The extremal process in nested conformal loops

Elie Aïdékon [1]

Université Paris VI

Summary. We consider a countable collection of nested simple loops which is conformally invariant. We study the extremal process composed of points with high conformal radius. It gives a Poisson point process with some random intensity measure. This work is inspired by the construction of the 2-d Gaussian Free Field in terms of CLE_4 by Miller and Sheffield [31], and applies techniques used for branching random walks.

1 The model

Let $\mathcal{C}$ be a countable random family of simple loops in the unit disk $\mathbb{U} = \{|z| < 1\}$ of the plane. We suppose that the loops are disjoint and non-nested. Moreover, we suppose that $\mathcal{C}$ is conformally invariant, which means that the law of $\mathcal{C}$ under conformal self-automorphisms of the unit disk remains unchanged. An example is given by the outermost loops of the Conformal Loop Ensemble, with parameter $\kappa \in (8/3, 4]$, introduced by Werner [38], and constructed via an exploration mechanism by Sheffield [34] (which allows to extend the construction to $\kappa \in (4, 8)$). They were characterized by Sheffield and Werner in [35].

By the Riemann mapping Theorem, we can look at the image of $\mathcal{C}$ in any simply connected domain different from the whole plane. The family $\mathcal{C}$ may be empty with positive probability. A simple argument (see Appendix, Section 6.1) shows that $\mathcal{C}$ has actually 0 or an infinite number of loops. Let $\mathscr{D}$ be a bounded simply connected domain. Draw a family of loops in $\mathscr{D}$ according to the law of $\mathcal{C}$. Inside each loop, draw another family of loops according to independent copies of $\mathcal{C}$ and iterate. We obtain a family of nested disjoint simple loops which is again conformally invariant, see Figure 1. We denote it by Γ, and we

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let $\mathbb{P}_{\mathscr{D}}$, $\mathbb{E}_{\mathscr{D}}$ be the probability and expectation associated to the law of Γ taken in the domain $\mathscr{D}$. We call $\Omega_{\mathscr{D}}$ the set of all discrete subsets of simple loops in $\mathscr{D}$. It is a metric space, and we equip it with the Borel σ-algebra (see [34], Section 4.1). Observe that in this framework, we may have extinction of the process, which means that there are no more loops added from some step on. We naturally suppose that $\mathbb{P}(\mathcal{C} = \emptyset) < 1$, hence the process has positive probability of non-extinction.
A loop γ is said of generation k, denoted by $|\gamma| = k$, if it has $k-1$ loops surrounding it. For a loop γ, we call $\mathbf{int}(\gamma)$ the interior of the loop, and $\mathscr{D}_k$ the union of the interiors of the loops at generation k. For any $z \in \mathscr{D}_k$, we write $\mathbf{A}_k^z$ for the loop of generation k that surrounds z. For convenience, we write $\mathbf{int}(\mathbf{A}_0^z) := \mathscr{D}$.

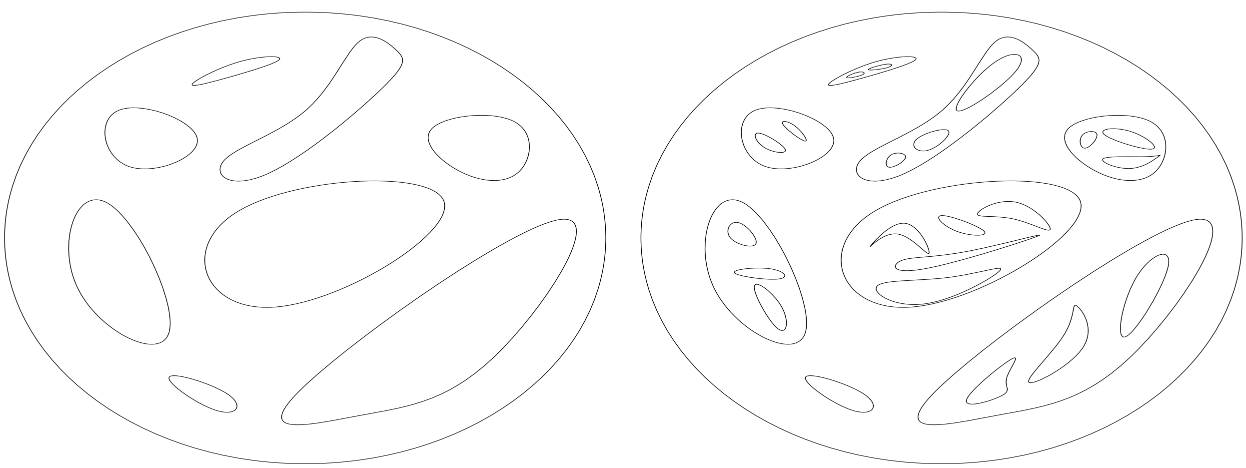

Figure 1: loops at generation 1 and 2 in the unit disk

The conformal radius of a simply connected domain $D \neq \mathbb{C}$ viewed from some point z, denoted by $\mathbf{R}(D, z)$, is defined by $\Psi_z'(0)$, where Ψ_z is the conformal bijection sending the unit disk $\mathbb{U}$ to D, such that $\Psi_z(0) = z$ and $\Psi_z'(0) > 0$. If $\mathrm{d}(D, z)$ is the Euclidean distance from z to the boundary of D, then it follows from Schwarz Lemma and Koebe quarter Theorem that $\mathbf{R}(D, z) \in [\mathrm{d}(D, z), 4\mathrm{d}(D, z)]$. Therefore, the conformal radius is comparable to the Euclidean distance. Moreover, it enjoys the following useful property: for any conformal bijection f, we have $\mathbf{R}(f(D), f(z)) = |f'(z)|\mathbf{R}(D, z)$.

For $z \in \mathscr{D}_k$, we write $\mathbf{R}_k^z$ for the conformal radius of $\mathbf{int}(\mathbf{A}_k^z)$ viewed from z, and we set $\mathbf{R}_0^z := \mathbf{R}(\mathscr{D}, z)$. By construction of Γ, conditionally on a loop γ that surrounds z and the loops that are outside of it, the family of nested loops in $\mathbf{int}(\gamma)$ has the law of Γ taken in the domain $\mathbf{int}(\gamma)$. In particular, the conformal invariance property of Γ shows that, for any $z \in \mathscr{D}_k$, $\left(\ln \frac{\mathbf{R}^z_{\ell-1}}{\mathbf{R}^z_\ell}\right)_{\ell\in[1,k]}$ are i.i.d. random variables. We denote by $\Phi(\beta)$ the Laplace transform of $\ln \frac{1}{\mathbf{R}^0_1}$ under $\mathbb{P}_{\mathbb{U}}$ (with the notation $\ln \frac{1}{\mathbf{R}^0_1} := +\infty$ if no loop surrounds 0). We suppose that there exists $\beta_c > 0$ such that

$$\ln \Phi(\beta_c) = (\beta_c + 2)\frac{\Phi'(\beta_c)}{\Phi(\beta_c)} =: v_c. \tag{1.1}$$

It includes all CLE_κ, $\kappa \in (8/3, 4]$. Still, we believe that this assumption is not necessary, but we assume it for technical reasons. In the first part of the paper, we define some natural measures on $\mathscr{D}$ related to Γ. For $\beta > 0$, we consider the following process, indexed by $k \geq 0$:

$$W^{(\beta)}_k := \frac{1}{\Phi(\beta)^k}\int_{\mathscr{D}_k} (\mathbf{R}^z_k)^\beta dz.$$

Let $\mathcal{F}_k$ be the σ-algebra generated by $(\gamma, |\gamma| \leq k)$, and $\mathcal{F}_\infty := \sigma(\gamma,\, \gamma \in \Gamma)$.

Lemma 1.1 *Let $\beta > 0$. The process $(W^{(\beta)}_k)_{k\geq 0}$ is a $(\mathcal{F}_k)_{k\geq 0}$-martingale.*

Proof. We observe that

$$W^{(\beta)}_{k+1} = \Phi(\beta)^{-k-1}\int_{\mathscr{D}_{k+1}} (\mathbf{R}^z_k)^\beta \mathrm{e}^{-\beta \ln\left(\frac{\mathbf{R}^z_k}{\mathbf{R}^z_{k+1}}\right)} dz.$$

Taking conditional expectation yields that

$$\begin{aligned}\mathbb{E}_{\mathscr{D}}\left[W^{(\beta)}_{k+1} \mid \mathcal{F}_k\right] &= \Phi(\beta)^{-k-1}\int_{\mathscr{D}_k} (\mathbf{R}^z_k)^\beta \mathbb{E}_{\mathscr{D}}\left[\mathrm{e}^{-\beta \ln\left(\frac{\mathbf{R}^z_k}{\mathbf{R}^z_{k+1}}\right)} \mid \mathcal{F}_k\right] dz \\ &= W^{(\beta)}_k.\end{aligned}$$

It completes the proof. □

Since $(W^{(\beta)}_k)_{k\geq 0}$ is a nonnegative martingale, it converges almost surely, and we let $W^{(\beta)}_\infty$ be this limit. More generally, we can consider for any Borelian set $A \subset \mathscr{D}$ the martingale

$$W^{(\beta)}_k(A) := \frac{1}{\Phi(\beta)^k}\int_{\mathscr{D}_k\cap A} (\mathbf{R}^z_k)^\beta dz. \tag{1.2}$$

Theorem 1.2 *Let $\mathscr{D}$ be a bounded simply connected domain, and A a deterministic Borelian subset. If $0 < \beta < \beta_c$, then $W^{(\beta)}_k(A)$ converges in $L^1(\mathbb{P}_{\mathscr{D}})$. In this case, we have $W^{(\beta)}_\infty > 0$ almost surely on the event of non-extinction. If $\beta \geq \beta_c$, $W^{(\beta)}_\infty = 0$ $\mathbb{P}_{\mathscr{D}}$-almost surely.*

We deduce (see Section 6.3) that when $0 < \beta < \beta_c$, $W_\infty^{(\beta)}$ gives rise to a Radon measure on $\mathscr{D}$ and we will denote it by $W_\infty^{(\beta)}$.

The case $\beta = \beta_c$ is a critical case. Even if the additive martingale vanishes, it is still possible to exhibit a martingale that converges to a nontrivial limit, almost surely. This martingale is the so-called derivative martingale defined by

$$(1.3) \qquad D_k := -\frac{1}{\Phi(\beta_c)^k} \int_{\mathscr{D}_k} ((\beta_c + 2)\ln(\mathbf{R}_k^z) - kv_c)(\mathbf{R}_k^z)^{\beta_c} dz.$$

We can check again that it defines a martingale. However, it is not clear anymore that it converges, since it is no longer nonnegative. For any Borelian set $A \subset \mathscr{D}$, we define

$$D_k(A) := -\frac{1}{\Phi(\beta_c)^k} \int_{\mathscr{D}_k \cap A} ((\beta_c + 2)\ln(\mathbf{R}_k^z) - kv_c)(\mathbf{R}_k^z)^{\beta_c} dz.$$

Theorem 1.3 *Let $\mathscr{D}$ be a bounded simply connected domain, and A a deterministic Borelian subset. The derivative martingale $(D_k(A))_{k\geq 0}$ converges $\mathbb{P}_{\mathscr{D}}$-almost surely towards a nonnegative limit. We call $D_\infty(A)$ this limit, and $D_\infty := D_\infty(\mathscr{D})$. Furthermore $D_\infty > 0$ almost surely on the event of non-extinction.*

Again, D_∞ defines a measure on $\mathscr{D}$, see Section 3.

These results should be compared to the analogs in the case of the 2-d Gaussian Free Field, for which Duplantier and Sheffield [20] introduced the Liouville quantum gravity measure. If h is a Gaussian Free Field on some bounded planar domain, the Liouville quantum gravity measure on D is informally given by $\mathrm{e}^{\gamma h(z)} dz$. Even if h is a distribution, the authors manage to give a precise meaning to such a measure. The idea is to look at a regularized version h_ε of h, by taking for $h_\varepsilon(z)$ the mean value on the circle of radius ε centered at z, and then letting $\varepsilon \to 0$. These measures, which are conjectured to arise in various scaling limits, are used by Duplantier and Sheffield to prove a version of the famous KPZ equation. We refer to the concerned article [20] or to the survey of Garban [22] for more information. The correspondence with our setting is made through a result of Miller and Sheffield in [31] who prove that the Conformal Loop Ensemble with $\kappa = 4$, called CLE_4, describes the level lines of the Gaussian Free Field. To make the analogy precise, take a CLE_4, and put on each loop a Bernoulli random variables equal to π or $-\pi$, each with probability 1/2. Inside each loop with mark different from π, draw an independent CLE_4, and increase the mark by independent Bernoulli marks on the new loops. Iterate on loops with marks different from π. This procedure will give a random family of loops (which is no more CLE_4) in which all loops are marked with π. The analog of the Liouville quantum gravity measure is then the martingale $W_k^{(\beta)}$ in our setting, with $\beta = \gamma^2/2$. From [36], we can explicitly compute β_c satisfying (1.1), and we find that $\beta_c = 2$. Therefore, we recover a similar phase transition to what is known for the Gaussian Free Field, see [20] for the case $\beta < 2$ and [19] for the case $\beta = 2$. We stress that we do not prove that the measures $W_\infty^{(\beta)}$ are the Liouville measures constructed in [20], but we believe that it is the case.

Because of the tree structure of Γ, it is possible to apply techniques used in branching random walks to prove the convergence of the martingales. The strategy of this first part is therefore a clear path which is indebted to a longstanding tradition of papers in the context of multiplicative cascades and branching random walks (see Kahane and Peyriere [26], Biggins [8], Biggins and Kyprianou [9],[10],[11], Harris [23], Harris, Harris and Kyprianou [24]). The derivative martingale was first studied by Lalley and Sellke [28]. The simple proofs on the convergence of martingales using spine techniques comes from Lyons [29], and independently from Waymire and Williams [37]. When dealing with Gaussian Free Field, such a connection between branching random walks and Gaussian Free Fields has already been well studied in the literature ([14]), and was essential in the proof of the convergence in law of the recentered maximum of the discrete Gaussian Free Field by Bramson, Ding and Zeitouni [16]. In our case, there are still some technicalities to bridge the gap between the two models, but they are quite easily overcome with the help of properties of conformal maps.

The second part of the paper investigates the extremal process associated to the variables $(U_k^z := (\beta_c + 2)\ln(\mathbf{R}_k^z) - kv_c)_{z\in\mathscr{D}_k}$. The Hausdorff dimension of large loops has already been computed by Miller, Watson and Wilson [32] in the setting of CLE_κ. The analog for the Gaussian free field is the Hausdorff dimension of "thick points" and is given in a work of Hu, Miller and Peres [25]. The complete picture of the extremal process associated to the Gaussian free field is still open, but recent results towards it were obtained by Biskup and Louidor [13]. In our setting, we show that the extremal process is a decorated Poisson point process. At the coarse level, the extremal process is just a Poisson point process with some random intensity. In the fine field, the "landscape" seen from some extremal point gives rise to some decoration. Because looking at the landscape viewed from a true local extremum would induce some non-trivial conditioning on loops, we prefer to "root" our landscape at some randomly chosen point. Let us precise the setting.

Let $\delta > 0$ be a parameter that will be fixed once and for all. For any integers $n \geq k \geq 1$ and any loop γ at generation k, denote by $\mathfrak{g}_{n,\gamma}$ a point inside γ chosen with density proportional to

$$\nu_n^z := (\mathbf{R}_n^z)^{\beta_c} \Phi(\beta_c)^{-n} \mathrm{e}^{\delta U_n^z}. \tag{1.4}$$

Let $\mathfrak{G}_{n,\gamma} := U_n^{\mathfrak{g}_{n,\gamma}}$. We should at this point define what we mean by landscape. For any $\ell \geq 0$ and $z \in \mathscr{D}_\ell$, let Ψ_ℓ^z be the conformal map sending the unit disk $\mathbb{U}$ into $\mathbf{int}(\mathbf{A}_\ell^z)$, the interior of the ℓ-th loop in Γ that surrounds z, such that $\Psi_\ell^z(0) = z$ and $(\Psi_\ell^z)'(0) > 0$. For any ℓ, we can look at the image by $(\Psi_\ell^z)^{-1}$ of the loops of generation less than n which lie inside $\mathbf{int}(\mathbf{A}_\ell^z)$. Denote by $Q_{\ell,n}^z$ this set of loops, then write $\mathcal{Q}_n^z := (Q_{n-\ell,n}^z,\ \ell \in [1,n])$. In some sense, $\mathcal{Q}_n^z$ reconstruct the loops around z backwards in time. Finally, set

$$\mathcal{P}_{k,n} := \sum_{|\gamma|=k} \delta_{(\mathfrak{g}_{n,\gamma}, \mathfrak{G}_{n,\gamma} + \frac{3}{2}\ln n, \mathcal{Q}_n^{\mathfrak{g}_{n,\gamma}})} \tag{1.5}$$

where the sum runs over loops $\gamma \in \Gamma$ at generation k. The points in $\mathcal{P}_{k,n}$ are elements of $\mathscr{D} \times \mathbf{R} \times \Omega_{\mathbb{U}}^{\mathbb{N}}$, which is a metric space when $\Omega_{\mathbb{U}}^{\mathbb{N}}$ is endowed with the metric associated to the convergence of any finite subsequence. We work under the framework of Kallenberg [27]. We say that random Radon measures μ_n converges in distribution to μ if $\int f d\mu_n$ converges in distribution to $\int f d\mu$ as $n \to \infty$, for any continuous function f with compact support.

Theorem 1.4 *The point process $\mathcal{P}_{k,n}$ converges in distribution as $n \to \infty$ then $k \to \infty$ to a Poisson point process on $\mathscr{D} \times \mathbf{R} \times \Omega_{\mathbb{U}}^{\mathbb{N}}$ with intensity measure $C_2 \mathrm{e}^{-x} D_\infty(dz) dx \mathcal{M}(d\mathcal{Q})$ where $\mathcal{M}$ is some finite measure on the space $\Omega_{\mathbb{U}}^{\mathbb{N}}$.*

The constant C_2 is defined in (5.1) and the measure $\mathcal{M}$ is defined in (5.6). The fact that $\mathcal{P}_{k,n}$ encodes indeed the extremal process can be seen through Corollaries 4.11 and 4.13. The theorem is the analog of results in the branching random walk and branching Brownian motion cases, see the seminal work of Bramson [15], then subsequent works [1], [30], [2] , [5], [6], [7]. The techniques used here are inspired by these papers. It is worth mentioning that we were able to derive an exact (even if involved) representation of the decoration, giving the analog of the results in [2] using similar ideas. Translated into the branching random walk setting, this would also give a representation of the decoration, which could be considered as a new result (see Madaule [30] for the convergence in law of the extremal process in the branching random walk case). Finally, a paper of Biggins, Hambly and Jones [12] considers a very similar object than the one considered here. Using correspondence with branching random walks, the authors are able to give the multifractal spectrum of the sizes of the sets.

The paper is organized as follows. We prove Theorem 1.2 and Theorem 1.3 respectively in Section 2 and Section 3. In Section 4, we give some preliminary results on the extremal process, that allows us to prove Theorem 1.4 in Section 5.

We denote by $c, c', c_0, c_1, \ldots$ constants which can change value from line to line.

