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J. Math. Anal. Appl. 318 (2006) 507–516

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Quantization of probability distributions under norm-based distortion measures II: Self-similar distributions

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Received 24 November 2004

Available online 11 July 2005

Submitted by B. Bongiorno

Abstract

For a probability measure P on $\mathbb{R}^d$ and $n \in \mathbb{N}$ consider $e_n = \inf \int \min_{a \in \alpha} V(\|x - a\|)\, dP(x)$ where the infimum is taken over all subsets α of $\mathbb{R}^d$ with $\mathrm{card}(\alpha) \leqslant n$ and V is a nondecreasing function. Under certain conditions on V, we derive the precise n-asymptotics of e_n for self-similar distributions P and we find the asymptotic performance of optimal quantizers using weighted empirical measures.

Keywords: High-rate vector quantization; Norm-difference distortion; Self-similar measures; Empirical measure; Weak convergence; Local distortion; Point density measure

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.06.022

1. Introduction

Consider a random variable $X:(\Omega,\mathcal{A},\mathbb{P})\to\mathbb{R}^d$ with distribution $\mathbb{P}^X=P$ and let $V:\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing function with $V(0)=0$. For $n\in\mathbb{N}$ and any norm $\|\cdot\|$ on $\mathbb{R}^d$, the n-optimal V-quantization is the global minimization of

$$E\min_{a\in\alpha}V\big(\|X-a\|\big)$$

over all subsets $\alpha\subset\mathbb{R}^d$ with $\mathrm{card}(\alpha)\leqslant n$. Such a set α is called n-codebook or n-quantizer. So the resulting error by using $a\in\alpha$ instead of X is measured by the norm-difference distortion based on the loss function V. The minimal nth V-quantization error is then defined by

$$e_{n,V}(X)=e_{n,V}(P):=\inf\Big\{\mathbb{E}\min_{a\in\alpha}V\big(\|X-a\|\big)\colon\ \alpha\subset\mathbb{R}^d,\ 1\leqslant\mathrm{card}(\alpha)\leqslant n\Big\}. \qquad (1.1)$$

This quantity is finite provided $\mathbb{E}V(\|X\|)<\infty$.

For a given n-codebook α one defines an associated closest neighbour projection

$$\pi_\alpha:=\sum_{a\in\alpha}a\mathbf{1}_{C_a(\alpha)}$$

and the induced α-quantization of X by

$$\widehat{X}^\alpha:=\pi_\alpha(X), \qquad (1.2)$$

where $\{C_a(\alpha)\colon\ a\in\alpha\}$ is a Voronoi partition of $\mathbb{R}^d$ w.r.t. α consisting of Borel sets, that is

$$C_a(\alpha)\subset\Big\{x\in\mathbb{R}^d\colon\ \|x-a\|=\min_{b\in\alpha}\|x-b\|\Big\} \qquad (1.3)$$

for every $a\in\alpha$. Then one easily checks that for any random variable $Y:\Omega\to\alpha\subset\mathbb{R}^d$,

$$\mathbb{E}V\big(\|X-Y\|\big)\geqslant\mathbb{E}V\big(\big\|X-\widehat{X}^\alpha\big\|\big)=\mathbb{E}\min_{a\in\alpha}V\big(\|X-a\|\big)$$

so that

$$\begin{aligned}e_{n,V}(X)&=\inf\big\{\mathbb{E}V\big(\|X-\widehat{X}\|\big)\colon\ \widehat{X}=f(X),\ f:\mathbb{R}^d\to\mathbb{R}^d\text{ measurable},\\&\qquad\mathrm{card}\,f\big(\mathbb{R}^d\big)\leqslant n\big\}\\&=\inf\big\{\mathbb{E}V\big(\|X-Y\|\big)\colon\ Y:\Omega\to\mathbb{R}^d\text{ measurable},\ \mathrm{card}\big(Y(\Omega)\big)\leqslant n\big\}.\end{aligned} \qquad (1.4)$$

