d}\ell_t^\varepsilon\right) \leq 0$$

which implies that

$$\varepsilon + \rho^\varepsilon = \gamma^\varepsilon \geq \gamma^{\varepsilon'} = \varepsilon' + \rho^{\varepsilon'}, \qquad \forall\, \varepsilon > \varepsilon' > 0,$$

i.e. $\varepsilon \mapsto \varepsilon + \rho^\varepsilon = \gamma^\varepsilon$ is monotone non-decreasing.

Proof

We obtain $\varepsilon + \rho^\varepsilon \geq \rho^{\varepsilon'} \geq \rho^\varepsilon$ for $\varepsilon > \varepsilon' > 0$, so that (ρ^ε) is a Cauchy family in the sup norm as $\varepsilon \downarrow 0$; we denote the limit by ρ. Moreover by monotone convergence

$$\int_0^t g(\varepsilon + \rho_s^\varepsilon)\, \mathrm{d}s \uparrow \int_0^t g(\rho_s)\, \mathrm{d}s, \qquad \varepsilon \downarrow 0.$$

We also obtain that, as $\varepsilon \downarrow 0$,

$$\ell_t^\varepsilon = \rho_t^\varepsilon - \mathrm{w}_t - \int_0^t g(\varepsilon + \rho_s^\varepsilon)\, \mathrm{d}s$$

converges to a continuous, monotone non-decreasing function ℓ_t with $\ell_0 = 0$. Since ρ^ε converges uniformly to ρ

$$0 = \int_0^t \rho_s^\varepsilon\, \mathrm{d}\ell_s^\varepsilon \to \int_0^t \rho_s\, \mathrm{d}\ell_s, \qquad t \in [0, T]. \qquad \square$$

Theorem

Let $\delta > 1$ and $x \geq 0$ and $(B_t)_{t\geq 0}$ a standard BM. Then the equation defining the δ-Bessel process enjoys pathwise uniqueness and existence of strong solutions.

Proof. Let us set $g(x) := \frac{\delta-1}{2} x^{-1}$, $x > 0$. By the Proposition, a.s. there exists a unique couple $(\rho, \ell) \in (C([0,T]))^2$ such that $\rho \geq 0$, $\rho^{-1} \in L^1(0,T)$, $\ell_0 = 0$, $\mathrm{d}\ell \geq 0$, $\rho \, \mathrm{d}\ell = 0$ and

$$\rho_t = x + B_t + \frac{\delta - 1}{2} \int_0^t \rho_s^{-1} \, \mathrm{d}s + \ell_t, \qquad t \in [0,T]. \tag{13}$$

By Proposition 14 if (ρ, ℓ) and $(\hat\rho, \hat\ell)$ with $\ell = \hat\ell \equiv 0$ satisfy (13), then $\rho \equiv \hat\rho$, which proves pathwise uniqueness for (11).

Proof

We must now prove that the solution (ρ, ℓ) satisfies a.s. $\ell \equiv 0$. Let

$$\rho_t^\varepsilon = \mathrm{w}_t + \int_0^t f_\varepsilon(\rho_s^\varepsilon)\, \mathrm{d}s + \ell_t^\varepsilon,$$

$$f_\varepsilon(x) := \frac{\delta - 1}{2}(\varepsilon + x)^{-1}, \qquad F_\varepsilon(x) = \frac{\delta - 1}{2}\log(\varepsilon + x).$$

We use (6) with $e^{2F_\varepsilon(x)} = (\varepsilon + x)^{\delta - 1}$ and obtain

$$\int_{\mathbb{R}_+} (\varepsilon + x)^{\delta - 1}\, \mathbb{E}(\ell_t^\varepsilon(x))\, \mathrm{d}x = \frac{t}{2}\, \varepsilon^{\delta - 1}.$$

As $\varepsilon \downarrow 0$ we obtain

$$\int_{\mathbb{R}_+} x^{\delta - 1}\, \mathbb{E}(\ell_t(x))\, \mathrm{d}x = 0.$$

Since $\mathbb{R}_+ \ni x \mapsto \mathbb{E}(\ell_t(x))$ is a monotone non-increasing function, we obtain that $\mathbb{E}(\ell_t(x)) = 0$ for all $x > 0$ and all $t \geq 0$. In order to prove that $\mathbb{E}(\ell_t(0)) = 0$, we can show that a.s. $\mathbb{R}_+ \ni x \mapsto \ell_t(x)$ is continuous. See the lecture notes.

Squared Bessel Processes

The classical construction of Bessel processes is different: first one defines the δ-squared Bessel process, namely the only non-negative solution to the SDE

$$Z_t = y + \delta t + 2 \int_0^t \sqrt{Z_s}\, \mathrm{d}B_s, \qquad t \geq 0, \tag{14}$$

where $y, \delta \geq 0$. By the classical Yamada-Watanabe theorem, this equation enjoys pathwise uniqueness, see e.g. [Revuz-Yor, Theorem IX.3.5]. One then defines the δ-Bessel $\rho_t := \sqrt{Z_t}$, and shows with the Ito formula that ρ solves (11) with $x = \sqrt{y}$, at least for $\delta \geq 2$.

With equation (14) one can introduce at once the whole family of δ-Bessel processes with $\delta \geq 0$, while the SDEs (11)-(12) yield the correct process only for $\delta \geq 1$: we shall discuss the case $\delta < 1$ later.

Theorem

Let $T_0 := \inf\{t > 0 : X_t = 0\}$. *Fix* $\delta \geq 1$. *Then we have*

1. $\mathbf{P}^\delta_x(T_0 < +\infty) = 1$ *for all* $x > 0$ *iff* $\delta < 2$.
2. $\mathbf{P}^\delta_x(T_0 < +\infty) = 0$ *for all* $x > 0$ *iff* $\delta \geq 2$.

Proof. By scaling and uniqueness in law, we have the following property for the solution to (11): for all $c > 0$

$$\text{the law of } (cX_{c^{-2}t})_{t\geq 0} \text{ under } \mathbf{P}^\delta_x \text{ is given by } \mathbf{P}^\delta_{cx},$$

namely the δ-Bessel process is self-similar of index $\frac{1}{2}$. In particular, $\mathbf{P}^\delta_x(T_0 < +\infty) = \mathbf{P}^\delta_1(T_0 < +\infty)$ for all $x > 0$.

Proof

Let $\delta \geq 1$ and $\nu := \frac{\delta}{2} - 1 \geq -\frac{1}{2}$. If we set for $t \geq 0$,

$$A_t := \int_0^t \exp(2(B_s + \nu s))\, ds, \qquad \tau_t := \inf\{u \geq 0 : A_u > t\},$$

then under $\mathbf{P}_1^\delta$ the process $(X_t, t \in [0, T_0[)$ has the same law as $(\exp(\xi_{\tau_t}), t \in [0, A_\infty[)$, where we set $\xi_t := B_t + \nu t$. This is a particular case of the so-called Lamperti formula.

Since $\frac{B_t}{t} \to 0$ a.s. as $t \to +\infty$, we obtain that a.s.

$$A_\infty := \int_0^{+\infty} \exp(2(B_s + \nu s))\, ds$$

is finite if $\nu < 0$ and it is infinite if $\nu > 0$. For the case $\nu = 0$, see the lecture notes. $\square$

Proof

Another possible proof: via the scale function of Bessel processes

$$s(x) := \begin{cases} -x^{2-\delta} & \textit{if } \delta > 2 \\ \log x & \textit{if } \delta = 2 \\ x^{2-\delta} & \textit{if } \delta < 2 \end{cases} \qquad x \geq 0,$$

then under $\mathbf{P}_x^\delta$ the process $s(X_{t \wedge T_0})_{t \geq 0}$ is a local martingale; moreover, setting and $T_a := \inf\{t \geq 0 : X_t = a\}$, we have

$$\mathbf{P}_x^\delta(T_a < T_b) = \frac{s(b) - s(x)}{s(b) - s(a)}, \qquad 0 < a \leq x \leq b.$$

Then letting $a \downarrow 0$ and $b \uparrow +\infty$ for $x > 0$, we find that $\mathbf{P}_x^\delta(T_0 < +\infty) = 1$ for $\delta < 2$ and $\mathbf{P}_x^\delta(T_0 < +\infty) = 0$ for $\delta \geq 2$. □

Local times at 0 for $\delta \in]1,2[$

Let $(\mathcal{L}_t^a)_{a,t\geq 0}$ be the local times a δ-Bessel process as a semi-martingale, i.e. in the sense of (10). By Tanaka's formula (see Proposition 12)

$$\rho_t = \rho_t^+ = \rho_0^+ + \int_0^t \mathbb{1}_{(\rho_s>0)}\,\mathrm{d}\rho_s + \frac{1}{2}\mathcal{L}_t^0 = \rho_0 + \rho_t - \rho_0 + \frac{1}{2}\mathcal{L}_t^0$$

which implies $L_t^0 \equiv 0$. However

$$\ell_t^a := a^{1-\delta}\,\mathcal{L}_t^a, \quad a > 0, \qquad \ell_t^0 := \lim_{a\downarrow 0} \ell_t^a,$$

yields a bi-continuous family $(\ell_t^a)_{a,t\geq 0}$ with the occupation-time formula

$$\int_0^t \psi(\rho_s)\,\mathrm{d}s = \int_{\mathbb{R}_+} \psi(a)\,\ell_t^a\,a^{\delta-1}\,\mathrm{d}a.$$

- $\tau_t := \inf\{s > 0 : \ell_s^0 > t\}$ is a stable subordinator of index $1 - \frac{\delta}{2}$
- $\{t \geq 0 : \rho_t = 0\}$ has Hausdorff measure equal to $1 - \frac{\delta}{2}$.

Diffusion local times

We have already seen the value of the scale function of Bessel processes. Another useful concept is the *speed measure*

$$m_\delta(\mathrm{d}x) := C_\delta \left(x^+\right)^{-2\frac{\delta-1}{\delta-2}} \mathrm{d}x, \qquad \delta < 2.$$

with $C_\delta > 0$. We define for $(B_t)_{t\geq 0}$ a BM with local times $(L_t^a)_{t,a}$

$$A_t := \int_0^t C_\delta \left((B_s)^+\right)^{-2\frac{\delta-1}{\delta-2}} \mathrm{d}s = \int L_t^a \, m_\delta(\mathrm{d}a)$$

$$\gamma_t := \inf\{u \geq 0 : A_u > t\}.$$

Theorem

Let ρ be a δ-Bessel process, $\delta > 0$. Then $(B_{\gamma_t})_{t\geq 0}$ and $(s(\rho_t))_{t\geq 0}$ have the same law. Moreover

$$\ell_t^a = L_{\gamma_t}^{a^{2-\delta}}.$$

See [Rogers-Williams, 2. vol., V.47-48]

Explicit computations

Let now $d \in \mathbb{N}$ and $B_t^{(d)} = (B_t^1, \ldots, B_t^d)$ a d-dimensional Brownian motion started at $0 \in \mathbb{R}^d$. Then we have the following

Proposition

Let $a \in \mathbb{R}^d$. Then $(|a + B_t^{(d)}|)_{t \geq 0}$ has the same law as the d-Bessel process started at $x = |a|$.

Proof.

Since for all $a \in \mathbb{R}^d$, $\mathbb{P}(\exists t > 0 : a + B_t^{(d)} = 0)$ is 1 for $d = 1$ and 0 for $d \geq 2$, then by the Ito formula for $\rho_t := |a + B_t^{(d)}|$ for $d \geq 2$

$$\rho_t = |a| + \int_0^t \frac{d-1}{2} \frac{1}{\rho_s} \, \mathrm{d}s + \int_0^t \frac{a + B_s^{(d)}}{|a + B_s^{(d)}|} \cdot \mathrm{d}B_s^{(d)}.$$

The last term is a continuous martingale with quadratic variation equal to t, and by Lévy's Theorem it is a standard Brownian motion $(\hat{B}_t)_{t \geq 0}$. □

Explicit computations

Proposition

Let $\delta \geq 1$. The transition semigroup $(p_t^\delta(x,y))_{t\geq 0, x\geq 0, y\geq 0}$ of the δ-Bessel process is given for $t > 0$ and $y \geq 0$ by

$$p_t^\delta(x,y) = \frac{1}{t}\left(\frac{y}{x}\right)^\nu y \exp\left(-\frac{x^2+y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right), \qquad x > 0,$$

$$p_t^\delta(0,y) = \frac{1}{2^\nu t^{\nu+1}\Gamma(\nu+1)} y^{2\nu+1} \exp\left(-\frac{y^2}{2t}\right),$$

where $\nu := \frac{\delta}{2} - 1$ and I_ν is the modified Bessel function of index ν

$$I_\nu(z) := \sum_{k=0}^\infty \frac{(z/2)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}, \quad z \geq 0.$$

In particular $p_t(x,y) \sim C y^{2\nu+1} = C y^{\delta-1}$ *as* $y \downarrow 0$.

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function with bounded first derivative f' and consider the following SDE in $\mathbb{R}$

$$Z_t = x + \int_0^t f(Z_s)\,\mathrm{d}s + B_t, \qquad t \geq 0.$$

Let $\mathbf{Q}_x$ be the law of $(Z_t)_{t\geq 0}$.

Exercise

Let $F : \mathbb{R} \mapsto \mathbb{R}$ such that $F' = f$. With the Girsanov theorem and the Ito formula show that

$$\mathbf{Q}_x|_{\mathscr{F}_t^0} = \exp\left(F(X_t) - F(x) - \frac{1}{2}\int_0^t (f'(X_s) + f^2(X_s))\,\mathrm{d}s \right) \cdot \mathbf{W}_x|_{\mathscr{F}_t^0}.$$

We recall that $\mathbf{W}_x$ is the law of $(x + B_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ is the canonical process.

Two exercises

We denote now by $(\mathbf{Q}_{x,y})_{y\in\mathbb{R}}$ a regular conditional distribution for $\mathbf{Q}_x[\cdot \,|\, X_1 = y]$, or in other words for the law of $(Z_t, t \in [0,1])$ given Z_1, i.e. for all Borel set $A \subset C([0,1])$

$$\mathbf{Q}_{x,y}(A) = \mathbf{Q}_x[A \,|\, X_1 = y].$$

Let $\mathbf{W}_{x,y}$ be the law of the Brownian bridge from x to y over the interval $[0,1]$, namely of $(x(1-t) + ty + B_t - tB_1)_{t\in[0,1]}$.

Exercise

We have for all $a, b \in \mathbb{R}$

$$\mathbf{Q}_{a,b} = \frac{1}{Z_{a,b}} \exp\left(-\frac{1}{2} \int_0^1 (f'(X_s) + f^2(X_s))\, \mathrm{d}s \right) \cdot \mathbf{W}_{a,b},$$

where $Z_{a,b} \in \,]0,+\infty[$ *is a normalisation constant.*

Back to Bessel processes

We want to consider for $c > 0$ and $\delta = 1 + 2c > 1$,

$$f(x) = \frac{c}{x}\,\mathbb{1}_{(x>0)} = \frac{\delta - 1}{2}\frac{1}{x}\,\mathbb{1}_{(x>0)},$$

which is of course not smooth and therefore requires some additional care. Let us note that the common term in the above exercises becomes

$$-\frac{1}{2}\int_0^1 (f'(X_s) + f^2(X_s))\,\mathrm{d}s = \frac{c - c^2}{2}\int_0^1 \frac{1}{X_s^2}\,\mathrm{d}s.$$

Note that

$$\frac{c - c^2}{2} = -\frac{(\delta - 1)(\delta - 3)}{8} \begin{cases} < 0 & \text{for} \quad \delta > 3 \\ = 0 & \text{for} \quad \delta = 3 \\ > 0 & \text{for} \quad \delta < 3. \end{cases}$$

We define $T_0 := \inf\{t > 0 : X_t = 0\}$.