2 The additive martingale

2.1 A change of measure

Fix $\beta > 0$. Since $W_k^{(\beta)}$ is a nonnegative martingale, the Kolmogorov extension Theorem implies that there exists a probability measure $\hat{\mathbb{P}}_{\mathscr{D}}$ on $\mathcal{F}_\infty$ such that, for any $k \geq 0$,

$$\frac{d\hat{\mathbb{P}}_{\mathscr{D}}}{d\mathbb{P}_{\mathscr{D}}}\Big|_{\mathcal{F}_k} := \frac{W_k^{(\beta)}}{W_0^{(\beta)}}.$$

This is the *Peyriere probability* introduced in [33] when dealing with multiplicative cascades. This change of measure was given in [17] for the branching Brownian motion, and in [29] for branching random walks. The aim of this section is to understand the law of (Γ, Ξ) under $\hat{\mathbb{P}}_{\mathscr{D}}$. Recall that $\mathcal{C}$ denotes our random non-nested loops in the unit disk $\mathbb{U}$. We call ψ the conformal bijection from $\mathbb{U}$ to the interior of the loop in $\mathcal{C}$ that surrounds 0, such that $\psi(0) = 0$ and $\psi'(0) > 0$. Let $(\mathcal{C}_i, \psi_i)_{i \geq 1}$ be i.i.d. random variables distributed as $(\mathcal{C}, \psi)$ under $\mathbb{P}_{\mathbb{U}}$. The loop in $\mathcal{C}_i$ that surrounds 0 is denoted by $\mathcal{C}_i(0)$ (when it exists).

Let $z \in \mathscr{D}$. We want to decompose Γ along the loops that surround z. For any $z \in \mathscr{D}$, call Ψ^z the conformal bijection from $\mathbb{U}$ to $\mathscr{D}$ such that $\Psi^z(0) = z$ and $(\Psi^z)'(0) > 0$. Let $\mathbf{A}_1^z := \Psi^z(\mathcal{C}_1(0))$, and for $k \geq 2$,

$$\mathbf{A}_k^z := \Psi^z \circ \psi_1 \circ \ldots \circ \psi_{k-1}(\mathcal{C}_k(0)).$$

We stop the procedure as soon as $\mathcal{C}_k(0)$ is not defined. In each layer $\mathbf{int}(\mathbf{A}_k^z) \backslash \mathbf{int}(\mathbf{A}_{k+1}^z)$, draw the "brother loops" $\Psi^z \circ \psi_1 \circ \ldots \circ \psi_{k-1}(\gamma)$ for any $\gamma \in \mathcal{C}_k$ different from $\mathcal{C}_k(0)$ (see Figure 2). Finally fill each of these "brother loops" with independent copies of Γ. Call $\Gamma(z)$ the result. By the conformal invariance property, we can check that $\Gamma(z)$ is indeed distributed as Γ under $\mathbb{P}$. Notice that in this setting, for any $z \in \mathscr{D}_k$,

$$\mathbf{R}_k^z = \mathbf{R}(\mathscr{D}, z) \prod_{i=1}^{k} \psi_i'(0).$$

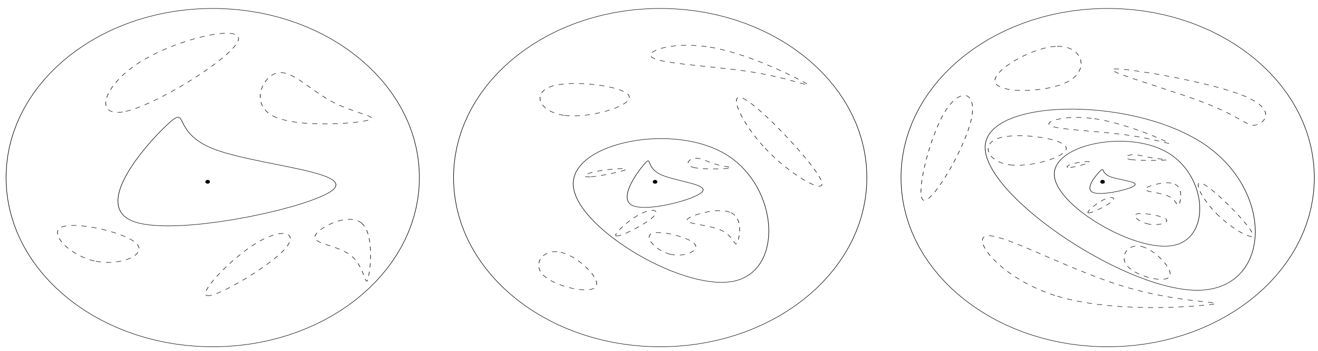

Figure 2: Backwards construction of the loops surrounding 0 (plain lines) and the brother loops (dashed lines) in the unit disk, here until generation 3. We conformally map $\mathcal{C}_3$ into $\mathcal{C}_2(0)$, then $\mathcal{C}_2$ into $\mathcal{C}_1(0)$.

Similarly, let $(\hat{\mathcal{C}}_i, \hat{\psi}_i)_{i\geq 1}$ be i.i.d. random variables distributed as $(\hat{\mathcal{C}}, \hat{\psi})$, whose law has Radon-Nikodym derivative $\Phi(\beta)^{-1}\psi_1'(0)^\beta$ with respect to the law of $(\mathcal{C}, \psi)$. The loop in $\hat{\mathcal{C}}_i$ that surrounds 0 is denoted by $\hat{\mathcal{C}}_i(0)$. Notice that it necessarily exists. Define $\hat{\mathbf{A}}_1^z := \Psi^z(\hat{\mathcal{C}}_1(0))$, and for $k \geq 2$,

$$\hat{\mathbf{A}}_k^z := \Psi^z \circ \hat{\psi}_1 \circ \ldots \circ \hat{\psi}_{k-1}(\hat{\mathcal{C}}_k(0)).$$

Draw the "brother loops" $\Psi^z \circ \hat{\psi}_1 \circ \ldots \circ \hat{\psi}_{k-1}(\gamma)$ for any $\gamma \in \hat{\mathcal{C}}_k$ different from $\hat{\mathcal{C}}_k(0)$ and fill them with independent copies of Γ. Call $\hat{\Gamma}(z)$ the result. If $\hat{\mathbf{R}}_k^z$ denotes the conformal radius of $\hat{\mathbf{A}}_k^z$ seen from z, then

$$\hat{\mathbf{R}}_k^z = \mathbf{R}(\mathscr{D}, z) \prod_{i=1}^{k} \hat{\psi}_i'(0). \tag{2.1}$$

Write $\hat{\Gamma}$ for $\hat{\Gamma}\left(\hat{\xi}\right)$ where $\hat{\xi}$ is a random point in $\mathscr{D}$ with density $\frac{1}{W_0^{(\beta)}}\mathbf{R}(\mathscr{D}, z)^\beta dz$. We write as well $\hat{\mathbf{R}}_k$, $\hat{\mathbf{A}}_k$ for $\hat{\mathbf{R}}_k^{\hat{\xi}}$, $\hat{\mathbf{A}}_k^{\hat{\xi}}$. Finally, we set $\hat{\psi}_0 := \Psi^{\hat{\xi}}$.

Proposition 2.1 *The family of nested loops Γ under $\hat{\mathbb{P}}_{\mathscr{D}}$ has the distribution of $\hat{\Gamma}$.*

Proof. Let F be a nonnegative measurable function on the space $\Omega_{\mathscr{D}}$ of loops in $\mathscr{D}$. By construction, the law of $\hat{\Gamma}_k(z) := \{\gamma \in \hat{\Gamma}(z), |\gamma| \leq k\}$ has Radon-Nikodym derivative

$$\Phi(\beta)^{-k} \prod_{i=1}^{k} \psi_i'(0)^{\beta} = \Phi(\beta)^{-k} \left(\frac{\mathbf{R}_k^z}{\mathbf{R}_0^z}\right)^{\beta} \mathbf{1}_{\{z \in \mathscr{D}_k\}}$$

with respect to the law of $\Gamma_k(z) := \{\gamma \in \Gamma(z), |\gamma| \leq k\}$. Let P be some probability measure under which $\hat{\Gamma}$ has the required distribution, and E the expectation associated. We get that

$$\begin{aligned}
\mathrm{E}\left[F\left(\hat{\Gamma}_k\right)\right] &= \frac{1}{W_0^{(\beta)}} \int_{\mathscr{D}} \mathrm{E}\left[F\left(\hat{\Gamma}_k(z)\right)\right] \mathbf{R}(\mathscr{D}, z)^{\beta} dz \\
&= \frac{\Phi(\beta)^{-k}}{W_0^{(\beta)}} \int_{\mathscr{D}} \mathrm{E}\left[F\left(\Gamma_k(z)\right)(\mathbf{R}_k^z)^{\beta} \mathbf{1}_{\{z \in \mathscr{D}_k\}}\right] dz \\
&= \frac{\Phi(\beta)^{-k}}{W_0^{(\beta)}} \int_{\mathscr{D}} \mathbb{E}_{\mathscr{D}}\left[F\left(\Gamma_k\right)(\mathbf{R}_k^z)^{\beta} \mathbf{1}_{\{z \in \mathscr{D}_k\}}\right] dz \\
&= \frac{\Phi(\beta)^{-k}}{W_0^{(\beta)}} \mathbb{E}_{\mathscr{D}}\left[F\left(\Gamma_k\right) W_k^{(\beta)}\right] = \hat{\mathbb{E}}_{\mathscr{D}}\left[F\left(\Gamma_k\right)\right].
\end{aligned}$$

This ends the proof of the proposition. □

Therefore, extending the filtration to make $\hat{\xi}$ measurable, we can identify, under $\hat{\mathbb{P}}_{\mathscr{D}}$, Γ with $\hat{\Gamma}$, and we will do so from now on. We let $\hat{\mathcal{F}}_k := \sigma(\hat{\xi}, \gamma, |\gamma| \leq k)$ and $\hat{\mathcal{F}}_\infty := \sigma(\hat{\xi}, \Gamma)$. Recall the notation $\mathbf{R}_k^z$, $\mathbf{A}_k^z$ for $z \in \mathscr{D}_k$. Then, when working under $\hat{\mathbb{P}}_{\mathscr{D}}$, we have $(\mathbf{R}_k^{\hat{\xi}}, \mathbf{A}_k^{\hat{\xi}}) = (\hat{\mathbf{R}}_k, \hat{\mathbf{A}}_k)$.

The remaining arguments to prove Theorem 1.2 follow the strategy of Lyons [29], see also Waymire [37] in the context of multiplicative cascades. Notice that under $\hat{\mathbb{P}}_{\mathscr{D}}$, the process $(W_k^{(\beta)})_{k\geq 0}$ is a positive super-martingale, hence it has a limit that we still denote by $W_\infty^{(\beta)}$. The convergence of the martingale $(W_k^{(\beta)})_{k\geq 0}$ in $L^1(\mathbb{P}_{\mathscr{D}})$ can be addressed via the following remark, which is Exercise 3.6 in Durrett [21].

(i) If $W_\infty^{(\beta)} < \infty$ $\hat{\mathbb{P}}_{\mathscr{D}}$-almost surely, then the convergence holds in $L^1(\mathbb{P}_{\mathscr{D}})$.

(ii) If $W_\infty^{(\beta)} = \infty$ $\hat{\mathbb{P}}_{\mathscr{D}}$-almost surely, then $W_\infty^{(\beta)} = 0$ $\mathbb{P}_{\mathscr{D}}$-almost surely.

We will show (i) in the case $0 < \beta < \beta_c$, and (ii) in the case $\beta \geq \beta_c$. By conformal invariance, it is not hard to see that the probability $\mathbb{P}_{\mathscr{D}}(W_\infty^{(\beta)} = 0)$ does not depend on $\mathscr{D}$. Reasoning on the loops at generation 1, we see that $\mathbb{P}_{\mathbb{U}}(W_\infty^{(\beta)} = 0)$ is a fixed point of the generating function of the number of loops at generation 1. Therefore, $\mathbb{P}_{\mathscr{D}}(W_\infty^{(\beta)}=0)$ is either 1 or the extinction probability.

It remains to show the convergence in $L^1(\mathbb{P}_{\mathscr{D}})$ of $W_\infty^{(\beta)}(A)$. We will restrict to the case $A=\mathscr{D}$. The only difference is that the spine $\hat\xi$ must be chosen with density $\frac{1}{W_0^{(\beta)}(A)}(\mathbf{R}_0^z)^\beta \mathbf{1}_A(z)dz$ in the case of an arbitrary Borelian set A. Before proceeding with the proof of the convergence in L^1, we state two preliminary lemma.

Lemma 2.2 *Under $\hat{\mathbb{P}}_{\mathscr{D}}$, $(\ln \hat R_\ell)_{\ell\geq 0}$ is a random walk with drift $\Phi'(\beta)/\Phi(\beta)$.*

Proof. This follows easily from the construction of $\hat{\mathbf{R}}_\ell$, and equation (2.1) . □

For any $k\geq 1$, let $\hat m(k)$ be the smallest integer $\ell\leq k-1$ such that $\sup_{z\in\mathbb{U}}|\hat\psi_{k-\ell}\circ\ldots\circ\hat\psi_k(z)|<1/2$ (set $\hat m(k):=k$ if no such ℓ exists). The constants in the following lemma do not depend on $\mathscr{D}$. We let $\hat\rho_\ell:=\sup_{|z|<1}|\hat\psi_\ell(z)|$.

Lemma 2.3 *(i) There exists a constant $c>0$ such that for any $k\geq 1$, on the event $\{m(k)\leq k-1\}$,*

$$\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_k)}|z-\hat\xi|\leq c\,\hat{\mathbf{R}}_k\left(\prod_{i=k-\hat m(k)}^{k}\hat\psi_i'(0)\right)^{-1}.$$

(ii) There exists a constant $c>0$ such that for any $k\geq 1$ and $\ell\leq k-1$,

$$\hat{\mathbb{P}}_{\mathscr{D}}(\hat m(k)\geq \ell)\leq 2\mathrm{e}^{-c\ell}.$$

(iii) There exist constants $c,c'>0$ such that for any $x\geq 1$,

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_k)}|z-\hat\xi|\geq x\,\hat{\mathbf{R}}_k\right)\leq c\mathrm{e}^{-c'k}+cx^{-c'}.$$

Proof. (i) Observe that $\mathbf{int}(\hat{\mathbf{A}}_k)=\hat\psi_0\circ\hat\psi_1\circ\ldots\circ\hat\psi_k(\mathbb{U})$. Therefore, we want to bound

$$|\hat\psi_0\circ\hat\psi_1\circ\ldots\hat\psi_k(z)-\hat\xi|$$

for any $z\in\mathbb{U}$. By Koebe's distortion Theorem, there exists a constant c such that $\max_{|z|<1/2}|f(z)-f(0)|\leq c|f'(0)|$ for any injective holomorphic function f on $\mathbb{U}$. It follows that, on the event $\{\hat m(k)\leq k-1\}$,

$$|\hat\psi_0\circ\hat\psi_1\circ\ldots\circ\hat\psi_k(z)-\hat\xi|\leq c\prod_{i=0}^{k-\hat m(k)-1}\hat\psi_i'(0)=c\,\hat{\mathbf{R}}_k\left(\prod_{i=k-\hat m(k)}^{k}\hat\psi_i'(0)\right)^{-1},$$

by equation (2.1). (ii) The maximum principle implies that $|\hat{\psi}_i(z)| \leq |z|\hat{\rho}_i$ for any $z \in \mathbb{U}$. We get that $\sup_{|z|<1} |\hat{\psi}_{k-\ell} \circ \ldots \circ \hat{\psi}_k(z)| \leq \prod_{i=k-\ell}^{k} \hat{\rho}_i$. This is a product of i.i.d. random variables which are strictly less than 1 almost surely. Write that $\hat{\mathbb{P}}_{\mathscr{D}}(\hat{m}(k) \geq \ell) \leq \hat{\mathbb{P}}_{\mathscr{D}}(\prod_{i=k-\ell}^{k} \hat{\rho}_i \geq 1/2)$ and use Markov inequality to complete the proof. (iii) By the Markov inequality, we have, for c_3 large enough, that $\hat{\mathbb{P}}_{\mathscr{D}}(\prod_{i=k-\ell}^{k} \hat{\psi}_i'(0) \leq \mathrm{e}^{-c_3\ell}) \leq c_4\mathrm{e}^{-c_5\ell}$ (we can check that $-\ln \hat{\psi}_1'(0)$ admits small exponential moments indeed). Combine this bound with (ii), for $\ell = \ln(x)/c_3$ to prove statement (iii). $\square$

2.2 The case $0 < \beta < \beta_c$

We need to show that $W_\infty^{(\beta)} < \infty$ $\hat{\mathbb{P}}_{\mathscr{D}}$-almost surely. We take the $\hat{\mathbb{P}}_{\mathscr{D}}$-expectation of $W_k^{(\beta)}$ conditionally on $\hat{\xi}$ and $(\hat{\mathbf{A}}_\ell)_{\ell\geq 0}$. Consider some outermost loop γ that lies inside the layer between $\hat{\mathbf{A}}_\ell$ and $\hat{\mathbf{A}}_{\ell+1}$, with $\ell < k$. The loops inside γ are a copy of Γ. Therefore, conditioning also on γ, we have

$$\Phi(\beta)^{-k}\hat{\mathbb{E}}_{\mathscr{D}}\left[\int_{\mathbf{int}(\gamma)} (\mathbf{R}_k^z)^\beta dz \mid \hat{\xi}, (\hat{\mathbf{A}}_\ell)_{\ell\geq 0}, \gamma\right] = \Phi(\beta)^{-\ell-1}\int_{\mathbf{int}(\gamma)} (\mathbf{R}_{\ell+1}^z)^\beta dz.$$

For any $z \in \mathbf{int}(\gamma)$, we have $\mathbf{R}_{\ell+1}^z \leq \mathbf{R}_\ell^z$ which is the conformal radius of $\mathbf{int}(\hat{A}_\ell)$ viewed from z. Hence,

$$\hat{\mathbb{E}}_{\mathscr{D}}\left[\int_{\mathbf{int}(\gamma)} (\mathbf{R}_k^z)^\beta dz \mid \hat{\xi}, (\hat{\mathbf{A}}_\ell)_{\ell\geq 0}, \gamma\right] \leq \int_{\mathbf{int}(\gamma)} (\mathbf{R}_\ell^z)^\beta dz.$$

Removing the conditioning on γ and summing over the outermost loops γ, we get

$$\begin{aligned}\Phi(\beta)^{-k}\hat{\mathbb{E}}_{\mathscr{D}}\left[\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)\setminus\mathbf{int}(\hat{\mathbf{A}}_{\ell+1})} (\mathbf{R}_k^z)^\beta dz \mid \hat{\xi}, (\hat{\mathbf{A}}_\ell)_{\ell\geq 0}\right] &\leq \Phi(\beta)^{-\ell-1}\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)\setminus\mathbf{int}(\hat{\mathbf{A}}_{\ell+1})} (\mathbf{R}_\ell^z)^\beta dz \\ &\leq \Phi(\beta)^{-\ell-1}\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)} (\mathbf{R}_\ell^z)^\beta dz.\end{aligned}$$

Summing over $\ell \leq k-1$, and adding the term $\Phi(\beta)^{-k}\int_{\mathbf{int}(\hat{\mathbf{A}}_k)} (\mathbf{R}_k^z)^\beta dz$, it follows that

$$\hat{\mathbb{E}}_{\mathscr{D}}\left[W_k^{(\beta)} \mid \hat{\xi}, (\hat{\mathbf{A}}_\ell)_{\ell\geq 0}\right] \leq \sum_{\ell=0}^{k} \Phi(\beta)^{-\ell-1}\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)} (\mathbf{R}_\ell^z)^\beta dz.$$

By Fatou's lemma, it comes down to proving that $\sum_{\ell\geq 0}\Phi(\beta)^{-\ell}\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)} (\mathbf{R}_\ell^z)^\beta dz$ is finite $\hat{\mathbb{P}}_{\mathscr{D}}$-almost surely. Recall that $\mathbf{R}(\hat{\mathbf{A}}_\ell, z) \leq 4\mathrm{d}(\hat{\mathbf{A}}_\ell, z)$, where $\mathrm{d}(\hat{\mathbf{A}}_\ell, z)$ is the Euclidean distance from z to $\hat{\mathbf{A}}_\ell$. Now observe that $\mathrm{d}(\hat{\mathbf{A}}_\ell, z) \leq |z - \hat{\xi}| + \mathrm{d}(\hat{\mathbf{A}}_\ell, \hat{\xi})$. Lemma 2.3 implies that, if ℓ

is large enough, then $|z-\hat{\xi}| \leq \ell^c \hat{\mathbf{R}}_\ell$ for any $z \in \hat{\mathbf{A}}_\ell$. Moreover, $\mathrm{d}(\hat{\mathbf{A}}_\ell, \hat{\xi}) \leq \hat{\mathbf{R}}_\ell$. Putting this together leads to

$$\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)} (\mathbf{R}_\ell^z)^\beta dz \leq (4\ell^c+1)^\beta (\hat{\mathbf{R}}_\ell)^\beta \int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)} dz.$$

Using again Lemma 2.3, we have $\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)} dz \leq \pi \ell^{2C} \hat{\mathbf{R}}_\ell^2$. Therefore, for ℓ large enough, we have

$$\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)} (\mathbf{R}_\ell^z)^\beta dz \leq \ell^{c'} (\hat{\mathbf{R}}_\ell)^{\beta+2}$$

$\hat{\mathbb{P}}_{\mathscr{D}}$-almost surely, for some big constant c'. In view of Lemma 2.2, we have that $\Phi(\beta)^{-\ell}(\hat{\mathbf{R}}_\ell)^{\beta+2}$ is exponentially small, and hence $\Phi(\beta)^{-\ell}\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)} (\mathbf{R}_\ell^z)^\beta dz$ has a convergent series. This ends the proof that $W_\infty^{(\beta)} < \infty$ $\hat{\mathbb{P}}_{\mathscr{D}}$-almost surely.