In electrical engineering this quantization problem arises in the context of coding signals effectively (see [4]). Most of the known results concern the case that the loss function V equals t^r for some $r>0$, so-called r-quantization (see, for instance, [6] for the mathematical aspects of the theory). Among others Gardner and Rao [3] and Li et al. [10] emphasize the need for more general loss functions (see also Lindner et al. [11]). For nonsingular probability distributions and loss functions V satisfying

$$\lim_{t\to0+}V(t)/t^r=1 \qquad (B_r)$$

the high resolution theory (i.e., the behaviour of $e_{n,V}$ for $n\to\infty$) has been studied in [2] (see also [8,9]). Examples different from r-quantization are exponential quantization with $V(t)=\exp(t^r)-1$ and log-quantization with $V(t)=(\log(1+t))^r$.

In the present paper we will investigate the high-resolution theory for self-similar probabilities P on $\mathbb{R}^d$ and loss functions V satisfying (B_r). Most of these probability measures are singular so that the results of [2] do not apply. We derive the precise n-asymptotics of the V-quantization errors $e_{n,V}(P)$. Then the performance of asymptotically V-optimal n-quantizers $(\alpha_n)_n$ and of local errors is investigated. We establish the weak convergence of the standard empirical measures

$$\frac{1}{n}\sum_{a\in\alpha_n}\delta_a,$$

of suitably normalized versions of the weighted empirical measures

$$\sum_{a\in\alpha_n}\int_{C_a(\alpha_n)} V\big(\|x-a\|\big)\,dP(x)\,\delta_a,$$

and of the finite measures

$$\int_{(\cdot)}\min_{a\in\alpha_n} V\big(\|x-a\|\big)\,dP(x).$$

In the nonarithmetic case and under distribution dependent rates one can achieve results as sharp as for nonsingular distributions.

Notations. $a_n \sim b_n$ means $a_n = b_n + o(b_n)$, $a_n \asymp b_n$ means $a_n = O(b_n)$ and $b_n = O(a_n)$, $\Rightarrow$ denotes weak convergence of finite measures on $\mathbb{R}^d$ and $d(x,A) := \inf_{y\in A}\|x-y\|$ for $A\subset\mathbb{R}^d$.

2. Main results

In what follows N is a natural number with $N \geqslant 2$ and $S_1,\dots,S_N:\mathbb{R}^d\to\mathbb{R}^d$ are contractive similitudes. Let s_i be the contraction number of S_i, i.e., $s_i\in(0,1)$ and $\|S_i(x)-S_i(y)\| = s_i\|x-y\|$ for all $x,y\in\mathbb{R}^d$. Sometimes the N-tuple $(S_1,\dots,S_N)$ is called an iterated function system. Its attractor A is the unique nonempty compact subset A of $\mathbb{R}^d$ with

$$A=\bigcup_{i=1}^N S_i(A). \tag{2.1}$$

For every probability vector $(p_1,\dots,p_N)$ there exists a unique Borel probability P on $\mathbb{R}^d$ which satisfies

$$P=\sum_{i=1}^N p_i P^{S_i}, \tag{2.2}$$

where P^{S_i} denotes the image measure of P under S_i. P is called the self-similar probability measure corresponding to $(S_1,\dots,S_N;p_1,\dots,p_N)$.

We will always assume that $p_i > 0$ for every i so that A equals the support $\mathrm{supp}(P)$ of P. $(S_1, \dots, S_N)$ is said to satisfy the open-set-condition (OSC) if there exists a non-empty open set $U \subset \mathbb{R}^d$ with $S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all i, j with $i \neq j$. From now on let $(S_1, \dots, S_N)$ satisfy the OSC. For $r \in (0, \infty)$ there is a unique number $D_r = D_r(P) \in (0, \infty)$ satisfying

$$\sum_{i=1}^{N} \left(p_i s_i^r\right)^{D_r/(D_r+r)} = 1 \tag{2.3}$$

(cf. [6, Lemma 14.4]). We will see (Theorem 1 below) that under condition (B_r) the number D_r equals the V-quantization dimension of P defined by

$$\lim_{n\to\infty} \frac{\log n}{-\frac{1}{r}\log e_{n,V}(P)},$$

which is bounded above by the space dimension d. In the sequel let P be the self-similar probability corresponding to $(S_1, \dots, S_N; p_1, \dots, p_N)$.