Proposition

Let $x > 0$ and $\delta \geq 2$. Then

$$\mathbf{P}^\delta_x\big|_{\mathscr{F}^0_t} = \left(\frac{X_{t\wedge T_0}}{x}\right)^{\frac{\delta-1}{2}} \exp\left(-\frac{(\delta-1)(\delta-3)}{8}\int_0^t \frac{1}{X_s^2}\,\mathrm{d}s\right)\cdot \mathbf{W}_x\big|_{\mathscr{F}^0_t}.$$

Proof. If $f(x) = \frac{\delta-1}{2}\frac{1}{x}\,\mathbb{1}_{(x>0)}$, then since $X_{T_0} = 0$

$$\exp\left(F(X_{t\wedge T_0}) - F(x) - \frac{1}{2}\int_0^{t\wedge T_0}(f'(X_s) + f^2(X_s))\,\mathrm{d}s\right) =$$

$$= \left(\frac{X_{t\wedge T_0}}{x}\right)^{\frac{\delta-1}{2}} \exp\left(-\frac{(\delta-1)(\delta-3)}{8}\int_0^t \frac{1}{X_s^2}\,\mathrm{d}s\right).$$

See the lecture notes for the full proof. □

Let $x, y \geq 0$. For all Borel set $A \subset C([0,1])$

$$\mathbf{P}^\delta_{x,y}(A) = \mathbf{P}^\delta_x[A \,|\, X_1 = y].$$

Let $\mathbf{W}_{x,y}$ the law of $(x(1-t) + ty + B_t - tB_1)_{t\in[0,1]}$, Brownian bridge from x to y over the interval $[0,1]$.

Corollary

Let $a, b > 0$. Then

$$\mathbf{P}^3_{a,b} = \frac{\mathbb{1}_{(T_0>1)}}{1-\exp(-2ab)} \cdot \mathbf{W}_{a,b} = \mathbf{W}_{a,b}(\,\cdot\,|\, T_0 > 1). \tag{15}$$

For $\delta \geq 2$

$$\mathbf{P}^\delta_{a,b} = \frac{1}{Z^\delta_{a,b}} \exp\left(-\frac{(\delta-1)(\delta-3)}{8}\int_0^1 \frac{1}{X_s^2}\,\mathrm{d}s\right) \cdot \mathbf{P}^3_{a,b}.$$

Since $a, b \mapsto \mathbf{P}^{\delta}_{a,b}$ is continuous in the weak topology, we obtain

Corollary

$$\mathbf{P}^{3}_{0,0} = \lim_{a \downarrow 0} \mathbf{W}_{a,a}(\,\cdot \mid T_0 > 1) = \lim_{a \downarrow 0} \mathbf{W}_{0,0}(\,\cdot \mid \inf X \geq -a). \qquad (16)$$

$\mathbf{P}^{3}_{0,0}$ is called the law of the normalised Brownian excursion.

Note that

$$\mathbf{W}_{a,a}(T_0 > 1) = \mathbf{W}_{0,0}(\inf X \geq -a) = 1 - \exp(-2a^2),$$

$$\mathbf{W}_{0,0}(T_0 > 1) = \mathbf{W}_{0,0}(\inf X \geq 0) = 0,$$

so that the above conditioning is singular.

Proposition

Let $\mathcal{H}$ be a separable Hilbert space. There exists a process $(W(h), h \in \mathcal{H})$ such that $h \mapsto W(h)$ is linear, $W(h)$ is a centered real Gaussian random variable and

$$\mathbb{E}(W(h)\,W(k)) = \langle h, k\rangle_{\mathcal{H}}, \qquad \forall\, h, k \in \mathcal{H}.$$

Proof.

Let $(Z_i)_i$ be a i.i.d. sequence of real standard Gaussian variables and $(h_i)_i$ a complete orthonormal system in $\mathcal{H}$ and set

$$W(h) := \sum_{i=1}^{+\infty} \langle h, h_i\rangle_{\mathcal{H}}\, Z_i, \qquad h \in \mathcal{H}.$$

This series converges in $L^2(\Omega)$. □

White noise

Let now $(T, \mathcal{B}, m)$ be a separable measurable space, with m a σ-finite measure. We apply the Proposition to $\mathcal{H} := L^2(T, \mathcal{B}, m)$, with canonical scalar product

$$\langle h, k \rangle_{\mathcal{H}} := \int_T h(x)\, k(x)\, m(\mathrm{d}x), \qquad h, k \in L^2(T, \mathcal{B}, m).$$

The process $(W(h), h \in \mathcal{H})$ is called a white noise over $(T, \mathcal{B}, m)$.

If $A \in \mathcal{B}$ and $m(A) < +\infty$, then $\mathbb{1}_A \in \mathcal{H}$ and we denote $W(A) := W(\mathbb{1}_A)$. If $A, B \in \mathcal{B}$ with $m(A) + m(B) < +\infty$ then

$$\mathbb{E}(W(A)\, W(B)) = m(A \cap B).$$

In particular, if $m(A \cap B) = 0$, then $\{W(A'), A' \subseteq A\}$ and $\{W(B'), B' \subseteq B\}$ are independent.

White noise

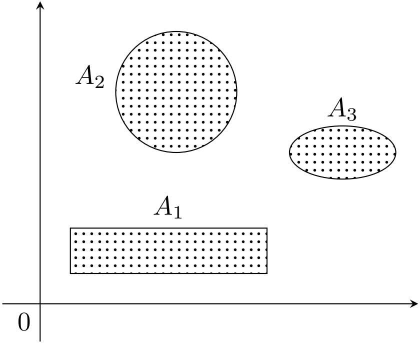


Figure : The family $(W(A_i))_i$ is Gaussian and independent since the sets A_i are pairwise disjoint. $W(A_i) \sim \mathcal{N}(0, m(A_i))$ and $W(\cup_i A_i) \sim \mathcal{N}(0, \sum_i m(A_i))$.

For all measurable set A, $W(A)$ is the amount of noise contained in A.

Finite dimensional white noise

Let us consider first the easiest case: $T = \{1, \ldots, d\}$ and m is the counting measure.

In this case $L^2(T, \mathcal{B}, m) = \mathbb{R}^d$ and the white noise $W(h)$ can be realised as $W(h) = \langle W, h \rangle_{\mathbb{R}^d}$, where $W \sim \mathcal{N}(0, I)$.

Brownian motion

Let now $T = \mathbb{R}$ endowed with the Borel σ-algebra and the Lebesgue measure λ_1. Then for all $[a, b]$ and $[c, d]$ in $\mathbb{R}$

$$\mathbb{E}(W([a,b])\, W([c,d])) = \lambda_1([a,b] \cap [c,d]).$$

Then the process

$$B_t := \begin{cases} W([0,t]), & t \geq 0, \\ W([t,0]), & t < 0. \end{cases} \tag{17}$$

is (a modification of) a two-sided standard Brownian motion, and $W(\mathrm{d}t)$ is simply called white noise over $\mathbb{R}$. In particular, the process $(W([0,t]),\ t \geq 0)$ is (a modification of) a standard BM.

Multi-dimensional Brownian motion

Let now $T = \mathbb{R} \times \{1, \ldots, d\}$ endowed with the Borel σ-algebra and the measure $\lambda_1 \otimes m$ where λ_1 is the Lebesgue measure and m is the counting measure.

Then for all $[a, b]$ and $[c, d]$ in $\mathbb{R}$ and for any $i, j \in \{1, \ldots, d\}$

$$\mathbb{E}(W([a,b] \times \{i\})\, W([c,d] \times \{j\})) = \lambda_1([a,b] \cap [c,d])\, \mathbb{1}_{(i=j)}.$$

Then the process $(B^1_t, \ldots, B^d_t)$, defined by

$$B^i_t := \begin{cases} W([0,t] \times \{i\}), & t \geq 0, \\ W([t,0] \times \{i\}), & t < 0. \end{cases}$$

is (a modification of) a two-sided standard Brownian motion in $\mathbb{R}^d$.

SDEs with additive noise

Then a standard SDE with additive noise

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}B_t$$

can be written, if $B_t = W([0,t])$, as

$$\dot{X}_t = b(X_t) + W = b(X_t) + \dot{B}_t.$$

This is just a notation and both formulae should be interpreted as

$$X_t = X_0 + \int_0^t b(X_s)\,\mathrm{d}s + B_t, \qquad t \geq 0.$$

Brownian sheet

If $T = \mathbb{R}_+^2 = [0, +\infty[^2$ endowed with the Borel σ-algebra and the Lebesgue measure λ_2, then

$$\mathbb{E}(W([0,t] \times [0,t'])\, W([0,s] \times [0,s'])) = (t \wedge s)\, (t' \wedge s')$$

The process $(B(t,s) := W([0,t] \times [0,s]), t, s \geq 0)$ is called a Brownian sheet and $W(\mathrm{d}t, \mathrm{d}s)$ a space-time white noise. One can also use the notations

$$W(\mathrm{d}t, \mathrm{d}s) = \frac{\partial^2 B}{\partial t \partial s} = W(t,s).$$

Notice that the same construction can be done if $T = \mathbb{R}_+^d$: this gives a space-time white noise with a d-dimensional space variable.

Lemma

Let $(e_i)_i$ be a complete orthonormal system in $L^2([0,+\infty[)$. Then

1. *Let $w_t^i := W(\mathbb{1}_{[0,t]} \otimes e_i)$, $t \geq 0$, $i \in \mathbb{N}$. Then $(w^i)_i$ is an iid sequence of standard Brownian motions.*
2. *For all $h \in L^2([0,+\infty[)$ and $t \geq 0$*

$$W(\mathbb{1}_{[0,t]} \otimes h) = \sum_i w_t^i \langle e_i, h \rangle$$

where the equality is in $L^2(\Omega)$.

A cylindrical Brownian motion in a separable Hilbert space H is

$$\langle W_t, h \rangle := \sum_i B_t^i \langle e_i, h \rangle, \qquad t \geq 0,$$

where $(e_i)_i$ is a complete orthonormal system in H and $(B^i)_i$ is an iid sequence of Brownian motions.

The stochastic heat equation

We want to study the stochastic PDE

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 v}{\partial x^2} + W, \\ v(t,0) = v(t,1) = 0, \qquad t \geq 0, \\ v(0,x) = v_0(x), \qquad x \in [0,1] \end{cases} \tag{18}$$

where $W(t,x)$ is a space-time white-noise over $[0,+\infty[\times[0,1]$.

This is the first SPDE we encounter. It is interpreted in the PDE-weak sense: for all $h \in C_c^2(0,1)$ and $t \geq 0$

$$\langle v_t, h\rangle = \langle v_0, h\rangle + \frac{1}{2}\int_0^t \langle v_s, h''\rangle \, \mathrm{d}s + \int_0^t \int_0^1 h(x)\, W(\mathrm{d}s, \mathrm{d}x).$$

The deterministic heat equation

Let us start from the heat equation without noise:

$$\begin{cases} \dfrac{\partial \mathrm{v}}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 \mathrm{v}}{\partial x^2}, \\ \\ \mathrm{v}(t,0) = \mathrm{v}(t,1) = 0, \qquad t > 0 \\ \\ \mathrm{v}(0,x) = v_0(x), \qquad x \in [0,1] \end{cases} \tag{19}$$

where $v_0 \in H := L^2(0,1)$. We set for all $k \geq 1$:

$$e_k(x) := \sqrt{2}\,\sin(k\pi x), \qquad x \in [0,1]. \tag{20}$$

Note that $\{e_k\}_{k\geq 1}$ is a complete basis of eigenvectors of the second derivative with homogeneous Dirichlet boundary conditions:

$$\frac{d^2}{dx^2} e_k = -(\pi k)^2 e_k, \qquad e_k(0) = e_k(1) = 0, \qquad k \geq 1.$$

Fourier decomposition

We set

$$\lambda_k := \frac{(\pi k)^2}{2}, \qquad k \geq 1,$$

The solution of the heat equation (19) is therefore

$$\mathrm{v}(t,x) = \sum_{k\geq 1} e^{-t\lambda_k} \langle e_k, v_0 \rangle \, e_k(x), \qquad t > 0,\ x \in [0,1].$$

Since $|e_k(x)| \leq \sqrt{2}$ and $\sum_{k\geq 1} e^{-t\lambda_k} k^m < +\infty$ for all $m \in \mathbb{N}$, the above series converges uniformly on $[\varepsilon, +\infty[\times[0,1]$ for all $\varepsilon > 0$, together with all its partial derivatives in t and x. One can write more compactly, using the semigroup notation,

$$\mathrm{v}_t := \mathrm{v}(t,\cdot) = e^{tA} v_0 := \sum_{k\geq 1} e^{-t\lambda_k} \langle e_k, v_0 \rangle \, e_k, \qquad t \geq 0.$$

Fourier expansion of the SHE

Let us consider the scalar product of both terms of (18) and e_k. Setting $v_t^k := \langle v(t,\cdot), e_k\rangle$ we obtain

$$\mathrm{d}v_t^k = -\frac{(k\pi)^2}{2}\, v_t^k\,\mathrm{d}t + \mathrm{d}B_t^k, \qquad v_0^k = \langle v_0, e_k\rangle$$

$$B_t^k := \int_{[0,t]\times[0,1]} e_k(x)\, W(\mathrm{d}s,\mathrm{d}x) = W\left(\mathbb{1}_{[0,t]}\otimes e_k\right).$$

We proved in Lemma 26 that $(B_t^k, t\geq 0)_{k\geq 1}$ is an independent sequence of Brownian motions. Then $(v_\cdot^k)_{k\geq 1}$ is an independent family of O-U processes of respective parameter $\lambda_k > 0$, with

$$v_t^k = e^{-\lambda_k t} v_0^k + \int_0^t e^{-\lambda_k(t-s)}\,\mathrm{d}B_s^k, \qquad t\geq 0.$$

An important remark is the following:

$$\sum_k \frac{1}{\lambda_k} = \sum_k \frac{2}{(\pi k)^2} < +\infty.$$

Fourier expansion of the SHE

Lemma

For all $t \geq 0$ and $v_0 \in H = L^2(0,1)$ the series

$$V_t(v_0) := \sum_{k=1}^{+\infty} v_t^k \, e_k = e^{tA} v_0 + \sum_{k=1}^{+\infty} \left(\int_0^t e^{-\lambda_k (t-s)} \, \mathrm{d}B_s^k \right) e_k \qquad (21)$$

converges in $L^2(\Omega; H)$ to a well-defined r.v. with values in H.

Proof.