2.3 The case $\beta \geq \beta_c$

Now we want to show that $W_\infty^{(\beta)} = \infty$ $\hat{\mathbb{P}}_{\mathscr{D}}$-almost surely. Notice that $W_k^{(\beta)} \geq \int_{\mathbf{int}(\hat{\mathbf{A}}_k)} (\mathbf{R}_k^z)^\beta dz$. We use the following estimate.

Lemma 2.4 *Let D be a simply connected domain different from the whole plane and $z \in D$. Then, for any z' at distance less than $\frac{1}{8}\mathbf{R}(D,z)$, we have $\mathbf{R}(D,z') \in [\mathbf{R}(D,z)/8, 9\mathbf{R}(D,z)/2]$.*

Proof. We know that $\mathbf{R}(D,z') \geq \mathrm{d}(D,z')$ where we recall that $\mathrm{d}(D,z')$ is the Euclidean distance from z' to the boundary of D. The triangular inequality implies that $\mathrm{d}(D,z') \geq \mathrm{d}(D,z) - |z-z'| \geq \frac{\mathbf{R}(D,z)}{4} - |z-z'|$, hence the lower bound. Similarly, $\mathbf{R}(D,z') \leq 4d(D,z') \leq 4(|z-z'| + \mathrm{d}(D,z)) \leq 4(|z-z'| + \mathbf{R}(D,z))$ which yields the upper bound. □

The lemma implies that

$$W_k^{(\beta)} \geq \pi 8^{-\beta-2} \Phi(\beta)^{-k} (\hat{\mathbf{R}}_k)^{\beta+2}. \tag{2.2}$$

By Lemma 2.2, $-k \ln \Phi(\beta) + (\beta+2)\ln(\hat{\mathbf{R}}_k)$ is a random walk with nonnegative drift. We deduce that $W_\infty^{(\beta)} = \limsup_{k\to\infty} W_k^{(\beta)} = +\infty$, $\hat{\mathbb{P}}_{\mathscr{D}}$-almost surely.

3 The derivative martingale

We consider the case $\beta = \beta_c$. Recall the notation $\hat{\mathbb{P}}_{\mathscr{D}}$ and $\hat{\mathbf{R}}_k$ in Section 2.1. Define

$$\hat{U}_k := (\beta_c + 2) \ln \hat{\mathbf{R}}_k - k v_c.$$

Lemma 2.2 shows that $(\hat{U}_k)_{k\geq 0}$ is under $\hat{\mathbb{P}}_{\mathscr{D}}$ a centered random walk. Let $h_1 : \mathbb{R} \to \mathbb{R}$ be the renewal function associated to the random walk $(-\hat{U}_k)_{k\geq 0}$. Denote by $\hat{\mathbb{P}}^0_{\mathbb{U}}$ the probability measure $\hat{\mathbb{P}}_{\mathbb{U}}(\cdot \mid \hat{\xi} = 0)$. Under $\hat{\mathbb{P}}^0_{\mathbb{U}}$, $\hat{U}_0 = 0$ almost surely. The function h_1 is defined by $h_1(x) = 0$ if $x < 0$, $h_1(0) = 1$ and

$$h_1(x) := \sum_{k\geq 0} \hat{\mathbb{P}}^0_{\mathbb{U}} \left(\min_{i\leq k} -\hat{U}_i \geq -x \right) \qquad \text{if } x > 0. \tag{3.1}$$

It is known that $h_1(-\hat{U}_k)\mathbf{1}_{\{\min_{i\leq k} -\hat{U}_i \geq 0\}}$ gives rise to a martingale. Let $p(x, dy) := \hat{\mathbb{P}}^0_{\mathbb{U}}(x - \hat{U}_1 \in dy)$. The Markov chain with transition probabilities from $x \geq -\alpha$ to y given by $\frac{h_1(\alpha+y)}{h_1(\alpha+x)} p(x, dy)$ is called the *random walk conditioned to stay above* $-\alpha$. Finally, we will use the fact that there exists a constant $c_- > 0$ such that $h_1(x) \sim c_- x$ as $x \to \infty$.

Let $\alpha \geq 0$. For $z \in \mathscr{D}$, write $U^z_k := (\beta_c + 2) \ln \mathbf{R}^z_k - kv_c$, and let $\tau^\alpha_z := \min\{k \geq 0 : U^z_k > \alpha\}$. We define

$$D^{(\alpha)}_k := \Phi(\beta_c)^{-k} \int_{\mathscr{D}_k} h_1 \left(\alpha - U^z_k\right) (\mathbf{R}^z_k)^{\beta_c} \mathbf{1}_{\{\tau^\alpha_z > k\}} dz. \tag{3.2}$$

Lemma 3.1 *Under $\mathbb{P}_{\mathscr{D}}$, the process $(D^{(\alpha)}_k)_{k\geq 0}$ is a $(\mathcal{F}_k)_{k\geq 0}$-martingale.*

Proof. We have

$$D^{(\alpha)}_{k+1} = \Phi(\beta_c)^{-k-1} \int_{\mathscr{D}_k} (\mathbf{R}^z_k)^{\beta_c} h_1 \left(\alpha - U^z_{k+1}\right) \mathrm{e}^{-\beta_c \ln\left(\frac{\mathbf{R}^z_k}{\mathbf{R}^z_{k+1}}\right)} \mathbf{1}_{\{\tau^\alpha_z > k+1\}} dz.$$

Conditioning on $\mathcal{F}_k$, we get that $\mathbb{E}_{\mathscr{D}} \left[D^{(\alpha)}_{k+1} \mid \mathcal{F}_k \right]$ is

$$\Phi(\beta_c)^{-k-1} \int_{\mathscr{D}_k} (\mathbf{R}^z_k)^{\beta_c} \mathbf{1}_{\{\tau^\alpha_z > k\}}\, \mathbb{E}_{\mathscr{D}} \left[h_1 \left(\alpha - U^z_{k+1}\right) \mathrm{e}^{-\beta_c \ln\left(\frac{\mathbf{R}^z_k}{\mathbf{R}^z_{k+1}}\right)} \mathbf{1}_{\{U^z_{k+1} \leq \alpha\}} \mid \mathcal{F}_k \right] dz.$$

Observe that

$$\begin{aligned} \mathbb{E}_{\mathscr{D}} \left[h_1 \left(\alpha - U^z_{k+1}\right) \mathrm{e}^{-\beta_c \ln\left(\frac{\mathbf{R}^z_k}{\mathbf{R}^z_{k+1}}\right)} \mathbf{1}_{\{U^z_{k+1} \leq \alpha\}} \mid \mathcal{F}_k \right] &= \Phi(\beta_c) \hat{\mathbb{E}}^0_{\mathbb{U}} \left[h_1 \left(\alpha_k - \hat{U}_1 \right) \mathbf{1}_{\{\alpha_k - \hat{U}_1 \geq 0\}} \right] \\ &= \Phi(\beta_c) h_1(\alpha_k) \end{aligned}$$

where $\alpha_k := \alpha - U^z_k \geq 0$ when $\tau^\alpha_z > k$. This ends the proof of the lemma. $\square$

Similarly, for any Borelian set $A \subset \mathscr{D}$, we define the martingale

$$D_k^{(\alpha)}(A) := \Phi(\beta_c)^{-k} \int_{\mathscr{D}_k \cap A} h_1\left(\alpha - U_k^z\right)(\mathbf{R}_k^z)^{\beta_c} \mathbf{1}_{\{\tau_z^\alpha > k\}} dz.$$

We are going to prove the following proposition.

Proposition 3.2 *Let $\mathscr{D}$ be a bounded simply connected domain and A a deterministic Borelian set in $\mathscr{D}$. The martingale $D_k^{(\alpha)}(A)$ converges almost surely and in $L^1(\mathbb{P}_\mathscr{D})$ to some limit $D_\infty^{(\alpha)}(A)$. We write $D_\infty^{(\alpha)}$ for $D_\infty^{(\alpha)}(\mathscr{D})$.*

Let us first admit the proposition. Recall that $W_k^{(\beta_c)} = \Phi(\beta_c)^{-k} \int_{\mathscr{D}_k} (\mathbf{R}_k^z)^{\beta_c} dz$. We proved that $W_k^{(\beta_c)}$ vanishes when $k \to \infty$. By Lemma 2.4, we can see that $W_k^{(\beta_c)} = \Phi(\beta_c)^{-k} \int_{\mathscr{D}_k} (\mathbf{R}_k^z)^{\beta_c} dz \geq \Phi(\beta_c)^{-k} \pi 8^{-\beta_c - 2} (\mathbf{R}_k^{z_0})^{\beta_c + 2}$ for any $z_0 \in \mathscr{D}$. It readily implies that

$$\sup_{z \in \mathscr{D}_k} U_k^z \xrightarrow{k \to \infty} -\infty,$$

hence

$$\sup_{k \geq 0} \sup_{z \in \mathscr{D}_k} U_k^z < \infty$$

almost surely. Let $\alpha \geq 0$. Let A be a Borelian set with positive Lebesgue measure. On the event $\sup_{k \geq 0} \sup_{z \in \mathscr{D}_k} U_k^z \leq \alpha$, we see that $D_k^{(\alpha)}(A) \sim c_- D_k(A)$ as $k \to \infty$. In particular, $D_k(A)$ converges to a nonnegative limit on this event. Letting $\alpha \to \infty$, we deduce that $D_k(A)$ converges to a nonnegative limit almost surely. As was the case for $W_\infty^{(\beta)}$, we see that $D_\infty^{(\alpha)}$ defines a measure on $\mathscr{D}$, see Section 6.3. It implies that D_∞ also defines a measure on $\mathscr{D}$. We naturally call D_∞ this measure. We see again that $\mathbb{P}_\mathscr{D}(D_\infty = 0)$ does not depend on $\mathscr{D}$, hence, reasoning on the first generation of loops, that it is either 1 or the extinction probability.

The rest of the section is devoted to the proof of Proposition 3.2. Again, we will only prove it for $A = \mathscr{D}$.

3.1 Another change of measure

Fix $\alpha \geq 0$. Define a probability measure $\overline{\mathbb{P}}_\mathscr{D}$ on $\mathcal{F}_\infty$ such that, for any $k \geq 0$,

$$\frac{d\overline{\mathbb{P}}_\mathscr{D}}{d\mathbb{P}_\mathscr{D}}\Big|_{\mathcal{F}_k} := \frac{D_k^{(\alpha)}}{D_0^{(\alpha)}}.$$

We describe the law of (Γ,Ξ) under $\overline{\mathbb{P}}_{\mathscr{D}}$. As before, $\mathcal{C}$ is the family of non-nested loops in the unit disk $\mathbb{U}$ and ψ is the conformal bijection from $\mathbb{U}$ to the interior of the loop in $\mathcal{C}$ that surrounds 0, such that $\psi(0)=0$ and $\psi'(0)>0$.

Let $z\in\mathscr{D}$. We would like to construct a collection of loops $\overline{\Gamma}(z)$. Define $\overline{\psi}_0^z := \Psi^z$ and $\overline{\mathbf{R}}_0^z := \mathbf{R}(\mathscr{D},z)$. Suppose that $\overline{\psi}_k^z$ and $\overline{\mathbf{R}}_k^z$ are known. Conditionally on it, take $(\overline{\mathcal{C}}_{k+1}^z,\overline{\psi}_{k+1}^z)$ a random variable whose law has Radon-Nikodym derivative

$$\Phi(\beta_c)^{-1}\frac{h_1(\alpha-\overline{U}_k^z-(\beta_c+2)\ln\psi'(0)+v_c)}{h_1(\alpha-\overline{U}_k^z)}\psi'(0)^{\beta_c}$$

with respect to the law of $(\mathcal{C},\psi)$, where $\overline{U}_k^z := (\beta_c+2)\ln\overline{\mathbf{R}}_k^z - kv_c$. We call $\overline{\mathcal{C}}_{k+1}^z(0)$ the loop in $\overline{\mathcal{C}}_{k+1}^z$ that surrounds 0. Then, let

$$\overline{\mathbf{A}}_{k+1}^z := \overline{\psi}_0^z\circ\ldots\circ\overline{\psi}_k^z(\overline{\mathcal{C}}_{k+1}^z(0)),$$

and $\overline{\mathbf{R}}_{k+1}^z := \overline{\mathbf{R}}_k^z(\overline{\psi}_{k+1}^z)'(0)$. In words, $\overline{\mathbf{A}}_k^z$ stands for the loop of generation k that surrounds z and $\overline{\mathbf{R}}_k^z$ is the conformal radius of its interior seen from z. Draw the "brother loops" by taking $\Psi^z\circ\overline{\psi}_1^z\circ\ldots\circ\overline{\psi}_{k-1}^z(\gamma)$ for any $\gamma\in\overline{\mathcal{C}}_k^z$ different from $\overline{\mathcal{C}}_k^z(0)$ and fill them with independent copies of Γ. Call $\overline{\Gamma}(z)$ the result.

Let $\overline{\xi}$ be a random point in $\mathscr{D}$ with density $\frac{1}{D_0^{(\alpha)}}h_1(\alpha-(\beta_c+2)\ln\mathbf{R}(\mathscr{D},z))\mathbf{R}(\mathscr{D},z)^{\beta_c}dz$. We write $\overline{\Gamma}$ for $\overline{\Gamma}\left(\overline{\xi}\right)$, and we will forget the upper script z in $\overline{\psi}_i^z$, $\overline{\mathbf{R}}_k^z$, $\overline{\mathbf{A}}_k^z$, $\overline{U}_k^z$ when z is $\overline{\xi}$.

Proposition 3.3 *The distribution of* Γ *under* $\overline{\mathbb{P}}_{\mathscr{D}}$ *is the one of* $\overline{\Gamma}$.

Lemma 3.4 *Under* $\overline{\mathbb{P}}_{\mathscr{D}}$, $(-\overline{U}_\ell)_{\ell\geq 0}$ *has the distribution of the random walk conditioned to stay above* $-\alpha$.

We omit the proof of the proposition since it follows the lines of Proposition 2.1. The lemma easily comes from the construction of $(\overline{U}_\ell)_{\ell\geq 0}$. We finally prove an analog of Lemma 2.3.

Lemma 3.5 *There exists a constant* $c>0$ *such that we have for* k *large enough*

$$\max_{z\in\mathbf{int}(\overline{\mathbf{A}}_k)}|z-\overline{\xi}|\leq k^c\,\overline{\mathbf{R}}_k$$

$\overline{\mathbb{P}}_{\mathscr{D}}$*-almost surely.*

Proof. We follow the lines of the proof of Lemma 2.3 applied for $\beta = \beta_c$. Let $\overline{m}(k)$ be the smallest integer $\ell \leq k-1$ such that $\sup_{z \in \mathbb{U}} |\overline{\psi}_{k-\ell} \circ \ldots \circ \overline{\psi}_k(z)| < 1/2$ (with again $\overline{m}(k) := k$ if no such ℓ exists). It is enough to prove that $\overline{\mathbb{P}}_{\mathscr{D}}(\overline{m}(k) \geq \ell) \leq c_6\sqrt{k}\mathrm{e}^{-c_7\ell}$ and $\overline{\mathbb{P}}_{\mathscr{D}}(\prod_{i=k-\ell+1}^{k} \overline{\psi}_i'(0) \geq \mathrm{e}^{-c_3\ell}) \leq c_8\sqrt{k}\mathrm{e}^{-c_9\ell}$. Notice that, the law of $(\overline{\psi}_i, \overline{\mathbf{R}}_i, i \leq k)$ has Radon-Nikodym derivative

$$\frac{h_1(\alpha + kv_c - (\beta_c+2)\ln \hat{\mathbf{R}}_k)}{h_1(\alpha - (\beta_c+2)\ln \hat{\mathbf{R}}_0)} \mathbf{1}\{\tau_{\hat{\xi}}^{\alpha} > k\}$$

with respect to the law of $(\hat{\psi}_i, \hat{\mathbf{R}}_i, i \leq k)$ (taken in the case $\beta = \beta_c$). Therefore,

$$\overline{\mathbb{P}}_{\mathscr{D}}(\overline{m}(k) \geq \ell) = \hat{\mathbb{E}}_{\mathscr{D}}\left[\frac{h_1(\alpha - (\beta_c+2)\ln \hat{\mathbf{R}}_k + kv_c)}{h_1(\alpha - (\beta_c+2)\ln \hat{\mathbf{R}}_0)} \mathbf{1}\{\tau_{\hat{\xi}}^{\alpha} > k, \hat{m}(k) \geq \ell\}\right].$$

We know from Lemma 2.3 that $\hat{\mathbb{P}}_{\mathscr{D}}(\hat{m}(k) \geq \ell) \leq c_1\mathrm{e}^{-c_2\ell}$, so we just need to use Cauchy-Schwarz inequality. We proceed similarly for $\overline{\mathbb{P}}_{\mathscr{D}}(\prod_{i=k-\ell+1}^{k} \overline{\psi}_i'(0) \geq \mathrm{e}^{-c_3\ell})$. □