Observe first that the existence of V-optimal n-quantizers α for P, i.e.,

$$\mathbb{E} \min_{a\in\alpha} V\left(\|X - a\|\right) = e_{n,V}(P)$$

is ensured if V is continuous on the left (see [1,2]) and without any condition on V if the underlying norm on $\mathbb{R}^d$ is the l_2-norm. This follows again from [2] and the fact that P vanishes on l_2-spheres (see [5]). Notice further that condition (B_r) implies $V(0+) = V(0) = 0$ and $V(t) > 0$ for every $t > 0$ so that, in particular, under (B_r),

$$\lim_{n\to\infty} e_{n,V}(P) = 0 \tag{2.4}$$

(see [2]).

The precise asymptotic behaviour of the quantization errors $e_{n,V}$ and the solution of the empirical measure problem for V-quantization can be deduced from recent results on the r-quantization problem which corresponds to $V(t) = t^r$. Set

$$e_{n,r}(P) := \inf\left\{\mathbb{E} \min_{a\in\alpha} \|X - a\|^r \colon \alpha \subset \mathbb{R}^d,\ 1 \leqslant \mathrm{card}(\alpha) \leqslant n\right\}. \tag{2.5}$$

Define the point density measure P_r as the self-similar probability corresponding to $(S_1, \dots, S_N; q_1, \dots, q_N)$ where

$$q_i = \left(p_i s_i^r\right)^{D_r/(D_r+r)}. \tag{2.6}$$

A vector $(a_1, \dots, a_N) \in (\mathbb{R} \setminus \{0\})^N$ is called arithmetic if $(a_1, \dots, a_N) \in a\mathbb{Z}^N$ for some $a \in \mathbb{R}$. We need the following condition:

$$\left(\log\left(p_1 s_1^r\right), \dots, \log\left(p_N s_N^r\right)\right) \quad \text{is not arithmetic.} \tag{C_r}$$

A sequence $(\alpha_n)_{n\in\mathbb{N}}$ of n-quantizers is called an asymptotically V-optimal n-quantizer for P if

$$\mathbb{E} \min_{a\in\alpha_n} V\left(\|X - a\|\right) \sim e_{n,V}(P) \quad \text{as } n \to \infty. \tag{2.7}$$

Theorem 1. *Assume* (B_r).

(a) $e_{n,V}(P) \asymp n^{-r/D_r}$ *as* $n \to \infty$.
(b) *Assume* (C_r). *Then*

$$Q_r(P) := \lim_{n\to\infty} n^{r/D_r} e_{n,V}(P)$$

exists in $(0, \infty)$.
(c) *Assume* (C_r). *Let* $(\alpha_n)_n$ *be an asymptotically* V*-optimal* n*-quantizer for* P. *Then*

$$\frac{1}{n} \sum_{a\in\alpha_n} \delta_a \quad \Rightarrow \quad P_r \quad \text{as } n \to \infty.$$

Remark. For $V(t) = t^r$ Theorem 1(a) and (b) were first proved by Pötzelberger [12] in the case that $S_1, \ldots, S_N$ satisfy the strong separation condition, i.e., $S_i(A) \cap S_j(A) = \emptyset$ for $i \neq j$. Under the same assumptions Pötzelberger [12] also considered the arithmetic case.

We will use the following lemmas for the proof of Theorem 1 which hold for arbitrary distributions P with nonfinite compact support.