Since $v_t^k \sim \mathcal{N}\left(e^{-\lambda_k t} v_0^k, \frac{1}{2\lambda_k}(1 - e^{-2\lambda_k t})\right)$, then

$$\mathbb{E}\left(\left\| \sum_{k=n+1}^{m} v_t^k \, e_k \right\|^2 \right) = \sum_{k=n+1}^{m} \left[e^{-2\lambda_k t} \left(v_0^k\right)^2 + \frac{1}{2\lambda_k}(1 - e^{-2\lambda_k t}) \right] \to 0$$

as $n, m \to +\infty$, since $\lambda_k = Ck^2$. □ □

Path continuity

Until now we have considered V_t as a $L^2(0,1)$-valued random variable. However, if $x \in [0,1]$ is fixed, then

$$v_n(t,x) := \sum_{k=1}^{n} v_t^k \, e_k(x) = \sum_{k=1}^{n} \langle v_t, e_k \rangle \, e_k(x) \in \mathbb{R}$$

is well defined and continuous. The deterministic part $e^{tA} v_0$ is simple to analyse, therefore let us suppose that $v_0 = 0$. Then since $e_k^2(\cdot) \leq 2$

$$\mathbb{E}\left(|v_n(t,x) - v_m(t,x)|^2\right) = \sum_{k=n+1}^{m} \mathbb{E}\left(\left(v_t^k\right)^2\right) e_k^2(x)$$

$$\leq 2 \sum_{k=n+1}^{m} \int_0^t e^{-2\lambda_k(t-s)} \, \mathrm{d}s = 2 \sum_{k=n+1}^{m} \frac{1 - e^{-2\lambda_k t}}{2\lambda_k} \to 0$$

as $n, m \to +\infty$. Therefore, there exists a well defined stochastic process $(v(t,x), t \geq 0, x \in [0,1])$, limit in $L^2(\mathbb{P})$ of $(v_n(t,x), t \geq 0, x \in [0,1])$ as $n \to \infty$.

Lemma

Let $v_0 = 0$*. For all* $m \in \mathbb{N}$ *there exists a constant* $C_m < +\infty$ *such that*

$$\mathbb{E}\left(|v(t,x) - v(s,y)|^{2m}\right) \leq C_m \left(|t-s|^{\frac{m}{2}} + |x-y|^m\right),$$

for all $t, s \geq 0$*,* $x, y \in [0,1]$*.*

Proof. Since $(v(t,x) - v(s,y))$ is a real Gaussian variable with 0 mean, in order to estimate its moments it is enough to compute the second one, i.e. it is enough to prove that for some constant C

$$\mathbb{E}\left(|v(t,x) - v(s,y)|^2\right) \leq C \left(|t-s|^{\frac{1}{2}} + |x-y|\right).$$

First we have

$$|v_n(t,x) - v_n(s,y)|^2 \leq 2\,|v_n(t,x) - v_n(s,x)|^2 + 2\,|v_n(s,x) - v_n(s,y)|^2.$$

Proof

Now (recall that $v_0 = 0$)

$$\mathbb{E}\left(|v_n(s,x) - v_n(s,y)|^2\right) =$$

$$= \mathbb{E}\left(\left|\sum_{k=1}^{n} v_s^k\,(e_k(x) - e_k(y))\right|^2\right)$$

$$= \sum_{k=1}^{n} \frac{1 - e^{-2\lambda_k s}}{2\lambda_k}(e_k(x) - e_k(y))^2 \leq \sum_{k=1}^{n} \frac{1 \wedge (|x-y|\,k)^2}{k^2}$$

$$\leq 1 \wedge |x-y| + \int_1^\infty \frac{1 \wedge (|x-y|\,k)^2}{k^2}\,dk \leq 3\,|x-y|.$$

Proof

With similar computations

$$\mathbb{E}\left(|v_n(t,x) - v_n(s,x)|^2\right) =$$

$$= \sum_{k=1}^{n} e_k^2(x) \left[(1 - e^{-\lambda_k(t-s)})^2 \frac{1 - e^{-2\lambda_k s}}{2\lambda_k} + e^{-2\lambda_k s} \frac{1 - e^{-2\lambda_k (t-s)}}{2\lambda_k} \right]$$

$$\leq 2 \sum_{k=1}^{n} \frac{1 \wedge (|t-s|\, k^2)}{k^2} \leq 2 \left(1 \wedge |t-s| + \int_1^\infty \frac{1 \wedge (|t-s|\, k^2)}{k^2}\, dk \right)$$

$$\leq 6 \sqrt{|t-s|}.$$

We have used the fact that

$$(1 - e^{-\lambda_k(t-s)})^2 \leq (1 \wedge [\lambda_k(t-s)])^2 \leq 1 \wedge [\lambda_k(t-s)].$$

The desired result follows if we let $n \to \infty$. □

Proposition

Let $v_0 = 0$. There exists a continuous modification of $(v(t,x), t \geq 0, x \in [0,1])$, that we call v again. Moreover a.s. for all $\varepsilon \in]0,1[$ and $T < +\infty$

$$\sup_{x,y \in [0,1],\, t,s \in [0,T]} \frac{|v(t,x) - v(s,y)|}{|t-s|^{\frac{1-\varepsilon}{4}} + |x-y|^{\frac{1-\varepsilon}{2}}} < +\infty.$$

This proposition is a consequence of the Kolmogorov criterion stated for a inhomogeneous distance on $\mathbb{R}^d$.

Two points of view

The solution to the stochastic heat equation can be seen:

- as a continuous $L^2(0,1)$-valued process $(V_t)_{t\geq 0}$, solving the SDE

$$dV = AV\,dt + dW_t$$

where A is ∂_x^2 with homogeneous Dirichlet boundary condition and

$$W_t := \sum_i B_t^i\, e_i$$

is a cylindrical Brownian motion in $L^2(0,1)$. This is the Da Prato-Zabczyk approach.

- as a bi-continuous real valued process $(v(t,x), t \geq 0, x \in [0,1])$. This is the Walsh approach.

A probability measure μ on H is said to be invariant for $(V_t)_{t\geq 0}$, see (27), if, given a r.v. $\mathcal{Z}$ with law μ and independent of W, the process $(V_t(\mathcal{Z}))_{t\geq 0}$ is stationary.

An invariant probability measure μ of $(V_t)_{t\geq 0}$ is reversible if for all $T \geq 0$, the processes $(V_t(\mathcal{Z}))_{t\in[0,T]}$ and $(V_{T-t}(\mathcal{Z}))_{t\in[0,T]}$ have the same law.

Proposition

The law $\mathbf{W}_{0,0}$ *of the brownian bridge* $(B_x - xB_1, x \in [0,1])$*, with* B *a standard BM, is the unique invariant probability measure of* $(V_t)_{t\geq 0}$*. Moreover* $\mathbf{W}_{0,0}$ *is reversible for* $(V_t)_{t\geq 0}$*.*

The proof of this proposition is split into several (simple) lemmas.

The invariant measure

First we have the *Karhunen-Loève* decomposition of the Brownian bridge.

Lemma

Let μ_{t,v_0} *the law on* $H = L^2(0,1)$ *of* $V_t(v_0)$*, solution to* (18)*. Then* $\mu_{t,v_0} \Longrightarrow \mu$*, law of the* H*-valued r.v.*

$$\beta := \sum_{k=1}^{+\infty} \frac{1}{\pi k} Z_k \, e_k = \sum_{k=1}^{+\infty} \frac{1}{\sqrt{2\lambda_k}} Z_k \, e_k, \qquad (22)$$

where $(Z_k)_{k\geq 1}$ *is an i.i.d. sequence of* $\mathcal{N}(0,1)$ *variables and* $(e_k)_k$ *is the complete orthonormal system of* H *defined in* (20).

Proof.

Recall that $v_t^k \sim \mathcal{N}\left(e^{-\lambda_k t} v_0^k, \frac{1}{2\lambda_k}(1 - e^{-2\lambda_k t})\right)$. □

Lemma

The $L^2(0,1)$*-valued r.v.* β *has law* $\mathbf{W}_{0,0}$.

Proof.

$$\mathbb{E}\left(\beta_x\,\beta_y\right) = \sum_{k=1}^{+\infty} \frac{1}{(\pi k)^2}\, e_k(x)\, e_k(y) =: \alpha(x,y), \qquad x, y \in [0,1].$$

$$\begin{cases} -\dfrac{\partial^2 \alpha}{\partial x^2} = \delta_y(\mathrm{d}x), \\ \\ \alpha(0,y) = \alpha(1,y) = 0. \end{cases}$$

The unique solution to this equation is

$$\alpha(x,y) = x \wedge y - xy \qquad x, y \in [0,1]$$

which is the covariance function of a Brownian bridge on $[0,1]$. □

What we have done so far

- We have seen that a reflected dynamics is associated with a conditioning of the invariant measure.
- The stochastic heat equation has the law of the Brownian bridge as invariant measure
- The Brownian bridge conditioned to be non-negative is the 3-Bessel bridge, i.e. the normalised Brownian excursion.

Therefore we expect the reflected stochastic heat equation to have the law of 3-Bessel bridge as invariant measure.

The next step is the construction of the solution u to this SPDE with reflection.

Then we shall see the impact that the fine properties of these Brownian processes have on u.

Ito formula

Since $W = \sum_k \mathrm{d}B_t^k\, e_k$, we understand why the classical Ito calculus runs into trouble: if $\varphi \in C_c^2(\mathbb{R})$ then formally

$$\mathrm{d}\varphi(v(t,x)) = \left(\frac{1}{2}\, \frac{\partial^2 v}{\partial x^2}(t,x)\, \mathrm{d}t + \sum_k \mathrm{d}B_t^k\, e_k(x) \right) \varphi'(v(t,x)) + \frac{1}{2}\varphi''(v(t,x)) \sum_k \langle e_k, e_k \rangle\, \mathrm{d}t$$

and $\sum_k \langle e_k, e_k \rangle = +\infty$.

Ito formula

In fact the two terms in red

$$\mathrm{d}\varphi(v(t,x)) = \left(\frac{1}{2} \frac{\partial^2 v}{\partial x^2}(t,x)\,\mathrm{d}t + \sum_k \mathrm{d}B_t^k\, e_k(x) \right) \varphi'(v(t,x)) + \frac{1}{2}\varphi''(v(t,x)) \sum_k \langle e_k, e_k \rangle\,\mathrm{d}t$$

are ill-defined and compensate each other.

This is a renormalisation phenomenon.

Semilinear SPDEs

Let $f : \mathbb{R} \mapsto \mathbb{R}$ such that there exists a constant $L > 0$ with

$$|f(u) - f(v)| \leq L|u - v|, \qquad \forall\, u, v \in \mathbb{R}.$$

We study the following SPDE in $(t, x) \in \mathbb{R}_+ \times [0, 1]$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + f(u) + W \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Interpreted in the PDE-weak sense

$$\begin{aligned} \langle u_t, h\rangle = \langle u_0, h\rangle &+ \frac{1}{2}\int_0^t \langle u_s, h''\rangle\, \mathrm{d}s + \int_0^t \langle f(u_s), h\rangle\, \mathrm{d}s \\ &+ \int_0^t \int_0^1 h(x)\, W(\mathrm{d}s, \mathrm{d}x) \end{aligned} \tag{23}$$

Semilinear SPDEs

We set $U_t := u(t,\cdot)$, $\quad U_t^k := \langle U_t, e_k \rangle$, $\quad f_t^k := \langle f(U_t), e_k \rangle$,

$$B_t^k := \int_{[0,t]\times[0,1]} e_k(x)\, W(\mathrm{d}s, \mathrm{d}x) = W\left(\mathbb{1}_{[0,t]} \otimes e_k\right).$$

Then

$$\mathrm{d}U_t^k = \left(-\lambda_k\, U_t^k + f_t^k\right)\, \mathrm{d}t + \mathrm{d}B_t^k, \qquad U_0^k = \langle u_0, e_k \rangle$$

and, by applying the Ito formula to the process $t \mapsto e^{\lambda_k t} U_t^k$ we obtain

$$U_t^k = e^{-\lambda_k t} U_0^k + \int_0^t e^{-\lambda_k (t-s)} f_s^k \,\mathrm{d}s + \int_0^t e^{-\lambda_k (t-s)}\, \mathrm{d}B_s^k.$$

The $(B^k)_k$ are still iid BMs but the non-linearity f couples the U^k's.

Semilinear SPDEs

If we sum again over the Fourier basis

$$U_t = \sum_k \langle U_t^k, e_k \rangle \, e_k = e^{tA} u_0 + \int_0^t e^{(t-s)A} f(U_s) \, \mathrm{d}s + W_A(t),$$

where the stochastic convolution

$$W_A(t) := \sum_{k=1}^{+\infty} \left(\int_0^t e^{-\lambda_k (t-s)} \, \mathrm{d}B_s^k \right) e_k$$

is the solution of the SHE (18) with null initial condition, see (21).

If f is Lipschitz, then we can write a fixed point argument in $E_T := C([0,T]; H)$ with $\Gamma : E_T \mapsto E_T$ defined by $\Gamma(g) := h$, where

$$h_t := e^{tA} u_0 + \int_0^t e^{(t-s)A} f(g_s) \, \mathrm{d}s + W_A(t), \qquad t \in [0,T].$$

Higher space dimension

One can study the stochastic PDE for $t \geq 0$, $x \in \mathbb{R}^d$

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v + W,$$

where $W(t,x)$ is a space-time white-noise over $[0,+\infty[\times\mathbb{R}^d$.

The solution is well-defined as a Gaussian field, but has no continuous modification.

In fact, its solutions live in distribution spaces: the higher the dimension, the more irregular the solution.

Non-linear SPDEs become much more difficult and one needs the theory of Regularity Structures.

Reflection seems out of reach.

Let $a \geq 0$. We fix a space-time white noise W on $[0, +\infty[\times[0, 1]$. We study the following SPDE with reflection:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + f(u) + W + \eta \\ u(0,x) = u_0(x),\ u(t,0) = u(t,1) = a \\ u \geq 0,\ \mathrm{d}\eta \geq 0,\ \int u\,\mathrm{d}\eta = 0 \end{cases} \tag{24}$$

where we assume that:

1. $u_0 : [0, 1] \mapsto \mathbb{R}$ is continuous and $u_0 \geq 0$.
2. $f : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz and bounded.

The Nualart-Pardoux equation

Definition

A pair (u, η) is said to be a solution of equation (24) if a.s.

1. $(u(t,x),\ t \geq 0, x \in [0,1])$ is continuous
2. $u \geq 0$, $u(t,0) = u(t,1) = a$ for all $t \geq 0$ and $u(0,\cdot) = u_0$
3. $\eta(\mathrm{d}t, \mathrm{d}x)$ is a measure on $]0,+\infty[\,\times\,]0,1[$ such that $\eta(]0,T] \times [\delta, 1-\delta]) < +\infty$ for all $T, \delta > 0$
4. for all $t \geq 0$ and $h \in C_c^\infty(0,1)$

$$\begin{aligned} \langle u_t, h\rangle = \langle u_0, h\rangle \ &+ \frac{1}{2}\int_0^t \langle u_s, h''\rangle\,\mathrm{d}s + \int_0^t \langle f(u_s), h\rangle\,\mathrm{d}s \\ &+ \int_0^t\int_0^1 h(x)\,W(\mathrm{d}s,\mathrm{d}x) \ + \int_0^t\int_0^1 h(x)\,\eta(\mathrm{d}s,\mathrm{d}x) \end{aligned} \qquad (25)$$

5. $\int u\,\mathrm{d}\eta \ = \ 0$ or, equivalently, the support of η is contained in $\{(t,x) :\ u(t,x) = 0\}$.