3.2 Proof of Proposition 3.2

Fix $\alpha \geq 0$. We need to prove that $\lim_{k\to\infty} D_k^{(\alpha)} < \infty$ $\overline{\mathbb{P}}_{\mathscr{D}}$-almost surely. We follow the proof of Biggins and Kyprianou [10]. We take the $\overline{\mathbb{P}}_{\mathscr{D}}$-expectation conditionally on $\overline{\xi}$ and $(\overline{\mathbf{A}}_\ell)_{\ell \geq 0}$. For some outermost loop γ that lies inside the layer between $\overline{\mathbf{A}}_\ell$ and $\overline{\mathbf{A}}_{\ell+1}$, we have

$$\begin{aligned} (3.3) \qquad & \Phi(\beta_c)^{-k}\overline{\mathbb{E}}_{\mathscr{D}}\left[\int_{\mathbf{int}(\gamma)} h_1\left(\alpha - U_k^z\right)(\mathbf{R}_k^z)^{\beta_c}\mathbf{1}_{\{\tau_z^\alpha > k\}}dz \mid \overline{\xi}, (\overline{\mathbf{A}}_\ell)_{\ell\geq 0}, \gamma\right] \\ = \; & \Phi(\beta_c)^{-\ell-1}\int_{\mathbf{int}(\gamma)} h_1\left(\alpha - U_{\ell+1}^z\right)(\mathbf{R}_{\ell+1}^z)^{\beta_c}\mathbf{1}_{\{\tau_z^\alpha > \ell+1\}}dz. \end{aligned}$$

Write $y_+ := \max(0, y)$. There exists a constant $c_1 > 0$ such that $h_1(x) \leq c_1(1+x)_+$ for any $x \in \mathbb{R}$. It gives that

$$h_1\left(\alpha - U_{\ell+1}^z\right) \leq c_1(1+\alpha - U_\ell^z)_+ + (\beta_c+2)c_1\left(\ln\frac{\mathbf{R}_\ell^z}{\mathbf{R}_{\ell+1}^z}\right)_+ - c_1 v_c.$$

We deduce that

$$h_1\left(\alpha - U_{\ell+1}^z\right)(\mathbf{R}_{\ell+1}^z)^{\beta_c} \leq c_1(1+\alpha - U_\ell^z)_+(\mathbf{R}_{\ell+1}^z)^{\beta_c} + c_2(\mathbf{R}_\ell^z)^{\beta_c} + c_3(\mathbf{R}_{\ell+1}^z)^{\beta_c}.$$

For any $z \in \mathbf{int}(\gamma)$, $\mathbf{R}_{\ell+1}^z \leq \mathbf{R}_\ell^z$. We get that, for some $c_4 = c_4(\alpha)$,

$$h_1\left(\alpha - U_{\ell+1}^z\right)(\mathbf{R}_{\ell+1}^z)^{\beta_c} \leq c_4(1+(-U_\ell^z)_+)(\mathbf{R}_\ell^z)^{\beta_c}.$$

Equation (3.3) becomes

$$\begin{aligned}
&\Phi(\beta_c)^{-k}\overline{\mathbb{E}}_{\mathscr{D}}\left[\int_{\mathbf{int}(\gamma)} h_1\left(\alpha - U_k^z\right)\left(\mathbf{R}_k^z\right)^{\beta_c}\mathbf{1}_{\{\tau_z^\alpha>k\}}dz \mid \overline{\xi}, (\overline{\mathbf{A}}_\ell)_{\ell\geq 0}, \gamma\right]\\
\leq\ & c_5\Phi(\beta_c)^{-\ell}\int_{\mathbf{int}(\gamma)}(1+(-U_\ell^z)_+)(\mathbf{R}_\ell^z)^{\beta_c}dz.
\end{aligned}$$

It yields that

$$\begin{aligned}
&\Phi(\beta_c)^{-k}\overline{\mathbb{E}}_{\mathscr{D}}\left[\int_{\mathbf{int}(\overline{\mathbf{A}}_\ell)\setminus\mathbf{int}(\overline{\mathbf{A}}_{\ell+1})} h_1\left(\alpha - U_k^z\right)\left(\mathbf{R}_k^z\right)^{\beta_c}\mathbf{1}_{\{\tau_z^\alpha>k\}}dz \mid \overline{\xi}, (\overline{\mathbf{A}}_\ell)_{\ell\geq 0}\right]\\
\leq\ & c_5\Phi(\beta_c)^{-\ell}\int_{\mathbf{int}(\overline{\mathbf{A}}_\ell)\setminus\mathbf{int}(\overline{\mathbf{A}}_{\ell+1})}(1+(-U_\ell^z)_+)(\mathbf{R}_\ell^z)^{\beta_c}dz.
\end{aligned}$$

Summing over $\ell\leq k-1$, and adding the term $\Phi(\beta_c)^{-k}\int_{\mathbf{int}(\overline{\mathbf{A}}_k)} h_1\left(\alpha-U_k^z\right)\left(\mathbf{R}_k^z\right)^{\beta_c}dz$, it follows that

$$\overline{\mathbb{E}}_{\mathscr{D}}\left[D_k^{(\alpha)} \mid \overline{\xi}, (\overline{\mathbf{A}}_\ell)_{\ell\geq 0}\right] \leq c_6\sum_{\ell=0}^{k}\Phi(\beta_c)^{-\ell}\int_{\mathbf{int}(\overline{\mathbf{A}}_\ell)}\left(1+(-U_\ell^z)_+\right)\left(\mathbf{R}_\ell^z\right)^{\beta_c}dz$$

hence

$$\overline{\mathbb{E}}_{\mathscr{D}}\left[D_\infty^{(\alpha)} \mid \overline{\xi}, (\overline{\mathbf{A}}_\ell)_{\ell\geq 0}\right] \leq c_6\sum_{\ell\geq 0}\Phi(\beta_c)^{-\ell}\int_{\mathbf{int}(\overline{\mathbf{A}}_\ell)}\left(1+(-U_\ell^z)_+\right)\left(\mathbf{R}_\ell^z\right)^{\beta_c}dz. \tag{3.4}$$

By Lemma 3.5, we get for ℓ large enough

$$\left(1+(-U_\ell^z)_+\right)\left(\mathbf{R}_\ell^z\right)^{\beta_c} = \left(1+\left(-\overline{U}_\ell-(\beta_c+2)\ln\frac{\mathbf{R}_\ell^z}{\overline{\mathbf{R}}_\ell}+v_c\right)_+\right)\left(\mathbf{R}_\ell^z\right)^{\beta_c} \leq \ell^{c_7}\left(1+(-\overline{U}_\ell)_+\right)\left(\overline{\mathbf{R}}_\ell\right)^{\beta_c}.$$

It implies that

$$\int_{\mathbf{int}(\overline{\mathbf{A}}_\ell)}\left(1+(-U_\ell^z)_+\right)\left(\mathbf{R}_\ell^z\right)^{\beta_c}dz \leq \ell^{c_7}\left(1+(-\overline{U}_\ell)_+\right)\left(\overline{\mathbf{R}}_\ell\right)^{\beta_c}\int_{\mathbf{int}(\overline{\mathbf{A}}_\ell)}dz.$$

By Lemma 3.5, we get for ℓ large enough,

$$\int_{\mathbf{int}(\overline{\mathbf{A}}_\ell)}\left(1+(-U_\ell^z)_+\right)\left(\mathbf{R}_\ell^z\right)^{\beta_c}dz \leq \ell^{c_8}\left(1+(-\overline{U}_\ell)_+\right)\left(\overline{\mathbf{R}}_\ell\right)^{\beta_c+2}.$$

We know that $-\overline{U}_\ell$ is a random walk conditioned to stay above $-\alpha$, hence is greater than $\ell^{1/3}$ for example for ℓ large enough. Hence

$$\sum_{\ell\geq 1}\ell^{c_8}\Phi(\beta_c)^{-\ell}\left(1+(-\overline{U}_\ell)_+\right)\left(\overline{\mathbf{R}}_\ell\right)^{\beta_c+2} = \sum_{\ell\geq 1}\ell^{c_8}\mathrm{e}^{\overline{U}_\ell}\left(1+(-\overline{U}_\ell)_+\right) < \infty$$

$\overline{\mathbb{P}}_{\mathscr{D}}$-almost surely. In view of (3.4), it completes the proof of the proposition.

4 First estimates on the extremal process

Let $\mathscr{D}$ be a bounded simply connected domain. Recall the notation of Section 2.1. We consider the case $\beta = \beta_c$. We will write (instead of $W_k^{(\beta_c)}$),

$$W_k := \Phi(\beta_c)^{-k} \int_{\mathscr{D}_k} (\mathbf{R}_k^z)^{\beta_c} dz.$$

Recall that $\hat{\mathbb{P}}_{\mathscr{D}}$ is the change of measure associated to the martingale W_k. The following lemma is the analog of the many-to-one lemma in the branching random walk setting.

Lemma 4.1 *Let $k \geq 1$. Let F be a nonnegative function from $\mathscr{D} \times \Omega_{\mathscr{D}}$, $(z, \Gamma) \to F(z, \Gamma)$ which is measurable with respect to the product σ-algebra of the Borel σ-algebra on $\mathscr{D}$ and $\mathcal{F}_k$. Then*

$$\Phi(\beta_c)^{-k} \mathbb{E}_{\mathscr{D}} \left[\int_{\mathscr{D}_k} (\mathbf{R}_k^z)^{\beta_c} F(z, \Gamma) dz \right] = W_0 \hat{\mathbb{E}}_{\mathscr{D}} \left[F(\hat{\xi}, \hat{\Gamma}) \right].$$

Proof. The proof is similar to the proof of Proposition 2.1. We observe that

$$\begin{aligned} \hat{\mathbb{E}}_{\mathscr{D}} \left[F(\hat{\xi}, \hat{\Gamma}) \right] &= \int_{\mathscr{D}} \frac{1}{W_0} \hat{\mathbb{E}}_{\mathscr{D}} \left[F(z, \hat{\Gamma}(z)) \right] (\mathbf{R}_0^z)^{\beta_c} dz \\ &= \Phi(\beta_c)^{-k} \int_{\mathscr{D}} \frac{1}{W_0} \mathbb{E}_{\mathscr{D}} \left[F(z, \Gamma(z)) (\mathbf{R}_k^z)^{\beta_c} \mathbf{1}_{\{z \in \mathscr{D}_k\}} \right] dz \\ &= \Phi(\beta_c)^{-k} \int_{\mathscr{D}} \frac{1}{W_0} \mathbb{E}_{\mathscr{D}} \left[F(z, \Gamma) (\mathbf{R}_k^z)^{\beta_c} \mathbf{1}_{\{z \in \mathscr{D}_k\}} \right] dz \end{aligned}$$

which yields the lemma. □

Recall that $U_n^z := (\beta_c + 2) \ln \mathbf{R}_n^z - n v_c$. This section gives some estimates on the extremal process at time n. In Section 4.1, we look at the tail distribution of the maximum. In Section 4.2, we show that when the maximum is atypically large, there is only one cluster in the extremal process. Section 4.3 says that the extremal process is related to the process $\mathcal{P}_{k,n}$ defined in (1.5).

Remark 4.2 *In the course of the proofs, we will use the following fact. Let $k \geq 1$ be an integer, and γ a loop at generation k. Then, for any nonnegative measurable function f, and any $n \geq k$,*

$$\mathbb{E}_{\mathscr{D}} \left[f(U_n^z,\, z \in \mathbf{int}(\gamma)) \mid \mathcal{F}_k \right] = \mathbb{E}_{\mathrm{e}^{-k\tilde{v}_c} \mathbf{int}(\gamma)} \left[f(U_{n-k}^z,\, z \in \mathbf{int}(\gamma)) \right]$$

where $\mathrm{e}^{-k\tilde{v}_c} \mathbf{int}(\gamma)$ denotes the set $\{\mathrm{e}^{-k v_c/(\beta_c+2)} z,\, z \in \mathbf{int}(\gamma)\}$.

4.1 Upper bound on the tail distribution of the maximum

Lemma 4.3 *There exists a constant $c > 0$ (which does not depend on $\mathscr{D}$) such that for any real x,*

$$\mathbb{P}_{\mathscr{D}}\left(\max_{k\geq 0, z\in\mathscr{D}_k} U_k^z > x\right) \leq c\mathrm{e}^{-x}W_0.$$

Proof. Let τ be the first integer $k \geq 0$ for which there exists $z \in \mathscr{D}_k$ such that $U_k^z > x$. Since τ is a stopping time, we have $\mathbb{E}_{\mathscr{D}}[W_\tau] \leq W_0$. Moreover, Lemma 2.4 proves that $W_k \geq \pi 8^{-\beta_c-2}\mathrm{e}^{U_k^z}$ for any $z \in \mathscr{D}_k$. Taking $k = \tau$ and z such that $U_k^z > x$, we find that $W_\tau \geq \pi 8^{-\beta_c-2}\mathrm{e}^x$ if $\tau < \infty$. Therefore $\mathbb{P}_{\mathscr{D}}(\tau < \infty) \leq 8^{\beta_c+2}\pi^{-1}\mathrm{e}^{-x}\mathbb{E}_{\mathscr{D}}[W_\tau] \leq 8^{\beta_c+2}\pi^{-1}\mathrm{e}^{-x}W_0$ indeed. □

Following the notation of [1], we write a_n for $-\frac{3}{2}\ln n$ and $I_n(x) := [x + a_n, x + a_n + 1)$.

Lemma 4.4 *There exist constants $c, c' > 0$ (which do not depend on $\mathscr{D}$) such that for any integer $n \geq 1$, any real numbers α, x, and any $L \in [0, O(\ln(n))]$,*

$$\begin{aligned}
&\mathbb{P}_{\mathscr{D}}\left(\exists\, z \in \mathscr{D}_n \,:\, U_n^z \in I_n(x),\, \max_{j\leq n} U_j^z \leq \alpha,\, \max_{\ell\in[n/2,n]} U_\ell^z \in I_n(x+L)\right)\\
\leq\;& c\mathrm{e}^{-c'L}\mathrm{e}^{-x}\int_{\mathscr{D}}(1+\max(0,\alpha - U_0^z))(\mathbf{R}_0^z)^{\beta_c}dz.
\end{aligned}$$

Proof. For any $z \in \mathscr{D}_n$, let $\chi_{\ell,n}(z)$ be the indicator of the event

$$\begin{aligned}
&U_n^z \in [a_n + x - c_1, a_n + x + c_1),\\
&\max_{j\leq n} U_j^z \leq \alpha + c_1,\\
&\max_{j\in[n/2,n]} U_j^z \leq a_n + x + L + c_1,\\
&U_\ell^z \in [a_n + x + L - c_1, a_n + x + L + c_1],
\end{aligned}$$

for some constant c_1 such that the following holds: if z is such that $U_n^z \in I_n(x)$, $\max_{j\leq n} U_j^z \leq \alpha$, $\max_{j\in[n/2,n]} U_j^z = U_\ell^z \in I_n(x+L)$, then $\chi_{\ell,n}(z') = 1$ for any $|z' - z| \leq \mathbf{R}_n(z)/8$ (this is possible by Lemma 2.4). We have $\Phi(\beta_c)^{-n}\int_{\mathscr{D}_n}(\mathbf{R}_n^z)^{\beta_c}\chi_{\ell,n}(z)dz \geq \pi 8^{-\beta_c-2}\mathrm{e}^{U_n^z} \geq cn^{-3/2}\mathrm{e}^x$ for any z such that $U_n^z \in I_n(x)$, $\max_{j\leq n} U_j^z \leq \alpha$, $\max_{j\in[n/2,n]} U_j^z = U_\ell^z \in I_n(x+L)$. Therefore,

$$\begin{aligned}
&\mathbb{P}_{\mathscr{D}}\left(\exists\, z \in \mathscr{D}_n \,:\, U_n^z \in I_n(x),\, \max_{j\leq n} U_j^z \leq \alpha,\, \max_{j\in[n/2,n]} U_j^z = U_\ell^z \in I_n(x+L)\right)\\
\leq\;& cn^{3/2}\mathrm{e}^{-x}\Phi(\beta_c)^{-n}\mathbb{E}_{\mathscr{D}}\left[\int_{\mathscr{D}}(\mathbf{R}_n^z)^{\beta_c}\chi_{\ell,n}(z)dz\right].
\end{aligned}$$

By Lemma 4.1, we have

$$\Phi(\beta_c)^{-n}\mathbb{E}_{\mathscr{D}}\left[\int_{\mathscr{D}_n}(\mathbf{R}_n^z)^{\beta_c}\chi_{\ell,n}(z)dz\right] = W_0\hat{\mathbb{E}}_{\mathscr{D}}[\chi_{\ell,n}(\hat{\xi})].$$

Equations (6.2) and (6.3) give that

$$\hat{\mathbb{E}}_{\mathscr{D}}[\chi_{\ell,n}(\hat{\xi}) \mid \hat{U}_0] \leq c(1+\max(0,\alpha-\hat{U}_0))(1+L)n^{-3/2}(n-\ell+1)^{-3/2}$$

if $\ell \geq 2n/3$, and

$$\hat{\mathbb{E}}_{\mathscr{D}}[\chi_{\ell,n}(\hat{\xi}) \mid \hat{U}_0] \leq c(1+\max(0,\alpha-\hat{U}_0))(1+L)n^{-3}\ln n$$

if $\ell \in [n/2, 2n/3]$. By the density of $\hat{\xi}$ in $\mathscr{D}$, we have

$$\hat{\mathbb{E}}_{\mathscr{D}}[\max(0,\alpha-\hat{U}_0)] = \frac{1}{W_0}\int_{\mathscr{D}}\max(0,\alpha-U_0^z)(\mathbf{R}_0^z)^{\beta_c}dz.$$

Consequently, we have

$$\begin{aligned}&\mathbb{P}_{\mathscr{D}}\left(\exists\, z \in \mathscr{D}_n \,:\, U_n^z \in I_n(x),\, \max_{j\leq n} U_j^z \leq \alpha,\, \max_{j\in[n/2,n]} U_j^z = U_\ell^z \in I_n(x+L)\right)\\ \leq\;& c(1+L)\mathrm{e}^{-x}(n-\ell+1)^{-3/2}\int_{\mathscr{D}}(1+\max(0,\alpha-U_0^z))(\mathbf{R}_0^z)^{\beta_c}dz\end{aligned}$$

if $\ell \geq 2n/3$ and

$$\begin{aligned}&\mathbb{P}_{\mathscr{D}}\left(\exists\, z \in \mathscr{D}_n \,:\, U_n^z \in I_n(x),\, \max_{j\leq n} U_j^z \leq \alpha,\, \max_{j\in[n/2,n]} U_j^z = U_\ell^z \in I_n(x+L)\right)\\ \leq\;& c(1+L)\mathrm{e}^{-x}n^{-3/2}\ln(n)\int_{\mathscr{D}}(1+\max(0,\alpha-U_0^z))(\mathbf{R}_0^z)^{\beta_c}dz\end{aligned}$$

if $\ell \in [n/2, 2n/3]$. It implies that, for any $a \geq 1$,

$$\begin{aligned}\text{(4.1)}\quad &\mathbb{P}_{\mathscr{D}}\left(\exists\, z,\, \exists\, \ell \in [\frac{n}{2}, n-a] \,:\, U_n^z \in I_n(x),\, \max_{j\leq n} U_j^z \leq \alpha,\, \max_{j\in[n/2,n]} U_j^z = U_\ell^z \in I_n(x+L)\right)\\ \leq\;& c(1+L)\mathrm{e}^{-x}\left(a^{-1/2}+\frac{\ln n}{\sqrt{n}}\right)\int_{\mathscr{D}}(1+\max(0,\alpha-U_0^z))(\mathbf{R}_0^z)^{\beta_c}dz.\end{aligned}$$