Lemma 1. *Assume* (B_r). *Let* $(s_n)_n$ *be a sequence in* $(0, \infty)$ *such that* $\lim_{n\to\infty} s_n/s_{n+k} = 1$ *for every* $k \in \mathbb{N}$. *Then*

$$\limsup_{n\to\infty} s_n e_{n,V}(P) \leqslant \limsup_{n\to\infty} s_n e_{n,r}(P)$$

and

$$\liminf_{n\to\infty} s_n e_{n,V}(P) \geqslant \liminf_{n\to\infty} s_n e_{n,r}(P).$$

Proof. Let $c \in (0, 1)$. Choose $t_0 \in (0, \infty)$ such that $V(t) \geqslant ct^r$ for every $t \in [0, t_0]$ and then choose a finite set $\beta \subset \mathbb{R}^d$ with $\max\{d(x, \beta): x \in A\} \leqslant t_0$. Let $m := \operatorname{card}(\beta)$. For $\alpha \subset \mathbb{R}^d$ with $\operatorname{card}(\alpha) \leqslant n$ one obtains

$$\begin{aligned}\int V\big(d(x,\alpha)\big)\,dP(x) &\geqslant \int_A V\big(d(x,\alpha\cup\beta)\big)\,dP(x) \geqslant c\int_A d(x,\alpha\cup\beta)^r\,dP(x)\\ &\geqslant c e_{n+m,r}(P)\end{aligned}$$

and hence

$$e_{n,V}(P) \geqslant c e_{n+m,r}(P).$$

Consequently,

$$\liminf_{n\to\infty} s_n e_{n,V}(P) \geqslant c \liminf_{n\to\infty} \frac{s_n}{s_{n+m}} s_{n+m} e_{n+m,r}(P) \geqslant c \liminf_{n\to\infty} s_n e_{n,r}(P).$$

Letting $c \to 1$ yields the lower estimate. As for the upper estimate, let $c > 1$ and choose $t_0 \in (0, \infty)$ such that $V(t) \leqslant ct^r$ for every $t \in [0, t_0]$. Choose β as above. One obtains for $\alpha \subset \mathbb{R}^d$ with $\operatorname{card}(\alpha) \leqslant n - m$, $n > m$,

$$e_{n,V}(P) \leqslant \int_A V\bigl(d(x, \alpha \cup \beta)\bigr)\, dP(x) \leqslant c \int_A d(x, \alpha \cup \beta)^r \, dP(x)$$
$$\leqslant c \int d(x, \alpha)^r \, dP(x)$$

and hence

$$e_{n,V}(P) \leqslant c e_{n-m,r}(P).$$

Consequently,

$$\limsup_{n\to\infty} s_n e_{n,V}(P) \leqslant c \limsup_{n\to\infty} \frac{s_n}{s_{n-m}} s_{n-m} e_{n-m,r}(P) \leqslant c \limsup_{n\to\infty} s_n e_{n,r}(P)$$

and letting $c \to 1$ gives the upper estimate. □

In particular, if $\lim_{n\to\infty} s_n e_{n,r}(P)$ exists (in $[0, \infty]$), then $\lim_{n\to\infty} s_n e_{n,V}(P)$ exists and

$$\lim_{n\to\infty} s_n e_{n,V}(P) = \lim_{n\to\infty} s_n e_{n,r}(P).$$

Lemma 2. *Assume* (B_r). *Assume that* $\lim_{n\to\infty} e_{n,r}(P)/e_{n+k,r}(P) = 1$ *for every* k. *Let* (α_n) *be an asymptotically* V*-optimal* n*-quantizer for* P. *Then* (α_n) *is an asymptotically* r*-optimal* n*-quantizer for* P.