Reduction to a PDE with random obstacle

Let $a \geq 0$ and v be the unique solution to

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2} + W, \\ v(t,0) = v(t,1) = a, \quad v(0,x) = u_0(x). \end{cases} \tag{26}$$

Then the function $z := u - v$ solves

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 z}{\partial x^2} + f(z+v) + \eta \\ z(0,x) = 0,\ z(t,0) = z(t,1) = 0 \\ z \geq -v,\ \mathrm{d}\eta \geq 0,\ \int (z+v)\,\mathrm{d}\eta = 0. \end{cases} \tag{27}$$

The important remark here is that equation (27) is a PDE (rather than a SPDE) with random obstacle $-v$.

Existence and uniqueness

Theorem

Let $\mathrm{w} \in C([0,T] \times [0,1])$ *with* $\mathrm{w}(0,\cdot) \geq 0$, $\mathrm{w}(\cdot,0) \geq 0$, $\mathrm{w}(\cdot,1) \geq 0$.
Then there exists a unique pair (z,η) *such that*

- $z \in C([0,T] \times [0,1])$, $\quad z(0,\cdot) = 0$, $\quad z(\cdot,0) = z(\cdot,1) = 0$
- $\eta(\mathrm{d}t,\mathrm{d}x)$ *is a measure on* $]0,T] \times]0,1[$ *such that* $\eta(]0,T] \times [\delta, 1-\delta]) < +\infty$ *for all* $\delta > 0$
- *For all* $t \in [0,T]$ *and* $h \in C_c^\infty(0,1)$

$$\langle z_t, h\rangle = \frac{1}{2}\int_0^t \langle z_s, h''\rangle \,\mathrm{d}s + \int_0^t \langle f(z_s + \mathrm{w}_s), h\rangle \,\mathrm{d}s + \int_0^t \int_0^1 h(x)\,\eta(\mathrm{d}s,\mathrm{d}x) \tag{28}$$

- $z \geq -\mathrm{w}$, $\quad \int (z + \mathrm{w})\,\mathrm{d}\eta = 0.$

Uniqueness

Let (z, η) and $(\bar{z}, \bar{\eta})$ two solutions. If $z - \bar{z}$ is regular enough

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|z-\bar{z}\|^2 = \\
&= \langle z-\bar{z}, \partial_{xx}(z-\bar{z}) + 2(f(z)-f(\bar{z}))\rangle + 2\int_0^1 (z-\bar{z})(\mathrm{d}\eta - \mathrm{d}\bar{\eta}) \\
&\le -\|\partial_x(z-\bar{z})\|^2 + 2L\|z-\bar{z}\|^2 - 2\int_0^1 (z\,\mathrm{d}\bar{\eta} + \bar{z}\,\mathrm{d}\eta) \\
&\le 2L\|z-\bar{z}\|^2
\end{aligned}
$$

where $\|\cdot\|$ is the norm in $L^2(0,1)$. Since $\|z-\bar{z}\|^2$ is zero at $t=0$, we conclude that $z \equiv \bar{z}$. This argument is not fully rigorous since $z - \bar{z}$ does not necessary enjoy the necessary regularity for the previous computations. However Nualart and Pardoux proved that this argument can be made rigorous. □

Existence: penalisation

We introduce the following approximating problem:

$$\begin{cases} \dfrac{\partial z^\varepsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 z^\varepsilon}{\partial x^2} + f(z^\varepsilon + \mathrm{w}) + \dfrac{(z^\varepsilon + \mathrm{w})^-}{\varepsilon} \\[2ex] z^\varepsilon(0,\cdot) = 0, \quad z^\varepsilon(t,0) = z^\varepsilon(t,1) = 0, \end{cases} \tag{29}$$

with $\varepsilon > 0$. We set

$$f_\varepsilon(r) := f(r) + \frac{(r^-)}{\varepsilon}, \qquad r \in \mathbb{R}.$$

We suppose as usually that f is monotone non-increasing, so that

$$f_\varepsilon(r) - f_\varepsilon(r') \leq 0, \qquad \forall\, r > r'.$$

Monotonicity

We show that $z^\varepsilon \le z^{\varepsilon'}$ if $\varepsilon > \varepsilon'$. Since $f_\varepsilon(\cdot) \le f_{\varepsilon'}(\cdot)$ and with an integration by parts:

$$\frac{d}{dt}\left\|(z^\varepsilon - z^{\varepsilon'})^+\right\|^2 =$$

$$= 2\left\langle (z^\varepsilon - z^{\varepsilon'})^+, \frac{1}{2}\partial_{xx}(z^\varepsilon - z^{\varepsilon'}) + f_\varepsilon(z^\varepsilon + \mathrm{w}) - f_{\varepsilon'}(z^{\varepsilon'} + \mathrm{w}) \right\rangle$$

$$\le -\left\|\partial_x(z^\varepsilon - z^{\varepsilon'})^+\right\|^2 + 2\left\langle (z^\varepsilon - z^{\varepsilon'})^+, f_{\varepsilon'}(z^\varepsilon + \mathrm{w}) - f_{\varepsilon'}(z^{\varepsilon'} + \mathrm{w}) \right\rangle$$

$$\le 0$$

where $\|\cdot\|$ is the norm in $L^2(0,1)$.

Since $z_0^\varepsilon - z_0^{\varepsilon'} = 0$, we conclude that $z^\varepsilon \le z^{\varepsilon'}$.

The rest of the proof

Three more steps, as for one-dimensional diffusions:

- The case of a smooth obstacle
- Continuity of the map w $\mapsto z^\varepsilon$ uniformly in $\varepsilon > 0$
- Conclusion.

The crucial point is the maximum principle of ∂^2_{xx}, i.e. the fact that monotonicity is preserved.

The rest of the proof

Three more steps, as for one-dimensional diffusions:

- The case of a smooth obstacle
- Continuity of the map $\mathrm{w} \mapsto z^\varepsilon$ uniformly in $\varepsilon > 0$
- Conclusion.

The crucial point is the maximum principle of ∂^2_{xx}, i.e. the fact that monotonicity is preserved.

In our computations it was the simple formula:

$$\langle (z^\varepsilon - z^{\varepsilon'})^+, \partial_{xx}(z^\varepsilon - z^{\varepsilon'}) \rangle = - \left\| \partial_x (z^\varepsilon - z^{\varepsilon'})^+ \right\|^2 \leq 0.$$

The penalised SPDE

Setting w := v and $u^\varepsilon := z^\varepsilon + v$, where v is defined by (26) and z^ε by (29), then u^ε solves the SPDE

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial x^2} + f(u^\varepsilon) + \dfrac{(u^\varepsilon)^-}{\varepsilon} + W \\ \\ u^\varepsilon(0,\cdot) = u_0, \quad u^\varepsilon(t,0) = u^\varepsilon(t,1) = a. \end{cases}$$

This is a particular example of (23).

It will be useful in order to compute the invariant measure of the reflected dynamics.

The invariant measure

We already expect the invariant measure of $(u_t)_{t\geq 0}$ to be $\mathbf{P}^3_{a,a}$, law of the 3-Bessel bridge $a \to a \geq 0$ (for $f = 0$).

Let us consider a Brownian motion $(B^{(d)}_t)_{t\geq 0}$ in $\mathbb{R}^d$ where $B^{(d)} = (B^1, \ldots, B^d)$ and the B^i's are iid standard BMs.

Let $V : \mathbb{R}^d \to \mathbb{R}$ be a smooth function and let

$$\mathrm{d}X_t = \nabla V(X_t)\,\mathrm{d}t + \mathrm{d}B^{(d)}_t, \qquad X_0 = x \in \mathbb{R}^d.$$

Then one can show that an invariant measure for $(X_t)_{t\geq 0}$ is given by

$$\exp(2V(x))\,\mathrm{d}x.$$

If this measure is finite on $\mathbb{R}^d$, we obtain an invariant probability measure.

Gradient systems

Let us consider again a semi-linear SPDE

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + f(u) + W.$$

Then $W = \sum_k \dot{B}^k e_k$ where $(e_k)_k$ is a c.o.s. in $H := L^2(0,1)$ such that

$$\frac{d^2}{dx^2} e_k = -(\pi k)^2 e_k, \qquad e_k(0) = e_k(1) = 0, \qquad k \geq 1,$$

recall (20). Question: can we find $V : H \mapsto \mathbb{R}$ such that

$$\mathrm{d}u = \nabla V(u)\,\mathrm{d}t + \mathrm{d}W.$$

Here the gradient ∇ is in the Hilbert-structure of H:

$$\langle \nabla V(\zeta), h\rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(V(\zeta + \varepsilon h) - V(\zeta)).$$

Gradient systems

We set $A : D(A) \subset H = L^2(0,1) \to H$

$$D(A) = \left\{ h \in H : \sum_{k \geq 1} \lambda_k^2 \langle e_k, h \rangle^2 < +\infty \right\},$$

$$Ah = -\sum_{k \geq 1} \lambda_k \, \langle e_k, h \rangle \, e_k, \qquad h \in D(A),$$

recall that $\sum_k \lambda_k^{-1} < +\infty$. If $h \in C_c^2(0,1)$ then

$$Ah = \frac{1}{2} \frac{d^2 h}{dx^2}.$$

Then we set for $\zeta \in D(A)$

$$V(\zeta) := \frac{1}{2} \langle A\zeta, \zeta \rangle + \langle F(\zeta), 1 \rangle$$

where $F : \mathbb{R} \to \mathbb{R}$, $F' = f$ and

$$\langle F(\zeta), 1 \rangle = \int_0^1 F(\zeta_x) \, \mathrm{d}x.$$

Then, if $\mathbb{R} \ni \varepsilon \to 0$ and $h \in D(A)$,

$$\frac{1}{\varepsilon}\left(\langle A(\zeta+\varepsilon h), \zeta+\varepsilon h\rangle - \langle A\zeta, \zeta\rangle\right) = 2\langle A\zeta, h\rangle + \varepsilon\langle Ah, h\rangle,$$

where we have used $\langle A\zeta, h\rangle = \langle \zeta, Ah\rangle$, and

$$\frac{1}{\varepsilon}\langle F(\zeta+\varepsilon h) - F(\zeta), 1\rangle = \int_0^1 \frac{1}{\varepsilon}(F(\zeta_x+\varepsilon h_x) - F(\zeta_x))\,\mathrm{d}x$$
$$\to \int_0^1 F'(\zeta_x)\, h_x\, \mathrm{d}x = \langle f(\zeta), h\rangle.$$

Therefore

$$\langle \nabla V(\zeta), h\rangle = \langle A\zeta + f(\zeta), h\rangle,$$
$$\nabla V(\zeta) = A\zeta + f(\zeta).$$

Gradient systems: finite-dimensional approximation

Let now $n \geq 1$. We note H_n the linear span of $\{e_1, \ldots, e_n\}$ and $B_t^{(n)} := \sum_{k=1}^n B_t^k \, e_k$. Let $\Pi_n : H \mapsto H_n$ be the orthogonal projection

$$\Pi_n h := \sum_{k=1}^{n} \langle h, e_k \rangle \, e_k, \qquad h \in H.$$

Then we set $A_n : H \mapsto H_n$ and $f^n : H \mapsto H_n$:

$$A_n := \Pi_n A \Pi_n = A \Pi_n = \Pi_n A, \qquad f^n := \Pi_n \circ f \circ \Pi_n.$$

Let ζ^n be the solution to the SDE in H_n

$$\zeta_t^n = \zeta_t^n(z) = z + \int_0^t \left(A \zeta_s^n + f^n(\zeta_s^n) \right) \, \mathrm{d}s + B_t^{(n)}, \qquad z \in H_n.$$

Theorem

Fix $u_0 \in H$. Then a.s. $(\zeta_t^n(\Pi_n u_0))_{t \geq 0}$ converges to u in $C([0,T]; H)$.

Gradient systems: finite-dimensional approximation

We define the probability measure on H_n

$$\nu_n(\mathrm{d}\zeta_n) := \frac{1}{Z_n} \exp\left(2V_n(\zeta_n)\right) \mathrm{d}\zeta_n, \qquad \zeta_n \in H_n,$$

where $\mathrm{d}\zeta_n$ is the n-dimensional Lebesgue measure on the Hilbert space H_n and Z_n is a normalisation constant.

Then it is well known that ν_n is invariant and reversible for $(\zeta_t^n)_{t\geq 0}$, namely if $\mathcal{Z}_n$ has law ν_n and is independent of W then $(\zeta_t^n(\mathcal{Z}_n))_{t\geq 0}$ is stationary and for all $T \geq 0$, the processes $(\zeta_t^n(\mathcal{Z}_n))_{t\in[0,T]}$ and $(\zeta_{T-t}^n(\mathcal{Z}_n))_{t\in[0,T]}$ have the same law.

Can we pass to the limit as $n \to +\infty$?

Invariant measures

Since $2V(\zeta) = \langle A\zeta, \zeta\rangle + 2\langle F(\zeta), 1\rangle$

$$\nu_n(\mathrm{d}\zeta_n) \; := \; \frac{1}{\hat{Z}_n} \exp\left(2\langle F(\zeta_n), 1\rangle\right) \mathcal{N}(0, Q_n)(\mathrm{d}\zeta_n)$$

where $Q_n = \mathrm{diag}\left(\frac{1}{2\lambda_1}, \ldots, \frac{1}{2\lambda_n}\right)$.

Recall now (22). Then $\mathcal{N}(0, Q_n) \Longrightarrow \mathbf{W}_{0,0}$, law of β. It is therefore simple to show that

$$\nu_n \Longrightarrow \nu := \frac{1}{\hat{Z}} \exp\left(2\langle F(\zeta), 1\rangle\right) \mathbf{W}_{0,0}(\mathrm{d}\zeta).$$

By Skorokhod's representation theorem we can suppose that $\mathcal{Z}_n$ converges a.s. to $\mathcal{Z}$ in H.

A technical estimate

Under the usual one sided bound

$$f(r) - f(r') \le L(r - r'), \qquad \forall\, r > r',$$

we have the following estimate:

$$\|\zeta_t^n(z) - \zeta_t^n(\hat{z})\| \le e^{Lt}\|z - \hat{z}\|.$$

This can be proved in the usual way, denoting $\zeta_t := \zeta_t^n(z)$ and $\hat{\zeta}_t := \zeta_t^n(\hat{z})$

$$\begin{aligned}
\frac{d}{dt}\|\zeta_t - \hat{\zeta}_t\|^2 &= 2\left\langle \zeta_t - \hat{\zeta}_t, A\left(\zeta_t - \hat{\zeta}_t\right) + f_\varepsilon(\zeta_t) - f_\varepsilon(\hat{\zeta}_t)\right\rangle \\
&\le -\left\|\partial_x\left(\zeta_t - \hat{\zeta}_t\right)\right\|^2 + 2\left\langle \zeta_t - \hat{\zeta}_t, f(\zeta_t) - f(\hat{\zeta}_t)\right\rangle \\
&\le 2L\|\zeta_t - \hat{\zeta}_t\|^2.
\end{aligned}$$

Therefore we have

- $(\zeta_t^n(\Pi_n u_0))_{t\geq 0}$ converges a.s. to $(u_t)_{t\geq 0}$ as $n \to +\infty$
- $z \mapsto \zeta_t^n(z)$ is Lipschitz, uniformly in n
- $\mathcal{Z}_n$ converges a.s.
- $(\zeta_t^n(\mathcal{Z}_n))_{t\geq 0}$ is stationary

Then it is easy to see that $(\zeta_t^n(\mathcal{Z}_n))_{t\geq 0}$ converges a.s. to a stationary solution of the SPDE with reflection.