We need to deal with $\ell \in [n-a, n]$. We repeat the same strategy. Let $\chi'_{\ell,n}(z)$ be the indicator of the event

$$\begin{aligned}&\max_{j\leq n} U_j^z \leq \alpha + c_2,\\ &\max_{j\in[n/2,\ell]} U_j^z \leq a_n + x + L + c_2,\\ &U_\ell^z \in [a_n + x + L - c_2, a_n + x + L + c_2]\end{aligned}$$

for c_2 big enough such that $\Phi(\beta_c)^{-\ell}\int_{\mathscr{D}_\ell}(\mathbf{R}_\ell^z)^{\beta_c}\chi'_{\ell,n}(z)dz \geq cn^{-3/2}\mathrm{e}^{x+L}$ as long as there is some $z\in\mathscr{D}_\ell$ such that $\max_{j\leq \ell} U_j^z \leq \alpha$, $\max_{j\in[n/2,\ell]} U_j^z = U_\ell^z \in I_n(x+L)$. We get that

$$\begin{aligned}&\mathbb{P}_{\mathscr{D}}\left(\exists\, z\in\mathscr{D}_n\,:\, U_n^z\in I_n(x),\, \max_{j\leq n} U_j^z \leq \alpha,\, \max_{j\in[n/2,n]} U_j^z = U_\ell^z \in I_n(x+L)\right)\\ \leq\;& c\mathrm{e}^{-x-L}n^{3/2}\mathbb{E}_{\mathscr{D}}\left[\Phi(\beta_c)^{-\ell}\int_{\mathscr{D}_\ell}(\mathbf{R}_\ell^z)^{\beta_c}\chi'_{\ell,n}(z)dz\right].\end{aligned}$$

Notice that, by Lemma 4.1 then equation (6.3), we have for $\ell\geq 2n/3$,

$$\Phi(\beta_c)^{-\ell}\mathbb{E}_{\mathscr{D}}\left[\int_{\mathscr{D}}(\mathbf{R}_\ell^z)^{\beta_c}\chi'_{\ell,n}(z)dz\right] = W_0\hat{\mathbb{E}}_{\mathscr{D}}[\chi'_{\ell,n}(\hat{\xi})] \leq cn^{-3/2}\hat{\mathbb{E}}_{\mathscr{D}}[\max(0,\alpha-\hat{U}_0)].$$

Therefore, for any $a\in[1,\frac{n}{3}]$, we have

$$\begin{aligned}(4.2)\quad&\mathbb{P}_{\mathscr{D}}\left(\exists\, z,\,\exists\,\ell\in[n-a,n]\,:\, U_n^z\in I_n(x),\, \max_{j\leq n} U_j^z\leq\alpha,\, \max_{j\in[n/2,n]} U_j^z = U_\ell^z\in I_n(x+L)\right)\\ \leq\;& ca\mathrm{e}^{-x-L}\int_{\mathscr{D}}(1+\max(0,\alpha-U_0^z))(\mathbf{R}_0^z)^{\beta_c}dz.\end{aligned}$$

Combine (4.1) with (4.2), and take $a=\mathrm{e}^{cL}\in[1,\frac{n}{3}]$ with some constant $c>0$ small enough to complete the proof. $\square$

We get an upper bound on the tail distribution of the maximum.

Corollary 4.5 *There exists some constant $c>0$ (which does not depend on $\mathscr{D}$) such that for any integer $n\geq 1$ and any real x,*

$$\mathbb{P}_{\mathscr{D}}\left(\max_{z\in\mathscr{D}_n} U_n^z \geq x-\frac{3}{2}\ln n\right) \leq c\mathrm{e}^{-x}\int_{\mathscr{D}}(1+\max(0,x-U_0^z))(\mathbf{R}_0^z)^{\beta_c}dz.$$

We finish this section with a result that will be useful later.

Corollary 4.6 *Let $\mathscr{D}$ be a bounded simply connected domain. Let $\varepsilon>0$ and $x\in\mathbf{R}$. There exists B large enough such that for any $k\geq 1$ and $n\geq 2k$,*

$$\mathbb{P}_{\mathscr{D}}\left(\exists\,|\gamma|=k,\,\exists\, z\in\mathbf{int}(\gamma)\,:\, U_n^z\geq a_n+x,\, \max_{z'\in\mathbf{int}(\gamma)}|z-z'|\geq B\mathbf{R}_k^z\right)\leq \varepsilon + cW_0\mathrm{e}^{-c'k}.$$

Proof. Let $\mathscr{D}$ be a bounded simply connected domain, $\varepsilon>0$ and $x\in\mathbf{R}$. By Lemma 4.3, we can take x_1 large enough such that

$$\mathbb{P}_{\mathscr{D}}\left(\max_{k\geq 0, z\in\mathscr{D}_k} U_k^z > x_1\right)\leq\varepsilon.$$

Fix x_1. The constants in this proof can depend on x and x_1. By Corollary 4.5 and Remark 4.2, for any loop γ at generation k,

$$\mathbb{P}_{\mathscr{D}}\left(\exists\, z \in \mathbf{int}(\gamma)\,:\, U_n^z \geq a_n + x \mid \mathcal{F}_k\right) \leq c\Phi(\beta_c)^{-k} \int_{\mathbf{int}(\gamma)} (1+\max(0,-U_k^z))(\mathbf{R}_k^z)^{\beta_c} dz.$$

By Lemma 4.1, we get

$$\begin{aligned}
(4.3)\quad & \mathbb{P}_{\mathscr{D}}\left(\exists\, |\gamma| = k,\, \exists\, z \in \mathbf{int}(\gamma)\,:\, U_n^z \geq a_n + x,\, \max_{z' \in \mathbf{int}(\gamma)} |z - z'| \geq B\mathbf{R}_k^z,\, \max_{\ell \leq k} U_\ell^z \leq x_1\right) \\
\leq\ & c\mathbb{E}_{\mathscr{D}}\left[\Phi(\beta_c)^{-k} \int_{\mathscr{D}} (1+\max(0,-U_k^z))(\mathbf{R}_k^z)^{\beta_c} \mathbf{1}_{\{\max_{z' \in \mathbf{int}(\mathbf{A}_k^z)} |z-z'| \geq B\mathbf{R}_k^z, \max_{\ell \leq k} U_\ell^z \leq x_1\}} dz\right] \\
=\ & cW_0 \hat{\mathbb{E}}_{\mathscr{D}}\left[(1+\max(0,-\hat{U}_k))\mathbf{1}_{\{\max_{z' \in \mathbf{int}(\hat{\mathbf{A}}_k)} |z'-\hat{\xi}| \geq B\hat{\mathbf{R}}_k, \max_{\ell \leq k} \hat{U}_\ell \leq x_1\}}\right].
\end{aligned}$$

Recall Lemma 2.3. Write $k(B)$ for $\min(k/2, \ln(B)^{1/2})$. We see that for any $j \in [k(B), k]$,

$$\begin{aligned}
& \hat{\mathbb{E}}_{\mathscr{D}}\left[(1+\max(0,-\hat{U}_k))\mathbf{1}_{\{\hat{m}(k)=j, \max_{\ell \leq k} \hat{U}_\ell \leq x_1\}}\right] \\
\leq\ & \hat{\mathbb{E}}_{\mathscr{D}}\left[|\hat{U}_{k-j}|\mathbf{1}_{\{\hat{m}(k)=j, \max_{\ell \leq k-j} \hat{U}_\ell \leq x_1\}}\right] + \hat{\mathbb{E}}_{\mathscr{D}}\left[(1+|\hat{U}_{k-j} - \hat{U}_k|)\mathbf{1}_{\{\hat{m}(k)=j, \max_{\ell \leq k-j} \hat{U}_\ell \leq x_1\}}\right].
\end{aligned}$$

By the Markov property at time $k-j$, we have

$$\hat{\mathbb{E}}_{\mathscr{D}}\left[|\hat{U}_{k-j}|\mathbf{1}_{\{\hat{m}(k)=j, \max_{\ell \leq k-j} \hat{U}_\ell \leq x_1\}}\right] \leq c_1 \mathrm{e}^{-c_2 j} \hat{\mathbb{E}}_{\mathscr{D}}\left[|\hat{U}_{k-j}|\mathbf{1}_{\{\max_{\ell \leq k-j} \hat{U}_\ell \leq x_1\}}\right] \leq c_3 \mathrm{e}^{-c_4 j}.$$

The Cauchy-Schwarz inequality shows that

$$\hat{\mathbb{E}}_{\mathscr{D}}\left[(1+|\hat{U}_{k-j} - \hat{U}_k|)\mathbf{1}_{\{\hat{m}(k)=j, \max_{\ell \leq k-j} \hat{U}_\ell \leq x_1\}}\right] \leq c_5 \mathrm{e}^{-c_6 j}.$$

Therefore, $\hat{\mathbb{E}}_{\mathscr{D}}\left[(1+\max(0,-\hat{U}_k))\mathbf{1}_{\{\hat{m}(k) \geq k(B), \max_{\ell \leq k} \hat{U}_\ell \leq x_1\}}\right] \leq c\mathrm{e}^{-c' k(B)}$. On the other hand,

$$\begin{aligned}
& \hat{\mathbb{E}}_{\mathscr{D}}\left[(1+\max(0,-\hat{U}_k))\mathbf{1}_{\{\max_{z' \in \mathbf{int}(\hat{\mathbf{A}}_k)} |z'-\hat{\xi}| \geq B\hat{\mathbf{R}}_k, \max_{\ell \leq k} \hat{U}_\ell \leq x_1, \hat{m}(k) \leq k(B)\}}\right] \\
\leq\ & \hat{\mathbb{E}}_{\mathscr{D}}\left[|\hat{U}_{k-k(B)}|\mathbf{1}_{\{\max_{z' \in \mathbf{int}(\hat{\mathbf{A}}_k)} |z'-\hat{\xi}| \geq B\hat{\mathbf{R}}_k, \max_{\ell \leq k-k(B)} \hat{U}_\ell \leq x_1, \hat{m}(k) \leq k(B)\}}\right] \\
& + \hat{\mathbb{E}}_{\mathscr{D}}\left[(1+|\hat{U}_k - \hat{U}_{k-k(B)}|)\mathbf{1}_{\{\max_{z' \in \mathbf{int}(\hat{\mathbf{A}}_k)} |z'-\hat{\xi}| \geq B\hat{\mathbf{R}}_k, \max_{\ell \leq k-k(B)} \hat{U}_\ell \leq x_1, \hat{m}(k) \leq k(B)\}}\right]
\end{aligned}$$

The Markov property at time $k-k(B)$ implies that

$$\begin{aligned}
& \hat{\mathbb{E}}_{\mathscr{D}}\left[|\hat{U}_{k-k(B)}|\mathbf{1}_{\{\max_{z' \in \mathbf{int}(\hat{\mathbf{A}}_k)} |z'-\hat{\xi}| \geq B\hat{\mathbf{R}}_k, \max_{\ell \leq k-k(B)} \hat{U}_\ell \leq x_1, \hat{m}(k) \leq k(B)\}}\right] \\
\leq\ & \hat{\mathbb{E}}_{\mathscr{D}}\left[|\hat{U}_{k-k(B)}|\mathbf{1}_{\{\max_{\ell \leq k-k(B)} \hat{U}_\ell \leq x_1\}}\right] \hat{\mathbb{P}}\left(\prod_{i=k-k(B)}^{k} \psi_i'(0)^{-1} \geq cB\right) \leq \frac{c_7 \mathrm{e}^{c_8 k(B)}}{B}.
\end{aligned}$$

Similarly, we have by the Cauchy-Schwarz inequality

$$\begin{aligned}&\hat{\mathbb{E}}_{\mathscr{D}}\left[(1+|\hat{U}_k-\hat{U}_{k-k(B)}|)\mathbf{1}_{\{\max_{z'\in \mathbf{int}(\hat{\mathbf{A}}_k)}|z'-\hat{\xi}|\geq B\hat{\mathbf{R}}_k, \max_{\ell\leq k-k(B)}\hat{U}_\ell\leq x_1, \hat{m}(k)\leq k(B)\}}\right]\\ \leq\ & c_9\frac{\mathrm{e}^{c_{10}k(B)}}{B}.\end{aligned}$$

We get

$$\begin{aligned}&\hat{\mathbb{E}}_{\mathscr{D}}\left[(1+\max(0,-\hat{U}_k))\mathbf{1}_{\{\max_{z'\in \mathbf{int}(\hat{\mathbf{A}}_k)}|z'-\hat{\xi}|\geq B\hat{\mathbf{R}}_k, \max_{\ell\leq k}\hat{U}_\ell\leq x_1\}}\right]\\ \leq\ & c\mathrm{e}^{-c'k(B)}+\frac{c_{11}\mathrm{e}^{c_{12}k(B)}}{B}.\end{aligned}\tag{4.4}$$

In view of (4.3), it remains to choose B large enough to complete the proof. □

4.2 Cluster of the extrema

This section is devoted to the proof of the fact that, conditionally on the maximum being unusually high, the other points with values close to the maximum are necessarily at euclidean distance $o(1)$ of the maximum. For future purposes, we will work under the measure $\hat{\mathbb{P}}_{\mathscr{D}}$. Under this measure, the claim is that, on the event in which the spine $\hat{\xi}$ has a high conformal radius at generation n, there is only one cluster of points with high conformal radius. We need some preliminary results. Recall that $a_n=-\frac{3}{2}\ln n$. Denote by $\mathcal{Z}_n^{x,L,B}$ the set of points $z\in\mathscr{D}_n$ such that

$$U_n^z\geq a_n+x,\qquad \max_{\ell\leq n}U_\ell^z\leq 0,\qquad \max_{\ell\in[n/2,n]}U_\ell^z\leq a_n+L,\qquad \max_{z'\in\mathscr{D}}|z'-z|\leq B\mathbf{R}_0^z.\tag{4.5}$$

We let $\mathcal{Z}_n^{x,L}$ be the set of points such that the three first conditions are satisfied.

Recall from (3.2) that

$$D_0^{(0)}=\int_{\mathscr{D}}h_1(-U_0^z)(\mathbf{R}_0^z)^{\beta_c}dz.$$

We know that $\hat{\mathbb{E}}_{\mathscr{D}}\left[\max(0,-\hat{U}_0)\right]=\frac{1}{W_0}\int_{\mathscr{D}}\max(0,-U_0^z)(\mathbf{R}_0^z)^{\beta_c}dz$. Since $h_1(x)\geq c\max(x,0)$, we see that

$$\hat{\mathbb{E}}_{\mathscr{D}}\left[\max(0,-\hat{U}_0)\right]\leq c\frac{D_0^{(0)}}{W_0}.\tag{4.6}$$

Lemma 4.7 *Let $c_0 := (4+2\beta_c)^{-1}$. For any $\varepsilon > 0$, $B \geq 1$, $L \geq 0$ and $x \leq L$, there exists a constant $d(\varepsilon, L, x, B)$ such that for any integer $n \geq 1$ and any bounded simply connected domain $\mathscr{D}$,*

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\exists\,\ell \in [0, \frac{2n}{3}] \,:\, \max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq d\,\hat{\mathbf{R}}_\ell \mathrm{e}^{-c_0 \hat{U}_\ell}, \hat{\xi} \in \mathcal{Z}_n^{x,L,B}\right) \leq \varepsilon n^{-3/2}\left(1 + \frac{D_0^{(0)}}{W_0}\right).$$

Proof. Fix $B \geq 1$, $L \geq 0$ and $x \leq L$. Let $\varepsilon > 0$. Let $\mathscr{D}$ be a bounded simply connected domain. We want to give an upper bound to

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq d\,\hat{\mathbf{R}}_\ell \mathrm{e}^{-c_0 \hat{U}_\ell}, \hat{\xi} \in \mathcal{Z}_n^{x,L,B}\right)$$

for any $d > 1$ and $\ell \leq 2n/3$. We discuss on the value of $-\hat{U}_\ell$. We have

$$\begin{aligned} &\hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq d\,\hat{\mathbf{R}}_\ell \mathrm{e}^{-c_0 \hat{U}_\ell}, \hat{\xi} \in \mathcal{Z}_n^{x,L,B}\right) \\ \leq\; &\sum_{u\in[1,u_c]} \hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq d\,\hat{\mathbf{R}}_\ell \mathrm{e}^{c_0 u}, \hat{\xi} \in \mathcal{Z}_n^{x,L},\, \hat{U}_\ell \in [-u,0]\right) \\ &+ \sum_{u>u_c} \hat{\mathbb{P}}_{\mathscr{D}}\left(B \geq d\,\frac{\hat{\mathbf{R}}_\ell}{\hat{\mathbf{R}}_0} \mathrm{e}^{c_0 u}, \hat{\xi} \in \mathcal{Z}_n^{x,L},\, \hat{U}_\ell \in [-u,0]\right) \end{aligned} \tag{4.7}$$

where $u_c := \max\{u \geq 1 \,:\, \ln(d)^{1/2} + u^{1/2} \leq \ell - 1\}$. Write $u(d)$ for $\ln(d)^{1/2} + u^{1/2}$. First, we find an upper bound for

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq d\,\hat{\mathbf{R}}_\ell \mathrm{e}^{c_0 u}, \hat{\xi} \in \mathcal{Z}_n^{x,L},\, \hat{U}_\ell \in [-u,0]\right),$$

with $u \leq u_c$. By the Markov property at time ℓ and equation (6.3),

$$\begin{aligned} &\hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq d\,\hat{\mathbf{R}}_\ell \mathrm{e}^{c_0 u}, \hat{\xi} \in \mathcal{Z}_n^{x,L},\, \hat{U}_\ell \in [-u,0]\right) \\ \leq\; & c'(x,L) n^{-3/2}(1+u)\hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq d\hat{\mathbf{R}}_\ell \mathrm{e}^{c_0 u},\, \max_{i\leq\ell} \hat{U}_i \leq 0,\, \hat{U}_\ell \in [-u,0]\right) \\ \leq\; & c' n^{-3/2}(1+u)\hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq d\hat{\mathbf{R}}_\ell \mathrm{e}^{c_0 u},\, \max_{i\leq\ell-u(d)} \hat{U}_i \leq 0,\, \hat{U}_{\ell-u(d)} \in [-u-cu(d),0]\right) \end{aligned}$$

for $c = 1 - v_c$ since $(\hat{U}_i)_{i\geq 0}$ has positive jumps bounded by $-v_c$. First, we observe that by the Markov property at time $\ell - u(d)$ and Lemma 2.3,

$$\begin{aligned} &\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{m}(\ell) \geq u(d),\, \max_{i\leq\ell-u(d)} \hat{U}_i \leq 0,\, \hat{U}_{\ell-u(d)} \in [-u-cu(d),0]\right) \\ \leq\; & c_1 \mathrm{e}^{-c_2 u(d)} \hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{i\leq\ell-u(d)} \hat{U}_i \leq 0,\, \hat{U}_{\ell-u(d)} \in [-u-cu(d),0]\right). \end{aligned}$$