Proof. An application of Lemma 1 with $s_n = 1/e_{n,r}(P)$ yields

$$e_{n,V}(P) \sim e_{n,r}(P) \quad \text{as } n \to \infty.$$

Since $\lim_{n\to\infty} e_{n,V}(P) = 0$, one obtains $\lim_{n\to\infty} \int V(d(x, \alpha_n))\, dP(x) = 0$. Therefore, by Proposition 2.2(v) in [2],

$$\lim_{n\to\infty} \max_{x\in A} d(x, \alpha_n) = 0.$$

Let $c \in (0, 1)$ and choose $t_0 \in (0, \infty)$ with $V(t) \geqslant ct^r$ for every $t \in [0, t_0]$. Then for all large enough n (such that $\max_{x\in A} d(x, \alpha_n) \leqslant t_0$),

$$e_{n,r}(P) \leqslant \int d(x, \alpha_n)^r \, dP(x) \leqslant \frac{1}{c} \int V\bigl(d(x, \alpha_n)\bigr)\, dP(x).$$

Letting $c \to 1$ yields

$$e_{n,r}(P) \sim \int d(x, \alpha_n)^r \, dP(x). \qquad \square$$

Proof of Theorem 1. It is known that

$$e_{n,r}(P) \asymp n^{-r/D_r}$$

and under (C_r),

$$\lim_{n\to\infty} n^{r/D_r} e_{n,r}(P) \quad \text{exists in } (0, \infty)$$

and

$$\frac{1}{n}\sum_{b\in\beta_n}\delta_b \quad\Rightarrow\quad P_r$$

for any asymptotically r-optimal n-quantizer $(\beta_n)_n$ (cf. Graf and Luschgy [7]). Therefore, the assertions (a) and (b) follow from Lemma 1 and (c) follows from Lemma 2. □

It is known that without the condition (C_r) parts (b) and (c) of the preceding theorem are not true. An example is the classical Cantor distribution (see [7]).

Notice that by Lemma 1, $Q_r(P)$ in fact depends on r and not on the exact form of V.

As for an application to numerical integration w.r.t. self-similar distributions, consider the Hölder class H^V of measurable functions $f:\mathbb{R}^d\to\mathbb{R}$ such that $|f(x)-f(y)|\leqslant V(\|x-y\|)$ for every $x,y\in\mathbb{R}^d$. For every $f\in H^V$ and every finite set $\alpha\subset\mathbb{R}^d$ one obtains

$$\begin{aligned}\left|\int f\,dP-\sum_{a\in\alpha}P\big(C_a(\alpha)\big)f(a)\right| &= \big|\mathbb{E}f(X)-\mathbb{E}f\big(\widehat{X}^\alpha\big)\big| \leqslant \mathbb{E}\big|f(X)-f\big(\widehat{X}^\alpha\big)\big|\\ &\leqslant \mathbb{E}V\big(\big\|X-\widehat{X}^\alpha\big\|\big)=\mathbb{E}\min_{a\in\alpha}V\big(\|X-a\|\big)\end{aligned}$$

so that under (B_r),

$$\inf\left\{\sup_{f\in H^V}\left|\int f\,dP-\sum_{a\in\alpha}P\big(C_a(\alpha)\big)f(a)\right|:\ \alpha\subset\mathbb{R}^d,\ \operatorname{card}(\alpha)\leqslant n\right\}=O\big(n^{-r/D_r}\big). \tag{2.8}$$

Another application concerns the a.e. reconstruction of X by α-quantizers of X. Assume (B_r). If $(\alpha_n)_n$ is rate-optimal, i.e., $\mathbb{E}\min_{a\in\alpha_n}V(\|X-a\|)=O(e_{n,V}(P))$, then

$$n^\vartheta\big\|X-\widehat{X}^{\alpha_n}\big\|\to 0\quad\text{a.e. as } n\to\infty \tag{2.9}$$

for every $\vartheta\in\{0\}\cup(0,(\frac{1}{D_r}-\frac{1}{r})^+)$. Moreover, the sequence $(n^{1/D_r}\|X-\widehat{X}^{\alpha_n}\|)_n$ is uniformly tight.

Next we will investigate the local quantization errors of asymptotically V-optimal n-quantizers $(\alpha_n)_n$ for the self-similar distribution P.