The penalised invariant measure

We consider now the penalised SPDE

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial x^2} + f(u^\varepsilon) + \dfrac{(u^\varepsilon)^-}{\varepsilon} + W \\ u^\varepsilon(0,\cdot) = u_0, \quad u^\varepsilon(t,0) = u^\varepsilon(t,1) = a. \end{cases}$$

The invariant measure is

$$\nu_\varepsilon^a(\mathrm{d}\zeta) := \frac{1}{Z_\varepsilon^a} \exp\left(2\langle F_\varepsilon(\zeta), 1\rangle\right) \mathbf{W}_{a,a}(\mathrm{d}\zeta),$$

where F_ε satisfies

$$F_\varepsilon(0) = 0, \qquad F'_\varepsilon(y) := f(y) + \frac{y^-}{\varepsilon} = f_\varepsilon(y).$$

The penalised invariant measure

Note that

$$\frac{d}{dr}\left(r^{-}\right)^{2} = -2r^{-}.$$

Then for $a > 0$

$$\nu_{\varepsilon}^{a}(\mathrm{d}\zeta) \;=\; \frac{1}{Z_{\varepsilon}^{a}}\,\exp\left(2\langle F(\zeta),1\rangle - \frac{1}{\varepsilon}\langle\left(\zeta^{-}\right)^{2},1\rangle\right)\mathbf{W}_{a,a}(\mathrm{d}\zeta),$$

converges as $\varepsilon \downarrow 0$ to

$$\nu^{a}(\mathrm{d}\zeta) \;:=\; \frac{1}{Z^{a}}\,\exp\left(2\langle F(\zeta),1\rangle\right)\mathbb{1}_{K}(\zeta)\,\mathbf{W}_{a,a}(\mathrm{d}\zeta),$$

where $K := \{u_0 : [0,1] \to \mathbb{R} :\; u_0 \in L^2(0,1),\; u_0 \geq 0\}$.

$a \downarrow 0$

We have seen in (15) that

$$\mathbf{P}^3_{a,a} = \mathbf{W}_{a,a}(\,\cdot\,|\,T_0 > 1) = \mathbf{W}_{a,a}(\,\cdot\,|\,K).$$

Then

$$\nu^a(\mathrm{d}\zeta) \;:=\; \frac{1}{\hat{Z}^a}\,\exp\left(2\langle F(\zeta),1\rangle\right)\,\mathbf{P}^3_{a,a}(\mathrm{d}\zeta),$$

and as $a \downarrow 0$

$$\nu^a(\mathrm{d}\zeta) \Longrightarrow \nu^0(\mathrm{d}\zeta) \;:=\; \frac{1}{\hat{Z}^a}\,\exp\left(2\langle F(\zeta),1\rangle\right)\,\mathbf{P}^3_{0,0}(\mathrm{d}\zeta).$$

In particular if $f \equiv 0$ then the invariant measure of the SPDE with reflection is simply $\mathbf{P}^3_{a,a}$.

The situation

The situation

- We have constructed the solution (u, η) to the SPDE with reflection (24)
- We have computed the (unique) probability invariant measure

$$\nu^a(\mathrm{d}\zeta) := \frac{1}{\hat{Z}^a} \exp\left(2\langle F(\zeta), 1\rangle\right) \mathbf{P}^3_{a,a}(\mathrm{d}\zeta),$$

for $a \geq 0$.

The situation

- We have constructed the solution (u, η) to the SPDE with reflection (24)
- We have computed the (unique) probability invariant measure

$$\nu^a(d\zeta) := \frac{1}{\hat{Z}^a} \exp\left(2\langle F(\zeta), 1\rangle\right) \mathbf{P}^3_{a,a}(d\zeta),$$

for $a \geq 0$.

Comments:

- Walsh and Da Prato-Zabczyk reconciled

The situation

- We have constructed the solution (u, η) to the SPDE with reflection (24)
- We have computed the (unique) probability invariant measure

$$\nu^a(\mathrm{d}\zeta) \ := \ \frac{1}{\hat{Z}^a} \exp\left(2\langle F(\zeta), 1\rangle\right) \mathbf{P}^3_{a,a}(\mathrm{d}\zeta),$$

for $a \geq 0$.

Comments:

- Walsh and Da Prato-Zabczyk reconciled
- In the slides we only prove results on stationary u, in the lecture notes the general case is treated

The situation

- We have constructed the solution (u, η) to the SPDE with reflection (24)
- We have computed the (unique) probability invariant measure

$$\nu^a(\mathrm{d}\zeta) \;:=\; \frac{1}{\hat{Z}^a}\,\exp\left(2\langle F(\zeta), 1\rangle\right)\,\mathbf{P}^3_{a,a}(\mathrm{d}\zeta),$$

for $a \geq 0$.

Comments:

- Walsh and Da Prato-Zabczyk reconciled
- In the slides we only prove results on stationary u, in the lecture notes the general case is treated
- u is the weak limit of discrete interfaces: ask Cyril Labbé for a confirmation.

Let $(\rho_t, \ell_t)_{t\geq 0}$ be the solution to the SDE with reflection (4)

$$\rho_t = x + B_t + \int_0^t f(\rho_s)\,\mathrm{d}s + \ell_t.$$

Then we proved in (6) that

$$\int_{\mathbb{R}_+} e^{2F(x)}\,\mathbb{E}(\ell_t(x))\,\mathrm{d}x = \frac{t}{2}\,e^{2F(0)}, \qquad t \geq 0.$$

where $F' = f$.

This formula allowed us to prove that for $\delta > 1$ the equation

$$\rho_t = x + B_t + \frac{\delta - 1}{2}\int_0^t \rho_s^{-1}\,\mathrm{d}s + \ell_t$$

has reflection term $\ell \equiv 0$ and the solution is a δ-Bessel process.

The Revuz measure of η

For reasons which will be clear later, it would be very useful to have a similar formula for (u, η).

In this case, such the analogous result would be an explicit formula for

$$\int \nu^a(\mathrm{d}u_0)\, \mathbb{E}\left[\int_0^t \int_0^1 h(x)\, \varphi(u_s)\, \eta(\mathrm{d}s, \mathrm{d}x)\right]$$

where $h : [0, 1] \to \mathbb{R}$ and $\varphi : H \to \mathbb{R}$ are bounded Borel.

The Revuz measure of η

For reasons which will be clear later, it would be very useful to have a similar formula for (u, η).

In this case, such the analogous result would be an explicit formula for

$$\int \nu^a(\mathrm{d}u_0)\, \mathbb{E}\left[\int_0^t \int_0^1 h(x)\, \varphi(u_s)\, \eta(\mathrm{d}s, \mathrm{d}x)\right]$$

where $h : [0, 1] \to \mathbb{R}$ and $\varphi : H \to \mathbb{R}$ are bounded Borel.

The one-dimensional formula was simpler since there was no spatial variable x, no $h(x)$ and no φ, since

$$\int_0^t \varphi(\rho_s)\, \mathrm{d}\ell_s = \varphi(0)\, \ell_t,$$

$$\int_{\mathbb{R}_+} e^{2F(x)}\, \mathbb{E}\left[\int_0^t \varphi(\rho_s(x))\, \mathrm{d}\ell_s(x)\right] \mathrm{d}x = \frac{t}{2}\, \varphi(0)\, e^{2F(0)}.$$

Integration by parts

Penalisation:

$$\rho_t^n = x + B_t + n\int_0^t (\rho_s^n)^-\,\mathrm{d}s + \int_0^t f(\rho_s^n)\,\mathrm{d}s, \qquad t \geq 0.$$

One can prove that

$$\lim_{\varepsilon\downarrow 0}\int \mu_n(\mathrm{d}x)\,\mathbb{E}\left[\int_0^t \varphi(\rho_s^n)\,n\,(\rho_s^n)^-\,\mathrm{d}s\right],$$
$$= \int \mu(\mathrm{d}x)\,\mathbb{E}\left[\int_0^t \varphi(\rho_s)\,\mathrm{d}\ell_s\right].$$

By stationarity

$$\int \mu_n(\mathrm{d}x)\,\mathbb{E}\left[\int_0^t \varphi(\rho_s^n)\,n\,(\rho_s^n)^-\,\mathrm{d}s\right] = t\int \mu_n(\mathrm{d}x)\,\varphi(x)\,n\,x^-.$$

We can prove with a change of variable as in (9) that

$$\lim_{n\to+\infty}\int \mu_n(\mathrm{d}x)\,\varphi(x)\,n\,x^- = \frac{1}{2}\,\varphi(0)\,e^{2F(0)}.$$

Integration by parts

The difficulty comes from

$$\mu_n(\mathrm{d}x)\, n\, x^- = e^{-n\,(x^-)^2 + 2F(x)}\, n\, x^-\, \mathrm{d}x.$$

However we can write

$$e^{-n\,(x^-)^2}\, n\, x^- = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}x}\, e^{-n\,(x^-)^2}.$$

Integration by parts

The difficulty comes from

$$\mu_n(\mathrm{d}x)\, n\, x^- = e^{-n\,(x^-)^2+2F(x)}\, n\, x^-\, \mathrm{d}x.$$

However we can write

$$e^{-n\,(x^-)^2}\, n\, x^- = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}x}\, e^{-n\,(x^-)^2}.$$

Therefore, if $\varphi \in C^1_c(\mathbb{R})$, we can integrate by parts and obtain

$$\int \mu_n(\mathrm{d}x)\, \varphi(x)\, n\, x^- = \int e^{2F(x)}\, \varphi(x)\, \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}x}\, e^{-n\,(x^-)^2}\, \mathrm{d}x$$
$$= -\frac{1}{2}\int \frac{\mathrm{d}}{\mathrm{d}x}\left(\varphi\, e^{2F}\right) e^{-n\,(x^-)^2}\, \mathrm{d}x \to -\frac{1}{2}\int_{\mathbb{R}_+} \frac{\mathrm{d}}{\mathrm{d}x}\left(\varphi\, e^{2F}\right)\, \mathrm{d}x.$$

If we integrate by parts again, we find now

$$\lim_{n\to+\infty} \int \mu_n(\mathrm{d}x)\, \varphi(x)\, n\, x^- = \frac{1}{2}\, \varphi(0)\, e^{2F(0)}, \qquad \forall\, \varphi \in C^1_c(\mathbb{R}).$$

Integration by parts

Can we repeat the same trick for (u, η)?

This means integrating by parts w.r.t. to

$$\mathbf{P}^3_{a,a} = \mathbf{W}_{a,a}(\,\cdot\,|\,T_0 > 1) = \mathbf{W}_{a,a}(\,\cdot\,|\,K)$$

where $K = \{u_0 : [0,1] \to \mathbb{R} : \ u_0 \in L^2(0,1), \ u_0 \geq 0\}$.

Integration by parts

Consider a regular bounded open set $O \subset \mathbb{R}^d$. Then the classical Gauss-Green formula states that for all $h \in \mathbb{R}^d$

$$\int_O (\partial_h \varphi)\, \rho \, \mathrm{d}x \;=\; -\int_O \varphi \, \frac{\partial_h \rho}{\rho}\, \rho \, \mathrm{d}x \;-\; \int_{\partial O} \varphi \, \langle \hat{n}, h \rangle \, \rho \, \mathrm{d}\sigma$$

- $\varphi, \rho \in C^1_b(O)$ with $\lambda \le \rho \le \lambda^{-1}$, $\lambda \in]0,1]$ is a constant,
- $\hat{n}$ is the inward-pointing normal vector to the boundary ∂O
- σ is the surface measure on ∂O
- $\partial_h \varphi$ is the directional derivative of φ along h
- $\partial_h \log \rho = (\partial_h \rho)/\rho$.

Integration by parts

Consider a regular bounded open set $O \subset \mathbb{R}^d$. Then the classical Gauss-Green formula states that for all $h \in \mathbb{R}^d$

$$\int_O (\partial_h \varphi)\, \rho\, \mathrm{d}x \;=\; -\int_O \varphi\, \frac{\partial_h \rho}{\rho}\, \rho\, \mathrm{d}x \;-\; \int_{\partial O} \varphi\, \langle \hat{n}, h\rangle\, \rho\, \mathrm{d}\sigma$$

- $\varphi, \rho \in C^1_b(O)$ with $\lambda \le \rho \le \lambda^{-1}$, $\lambda \in]0,1]$ is a constant,
- $\hat{n}$ is the inward-pointing normal vector to the boundary ∂O
- σ is the surface measure on ∂O
- $\partial_h \varphi$ is the directional derivative of φ along h
- $\partial_h \log \rho = (\partial_h \rho)/\rho$.

For us, $\mathbf{W}_{a,a} = \rho\, \mathrm{d}x, \quad K = O.$

What is the analog of $\rho\, \mathrm{d}\sigma$? and of $\hat{n}$? and of ∂O?