By Lemma 2.3, we know that on the event $\hat{m}(\ell) \le \ell - 1$, $\max_{z \in \mathbf{int}(\hat{\mathbf{A}}_\ell)} |z - \hat{\xi}| \le c_3 \hat{\mathbf{R}}_\ell \left(\prod_{j=\ell-\hat{m}(\ell)}^{\ell} \psi_j'(0) \right)^{-1}$. Therefore, by our choice of $u \le u_c$, we have

$$\begin{aligned} & \hat{\mathbb{P}}_{\mathscr{D}} \left(\max_{z \in \mathbf{int}(\hat{\mathbf{A}}_\ell)} |z - \hat{\xi}| \ge d\, \hat{\mathbf{R}}_\ell \mathrm{e}^{c_0 u},\ \max_{i \le \ell - u(d)} \hat{U}_i \le 0,\ \hat{U}_{\ell - u(d)} \in [-u - cu(d), 0], \hat{m}(\ell) \le u(d) \right) \\ \le\ & \hat{\mathbb{P}}_{\mathscr{D}} \left(\prod_{j=\ell-\hat{m}(\ell)}^{\ell} \psi_j'(0) \le \frac{c_4}{d} \mathrm{e}^{-c_0 u},\ \max_{i \le \ell - u(d)} \hat{U}_i \le 0,\ \hat{U}_{\ell - u(d)} \in [-u - cu(d), 0] \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \hat{\mathbb{P}}_{\mathscr{D}} \left(\prod_{j=\ell-u(d)}^{\ell} \psi_j'(0) \le \frac{c_4}{d} \mathrm{e}^{-c_0 u},\ \max_{i \le \ell - u(d)} \hat{U}_i \le 0,\ \hat{U}_{\ell - u(d)} \in [-u - cu(d), 0] \right) \\ \le\ & \frac{c_4 \mathrm{e}^{c_5 u(d)}}{d} \mathrm{e}^{-c_0 u} \hat{\mathbb{P}}_{\mathscr{D}} \left(\max_{i \le \ell - u(d)} \hat{U}_i \le 0,\ \hat{U}_{\ell - u(d)} \in [-u - cu(d), 0] \right) \end{aligned}$$

by Markov property at time $\ell - u(d)$ and Markov inequality. Therefore, we get that

$$\begin{aligned} & \hat{\mathbb{P}}_{\mathscr{D}} \left(\max_{z \in \mathbf{int}(\hat{\mathbf{A}}_\ell)} \mathbf{R}_\ell^z \ge d\, \hat{\mathbf{R}}_\ell \mathrm{e}^{c_0 u},\ \max_{i \le \ell - u(d)} \hat{U}_i \le 0,\ \hat{U}_{\ell - u(d)} \in [-u - cu(d), 0] \right) \\ \le\ & \left(\frac{c_4 \mathrm{e}^{c_5 u(d)}}{d} \mathrm{e}^{-c_0 u} + c_1 \mathrm{e}^{-c_2 u(d)} \right) \hat{\mathbb{P}}_{\mathscr{D}} \left(\max_{i \le \ell - u(d)} \hat{U}_i \le 0,\ \hat{U}_{\ell - u(d)} \in [-u - cu(d), 0] \right). \end{aligned}$$

Equation (6.2) implies that

$$\hat{\mathbb{P}}_{\mathscr{D}} \left(\max_{i \le \ell - u(d)} \hat{U}_i \le 0,\ \hat{U}_{\ell - u(d)} \in [-u - cu(d), 0] \right) \le c' \hat{\mathbb{E}}[1 + \max(0, -\hat{U}_0)](u + u(d))^2 (1 + \ell - u(d))^{-3/2}.$$

It entails that

$$\begin{aligned} & \sum_{u \in [1, u_c]} \hat{\mathbb{P}}_{\mathscr{D}} \left(\max_{z \in \mathbf{int}(\hat{\mathbf{A}}_\ell)} |z - \hat{\xi}| \ge d\, \hat{\mathbf{R}}_\ell \mathrm{e}^{c_0 u}, \hat{\xi} \in \mathscr{Z}_n^{x,L},\ \hat{U}_\ell \in [-u, 0] \right) \\ \le\ & c_5 n^{-3/2} \hat{\mathbb{E}}_{\mathscr{D}}[1 + \max(0, -\hat{U}_0)](1 + \ell)^{-3/2} \left(\frac{\mathrm{e}^{c_4 \ln(d)^{1/2}}}{d} + \mathrm{e}^{-c_2 \ln(d)^{1/2}} \right) (\ln(d))^{1/2} \\ \le\ & \varepsilon n^{-3/2} \left(1 + \frac{D_0^{(0)}}{W_0} \right) (1 + \ell)^{-3/2} \end{aligned}$$

by taking d large enough. Similarly, the Markov property at time ℓ gives for $u > u_c$,

$$\begin{aligned} \hat{\mathbb{P}}_{\mathscr{D}} \left(B \ge d \frac{\hat{\mathbf{R}}_\ell}{\hat{\mathbf{R}}_0} \mathrm{e}^{c_0 u}, \hat{\xi} \in \mathscr{Z}_n^{x,L},\ \hat{U}_\ell \in [-u, 0] \right) & \le c'(x, L) n^{-3/2} (1 + u) \hat{\mathbb{P}}_{\mathscr{D}} \left(B \ge d \frac{\hat{\mathbf{R}}_\ell}{\hat{\mathbf{R}}_0} \mathrm{e}^{c_0 u} \right) \\ & \le c' B n^{-3/2} (1 + u) c_6^{\ell} \frac{\mathrm{e}^{-c_0 u}}{d} \end{aligned}$$

by Markov inequality. If $u > u_c$ then $u \geq ((\ell-1)^2 - \ln(d))/2$. We deduce that

$$\sum_{u>u_c} \hat{\mathbb{P}}_{\mathscr{D}}\left(B \geq d\,\frac{\hat{\mathbf{R}}_\ell}{\hat{\mathbf{R}}_0}\mathrm{e}^{c_0 u}, \hat{\xi} \in \mathcal{Z}_n^{x,L},\, \hat{U}_\ell \in [-u,0]\right) \;\leq\; Bc_7 n^{-3/2}\frac{\mathrm{e}^{-c_8\ell}}{d^{3/4}} \leq \varepsilon n^{-3/2}\mathrm{e}^{-c_8\ell}$$

for d large enough. In view of equation (4.7), it yields that

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq d\,\hat{\mathbf{R}}_\ell \mathrm{e}^{-c_0\hat{U}_\ell}, \hat{\xi} \in \mathcal{Z}_n^{x,L}\right) \leq \varepsilon n^{-3/2}\left(1+\frac{D_0^{(0)}}{W_0}\right)\left((1+\ell)^{-3/2}+\mathrm{e}^{-c_8\ell}\right).$$

Sum over $\ell \in [0, 2n/3]$ to complete the proof. □

For any $i \geq 0$, we recall that $\hat{\rho}_i := \sup_{|z|<1} |\hat{\psi}_i(z)|$.

Lemma 4.8 *Let $L \geq 0$, $x \leq L$ and $\varepsilon > 0$. There exist some $b(\varepsilon, L, x)$ and $d(\varepsilon, L, x)$ (which do not depend on $\mathscr{D}$) such that for any $n \geq 1$,*

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\exists\,\ell \in \left[\frac{2n}{3}, n-b\right] \,:\, \hat{U}_\ell \geq a_n - (n-\ell)^{1/8}, \hat{\xi} \in \mathcal{Z}_n^{x,L}\right) \leq \varepsilon n^{-3/2}\left(\frac{D_0^{(0)}}{W_0}+1\right), \tag{4.8}$$

(4.9)
$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\exists\,\ell \in \left[\frac{2n}{3}, n-b\right] \,:\, \max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq (n-\ell)^d \hat{\mathbf{R}}_\ell, \hat{\xi} \in \mathcal{Z}_n^{x,L}\right) \leq \varepsilon n^{-3/2}\left(\frac{D_0^{(0)}}{W_0}+1\right)$$

and

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\exists\,\ell \in \left[\frac{2n}{3}, n-2b\right] \,:\, \prod_{i=\ell}^{n-b}\hat{\rho}_i \geq \mathrm{e}^{-(n-b-\ell)^{1/2}},\, \hat{\xi} \in \mathcal{Z}_n^{x,L}\right) \leq \varepsilon n^{-3/2}\left(\frac{D_0^{(0)}}{W_0}+1\right).$$

Proof. We prove the first inequality. The probability in the left-hand side of (4.8) is less than

$$\sum_{\ell=2n/3}^{n-b} \hat{\mathbb{P}}\left(\hat{U}_\ell \geq a_n - (n-\ell)^{1/8}, \hat{\xi} \in \mathcal{Z}_n^{x,L}\right). \tag{4.10}$$

The Markov property at time ℓ and equation (6.2) yield that

$$\begin{aligned}
&\hat{\mathbb{P}}\left(\hat{U}_\ell \geq a_n - (n-\ell)^{1/8}, \hat{\xi} \in \mathcal{Z}_n^{x,L}\right)\\
\leq\;& c(L,x)(n-\ell)^{-11/8}\hat{\mathbb{P}}\left(\hat{U}_\ell \geq a_n - (n-\ell)^{1/8}, \max_{i\leq\ell}\hat{U}_i \leq 0, \max_{i\in[n/2,\ell]}\hat{U}_i \leq a_n - L\right)\\
\leq\;& c'(L,x)(n-\ell)^{-9/8}n^{-3/2}\hat{\mathbb{E}}\left[1+\max(0,-\hat{U}_0)\right]
\end{aligned}$$

where the last inequality comes from equation (6.3). Plugging it into (4.10) readily proves equation (4.8). Let us prove the second inequality. The probability in the left-hand side of (4.9) is less than

$$\sum_{\ell=2n/3}^{n-b} \hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq (n-\ell)^d \hat{\mathbf{R}}_\ell, \hat{\xi}\in\mathcal{Z}_n^{x,L}\right). \tag{4.11}$$

Using Lemma 2.3, we observe that

$$\begin{aligned} &\hat{\mathbb{P}}\left(\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} |z-\hat{\xi}| \geq (n-\ell)^d \hat{\mathbf{R}}_\ell,\, \hat{\xi}\in\mathcal{Z}_n^{x,L}\right) \\ \leq\ & \hat{\mathbb{P}}\left(\hat{m}(\ell)\geq d_1\ln(n-\ell), \hat{\xi}\in\mathcal{Z}_n^{x,L}\right) + \hat{\mathbb{P}}\left(\prod_{i=\ell-d_1\ln(n-\ell)}^{\ell} \hat{\psi}_i'(0)^{-1} \geq c(n-\ell)^d, \hat{\xi}\in\mathcal{Z}_n^{x,L}\right). \end{aligned} \tag{4.12}$$

Let us bound the probabilities of the right-hand side. Let E be either the event $\{\hat{m}(\ell)\geq d_1\ln(n-\ell)\}$ or the event $\{\prod_{i=\ell-d_1\ln(n-\ell)}^{\ell}\hat{\psi}_i'(0)^{-1}\geq c(n-\ell)^d\}$. We discuss on the value of $\hat{U}_\ell - a_n$. We see that

$$\hat{\mathbb{P}}\left(E,\hat{\xi}\in\mathcal{Z}_n^{x,L}\right) \leq \sum_{u\geq -\lfloor L\rfloor} \hat{\mathbb{P}}\left(E,\hat{\xi}\in\mathcal{Z}_n^{x,L},\, \hat{U}_\ell\in[a_n-u,a_n-u+1]\right).$$

By the Markov property at time ℓ, we have

$$\begin{aligned} &\hat{\mathbb{P}}\left(E,\hat{\xi}\in\mathcal{Z}_n^{x,L},\, \hat{U}_\ell\in[a_n-u,a_n-u+1]\right) \\ \leq\ & c(L,x)\mathrm{e}^{-c'\frac{u}{\sqrt{n-\ell}}}\hat{\mathbb{P}}\left(E,\max_{i\leq\ell}\hat{U}_i\leq 0,\, \max_{i\in[n/2,\ell]}\hat{U}_i\leq a_n+L,\, \hat{U}_\ell\geq a_n-u\right). \end{aligned}$$

Since the random walk $(\hat{U}_i)_{i\geq 0}$ has bounded positive jumps, we see that

$$\begin{aligned} &\hat{\mathbb{P}}\left(E,\max_{i\leq\ell}\hat{U}_i\leq 0,\, \max_{i\in[n/2,\ell]}\hat{U}_i\leq a_n-L,\, \hat{U}_\ell\geq a_n-u\right) \\ \leq\ & \hat{\mathbb{P}}\Big(E,\max_{i\leq\ell}\hat{U}_i\leq 0,\, \max_{i\in[n/2,\ell]}\hat{U}_i\leq a_n-L,\, \hat{U}_{\ell-d_1\ln(n-\ell)}\geq a_n-u-cd_1\ln(n-\ell)\Big) \\ =:\ & p(\ell,u). \end{aligned}$$

In the case $E=\{\hat{m}(\ell)\geq d_1\ln(n-\ell)\}$, use the Markov property at time $\ell-d_1\ln(n-\ell)$ and Lemma 2.3 to see that, choosing d_1 large enough, $p(\ell,u)$ is less than $(n-\ell)^{-2}$ times the probability

$$\hat{\mathbb{P}}\Big(\max_{i\leq\ell-d_1\ln(n-\ell)}\hat{U}_i\leq 0,\max_{i\in[n/2,\ell-d_1\ln(n-\ell)]}\hat{U}_i\leq a_n+L,\hat{U}_{\ell-d_1\ln(n-\ell)}\geq a_n-u-cd_1\ln(n-\ell)\Big)$$

which is in turn less than $c(L)n^{-3/2}\hat{\mathbb{E}}[1+\max(0,-\hat{U}_0)](1+u+cd_1\ln(n-\ell))^2$ by equation (6.3). In the case $E=\{\prod_{i=\ell-d_1\ln(n-\ell)}^{\ell}\hat{\psi}_i'(0)^{-1}\geq(n-\ell)^d\}$, using the Markov inequality, we can choose d large enough so that the same bounds hold. Putting this together implies that

$$\begin{aligned}&\hat{\mathbb{P}}\left(E,\hat{\xi}\in\mathcal{Z}_n^{x,L},\,\hat{U}_\ell\geq a_n-u\right)\\ \leq\;&\frac{c_1}{(n-\ell)^2}n^{-3/2}\hat{\mathbb{E}}[(1+\max(0,-\hat{U}_0)](1+u+cd_1\ln(n-\ell))^2\mathrm{e}^{-c'\frac{u}{\sqrt{n-\ell}}}.\end{aligned}$$

Summing over $u\geq-\lfloor L\rfloor$ leads to

$$\hat{\mathbb{P}}\left(E,\hat{\xi}\in\mathcal{Z}_n^{x,L}\right)\leq\frac{c}{(n-\ell)^{3/2}}n^{-3/2}\hat{\mathbb{E}}[1+\max(0,-\hat{U}_0)].$$

Recall (4.6). In view of (4.12) then (4.11), the inequality (4.9) is proved. The proof of the third inequality follows the same lines, taking $\ell\leq n-2b$, $E=\{\prod_{i=\ell}^{n-b}\hat{\rho}_i\geq\mathrm{e}^{-(n-b-\ell)^{1/2}}\}$, and applying Markov property at time $n-b$ then ℓ. $\square$

We can now prove the main result of this section.

Lemma 4.9 *(i) Let $\varepsilon>0$, $B\geq1$ $y\in\mathbf{R}$, $L\geq0$ and $x\leq L$. There exists $c(L,x,\varepsilon,y)>0$ (which does not depend on $\mathscr{D}$) such that for any integer $n\geq1$,*

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi}\in\mathcal{Z}_n^{x,L,B},\,\max_{z\notin\mathbf{int}(\hat{\mathbf{A}}_{2n/3})}U_n^z>a_n+y\right)\leq n^{-3/2}\left(\varepsilon\frac{D_0^{(0)}}{W_0}+c\right)$$

and

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi}\in\mathcal{Z}_n^{x,L,B},\,\max_{z\notin\mathbf{int}(\hat{\mathbf{A}}_{2n/3})}\max_{\ell\leq n}U_\ell^z>0\right)\leq n^{-3/2}\left(\varepsilon\frac{D_0^{(0)}}{W_0}+c\right).$$

(ii) Let $\varepsilon>0$, $L\geq0$ and $x\leq L$. We can find $b(\varepsilon,x,L)$ (which does not depend on $\mathscr{D}$) such that for any integer $n\geq1$,

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi}\in\mathcal{Z}_n^{x,L},\,\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_{2n/3})\backslash\mathbf{int}(\hat{\mathbf{A}}_{n-b})}U_n^z>a_n+y\right)\leq\varepsilon n^{-3/2}\left(\frac{D_0^{(0)}}{W_0}+1\right)$$

and

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi}\in\mathcal{Z}_n^{x,L},\,\max_{z\in\mathbf{int}(\hat{\mathbf{A}}_{2n/3})\backslash\mathbf{int}(\hat{\mathbf{A}}_{n-b})}\max_{\ell\leq n}U_\ell^z>0\right)\leq\varepsilon n^{-3/2}\left(\frac{D_0^{(0)}}{W_0}+1\right).$$

Proof. We will prove the lemma for $y = 0$. We first prove (i). By Lemma 4.7, choosing d large enough, we can restrict to the event $E := \{\forall\, \ell \le 2n/3 \,:\, \max_{z\in \mathbf{int}(\hat{\mathbf{A}}_\ell)} \mathbf{R}_\ell^z + |z-\hat{\xi}| \le d\,\hat{\mathbf{R}}_\ell \mathrm{e}^{-\hat{U}_\ell/(4+2\beta_c)}\}$ (recall that $\mathbf{R}_\ell^z \le 4|z-\hat{\xi}| + 4\hat{\mathbf{R}}_\ell$). Let $\ell \in [1, 2n/3]$, and γ a brother loop of $\hat{\mathbf{A}}_\ell$. By Remark 4.2 and Corollary 4.5 with $\mathscr{D} = \mathrm{e}^{-n\tilde{v}_c}\mathbf{int}(\gamma)$, $x = \frac{3}{2}\ln\frac{n-\ell}{n}$, the probability that there exists $z \in \mathbf{int}(\gamma)$ such that $U_n^z \ge a_n$ is less than $c\Phi(\beta_c)^{-\ell}\int_{\mathbf{int}(\gamma)}(1+\max(0,-U_\ell^z)(\mathbf{R}_\ell^z)^{\beta_c}dz$ which is, on the event E, less than $c'(1+\max(0,-\hat{U}_\ell))\mathrm{e}^{\hat{U}_\ell/2}$. Therefore, for any $\ell \in [1, 2n/3]$,

$$\begin{aligned}\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi}\in\mathcal{Z}_n^{x,L},\ \max_{z\in\mathbf{int}(\hat{\mathbf{A}}_{\ell-1})\backslash\mathbf{int}(\hat{\mathbf{A}}_\ell)} U_n^z > a_n,\ E\right) &\le c'\hat{\mathbb{E}}\left[(1-\hat{U}_\ell)\mathrm{e}^{\hat{U}_\ell/2}\mathbf{1}_{\{\hat{\xi}\in\mathcal{Z}_n^{x,L}\}}\right]\\ &\le c\hat{\mathbb{E}}\left[(1-\hat{U}_\ell)^2\mathrm{e}^{\hat{U}_\ell/2}\mathbf{1}_{\{\max_{j\le\ell}\hat{U}_j\le 0\}}\right]n^{-3/2},\end{aligned}$$

for some $c = c(L,x)$ by equation (6.3). Similarly, using Lemma 4.3 instead of Corollary 4.5, we find that

$$\begin{aligned}&\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi}\in\mathcal{Z}_n^{x,L},\ \max_{z\in\mathbf{int}(\hat{\mathbf{A}}_{\ell-1})\backslash\mathbf{int}(\hat{\mathbf{A}}_\ell)}\ \max_{i\in[\ell,n]} U_i^z > 0,\ E\right)\\ \le\ & c\hat{\mathbb{E}}\left[(1-\hat{U}_\ell)\mathrm{e}^{\hat{U}_\ell/2}\mathbf{1}_{\{\max_{j\le\ell}\hat{U}_j\le 0\}}\right]n^{-3/2}.\end{aligned}$$