Theorem 2. *Assume (B_r) and (C_r). For $n\in\mathbb{N}$, define a finite measure on $\mathbb{R}^d$ by*

$$\frac{d\nu_n}{dP}(x):=n^{r/D_r}\min_{a\in\alpha_n}V\big(\|x-a\|\big),\quad x\in\mathbb{R}^d.$$

Then

$$\nu_n\quad\Rightarrow\quad Q_r(P)P_r\quad\text{as } n\to\infty$$

with $Q_r(P)$ from Theorem 1(b).

In terms of the α_n-quantization of X the preceding theorem reads

$$\lim_{n\to\infty}n^{r/D_r}\mathbb{E}g(X)V\big(\big\|X-\widehat{X}^{\alpha_n}\big\|\big)=Q_r(P)\int g\,dP_r$$

for every bounded continuous function $g:\mathbb{R}^d\to\mathbb{R}$.

Proof. Let $\{1,\dots,N\}^*$ denote the set of all finite words (sequences) on the alphabet $1,\dots,N$ including the empty word $\emptyset$. For $\sigma \in \{1,\dots,N\}^*$ set

$$S_\sigma := \begin{cases} \mathrm{id}_{\mathbb{R}^d}, & \sigma = \emptyset, \\ S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}, & \sigma = \sigma_1 \dots \sigma_n, \end{cases}$$

and

$$s_\sigma := \begin{cases} 1, & \sigma = \emptyset, \\ \prod_{i=1}^n s_{\sigma_i}, & \sigma = \sigma_1 \dots \sigma_n. \end{cases}$$

p_σ is defined analogously.

Let $O \subset \mathbb{R}^d$ be an arbitrary open set. By [7, Lemma 5.4] there exists a finite or infinite (possibly empty) sequence $(\sigma^{(k)})_k$ in $\{1,\dots,N\}^*$ such that $(S_{\sigma^{(k)}}(A))$ is a sequence of pairwise disjoint subsets of O with $P_r(O) = \sum_k P_r(S_{\sigma^{(k)}}(A))$. For $\sigma \in \{1,\dots,N\}^*$ set $\alpha_n(\sigma) := \{a \in \alpha_n\colon\ W_a(\alpha_n) \cap S_\sigma(A) \neq \emptyset\}$ and $n(\sigma) := \mathrm{card}(\alpha_n(\sigma))$ where $W_a(\alpha_n)$ is the closed Voronoi region $W_a(\alpha_n) = \{x \in \mathbb{R}^d\colon\ \|x-a\| = d(x,\alpha_n)\}$. It follows that

$$\begin{aligned} \nu_n(O) &= n^{r/D_r} \int_O V\big(d(x,\alpha_n)\big)\,dP(x) \\ &\geqslant n^{r/D_r} \sum_k \int_{S_{\sigma^{(k)}}(A)} V\big(d\big(x,\alpha_n\big(\sigma^{(k)}\big)\big)\big)\,dP(x). \end{aligned}$$

By the self-similarity of P we obtain

$$\begin{aligned} \int_{S_{\sigma^{(k)}}(A)} V\big(d\big(x,\alpha_n\big(\sigma^{(k)}\big)\big)\big)\,dP(x) &= p_{\sigma^{(k)}} \int V\big(d\big(S_{\sigma^{(k)}}(x),\alpha_n\big(\sigma^{(k)}\big)\big)\big)\,dP(x) \\ &= p_{\sigma^{(k)}} \int V\big(s_{\sigma^{(k)}}\, d\big(x, S_{\sigma^{(k)}}^{-1}\alpha_n\big(\sigma^{(k)}\big)\big)\big)\,dP(x). \end{aligned}$$

For $s > 0$ let $V_s : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$V_s(t) := s^{-r} V(st).$$

Then V_s is nondecreasing and we deduce

$$\int_{S_{\sigma^{(k)}}(A)} V\big(d\big(x,\alpha_n\big(\sigma^{(k)}\big)\big)\big)\,dP(x) \geqslant p_{\sigma^{(k)}} s_{\sigma^{(k)}}^r e_{n(\sigma^{k})),V_{s_{\sigma^{(k)}}}}(P).$$