An example

Let $\mu = \mathcal{N}(0, \frac{1}{2\lambda})$. Then

$$\int \varphi' \, \mathrm{d}\mu = \int 2\lambda \, x \, \varphi(x) \, \mu(\mathrm{d}x)$$

$$\int_{\mathbb{R}_+} \varphi' \, \mathrm{d}\mu = \int_{\mathbb{R}_+} \lambda \, x \, \varphi(x) \, \mu(\mathrm{d}x) - \sqrt{\frac{\lambda}{\pi}} \, \varphi(0).$$

Here $\mu(\mathrm{d}x) = \rho(x) \, \mathrm{d}x = \sqrt{\frac{\lambda}{\pi}} \, \exp(-\lambda \, x^2) \, \mathrm{d}x$,

$$\frac{\partial \rho}{\rho}(x) = -2\lambda \, x.$$

Integration by parts on the Gaussian measure

$$\int_H \partial_h \varphi \, d\mathbf{W}_{0,0} = - \int_H \langle \zeta, h'' \rangle \, \varphi(\zeta) \, \mathbf{W}_{0,0}(d\zeta)$$

- $\langle \cdot , \cdot \rangle$ is the scalar product in $H := L^2(0,1)$
- $\varphi : H \mapsto \mathbb{R}$ is bounded and smooth
- $h \in C^2_c(0,1)$, h'' is the second derivative of h
- $\partial_h \varphi$ is the directional derivative of φ along h:

$$\partial_h \varphi(\zeta) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\varphi(\zeta + \varepsilon h) - \varphi(\zeta)).$$

The boundary measure

$$\mathbf{P}^3_{a,a}\left[\partial_h\varphi\right] = -\mathbf{P}^3_{a,a}\left[\varphi(X)\,\langle X, h''\rangle\right] - \int_0^1 \mathrm{d}r\, h(r)\,\gamma(r,a)\,\mathbf{P}^3_{a,a}\left[\varphi(X)\,|\,X_r = 0\right].$$

where $\gamma(r,a) \geq 0$ is an explicit function of $r \in \,]0,1[$, $a \geq 0$.

$$\int_O (\partial_h\varphi)\,\rho\,\mathrm{d}x \;=\; -\int_O \varphi\,\frac{\partial_h\rho}{\rho}\,\rho\,\mathrm{d}x \;-\; \int_{\partial O} \varphi\,\langle \hat{n}, h\rangle\,\rho\,\mathrm{d}\sigma$$

By the Markov property and the Brownian scaling of $\mathbf{P}^3_{a,a}$

$$\mathbf{P}^\delta_{a,b}\left[\varphi(X)\,|\,X_r = c\right] = \int \varphi(T_r(k_1, k_2))\,\mathbf{P}^\delta_{a,c}(\mathrm{d}k_1)\,\mathbf{P}^\delta_{c,b}(\mathrm{d}k_2)$$

where for all $r \in]0,1[$ we have the scaling-plus-concatenation map $T_r : L^2(0,1) \times L^2(0,1) \mapsto L^2(0,1)$,

$$[T_r(k_1,k_2)](\tau) :=$$

$$:= \sqrt{r}\,k_1\left(\frac{\tau}{r}\right)\mathbb{1}_{(\tau\le r)} + \sqrt{1-r}\,k_2\left(\frac{\tau-r}{1-r}\right)\mathbb{1}_{(r<\tau\le 1)}.$$

The boundary measure

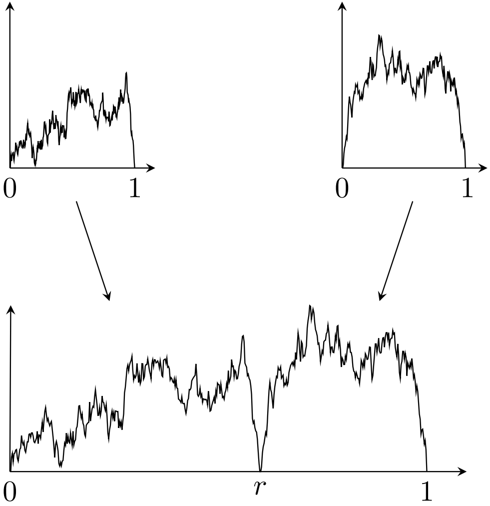


Figure : A typical path under the boundary measure and the action of T_r.

A random walk model

Let $(Y_i)_{i\in\mathbb{N}}$ be i.i.d. with $Y_i \sim \mathcal{N}(0,1)$. For $n \in \mathbb{N}$ and $y \in \mathbb{R}$ we set

$$S_n := Y_1 + \cdots + Y_n, \qquad g_t(y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right).$$

P_N is the law of $(S_1, \ldots, S_N)$ under the conditioning $\{S_{N+1} = 0\}$. For all Borel set $A \subseteq \mathbb{R}^N$ with $\phi_0 := \phi_{N+1} := 0$

$$\begin{aligned} P_N(A) &= \lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}\left[\mathbb{1}_A(S_1,\ldots,S_N)\,\mathbb{1}_{[0,\varepsilon]}(S_{N+1})\right]}{\mathbb{P}(S_{N+1}\in[0,\varepsilon])} \\ &= \frac{1}{C_N}\int_{\mathbb{R}^N} \mathbb{1}_A(\phi)\,\exp\{-H_N(\phi)\}\,\mathrm{d}\phi \end{aligned}$$

where

$$H_N(\phi) := \frac{1}{2}\sum_{i=1}^{N+1}(\phi_i - \phi_{i-1})^2.$$

A conditioned random walk

We set for $N \geq 1$, $\Omega_N^+ := \mathbb{R}_+^N = [0,\infty[^N$ and

$$dP_N^+ = \frac{1}{Z_N} \exp\{-H_N(\phi)\} \, \mathbb{1}_{(\phi \in \Omega_N^+)} \, d\phi$$

where Z_N is a normalisation constant. In particular

$$P_N^+ = P_N(\,\cdot \,|\, \Omega_N^+).$$

We define the Brownian scaling

$$\mathbb{R}^N \ni \phi \mapsto \Lambda_N(\phi)(x) := \frac{1}{\sqrt{N}} \phi_{\lceil Nx \rceil}, \quad x \in [0,1],$$

$$\mathbf{P}_N := \Lambda_N^*(P_N), \qquad \mathbf{P}_N^+ := \Lambda_N^*(P_N^+).$$

Proposition

As $N \to +\infty$, $\mathbf{P}_N$ converges weakly to $\mathbf{W}_{0,0}$ and $\mathbf{P}_N^+$ to $\mathbf{P}_{0,0}^3$.

Integration by parts in finite dimension

Lemma

For all $F \in C^1_b(\mathbb{R}^N)$ and $c \in \mathbb{R}^N$ with $c_0 := c_{N+1} := 0$

$$\int_{\mathbb{R}^N} \sum_{i=1}^N c_i \, \frac{\partial F}{\partial \phi_i} \, \mathrm{d}P_N = - \int_{\mathbb{R}^N} \sum_{i=1}^N c_i \, (\phi_{i+1} + \phi_{i-1} - 2\,\phi_i) \, F(\phi) \, P_N(\mathrm{d}\phi)$$

and passing to the limit $N \to +\infty$ under Brownian scaling

$$\int_H \partial_h \varphi \, \mathrm{d}\mathbf{W}_{0,0} \; = \; - \int_H \langle \zeta, h'' \rangle \, \varphi(\zeta) \, \mathbf{W}_{0,0}(\mathrm{d}\zeta)$$

for all $h \in C^2_c(0,1)$ and $\varphi \in C^1_b(H)$.

Integration by parts in finite dimension

Lemma

For all $F \in C_b^1(\mathbb{R}^N)$ *and* $c \in \mathbb{R}^N$ *with* $c_0 := c_{N+1} := 0$

$$\int_{\mathbb{R}_+^N} \sum_{i=1}^N c_i \frac{\partial F}{\partial \phi_i} \, \mathrm{d}P_N^+ =$$

$$= - \int_{\mathbb{R}_+^N} \sum_{i=1}^N c_i \left(\phi_{i+1} + \phi_{i-1} - 2\,\phi_i\right) F(\phi)\, P_N^+(\mathrm{d}\phi)$$

$$- \sum_{i=1}^N c_i \, \frac{Z_{i-1}\, Z_{N-i}}{Z_N} \int_{\mathbb{R}_+^{i-1} \times \mathbb{R}_+^{N-i}} F(\alpha \oplus \beta)\, P_{i-1}^+(\mathrm{d}\alpha)\, P_{N-i}^+(\mathrm{d}\beta)$$

where $\alpha \oplus \beta := (\alpha_1, \ldots, \alpha_{i-1}, 0, \beta_1, \ldots, \beta_{N-i})$.

Proof

The point is that the boundary of $\mathbb{R}_+^N$ is

$$\partial\left(\mathbb{R}_+^N\right) = \bigcup_{i=1}^{N} \mathbb{R}_+^{i-1} \times \{0\} \times \mathbb{R}_+^{N-i}.$$

For all $i \in \{1, \ldots, N\}$ we can apply a one-dimensional IbPF fixing ϕ_j for $j \neq i$

$$\int_{\mathbb{R}_+} \frac{\partial F}{\partial \phi_i} \, d\phi_i = -\int_{\mathbb{R}_+} (\phi_{i+1} + \phi_{i-1} - 2\,\phi_i)\, F(\phi)\, d\phi_i - \\ - \frac{1}{Z_N} F(\psi^i) \exp\left\{-H_N(\psi^i)\right\}$$

where $\psi^i = (\phi_1, \ldots, \phi_{i-1}, 0, \phi_{i+1}, \ldots, \phi_N)$.

Proof

Now

$$H_N(\phi_1,\dots,\phi_{i-1},0,\phi_{i+1},\dots,\phi_N) =$$
$$= H_{i-1}(\phi_1,\dots,\phi_{i-1}) + H_{N-i}(\phi_{i+1},\dots,\phi_N).$$

and therefore

$$\frac{1}{Z_N}\int_{\mathbb{R}_+^{\{1,\dots,N\}\setminus\{i\}}} F(\psi^i)\,\exp\left\{-H_N(\psi^i)\right\}\,\mathrm{d}\psi =$$
$$= \frac{1}{Z_N}\int_{\mathbb{R}_+^i\times\mathbb{R}_+^{N-i}} F(\alpha\oplus\beta)\,\exp\left\{-H_N(\alpha\oplus\beta)\right\}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$= \frac{Z_{i-1}\,Z_{N-i}}{Z_N}\int_{\Omega_{i-1}^+\times\Omega_{N-i}^+} F(\alpha\oplus\beta)\,P_{i-1}^+(\mathrm{d}\alpha)\,P_{N-i}^+(\mathrm{d}\beta).$$

This concludes the proof. □

$N \to +\infty$

$$\int_{\mathbb{R}_+^N} \sum_{i=1}^N c_i \, \frac{\partial F}{\partial \phi_i} \, \mathrm{d}P_N^+ =$$

$$= - \int_{\mathbb{R}_+^N} \sum_{i=1}^N c_i \, (\phi_{i+1} + \phi_{i-1} - 2\,\phi_i) \, F(\phi) \, P_N^+(\mathrm{d}\phi)$$

$$- \sum_{i=1}^N c_i \, \frac{Z_{i-1} \, Z_{N-i}}{Z_N} \int_{\mathbb{R}_+^{i-1} \times \mathbb{R}_+^{N-i}} F(\alpha \oplus \beta) \, P_{i-1}^+(\mathrm{d}\alpha) \, P_{N-i}^+(\mathrm{d}\beta)$$

Under Brownian rescaling as $N \to +\infty$:

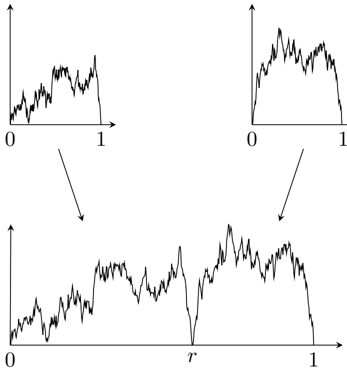
$$\mathbf{P}_{0,0}^3 \left[\partial_h \varphi\right] = -\mathbf{P}_{0,0}^3 \left[\varphi(X) \, \langle X, h'' \rangle\right]$$
$$- \int_0^1 \mathrm{d}r \, h(r) \, \gamma(r,0) \, \mathbf{P}_{0,0}^3 \otimes \mathbf{P}_{0,0}^3 \left[\varphi(T_r)\right].$$

The boundary measure

where for all $r \in]0,1[$, $T_r : L^2(0,1) \times L^2(0,1) \mapsto L^2(0,1)$,

$$[T_r(k_1,k_2)](\tau) :=$$

$$:= \sqrt{r}\, k_1\left(\frac{\tau}{r}\right) \mathbb{1}_{(\tau \le r)} + \sqrt{1-r}\, k_2\left(\frac{\tau - r}{1-r}\right) \mathbb{1}_{(r<\tau\le 1)}.$$



Theorem

For all bounded Borel $\varphi : H \mapsto \mathbb{R}$ *and* $h \in C_c(0,1)$

$$\int \nu^a(\mathrm{d}u_0)\, \mathbb{E}\left[\int_0^t \int_0^1 h(x)\, \varphi(u_s)\, \eta(\mathrm{d}s, \mathrm{d}x)\right] =$$

$$= \frac{t}{2Z^a} \int_0^1 \mathrm{d}r\, h(r)\, \gamma(r,a) \int \varphi(\zeta)\, e^{2F(\zeta)}\, \Sigma_a(r, \mathrm{d}\zeta).$$

where $\Sigma_a(r, \cdot) := \mathbf{P}^3_{a,a}[\,\cdot \mid X_r = 0]$.

Proof.

We set

$$\eta(\mathrm{d}s, h) := \int_0^1 h(x)\, \eta(\mathrm{d}s, \mathrm{d}x),$$

$$\eta^\varepsilon(\mathrm{d}s, h) := \int_0^1 h(x)\, \frac{1}{\varepsilon}(u^\varepsilon(s,x))^-\, \mathrm{d}x\, \mathrm{d}s = \frac{1}{\varepsilon}\langle h, (u_s^\varepsilon)^-\rangle\, \mathrm{d}s,$$

We need to prove two formulae:

$$\lim_{\varepsilon\downarrow 0} \int \nu_\varepsilon^a(\mathrm{d}u_0)\, \mathbb{E}\left[\int_0^t \varphi(u_s^\varepsilon)\, \eta^\varepsilon(\mathrm{d}s, h)\right] = $$

$$= \int \nu^a(\mathrm{d}u_0)\, \mathbb{E}\left[\int_0^t \varphi(u_s)\, \eta(\mathrm{d}s, h)\right],$$

$$\lim_{\varepsilon\downarrow 0} \int \nu_\varepsilon^a(\mathrm{d}u_0)\, \mathbb{E}\left[\int_0^t \varphi(u_s^\varepsilon)\, \eta^\varepsilon(\mathrm{d}s, h)\right] = $$

$$= \frac{1}{2Z^a} \int_0^1 \mathrm{d}r\, h(r)\, \gamma(r,a) \int \varphi(\zeta)\, e^{2F(\zeta)}\, \Sigma_a(r, \mathrm{d}\zeta).$$

Proof.

Let $\varphi \in C^1_b(H)$. By an IbPF

$$\int \nu^a_\varepsilon(\mathrm{d}\zeta)\,\varphi(\zeta)\,\langle h, \frac{1}{\varepsilon}\zeta^-\rangle =$$

$$= \frac{1}{Z^a_\varepsilon}\int \mathbf{W}_{a,a}(\mathrm{d}\zeta)\,\varphi(\zeta)\,\langle h, \frac{1}{\varepsilon}\zeta^-\rangle\ \exp\left(\langle 2F(\zeta) - \frac{1}{\varepsilon}(\zeta^-)^2, 1\rangle\right)$$

$$= \frac{1}{Z^a_\varepsilon}\int \mathbf{W}_{a,a}(\mathrm{d}\zeta)\,\varphi(\zeta)\ \exp\left(2\langle F(\zeta), 1\rangle\right)\left\langle h, \frac{1}{2}\nabla \exp\left(-\frac{1}{\varepsilon}\langle(\zeta^-)^2, 1\rangle\right)\right\rangle$$

$$= -\frac{1}{Z^a_\varepsilon}\int \mathbf{W}_{a,a}(\mathrm{d}\zeta)\left(\langle\nabla\varphi(\zeta), h\rangle + \varphi(\zeta)\left(\frac{1}{2}\langle h'', \zeta\rangle + \langle f(\zeta), h\rangle\right)\right)\cdot$$

$$\cdot \exp\left(\langle 2F(\zeta) - \frac{1}{\varepsilon}(\zeta^-)^2, 1\rangle\right)$$

$$= -\int \nu^a_\varepsilon(\mathrm{d}\zeta)\left(\langle\nabla\varphi(\zeta), h\rangle + \varphi(\zeta)\left(\frac{1}{2}\langle h'', \zeta\rangle + \langle f(\zeta), h\rangle\right)\right)$$

Proof.