We sum over $\ell \in [1, 2n/3]$ and we conclude by Lemma B.2 of [1]. Let us turn to the proof of (ii). It is enough to prove that

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi}\in\mathcal{Z}_n^{x,L},\ \max_{z\in\mathbf{int}(\hat{\mathbf{A}}_{2n/3})\backslash\mathbf{int}(\hat{\mathbf{A}}_{n-b})}\ \max_{\ell\le n} U_\ell^z > a_n\right) \le \varepsilon n^{-3/2}\left(\frac{D_0^{(0)}}{W_0}+1\right).$$

By Lemma 4.8, we can restrict to the event

$$E := \left\{\forall\,\ell\in\left[\frac{2n}{3}, n-b\right]\,:\,\hat{U}_\ell \le a_n - (n-\ell)^{1/8},\ \max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)}\mathbf{R}_\ell^z + |z-\hat{\xi}| \le (n-\ell)^d\hat{\mathbf{R}}_\ell\right\}.$$

Let $\ell \in [2n/3, n-b]$, and γ a brother loop of $\hat{\mathbf{A}}_\ell$. By Lemma 4.3, the probability that there is $z \in \mathbf{int}(\gamma)$ such that $\max_{i\in[\ell,n]} U_i^z \ge a_n$ is less than $c\mathrm{e}^{-a_n}\Phi(\beta_c)^{-\ell}\int_{\mathbf{int}(\gamma)}(\mathbf{R}_\ell^z)^{\beta_c}dz$. Therefore, conditionally on $\hat{\xi}$ and $(\hat{A}_i)_{i\ge 0}$, the probability that there is $z \in \mathbf{int}(\hat{\mathbf{A}}_{\ell-1})\backslash\mathbf{int}(\hat{\mathbf{A}}_\ell)$ such that $\max_{i\in[\ell,n]} U_i^z \ge a_n$ is less than $c\mathrm{e}^{-a_n}\Phi(\beta_c)^{-\ell}\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell-1)}(\mathbf{R}_{\ell-1}^z)^{\beta_c}dz$ which is less than $c'\mathrm{e}^{-a_n}\mathrm{e}^{\hat{U}_\ell}(n-\ell)^{d(\beta_c+2)} \le c'(n-\ell)^{c_1}\mathrm{e}^{-(n-\ell)^{1/8}}$ on the event E. It yields that taking b such that

$c' \sum_{i \geq b} i^{c_1} \mathrm{e}^{-i^{1/8}} \leq \varepsilon$, we have

$$\begin{aligned}\hat{\mathbb{P}}_{\mathscr{D}}\left(\max_{z\in \mathbf{int}(\hat{\mathbf{A}}_{2n/3})\backslash \mathbf{int}(\hat{\mathbf{A}}_{n-b})} \max_{i\in[\ell,n]} U_i^z > a_n, \hat{\xi} \in \mathcal{Z}_n^{x,L}, E\right) &\leq \varepsilon\, \hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi} \in \mathcal{Z}_n^{x,L}\right) \\ &\leq c(L,x)\varepsilon n^{-3/2}\left(\frac{D_0^{(0)}}{W_0}+1\right)\end{aligned}$$

by equation (6.3). It completes the proof. □

4.3 The extremal process is close to $\mathcal{P}_{k,n}$

The main results of this section are Corollaries 4.11 and 4.13. Recall that $a_n = -\frac{3}{2}\ln n$ and $I_n(x) := [x + a_n, x + a_n + 1)$. Recall the definition of ν_n^z in (1.4). We first prove that if there is a big maximum in some loop, then necessarily $\mathfrak{G}_{n,\gamma}$ is not far from the maximum. We will use the following lemma, then show the desired result in Corollary 4.11.

Lemma 4.10 *(i) There exists a constant $c > 0$ such that for any $n \geq 1$, any real numbers $x, \alpha \in \mathbf{R}$ with $x \geq \alpha - O(\ln n)$, and any bounded simply connected domain $\mathscr{D}$,*

$$\begin{aligned}&\mathbb{P}_{\mathscr{D}}\left(\exists\, z \in \mathscr{D}_n, \exists\, t \geq 1 \,:\, U_n^z \in I_n(x-t), \max_{\ell\leq n} U_\ell^z \leq \alpha, \max_{\ell\in[n/2,n]} U_\ell^z > a_n + x + t^2\right) \\ \leq\;& c\mathrm{e}^{-x}\int_{\mathscr{D}} (1+\max(0,\alpha - U_0^z))(\mathbf{R}_0^z)^{\beta_c} dz.\end{aligned}$$

(ii) There exists a constant $c > 0$ such that for any $n \geq 1$, any real numbers $x, \alpha \in \mathbf{R}$, $t \geq 1$ and any bounded simply connected domain $\mathscr{D}$,

$$\begin{aligned}&\mathbb{E}_{\mathscr{D}}\left[\int_{\mathscr{D}_n} \nu_n^z \mathbf{1}\left\{U_n^z \in I_n(x-t), \max_{\ell\leq n} U_\ell^z \leq \alpha, \max_{\ell\in[n/2,n]} U_\ell^z \leq a_n + x + t^2\right\} dz\right] \\ \leq\;& c\mathrm{e}^{\delta x}\mathrm{e}^{-\delta t/2}\mathrm{e}^{(1+\delta)a_n}\int_{\mathscr{D}} (1+\max(0,\alpha - U_0^z))(\mathbf{R}_0^z)^{\beta_c} dz.\end{aligned}$$

Proof. By Lemma 4.4, we have for any $t \geq 1$,

$$\begin{aligned}&\mathbb{P}_{\mathscr{D}}\left(\exists\, z \in \mathscr{D}_n \,:\, U_n^z \in I_n(x-t), \max_{\ell\leq n} U_n^z \leq \alpha, \max_{\ell\in[n/2,n]} U_\ell^z > a_n + x + t^2\right) \\ \leq\;& c\mathrm{e}^{-c't^2}\mathrm{e}^{t-x}\int_{\mathscr{D}} (1+\max(0,\alpha - U_0^z))(\mathbf{R}_0^z)^{\beta_c} dz.\end{aligned}$$

Sum over $t \geq 1$ to get (i). To bound the expectation in (ii), we use Lemma 4.1. We get that

$$\begin{aligned}
& \mathbb{E}_{\mathscr{D}}\left[\Phi(\beta_c)^{-n}\int_{\mathscr{D}_n}(\mathbf{R}_n^z)^{\beta_c}\mathrm{e}^{\delta U_n^z}\mathbf{1}_{\{U_n^z\in I_n(x-t),\max_{\ell\leq n}U_\ell^z\leq\alpha,\max_{\ell\in[n/2,n]}U_\ell^z\leq a_n+x+t^2\}}dz\right] \\
\leq\ & W_0\mathrm{e}^{\delta(a_n+x-t+1)}\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{U}_n\in I_n(x-t),\max_{\ell\leq n}\hat{U}_\ell\leq\alpha,\max_{\ell\in[n/2,n]}\hat{U}_\ell\leq a_n+x+t^2\right) \\
\leq\ & c\mathrm{e}^{\delta(a_n+x-t)}t^2n^{-3/2}\int_{\mathscr{D}}(1+\max(0,\alpha-U_0^z))(\mathbf{R}_0^z)^{\beta_c}dz \\
=\ & ct^2\mathrm{e}^{-\delta t}\mathrm{e}^{\delta x}\mathrm{e}^{(1+\delta)a_n}\int_{\mathscr{D}}(1+\max(0,\alpha-U_0^z))(\mathbf{R}_0^z)^{\beta_c}dz
\end{aligned}$$

where we used equation (6.3) in the second inequality. It completes the proof of (ii). □

Corollary 4.11 *Let $\mathscr{D}$ be a bounded simply connected domain. Let $x\in\mathbf{R}$ and $\varepsilon>0$. There exists α large enough such that for any $k\geq 1$ and $n\geq 2k$,*

$$\mathbb{P}_{\mathscr{D}}(\exists\,|\gamma|=k\,:\,\max_{z\in\mathbf{int}(\gamma)}U_n^z\geq a_n+x,\,\mathfrak{G}_{n,\gamma}\leq a_n-\alpha)\leq\varepsilon.$$

Proof. Fix $\mathscr{D}$ some bounded simply connected domain. Let $\varepsilon>0$ and $x\in\mathbf{R}$. By Lemma 4.3, let $x_1\geq 0$ be large enough such that

$$\mathbb{P}_{\mathscr{D}}\left(\max_{k\geq 0,z\in\mathscr{D}_k}U_k^z>x_1\right)\leq\varepsilon/2.$$

Let $k\geq 1$ and γ a loop at generation k. Then, by Lemma 4.10 (i) (and Remark 4.2), take x_2 large enough so that for any $n\geq k+1$,

$$\begin{aligned}
& \mathbb{P}_{\mathscr{D}}\Big(\exists\,z\in\mathbf{int}(\gamma),\,\exists\,t\geq 1\,:\,U_n^z\in I_n(x_2-t),\,\max_{\ell\leq n}U_\ell^z\leq x_1,\,\max_{\ell\in[\frac{n}{2},n]}U_\ell^z>a_n+x_2+t^2\mid\mathcal{F}_k\Big) \\
\leq\ & \eta D_k^+(\gamma)
\end{aligned}$$

where

$$D_k^+(\gamma):=\Phi(\beta_c)^{-k}\int_{\mathbf{int}(\gamma)}h_1(-U_k^z)(\mathbf{R}_k^z)^{\beta_c}dz.$$

Let E be the event which complementary is given by

$$E^c=\{\exists|\gamma|=k,\exists\,z\in\mathbf{int}(\gamma)\ \exists\,t\geq 1\,:\,U_n^z\in I_n(x_2-t),\,\max_{\ell\in[n/2,n]}U_\ell^z>a_n+x_2+t^2\}.$$

We have

$$\mathbb{P}_{\mathscr{D}}(E^c,\,\max_{\ell\leq n,z\in\mathscr{D}_\ell}U_\ell^z\leq x_1)\leq\eta\mathbb{E}_{\mathscr{D}}[D_k^+\mathbf{1}_{\{\max_{\ell\leq k,z\in\mathscr{D}_\ell}U_\ell^z\leq x_1\}}] \tag{4.13}$$

where

$$D_k^+ := \Phi(\beta_c)^{-k} \int_{\mathscr{D}_k} h_1(-U_k^z)(\mathbf{R}_k^z)^{\beta_c} dz.$$

Recall that $\mathfrak{g}_{n,\gamma}$ is chosen in each loop γ of generation k with density proportional to $\nu_n^z dz$. Therefore,

$$\begin{aligned}
&\mathbb{P}_{\mathscr{D}}\left(\exists\,|\gamma| = k \,:\, \max_{z\in \mathbf{int}(\gamma)} U_n^z \geq a_n + x,\, \mathfrak{G}_{n,\gamma} \leq a_n - \alpha, E, \max_{\ell \leq n, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1 \mid \mathcal{F}_k\right) \\
\leq\ & \sum_{|\gamma|=k} \mathbb{E}_{\mathscr{D}}\left[\mathbf{1}_E \mathbf{1}_{\{\max_{\ell\leq n, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1\}} \int_{\mathbf{int}(\gamma)} \frac{\nu_n^{z'}}{\int_{\mathbf{int}(\gamma)} \nu_n^z dz} \mathbf{1}_{\{\max_{z\in\mathbf{int}(\gamma)} U_n^z \geq a_n + x,\, U_n^{z'} \leq a_n - \alpha\}} dz' \mid \mathcal{F}_k\right].
\end{aligned}$$

By Lemma 2.4, if z is such that $U_n^z \geq a_n + x$, then $\int_{\mathbf{int}(\mathbf{A}_n^z)} \nu_n^z dz \geq c(x)\mathrm{e}^{(1+\delta)a_n}$. Therefore, for any loop γ at generation k,

$$\begin{aligned}
&\mathbb{E}_{\mathscr{D}}\left[\mathbf{1}_E \mathbf{1}_{\{\max_{\ell\leq n, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1\}} \int_{\mathbf{int}(\gamma)} \frac{\nu_n^{z'}}{\int_{\mathbf{int}(\gamma)} \nu_n^z dz} \mathbf{1}_{\{\max_{z\in\mathbf{int}(\gamma)} U_n^z \geq a_n + x,\, U_n^{z'} \leq a_n - \alpha\}} dz' \mid \mathcal{F}_k\right] \\
\leq\ & c' \mathrm{e}^{-(1+\delta)a_n} \mathbb{E}_{\mathscr{D}}\left[\mathbf{1}_E \mathbf{1}_{\{\max_{\ell\leq n, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1\}} \int_{\mathbf{int}(\gamma)} \nu_n^{z'} \mathbf{1}_{\{U_n^{z'} \leq a_n - \alpha\}} dz' \mid \mathcal{F}_k\right].
\end{aligned}$$

Lemma 4.10 (ii) and Remark 4.2 imply that we can find α large enough so that

$$\mathbb{E}_{\mathscr{D}}\left[\mathbf{1}_E \mathbf{1}_{\{\max_{\ell\leq n, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1\}} \int_{\mathbf{int}(\gamma)} \nu_n^{z'} \mathbf{1}_{\{U_n^{z'} \leq a_n - \alpha\}} dz' \mid \mathcal{F}_k\right] \leq \eta \mathbf{1}_{\{\max_{\ell\leq k, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1\}} D_k^+(\gamma).$$

Consequently,

$$\begin{aligned}
&\mathbb{P}_{\mathscr{D}}\left(\exists\,|\gamma| = k \,:\, \max_{z\in \mathbf{int}(\gamma)} U_n^z \geq a_n + x,\, \mathfrak{G}_{n,\gamma} \leq a_n - \alpha, E, \max_{\ell \leq n, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1\right) \\
\leq\ & \eta \mathbb{E}_{\mathscr{D}}[\mathbf{1}_{\{\max_{\ell\leq k, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1\}} D_k^+].
\end{aligned}$$

In view of (4.13), we find that

$$\begin{aligned}
&\mathbb{P}_{\mathscr{D}}\left(\max_{\ell\geq 0, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1,\, \exists\,|\gamma| = k \,:\, \max_{z\in \mathbf{int}(\gamma)} U_n^z \geq a_n + x,\, \mathfrak{G}_{n,\gamma} \leq a_n - \alpha\right) \\
\leq\ & 2\eta \mathbb{E}_{\mathscr{D}}[\mathbf{1}_{\{\max_{\ell\leq k, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1\}} D_k^+].
\end{aligned}$$

The expectation can be computed using Lemma 4.1. It gives that

$$\begin{aligned}
&\mathbb{P}_{\mathscr{D}}\left(\max_{\ell\geq 0, z\in\mathscr{D}_\ell} U_\ell^z \leq x_1,\, \exists\,|\gamma| = k \,:\, \max_{z\in \mathbf{int}(\gamma)} U_n^z \geq a_n + x,\, \mathfrak{G}_{n,\gamma} \leq a_n - \alpha\right) \\
\leq\ & 2\eta c(x_1)(W_0 + D_0^{(0)}).
\end{aligned}$$

We chose x_1 such that $\mathbb{P}_{\mathscr{D}}(\max_{k\geq 1, z\in\mathscr{D}_k} U_k^z > x_1) \leq \varepsilon/2$. It remains to take η small enough to complete the proof. $\square$

The next lemma is a preliminary step towards Corollary 4.13 which says that extrema are necessarily very close to some $\mathfrak{g}_{n,\gamma}$.

Lemma 4.12 *(i) Let $\varepsilon > 0$. Fix $B \geq 1$, $L \geq 0$ and $x \leq L$. There exists $c = c(\varepsilon, x, L, B) > 0$ (which does not depend on $\mathscr{D}$) such that for any $\eta > 0$ and any integer $n \geq 1$,*

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi} \in \mathcal{Z}_n^{x,L,B}, \int_{z\notin \mathbf{int}(\hat{\mathbf{A}}_{2n/3})} \nu_n^z \mathbf{1}_{\{\max_{\ell\leq n} U_\ell^z \leq 0, U_n^z \leq a_n + x\}} dz > \eta \mathrm{e}^{(1+\delta)a_n}\right)$$
$$\leq \; n^{-3/2}\left(\varepsilon \frac{D_0^{(0)}}{W_0} + c\frac{1+\eta}{\eta}\right).$$

(ii) Let $\varepsilon > 0$. Fix $L \geq 0$ and $x \leq L$. We can find $b(\varepsilon, L, x)$ (which does not depend on $\mathscr{D}$) such that for any real $\eta \in (0,1)$ and any integer $n \geq 1$,

$$\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi} \in \mathcal{Z}_n^{x,L}, \int_{\mathbf{int}(\hat{A}_{2n/3})\backslash \mathbf{int}(\hat{\mathbf{A}}_{n-b})} \nu_n^z \mathbf{1}_{\{U_n^z \leq a_n + x\}} dz > \eta \mathrm{e}^{(1+\delta)a_n}\right) \leq \frac{\varepsilon}{\eta} n^{-3/2}\left(\frac{D_0^{(0)}}{W_0} + 1\right).$$

Proof. We first prove (i). Fix $\varepsilon > 0$, $B \geq 1$, $L \geq 0$ and $x \leq L$. Let $\ell \in [1, 2n/3]$. By Lemma 4.10 (i) and (ii), and Remark 4.2, the probability that the integral

$$\int_{\mathbf{int}(\hat{\mathbf{A}}_{\ell-1})\backslash \mathbf{int}(\hat{\mathbf{A}}_{\ell})} \nu_n^z \mathbf{1}_{\{\max_{i\leq n} U_i^z \leq 0, U_n^z \leq a_n + x\}} dz$$

is bigger than $\eta \mathrm{e}^{(1+\delta)a_n} \mathrm{e}^{\hat{U}_\ell/4}$ is bounded by

$$c(x)\left(\Phi(\beta_c)^{-\ell} + \Phi(\beta_c)^{-\ell}\frac{\mathrm{e}^{-\hat{U}_\ell/4}}{\eta}\right)\int_{\mathbf{int}(\hat{\mathbf{A}}_\ell)} (1+\max(0, -U_\ell^z))(\mathbf{R}_\ell^z)^{\beta_c} dz.$$

By Lemma 4.7, choosing d large enough, we can restrict to the event $E := \{\forall\, \ell \leq 2n/3 : \max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)} \mathbf{R}_\ell^z + |z-\hat{\xi}| \leq d\hat{\mathbf{R}}_\ell \mathrm{e}^{-c_0\hat{U}_\ell}\}$. On this event, the last bound is smaller than

$$c'(1+\max(0, -\hat{U}_\ell))\mathrm{e}^{\hat{U}_\ell/2}\left(1 + \frac{\mathrm{e}^{-\hat{U}_\ell/4}}{\eta}\right).$$