Since V_s satisfies (B_r) and since

$$\lim_{n\to\infty} \frac{n(\sigma)}{n} = q_\sigma = \big(p_\sigma s_\sigma^r\big)^{D_r/(r+D_r)}$$

for every $\sigma \in \{1,\dots,N\}^*$ by [7, (24)] we obtain from Theorem 1(b) that

$$\lim_{n\to\infty} n\big(\sigma^{(k)}\big)^{r/D_r} e_{n(\sigma^{(k)}),V_{s_{\sigma^{(k)}}}}(P) = Q_r(P),$$

hence

$$\liminf_{n\to\infty} n^{r/Dr} \int\limits_{S_{\sigma(k)}(A)} V\big(d\big(x, \alpha_n\big(\sigma^{(k)}\big)\big)\big)\, dP(x)$$
$$\geqslant \liminf_{n\to\infty} p_{\sigma(k)} s^r_{\sigma(k)} \left(\frac{n}{n(\sigma^{(k)})}\right)^{r/D_r} n\big(\sigma^{(k)}\big)^{r/D_r} e_{n(\sigma^{(k)}),V_{s_{\sigma(k)}}}(P)$$
$$= p_{\sigma(k)} s^r_{\sigma(k)} \big(\big(p_{\sigma(k)} s^r_{\sigma(k)}\big)\big)^{-D_r/(r+D_r)r/D_r} Q_r(P)$$
$$= Q_r(P)\big(p_{\sigma(k)} s^r_{\sigma(k)}\big)^{D_r/(r+D_r)}.$$

We conclude that

$$\liminf_{n\to\infty} \nu_n(O) \geqslant Q_r(P) \sum_k \big(p_{\sigma(k)} s^r_{\sigma(k)}\big)^{D_r/(r+D_r)}.$$

Since $P_r(S_{\sigma(k)}(A)) = (p_{\sigma(k)} s^r_{\sigma(k)})^{D_r/(r+D_r)}$, this implies

$$\liminf_{n\to\infty} \nu_n(O) \geqslant Q_r(P) P_r(O).$$

Since $\nu_n(\mathbb{R}^d) = n^{r/D_r} \int V(d(x, \alpha_n))\, dP(x)$ and since (α_n) is asymptotically V-optimal we have $\lim_{n\to\infty} \nu_n(\mathbb{R}^d) = Q_r(P)$. This yields the conclusion of the theorem. □

Now the asymptotics for error localization at Voronoi regions can be deduced from Theorem 2 and [2, Lemma 4.7].

Theorem 3. *Assume (B_r) and (C_r). Let $(\alpha_n)_n$ be an asymptotically V-optimal n-quantizer for P. For $n \in \mathbb{N}$, let $\{C_a(\alpha_n)\colon a \in \alpha_n\}$ be a Voronoi partition of $\mathbb{R}^d$ w.r.t. α_n. Then*

$$n^{r/D_r} \sum_{a\in\alpha_n} \int\limits_{C_a(\alpha_n)} V\big(\|x-a\|\big)\, dP(x)\, \delta_a \quad \Rightarrow \quad Q_r(P) P_r \quad \textit{as } n\to\infty.$$

In terms of the α_n-quantization of X this reads

$$\lim_{n\to\infty} n^{r/D_r} \mathbb{E} g\big(\widehat{X}^{\alpha_n}\big) V\big(\big\|X - \widehat{X}^{\alpha_n}\big\|\big) = Q_r(P) \int g\, dP_r$$

for every bounded continuous function $g : \mathbb{R}^d \to \mathbb{R}$.

Combining the preceding theorem and Theorem 1(c) provides an indication that the uniformity feature

$$\int\limits_{C_{a_n}(\alpha_n)} V\big(\|x-a_n\|\big)\, dP(x) \sim \frac{e_{n,V}(P)}{n}$$

holds for self-similar probabilities P satisfying (C_r). However, as yet no rigorous proof is available.

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