Then, letting $\varepsilon \downarrow 0$

$$\int \nu_\varepsilon^a(\mathrm{d}\zeta)\,\varphi(\zeta)\,\langle h, \frac{1}{\varepsilon}\zeta^-\rangle \to$$

$$\to -\int \nu^a(\mathrm{d}\zeta)\left(\langle \nabla\varphi(\zeta), h\rangle + \varphi(\zeta)\left(\frac{1}{2}\langle h'', \zeta\rangle + \langle f(\zeta), h\rangle\right)\right)$$

$$= \frac{1}{2Z^a}\int_0^1 \mathrm{d}r\,h(r)\,\gamma(r,a)\int \varphi(\zeta)\,e^{2F(\zeta)}\,\Sigma_a(r,\mathrm{d}\zeta)$$

where we have integrated by parts again in the last equality. □

The contact set

We denote by $\pi : [0,+\infty[\,\times[0,1] \mapsto [0,+\infty[$ the projection $(t,x) \mapsto t$, and for a set $S \subset [0,+\infty[\,\times[0,1]$ we write

$$S_t := \{x \in [0,1] : (t,x) \in S\}, \qquad t \geq 0.$$

Theorem

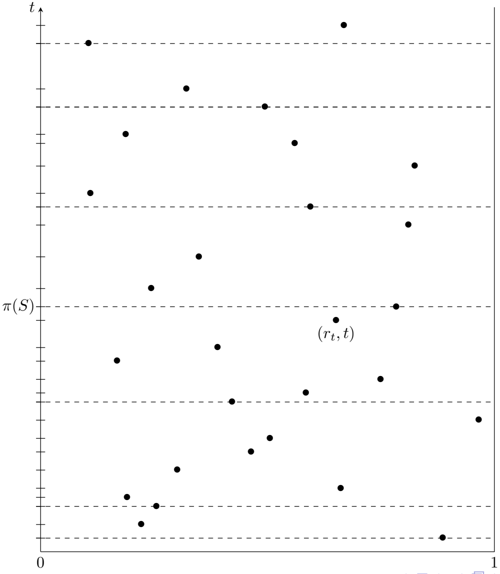
Let (u,η) be the stationary solution to equation (24). Let us denote by

$$\mathscr{C} := \{(t,x) : u(t,x) = 0,\, t > 0,\, x \in]0,1[\,\}$$

the contact set and let us recall that the support of η is contained in $\mathscr{C}$. Then a.s. the set $\pi(\mathscr{C})$ has zero Lebesgue measure and there exists a measurable set $S \subset \mathscr{C}$ such that

1. $\eta(\mathscr{C} \setminus S) = 0$
2. *for all $t > 0$, either $S_t = \emptyset$ or $S_t = \{r_t\}$, with $r_t \in]0,1[$.*
3. *if $S_t = \{r_t\}$, then $u(t,x) > 0$ for all $x \in]0,1[\,\setminus\{r_t\}$ and $u(t,r_t) = 0$.*

The contact set



Proof

Let $I_1, I_2 \subset]0,1[$ be closed intervals with $I_1 \cap I_2 = \emptyset$. Let $\varphi : C([0,1]) \to \mathbb{R}$ be defined by

$$\varphi(\zeta) := \mathbb{1}_{(\inf_{I_1} \zeta = \inf_{I_2} \zeta = 0)}, \qquad \zeta \in C([0,1]).$$

Then for all $h \geq 0$

$$\int \nu^a(\mathrm{d}u_0)\, \mathbb{E}\left[\int_0^t \int_0^1 h(x)\, \varphi(u_s)\, \eta(\mathrm{d}s, \mathrm{d}x)\right] =$$
$$= \frac{t}{2Z^a} \int_0^1 \mathrm{d}r\, h(r)\, \gamma(r,a) \int \varphi(\zeta)\, e^{2F(\zeta)}\, \Sigma_a(r, \mathrm{d}\zeta)$$
$$= 0.$$

If we consider $I_1 = [q_1, q_2]$, $I_2 = [q_3, q_4]$ with $q_i \in \mathbb{Q}$ then a.s. for $\eta(\mathrm{d}s \times [0,1])$-a.s. s, $u(s, \cdot)$ has only one zero over $]0,1[$. □

SPDEs with repulsion from 0

We study now the SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{c}{u^3} + W \\ \\ u(t,0) = u(t,1) = a, \quad t \geq 0 \\ \\ u(0,x) = u_0(x), \quad x \in [0,1] \end{cases}$$

where $a \geq 0$ and $c > 0$ are fixed, $u_0 \in C([0,1]) \cap K$ and we search for solutions $u \geq 0$.

Theorem

Let $a \geq 0$, $c > 0$ *and* $u_0 \in C([0,1]) \cap K$. *Then there exists a unique continuous* $u : [0,+\infty[\times [0,1] \mapsto [0,+\infty[$ *such that*

1. $u^{-3} \in L^1_{loc}([0,+\infty[\times]0,1[)$
2. *A.s. for all* $t \geq 0$ *and* $h \in C^\infty_c(0,1)$

$$\begin{aligned} \langle u_t, h\rangle = \langle u_0, h\rangle + \frac{1}{2}\int_0^t \langle u_s, h''\rangle \, \mathrm{d}s + \int_0^1 h(x)\, W(\mathrm{d}s, \mathrm{d}x) \\ + c\int_0^t \int_0^1 h(x)\, u^{-3}(s,x)\, \mathrm{d}s\, \mathrm{d}x. \end{aligned} \tag{30}$$

If $\delta > 3$ *is such that* $c = \frac{(\delta-3)(\delta-1)}{8}$, *then the only invariant probability measure of* (30) *is* $\mathbf{P}^\delta_{a,a}$.

Proof

The proof is based on the ideas we used in order to construct Bessel processes from SDEs with reflection. Let

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial x^2} + \dfrac{c}{(\varepsilon + u^\varepsilon)^3} + W + \eta^\varepsilon \\ u^\varepsilon(t,0) = u^\varepsilon(t,1) = a, \ u^\varepsilon(0,x) = u_0(x) \\ u^\varepsilon \geq 0, \, \mathrm{d}\eta^\varepsilon \geq 0, \ \int u^\varepsilon \, \mathrm{d}\eta^\varepsilon = 0 \end{cases}$$

where $\varepsilon > 0$.

By monotonicity arguments, we see that $\varepsilon \mapsto u^\varepsilon$ is monotone non-increasing, while $\varepsilon \mapsto \varepsilon + u^\varepsilon$ is monotone non-decreasing.

Proof

Then, as $\varepsilon \downarrow 0$, u^ε converges uniformly to u and for all non-negative $h \in C([0,1])$ and $t \geq 0$

$$\int_0^t \int_0^1 \frac{c}{(\varepsilon + u^\varepsilon(s,x))^3} \, h(x) \, \mathrm{d}x \, \mathrm{d}s \uparrow \int_0^t \int_0^1 \frac{c}{(u(s,x))^3} \, h(x) \, \mathrm{d}x \, \mathrm{d}s$$

as $\varepsilon \downarrow 0$. Moreover $\eta^\varepsilon \downarrow \eta$ and (u, η) solves

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{c}{u^3} + W + \eta \\ u(t,0) = u(t,1) = a, \; u(0,x) = u_0(x) \\ u \geq 0, \; \mathrm{d}\eta \geq 0, \; \int u \, \mathrm{d}\eta = 0, \; u^{-3} \in L^1_{loc} \end{cases}$$

We need to prove now that $\eta = 0$.

Proof

Note first that the invariant measure of u^ε is

$$\nu^{\varepsilon,a} := C(\varepsilon,a)\ \exp\left(-\frac{(\delta-1)(\delta-3)}{8}\int_0^1 \frac{1}{(\varepsilon+X_\tau)^2}\,\mathrm{d}\tau\right)\ \mathbf{P}^3_{a,a}.$$

which converges to $\mathbf{P}^\delta_{a,a}$ as $\varepsilon \downarrow 0$. By the IbPF for $\varphi \equiv 1$

$$\int \nu^{\varepsilon,a}(\mathrm{d}u_0)\,\mathbb{E}\left[\int_0^t \int_0^1 h(x)\,\eta^\varepsilon(\mathrm{d}s,\mathrm{d}x)\right] =$$

$$= t\,C(\varepsilon,a)\int_0^1 \mathrm{d}r\,h(r)\,\gamma(r,a)\int e^{2F_\varepsilon(\zeta)}\,\Sigma_a(r,\mathrm{d}\zeta).$$

Proof

Now for $r \in]0,1[$ and $c = \frac{(\delta-1)(\delta-3)}{8} > 0$

$$\int e^{2F_\varepsilon(\zeta)}\, \Sigma_a(r, \mathrm{d}\zeta) =$$

$$= \mathbf{E}^3_{a,a}\left[\exp\left(-c\int_0^1 \frac{1}{(\varepsilon + X_\tau)^2}\,\mathrm{d}\tau\right)\,\middle|\, X_r = 0\right]$$

$$\downarrow \mathbf{E}^3_{a,a}\left[\exp\left(-c\int_0^1 \frac{1}{X_\tau^2}\,\mathrm{d}\tau\right)\,\middle|\, X_r = 0\right] = 0$$

by the law of the iterated logarithm or an explicit computation. □

Hitting of zero

We have now functions $u = u^\delta$ for $\delta \geq 3$, stationary solutions to equations with reflection ($\delta = 3$) or repulsion from 0 ($\delta > 3$).

One of the main results of this course is the following

Theorem (Dalang, Mueller, Z.)

Let $\delta \geq 3$. If $k \in \mathbb{N}$ satisfies

$$k > \frac{4}{\delta - 2},$$

the probability that there exist $t > 0$ and $x_1, \ldots, x_k \in [0, 1]$ such that $0 < x_1 < \cdots < x_k < 1$ and $u(t, x_i) = 0$ for all $i = 1, \ldots, k$, is zero.

Hitting of zero

In particular, setting for $\delta \geq 3$

$$\zeta(\delta) := \sup\{k : \exists\, (t, x_1, \ldots, x_k) \in]0,1] \times]0,1[, u(t, x_i) = 0\}$$

then

- for $\delta = 3$, a.s. $\zeta(\delta) \leq 4$
- for $\delta \in]3, 3+1/3]$, a.s. $\zeta(\delta) \leq 3$
- for $\delta \in]3+1/3, 4]$, a.s. $\zeta(\delta) \leq 2$
- for $\delta \in]4, 6]$, a.s. $\zeta(\delta) \leq 1$
- for $\delta > 6$, a.s. $\zeta(\delta) = 0$.

In any case $\zeta(\delta) \leq 4$ a.s. for all $\delta \geq 3$. The behavior at the transition points $\delta \in \{3, 3+1/3, 4, 6\}$ might be non-optimal. Indeed, we conjecture that a.s.

$$\zeta(3) \leq 3, \qquad \zeta(3+1/3) \leq 2, \qquad \zeta(4) \leq 1, \qquad \zeta(6) = 0.$$

Hölder properties

Lemma

Let $\delta \geq 3$. For all $\beta \in (0, 1/2)$ and $T > 0$, there exists a finite random variable $\gamma \geq 0$ such that

$$|u(t,x) - u(t,y)| \;\leq\; \gamma\, |x-y|^{\beta}, \qquad x,y \in [0,1],\; T \geq t \geq 0,$$

and

$$u(t,x) - u(s,x) \;\geq\; -\gamma\,(t-s)^{\beta/2}, \qquad T \geq t \geq s \geq 0,\; x \in [0,1].$$

Proof of the Theorem

For all $\{q_i : i = 1, \ldots, 2k\} \subset \mathbb{Q}$ with $0 < q_1 < \cdots < q_{2k} < 1$, we define

$$Q := [0,1] \times \prod_{i=1}^{k} [q_{2i-1}, q_{2i}]$$

and the random set

$$A := \{(t, x_1, \ldots, x_k) \in Q : \ u(t, x_i) = 0, \ i = 1, \ldots, k\}.$$

Then the claim follows if $\mathbb{P}(A \neq \emptyset) = 0$ for all such $(q_i)_i$.

Proof

Since $k > \frac{4}{\delta-2}$, we can fix $\alpha \in]0,1[$ such that

$$4 + 2k - \alpha\delta k < 0.$$

For such α, we define the random set

$$A_n := \{(t, x_1, \ldots, x_k) \in Q : \; u(t, x_i) \leq 2^{-\alpha n}, \; i = 1, \ldots, k\}.$$

For all $n \in \mathbb{N}$, let

$$G_n := \{(j\, 2^{-4n}, i_1\, 2^{-2n}, \ldots, i_k\, 2^{-2n}) : \; j, i_1, \ldots, i_k \in \mathbb{Z}\},$$

and consider the events

$$\mathcal{K}_n := \{A_n \cap G_n \neq \emptyset\}, \qquad \mathcal{L}_n := \{A \neq \emptyset, \; A_n \cap G_n = \emptyset\}.$$

Since $A \subset A_n$ a.s.,

$$\{A \neq \emptyset\} \subseteq \mathcal{K}_n \cup \mathcal{L}_n.$$

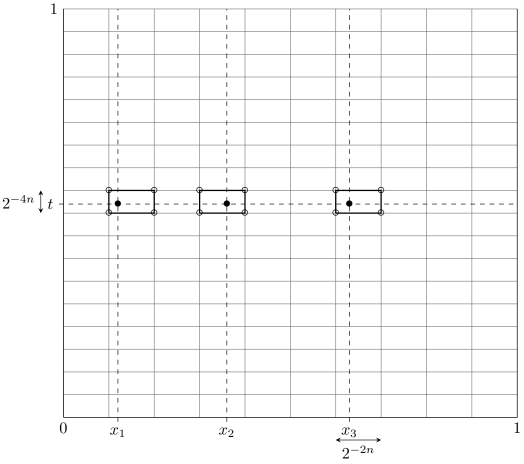


Figure : An element (t, x_1, x_2, x_3) of A with $k = 3$, namely $u(t, x_i) = 0$, $i = 1, 2, 3$. Either $u \leq 2^{-\alpha n}$ at k points $(s, y_1), \dots, (s, y_k) \in G_n$ (event $\mathcal{K}_n$) or u has a large oscillation in one of the three rectangles containing the (t, x_i)'s (event $\mathcal{L}_n$).

Proof

In order to prove that $\mathbb{P}(A \neq \emptyset) = 0$, we will show that the probabilities of $\mathcal{K}_n$ and $\mathcal{L}_n$ tend to 0 as $n \to \infty$.

Step 1. By definition, on the event $\mathcal{L}_n$, there exists $(t,x) \in [0,1] \times]0,1[$ such that $u(t,x) = 0$ but $A_n \cap G_n = \emptyset$. In particular, on $\mathcal{L}_n$, there exists $(s,y) \in G_n$ such that

$$u(s,y) > 2^{-\alpha n}, \quad 0 < t - s \leq 2^{-4n}, \quad |x-y| \leq 2^{-2n}.$$

Let $\beta \in]\alpha/2, 1/2[$. Then on $\mathcal{L}_n$

$$\begin{aligned} 2^{-\alpha n} < u(s,y) &= [u(s,y) - u(t,y)] + [u(t,y) - u(t,x)] \\ &\leq \gamma \left((t-s)^{\frac{\beta}{2}} + |y-x|^{\beta} \right) \leq \gamma\, 2^{-2\beta n + 1}. \end{aligned}$$

Therefore,

$$\mathbb{P}(\mathcal{L}_n) \leq \mathbb{P}(2^{-\alpha n} < \gamma\, 2^{-2\beta n+1}) = \mathbb{P}(\gamma > 2^{(2\beta - \alpha)n - 1}) \to 0$$

as $n \to \infty$, since $2\beta > \alpha$ and γ is a.s. finite.

Proof

Step 2. We set $I_n := G_n \cap Q$. Then, by definition,

$$\mathbb{P}(\mathcal{K}_n) = \mathbb{P}\left(\exists(t, x_1, \ldots, x_k) \in I_n : u(t, x_i) \le 2^{-\alpha n}, i = 1, \ldots, k\right).$$

Let $J_n := \{(x_1, \ldots, x_k) : (0, x_1, \ldots, x_k) \in I_n\}$. Then

$$\begin{aligned} \mathbb{P}(\mathcal{K}_n) &\le \sum_{j=1}^{2^{4n}} \sum_{(x_1,\ldots,x_k)\in J_n} \mathbb{P}\left(u(j\,2^{-4n}, x_i) \le 2^{-\alpha n},\ i = 1, \ldots, k\right) \\ &= 2^{4n} \sum_{(x_1,\ldots,x_k)\in J_n} \mathbf{P}^{\delta}_{a,a}\left(X_{x_i} \le 2^{-\alpha n},\ i = 1, \ldots, k\right), \end{aligned} \tag{31}$$

since we have chosen u to be stationary and therefore, for any $t \ge 0$, u_t has distribution $\mathbf{P}^{\delta}_{a,a}$.

Proof

For $\epsilon > 0$ and $x_0 := 0$, $y_0 := a$

$$\mathbf{P}^{\delta}_{a,a}\left(X_{x_i} \le \epsilon,\ i = 1,\dots,k\right) = \int_{[0,\epsilon)^k} \left[\prod_{i=1}^{k} p_{x_i - x_{i-1}}(y_{i-1}, y_i)\right] \frac{p_{1-x_k}(y_k, a)}{p_1(a,a)}\, \mathrm{d}y_1 \cdots \mathrm{d}y_k,$$

Since $p^{\delta}_t(x,y) \le K\, y^{\delta-1}$, for all $(x_i)_{i=1,\dots,k} \in J_n$

$$\mathbf{P}^{\delta}_{a,a}\left(X_{x_i} \le \epsilon,\ i = 1,\dots,k\right) \le C \left[\int_0^{\epsilon} y^{\delta-1}\, \mathrm{d}y\right]^k \le C\, \epsilon^{\delta k}, \qquad \epsilon > 0.$$

Therefore, by (31), since J_n has at most 2^{2kn} elements

$$\mathbb{P}(\mathcal{K}_n) \le C\, 2^{4n}\, 2^{2kn}\, (2^{-\alpha n})^{\delta k} = C\, 2^{(4+2k-\alpha\delta k)n} \longrightarrow 0$$

as $n \to \infty$. □

Hitting of zero

Theorem (Dalang, Mueller, Z.)

(a) *For all* $\delta \in [3,5]$, *with positive probability, there exist* $t > 0$ *and* $x \in (0,1)$ *such that* $u_t(x) = 0$.

(b) *For* $\delta = 3$, *with positive probability there exist* $t > 0$ *and* $\{x_1, x_2, x_3\} \subset (0,1)$, $x_1 < x_2 < x_3$, *such that* $u_t(x_i) = 0$, $i = 1, 2, 3$.

We conjecture that

1. $\zeta(\delta) < \dfrac{4}{\delta - 2}$ a.s. for all $\delta \geq 3$.
2. $\mathbb{P}(\zeta(\delta) = 1) > 0$ for $\delta \in [4,6)$, $\mathbb{P}(\zeta(\delta) = 2) > 0$ for $\delta \in [10/3, 4)$, and $\mathbb{P}(\zeta(\delta) = 3) > 0$ for $\delta \in [3, 10/3)$.

Integration by parts and Bessel processes

For $\delta = 1$ we have for all $\varphi \in C_c^1(\mathbb{R})$

$$\int_{\mathbb{R}_+} \varphi'(x)\, \mathrm{d}x = -\varphi(0).$$

The associated dynamics is

$$\begin{cases} \rho_t = x + B_t + \ell_t, & \rho_0 = x, \quad \ell_0 = 0, \\ \rho_t \geq 0, \quad \mathrm{d}\ell_t \geq 0, & \displaystyle\int_0^\infty \rho_t \, \mathrm{d}\ell_t = 0. \end{cases}$$

Integration by parts and Bessel processes

For $\delta > 1$ we have for all $\varphi \in C_c^1(\mathbb{R})$

$$\int_{\mathbb{R}_+} \varphi'(x)\, x^{\delta-1}\, \mathrm{d}x = -\int_{\mathbb{R}} \varphi(x)\, \frac{\delta-1}{x}\, x^{\delta-1}\, \mathrm{d}x.$$

The associated dynamics is

$$\rho_t = x + \frac{\delta-1}{2} \int_0^t \frac{1}{\rho_s}\, \mathrm{d}s + B_t, \qquad \rho_t \geq 0$$

limit as $\varepsilon \downarrow 0$ of

$$\rho_t^\varepsilon = x + \frac{\delta-1}{2} \int_0^t \frac{1}{\varepsilon + \rho_s^\varepsilon}\, \mathrm{d}s + B_t + \ell_t^\varepsilon.$$

There is a family of local times ℓ^a defined by the occupation time formula

$$\int_0^t \phi(\rho_s)\, \mathrm{d}s = \int_{\mathbb{R}_+} \phi(a)\, \ell_t^a\, a^{\delta-1}\, \mathrm{d}a.$$

Integration by parts and Bessel processes

For $\delta < 1$ we have for all $\varphi \in C_c^1(\mathbb{R})$

$$\int_{\mathbb{R}_+} \varphi'(x)\, x^{\delta-1}\, \mathrm{d}x = -\int_{\mathbb{R}} (\varphi(x) - \varphi(0))\, \frac{\delta-1}{x}\, x^{\delta-1}\, \mathrm{d}x.$$

The associated dynamics is the limit as $\varepsilon \downarrow 0$ of

$$\rho_t^\varepsilon = x + \frac{\delta-1}{2} \int_0^t \frac{1}{\varepsilon + \rho_s^\varepsilon}\, \mathrm{d}s + B_t + \ell_t^\varepsilon.$$

Integration by parts and Bessel processes

For $\delta < 1$ we have for all $\varphi \in C_c^1(\mathbb{R})$

$$\int_{\mathbb{R}_+} \varphi'(x)\, x^{\delta-1}\, \mathrm{d}x = -\int_{\mathbb{R}} (\varphi(x) - \varphi(0))\, \frac{\delta-1}{x}\, x^{\delta-1}\, \mathrm{d}x.$$

The associated dynamics is the limit as $\varepsilon \downarrow 0$ of

$$\rho_t^\varepsilon = x + \frac{\delta-1}{2} \int_0^t \frac{1}{\varepsilon + \rho_s^\varepsilon}\, \mathrm{d}s + B_t + \ell_t^\varepsilon.$$

The limit is equal to

$$\rho_t = \rho_0 + B_t + \frac{\delta-1}{2} \int_0^{+\infty} (\ell_t^a - \ell_t^0)\, a^{\delta-2}\, \mathrm{d}a$$

where the family of local times ℓ^a is defined by the occupation time formula

$$\int_0^t \phi(\rho_s)\, \mathrm{d}s = \int_{\mathbb{R}_+} \phi(a)\, \ell_t^a\, a^{\delta-1}\, \mathrm{d}a.$$

Integration by parts and SPDEs

For $\delta = 3$ we have

$$\mathbf{P}^3_{a,a}\left[\partial_h \varphi\right] = -\mathbf{P}^3_{a,a}\left[\varphi(X)\,\langle X, h''\rangle\right] - \int_0^1 \mathrm{d}r\, h(r)\,\gamma(r,a)\,\mathbf{P}^3_{a,a}\left[\varphi(X)\,|\,X_r = 0\right].$$

and the dynamics is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + W + \eta \\[2ex] u(0,x) = u_0(x),\; u(t,0) = u(t,1) = a \\[2ex] u \geq 0,\; \mathrm{d}\eta \geq 0,\; \int u\,\mathrm{d}\eta = 0 \end{cases}$$

Local times

Theorem

1. *For all $t \geq 0$:* $\eta([0,t], \mathrm{d}x) = \eta([0,t], x)\,\mathrm{d}x.$

Local times

Theorem

1. *For all* $t \geq 0$: $\eta([0,t],\mathrm{d}x) = \eta([0,t],x)\,\mathrm{d}x.$
2. *For all* $t \geq 0$:

$$\eta([0,t],x) = \frac{3}{4}\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon^3}\int_0^t \mathbb{1}_{[0,\epsilon]}(u(s,x))\,\mathrm{d}s.$$

Local times

Theorem

1. *For all* $t \geq 0$*:* $\qquad \eta([0,t], \mathrm{d}x) \,=\, \eta([0,t],x)\,\mathrm{d}x.$
2. *For all* $t \geq 0$*:*

$$\eta([0,t],x) \,=\, \frac{3}{4}\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon^3}\int_0^t \mathbb{1}_{[0,\epsilon]}(u(s,x))\,\mathrm{d}s.$$

3. *There exists a family of local times* $(\ell^a(\cdot,x))_{a\in[0,\infty),\,x\in(0,1)}$*, of* u *such that for all bounded Borel* $\varphi : \mathbb{R}_+ \to \mathbb{R}$*:*

$$\int_0^t \varphi(u(s,x))\,\mathrm{d}s \,=\, \int_0^\infty \varphi(a)\,\ell^a(t,x)\,a^2\,da, \quad t \geq 0.$$

Local times

Theorem

1. *For all* $t \geq 0$*:* $\eta([0,t],\mathrm{d}x) \;=\; \eta([0,t],x)\,\mathrm{d}x.$
2. *For all* $t \geq 0$*:*

$$\eta([0,t],x) \;=\; \frac{3}{4}\,\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon^3}\int_0^t \mathbb{1}_{[0,\epsilon]}(u(s,x))\,\mathrm{d}s.$$

3. *There exists a family of local times* $(\ell^a(\cdot,x))_{a\in[0,\infty),\,x\in(0,1)}$*, of* u *such that for all bounded Borel* $\varphi:\mathbb{R}_+\to\mathbb{R}$*:*

$$\int_0^t \varphi(u(s,x))\,\mathrm{d}s \;=\; \int_0^\infty \varphi(a)\,\ell^a(t,x)\,a^2\,da, \qquad t\geq 0.$$

4. *For all* $t \geq 0$*:* $\eta([0,t],x) \;=\; \dfrac{1}{4}\,\ell^0(t,x).$

Integration by parts and SPDEs

For $\delta > 3$ we have

$$\mathbf{E}^{\delta}_{a,a}\left[\partial_h \varphi(X)\right] = -\,\mathbf{E}^{\delta}_{a,a}\left[\varphi(X)\,\langle X, h''\rangle\right] - \frac{(\delta-1)(\delta-3)}{4}\,\mathbf{E}^{\delta}_{a,a}\left[\varphi(X)\,\langle X^{-3}, h\rangle\right].$$

and the dynamics is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{(\delta-1)(\delta-3)}{8} u^{-3} + W \\[2ex] u(0,x) = u_0(x),\ u(t,0) = u(t,1) = a \end{cases}$$

limit of

$$\frac{\partial u^{\varepsilon}}{\partial t} = \frac{1}{2}\frac{\partial^2 u^{\varepsilon}}{\partial x^2} + \frac{(\delta-1)(\delta-3)}{8}(u^{\varepsilon}+\varepsilon)^{-3} + W + \eta^{\varepsilon}.$$

Integration by parts and SPDEs

For $\delta \in [2,3[$ we set

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{1}{2}\frac{\partial^2 u^\varepsilon}{\partial x^2} + \frac{(\delta-1)(\delta-3)}{8}(u^\varepsilon + \varepsilon)^{-3} + W + \eta^\varepsilon.$$

and as $\varepsilon \downarrow 0$ we have again a renormalisation effect.

Open problems:

- IbPF ?
- dynamics ?

The non-linearity has the wrong sign.

$\delta \in [2,3[$

$$\mathbf{E}^{\delta}_{a,a}\left[\partial_h \varphi(X)\right] = -\mathbf{E}^{\delta}_{a,a}\left[\varphi(X)\,\langle X, h''\rangle\right]$$

$$- \kappa(\delta) \int_0^1 \mathrm{d}r\, h(r) \int_0^{+\infty} \frac{p^{\delta}_r(a,c) p^{\delta}_{1-r}(c,a)}{p^{\delta}_1(a,a)} \cdot$$

$$\cdot \frac{1}{c^3}\left[\mathbf{E}^{\delta}_{a,a}(\varphi(X)\,|\,X_r = c) - \mathbf{E}^{\delta}_{a,a}(\varphi(X)\,|\,X_r = 0)\right] c^{\delta-1}\,\mathrm{d}c$$

$\delta \in [2,3[$

$$
\begin{aligned}
&\mathbf{E}^{\delta}_{a,a}\left[\partial_h \varphi(X)\right] = -\mathbf{E}^{\delta}_{a,a}\left[\varphi(X)\,\langle X, h''\rangle\right] \\
&-\kappa(\delta)\int_0^1 \mathrm{d}r\, h(r)\int_0^{+\infty} \frac{p^{\delta}_r(a,c)p^{\delta}_{1-r}(c,a)}{p^{\delta}_1(a,a)}\,\cdot \\
&\cdot\,\frac{1}{c^3}\left[\mathbf{E}^{\delta}_{a,a}(\varphi(X)\,|\,X_r=c)-\mathbf{E}^{\delta}_{a,a}(\varphi(X)\,|\,X_r=0)\right] c^{\delta-1}\,\mathrm{d}c
\end{aligned}
$$

and the SPDE

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial t}\dfrac{\kappa(\delta)}{2}\displaystyle\int_0^{+\infty}\frac{\ell^c(t,x)-\ell^0(t,x)}{c^3}\,c^{\delta-1}\,\mathrm{d}c + W \\[2ex]
u(t,0)=u(t,1)=a,\ u(0,x)=u_0(x) \\[2ex]
\int_0^t \psi(u(s,x))\,\mathrm{d}s = \int_0^{+\infty}\psi(c)\,\ell^c(t,x)\,c^{\delta-1}\,\mathrm{d}c
\end{cases}
$$

For $\delta < 2$ the situation is even more complicated since 0 is hit by the stationary profile.

The case $\delta = 1$ is the most intriguing since it is the limit of homogeneous pinning models.

There is an IbPF for $\delta = 1$ but the form of the dynamics is hard even to conjecture.