It yields that

$$\begin{aligned}
&\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi}\in \mathcal{Z}_n^{x,L,B},\, \int_{z\notin \mathbf{int}(\hat{\mathbf{A}}_{2n/3})} \nu_n^z dz > \eta \mathrm{e}^{(1+\delta)a_n},\, E\right)\\
\leq\;& c\frac{1+\eta}{\eta}\sum_{\ell\leq 2n/3} \hat{\mathbb{E}}\left[\mathrm{e}^{\hat{U}_\ell/4}(1-\hat{U}_\ell),\, \hat{\xi}\in \mathcal{Z}_n^{x,L}\right] + \hat{\mathbb{P}}_{\mathscr{D}}\left(\sum_{\ell\leq 2n/3}\mathrm{e}^{\hat{U}_\ell/4}\geq 1,\, \hat{\xi}\in\mathcal{Z}_n^{x,L}\right)\\
\leq\;& c'\frac{1+\eta}{\eta}\sum_{\ell\leq 2n/3} \hat{\mathbb{E}}\left[\mathrm{e}^{\hat{U}_\ell/4}(1-\hat{U}_\ell),\, \hat{\xi}\in \mathcal{Z}_n^{x,L}\right]\\
\leq\;& c\frac{1+\eta}{\eta}\sum_{\ell\leq 2n/3} \hat{\mathbb{E}}\left[(1-\hat{U}_\ell)^2\mathrm{e}^{\hat{U}_\ell/4}\mathbf{1}_{\{\max_{j\leq \ell}\hat{U}_j\leq 0\}}\right] n^{-3/2}
\end{aligned}$$

by equation (6.3). We conclude by Lemma B.2 of [1]. Let us turn to the proof of (ii). By Lemma 4.8, we can restrict to the event

$$E := \left\{\forall\, \ell\in\left[\frac{2n}{3}, n-b\right]\,:\, \hat{U}_\ell \leq a_n-(n-\ell)^{1/8},\, \max_{z\in\mathbf{int}(\hat{\mathbf{A}}_\ell)}|z-\hat{\xi}| + \mathbf{R}_\ell^z \leq (n-\ell)^d\hat{\mathbf{R}}_\ell\right\}.$$

Let $\ell\in[2n/3, n-b]$. Conditionally on $\hat{\xi}$ and $(\hat{\mathbf{A}}_i)_{i\geq 0}$, the expectation of $\int_{\mathbf{int}(\hat{\mathbf{A}}_{\ell-1})\backslash\mathbf{int}(\hat{\mathbf{A}}_\ell)}\nu_n^z\mathbf{1}_{\{U_n^z\leq a_n+x\}}dz$ is less than that of $c\mathrm{e}^{\delta a_n}\Phi(\beta_c)^{-\ell}\int_{\mathbf{int}(\hat{\mathbf{A}}_{\ell-1})}(\mathbf{R}_\ell^z)^{\beta_c}dz$ (using the martingale W_n) which is in turn smaller than $c'\mathrm{e}^{\delta a_n+\hat{U}_\ell}(n-\ell)^{(\beta_c+2)d}$ on the event E. By the Markov inequality, we get that

$$\begin{aligned}
&\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi}\in\mathcal{Z}_n^{x,L},\, E,\, \int_{z\in\mathbf{int}(\hat{A}_{2n/3})\backslash\mathbf{int}(\hat{\mathbf{A}}_{n-b})}\nu_n^z\mathbf{1}_{\{U_n^z\leq a_n+x\}}dz > \eta\mathrm{e}^{(1+\delta)a_n}\right)\\
\leq\;& \frac{c}{\eta}\sum_{\ell=2n/3}^{n-b-1}(n-\ell)^{d'}\hat{\mathbb{E}}_{\mathscr{D}}\left[\mathrm{e}^{-(a_n-\hat{U}_\ell)}, \hat{\xi}\in\mathcal{Z}_n^{x,L},\, E\right]\\
\leq\;& \frac{c}{\eta}\sum_{\ell=2n/3}^{n-b-1}(n-\ell)^{d'}\mathrm{e}^{-(n-\ell)^{1/8}}\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi}\in\mathcal{Z}_n^{x,L}\right)
\end{aligned}$$

Take b such that $c\sum_{i\geq b} i^{2d}\mathrm{e}^{-i^{1/8}}\leq\varepsilon$ and use equation (6.3) to complete the proof. □

Corollary 4.13 *Let $\mathscr{D}$ be a bounded simply connected domain, and $x\in\mathbf{R}$. For any $\varepsilon>0$, there exists b large enough such that for any $k\geq 2b$, $n\geq 2k$,*

$$\mathbb{P}_{\mathscr{D}}\left(\exists\,|\gamma|=k,\, \exists\, z\in\mathbf{int}(\gamma)\backslash\mathbf{int}(\mathbf{A}_{n-b}^{\mathfrak{g}_{n,\gamma}})\,:\, U_n^z\geq a_n+x\right)\leq\varepsilon.$$

Proof. By Corollary 4.11, we can suppose that $\mathfrak{G}_{n,\gamma} \geq a_n - \alpha$ for any $|\gamma| = k$ for which there exists $z \in \mathbf{int}(\gamma)$ such that $U_n^z \geq a_n + x$. By Lemma 4.3 and Remark 4.2,

$$\mathbb{P}_{\mathscr{D}}\left(\max_{\ell \geq k, z \in \mathscr{D}_\ell} U_\ell^z > 0 \mid \mathcal{F}_k\right) \leq cW_k.$$

Since $W_k \to 0$ as $k \to \infty$, we can find k large enough such that

$$\mathbb{P}_{\mathscr{D}}\left(\max_{\ell \geq k, z \in \mathscr{D}_\ell} U_\ell^z > 0\right) \leq \varepsilon.$$

By Lemma 4.4 and Remark 4.2, there exists L large enough such that, for any $|\gamma| = k$,

$$\mathbb{P}_{\mathscr{D}}\left(\exists\, z \in \mathbf{int}(\gamma) \,:\, U_n^z \geq a_n - \alpha,\, \max_{\ell \in [k,n]} U_\ell^z \leq 0,\, \max_{\ell \in [n/2,n]} U_\ell^z \geq a_n + L \mid \mathcal{F}_k\right) \leq \eta D_k^+(\gamma)$$

with

$$D_k^+(\gamma) := \Phi(\beta_c)^{-k} \int_{\mathbf{int}(\gamma)} (1 + \max(0, -U_k^z))(\mathbf{R}_k^z)^{\beta_c} dz.$$

On the event $\{\max_{\ell \geq k, z \in \mathscr{D}_\ell} U_\ell^z \leq 0\}$, we see that $D_k^+ := \sum_{|\gamma|=k} D_k^+(\gamma) = W_k + D_k$ and we know that $\sup_{k \geq 0} W_k + D_k < \infty$, $\mathbb{P}_{\mathscr{D}}$-a.s. We deduce that , taking η small enough,

$$\mathbb{P}_{\mathscr{D}}\left(\exists\, |\gamma| = k, \exists\, z \in \mathbf{int}(\gamma) \,:\, U_n^z \geq a_n - \alpha,\, \max_{\ell \in [k,n]} U_\ell^z \leq 0,\, \max_{\ell \in [n/2,n]} U_\ell^z \geq a_n + L\right) \leq \varepsilon.$$

Together with Corollary 4.6, it implies that we just need to consider the probability

$$\mathbb{P}_{\mathscr{D}}\left(\exists |\gamma| = k, E^{\mathfrak{g}_{n,\gamma}}, \exists\, z \in \mathbf{int}(\gamma) \backslash \mathbf{int}(\mathbf{A}_{n-b}^{\mathfrak{g}_{n,\gamma}}) \,:\, U_n^z \geq a_n + x \mid \mathcal{F}_k\right)$$

where for any $z \in \mathbf{int}(\gamma)$, E^z is the event

$$\left\{\max_{\mathbf{int}(\gamma)} |z' - z| \leq B\mathbf{R}_k^z,\, \max_{\ell \in [k,n]} U_\ell^z \leq 0,\, \max_{\ell \in [n/2,n]} U_\ell^z \leq a_n + L, U_n^z \geq a_n - \alpha\right\}.$$

We have, reasoning on the value of $\mathfrak{g}_{n,\gamma}$,

$$\begin{aligned} &\mathbb{P}_{\mathscr{D}}\left(\exists |\gamma| = k, E^{\mathfrak{g}_{n,\gamma}}, \exists\, z \in \mathbf{int}(\gamma) \backslash \mathbf{int}(\mathbf{A}_{n-b}^{\mathfrak{g}_{n,\gamma}}) \,:\, U_n^z \geq a_n + x \mid \mathcal{F}_k\right) \\ \leq\ & \mathbb{E}_{\mathscr{D}}\left[\int_{\mathscr{D}_n} \frac{\nu_n^z}{\int_{\mathbf{int}(\mathbf{A}_k^z)} \nu_n^{z'} dz'} \mathbf{1}_{\{\exists\, z' \in \mathbf{int}(\mathbf{A}_k^z) \backslash \mathbf{int}(\mathbf{A}_{n-b}^z) : U_n^{z'} \geq a_n + x\}} \mathbf{1}_{E^z} dz \mid \mathcal{F}_k\right]. \end{aligned}$$

We notice that if there is $z' \in \mathbf{int}(\mathbf{A}_k^z)$ such that $U_n^{z'} \geq a_n + x$, then $\int_{\mathbf{int}(\mathbf{A}_k^z)} \nu_n^{z'} dz' \geq c(x)\mathrm{e}^{(1+\delta)(a_n)}$. On the event E^z, we have $\nu_n^z \leq c(L)\Phi(\beta_c)^{-n}(\mathbf{R}_n^z)^{\beta_c}\mathrm{e}^{\delta a_n}$. It yields that

$$\begin{aligned} &\mathbb{P}_{\mathscr{D}}\left(\exists |\gamma| = k, E^{\mathfrak{g}_{n,\gamma}}, \exists\, z \in \mathbf{int}(\gamma) \backslash \mathbf{int}(\mathbf{A}_{n-b}^{\mathfrak{g}_{n,\gamma}}) \,:\, U_n^z \geq a_n + x \mid \mathcal{F}_k\right) \\ \leq\ & c\mathrm{e}^{-a_n}\mathbb{E}_{\mathscr{D}}\left[\int_{\mathscr{D}_n} \Phi(\beta_c)^{-n}(\mathbf{R}_n^z)^{\beta_c} \mathbf{1}_{\{\exists\, z' \in \mathbf{int}(\mathbf{A}_k^z) \backslash \mathbf{int}(\mathbf{A}_{n-b}^z) : U_n^{z'} \geq a_n + x\}} \mathbf{1}_{E^z} dz \mid \mathcal{F}_k\right] \\ =\ & c\mathrm{e}^{-a_n} \sum_{|\gamma|=k} W_k(\gamma) \hat{\mathbb{P}}_{\mathbf{int}(\mathrm{e}^{-k\bar{v}_c}\gamma)}\left(\exists\, z' \notin \mathbf{int}(\hat{\mathbf{A}}_{n-k-b}) \,:\, U_{n-k}^{z'} \geq a_n + x, \tilde{E}^{\hat{\xi}}\right) \end{aligned}$$

by Remark 4.2 and Lemma 4.1, where we set

$$W_k(\gamma) := \Phi(\beta_c)^{-k} \int_{\mathbf{int}(\gamma)} (\mathbf{R}_k^z)^{\beta_c} dz$$

and for any $z \in \mathbf{int}(\mathrm{e}^{-k\tilde{v}_c}\gamma)$, $\tilde{E}^z$ is the event

$$\left\{ \max_{\mathbf{int}(\mathrm{e}^{-k\tilde{v}_c}\gamma)} |z' - z| \leq B\mathbf{R}_0^z,\ \max_{\ell \in [0,n-k]} U_\ell^z \leq 0,\ \max_{\ell \in [n/2-k,n-k]} U_\ell^z \leq a_n + L, U_{n-k}^z \geq a_n - \alpha \right\}.$$

By Lemma 4.9, we see that for b large enough

$$\hat{\mathbb{P}}_{\mathbf{int}(\mathrm{e}^{-k\tilde{v}_c}\gamma)} \left(\exists\, z' \notin \mathbf{int}(\hat{\mathbf{A}}_{n-k-b}) \,:\, U_{n-k}^{z'} \geq a_n + x, \tilde{E}^{\hat{\xi}} \right) \leq n^{-3/2} \left(\varepsilon \frac{D_k^+(\gamma)}{W_k(\gamma)} + c \right).$$

It gives that

$$\mathbb{P}_{\mathscr{D}} \left(\exists |\gamma| = k, E^{\mathfrak{g}_n,\gamma}, \exists\, z \in \mathbf{int}(\gamma) \backslash \mathbf{int}(\mathbf{A}_{n-b}^{\mathfrak{g}_n,\gamma}) \,:\, U_n^z \geq a_n + x \mid \mathcal{F}_k \right) \leq c \left(\varepsilon D_k^+ + c W_k \right)$$

Recall that $W_k \to 0$ as $k \to \infty$. Moreover, on the event $\{\max_{\ell \geq k, z \in \mathscr{D}_\ell} U_\ell^z \leq 0\}$, $D_k^+ = W_k + D_k$ which has a finite limit almost surely. We deduce the lemma. □

5 Convergence in law of the extremal process

For any integer $n \geq 1$, and any $z \in \mathscr{D}_n$, we defined the decoration $\mathcal{Q}_n^z$ in the introduction. Let $\mathfrak{g}_n$ be the random point in $\mathscr{D}_n$ with density proportional to

$$\nu_n^z = (\mathbf{R}_n^z)^{\beta_c} \Phi(\beta_c)^{-n} \mathrm{e}^{\delta U_n^z}.$$

Let $\mathfrak{G}_n := U_n^{\mathfrak{g}_n}$. Recall that $a_n := -\frac{3}{2} \ln n$. First, we want to know the distribution of $(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n})$ conditionally on the event $\{\mathfrak{G}_n \geq a_n + x\}$. Let F be some bounded continuous function on the space $\mathbf{R} \times \Omega_{\mathscr{D}}^p$ for some integer p. We extend F on $\mathbf{R} \times \Omega_{\mathscr{D}}^{\mathbb{N}}$ by looking at the projection on the space $\mathbf{R} \times \Omega_{\mathscr{D}}^p$. We are interested in the quantity

$$\mathbb{E}_{\mathscr{D}} \left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right) \mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x\}} \right] = \mathbb{E}_{\mathscr{D}} \left[\int_{\mathscr{D}_n} F\left(U_n^z - a_n, \mathcal{Q}_n^z\right) \mathbf{1}_{\{U_n^z \geq a_n + x\}} \frac{\nu_n^z}{\int_{\mathscr{D}_n} \nu_n^{z'} dz'} dz \right].$$

Consider the family of nested loops $\hat{\Gamma}(0)$ in the unit disk $\mathbb{U}$, as defined in Section 2.1. Recall that $\hat{\mathbb{P}}_{\mathbb{U}}^0$ is the probability distribution $\hat{\mathbb{P}}_{\mathbb{U}}$ conditioned on $\hat{\xi} = 0$. In particular, under $\hat{\mathbb{P}}_{\mathbb{U}}^0$, $\hat{\Gamma}$

has the distribution of $\hat{\Gamma}(0)$. Write $\hat{\mathcal{Q}}_n$ for $\mathcal{Q}_n^{\hat{\xi}}$. Recall that $\hat{\rho}_i := \max_{|z|<1} |\hat{\psi}_i(z)|$. We call σ^2 the variance of $\hat{U}_1$ under $\hat{\mathbb{P}}^0_{\mathbb{U}}$, and C_1, C_2 the constants which verify

$$\hat{\mathbb{P}}^0_{\mathbb{U}}\left(\max_{\ell\leq n} \hat{U}_\ell \leq 0\right) \sim_{n\to\infty} \frac{C_1}{n^{1/2}}, \qquad \hat{\mathbb{P}}^0_{\mathbb{U}}\left(\min_{\ell\leq n} \hat{U}_\ell \geq 0\right) \sim_{n\to\infty} \frac{C_2}{n^{1/2}}. \tag{5.1}$$

Define under $\hat{\mathbb{P}}^0_{\mathbb{U}}$, for any real $L \geq 0$, any integers $\zeta > b \geq p$, and any real y,

$$\begin{aligned} &K_{F,x}(L,\zeta,b,y) \\ := \ &\hat{\mathbb{E}}^0_{\mathbb{U}}\left[\frac{\mathrm{e}^{\delta \hat{U}_\zeta} F\left(y+\hat{U}_\zeta, \hat{\mathcal{Q}}_\zeta\right)}{\int_{\mathbf{int}(\hat{\mathbf{A}}_{\zeta-b})} \nu_\zeta^z dz} \mathbf{1}\left\{\max_{\ell\leq\zeta} \hat{U}_\ell \leq L-y, \hat{U}_\zeta \geq x-y, \prod_{j=1}^{\zeta-b} \hat{\rho}_j \leq \mathrm{e}^{-(\zeta-b)^{1/2}}\right\}\right] \end{aligned} \tag{5.2}$$

and

$$\tilde{K}_{F,x}\left(L,\zeta,b,y\right) := \frac{C_1 C_2 \sqrt{\pi}}{\sigma\sqrt{2}} K_{F,x}(L,\zeta,b,y). \tag{5.3}$$

Recall that $D_k^{(\alpha)}$ was defined in (3.2). We have

$$D_0^{(0)} = \int_{\mathscr{D}} h_1\left(-U_0^z\right)\left(\mathbf{R}_0^z\right)^{\beta_c} dz.$$

For any $B \geq 1$, we write

$$D_{0,B}^{(0)} = \int_{\mathscr{D}} h_1\left(-U_0^z\right)\left(\mathbf{R}_0^z\right)^{\beta_c} \mathbf{1}_{\{\max_{\mathscr{D}} |z'-z| \leq B\mathbf{R}_0^z\}} dz. \tag{5.4}$$

We let h_2 be the renewal function associated to $(\hat{U}_i)_{i\geq 0}$, i.e. $h_2(x) = 0$ if $x < 0$, $h_2(0) = 1$, and for any $x > 0$,

$$h_2(x) := \sum_{k\geq 0} \hat{\mathbb{P}}^0_{\mathscr{D}}\left(\min_{i\leq k} \hat{U}_i \geq -x\right). \tag{5.5}$$

We will prove the following proposition.

Proposition 5.1 *For any $\varepsilon > 0$, $x \in \mathbf{R}$ and $B \geq 1$, there exist $L_0 \geq \max(x,0)$ and $b_0, d \geq 1$ such that for any $L \geq L_0$, we can find $c(\varepsilon,x,L,B) > 0$ which verifies: for any bounded simply connected domain $\mathscr{D}$, any integers $b \geq b_0$ and $\zeta \geq d+b$,*

$$\begin{aligned} &\liminf_{n\to\infty} \mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x\}}\right] \\ \geq \ &(1-\varepsilon) D_{0,B}^{(0)} \int_{(-\infty,L]} \mathrm{e}^{-y} \tilde{K}_{F,x}(L,\zeta,b,y) h_2(L-y) dy - \left(\varepsilon D_0^{(0)} + cW_0\right) \end{aligned}$$

and

$$\limsup_{n\to\infty} \mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right) \mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x\}}\right]$$
$$\leq \quad (1+\varepsilon) D_0^{(0)} \int_{(-\infty, L]} \mathrm{e}^{-y} \tilde{K}_{F,x}(L, \zeta, b, y) h_2(L-y) dy + (\varepsilon D_0^{(0)} + cW_0).$$

The next step is to understand the limit of $\int_{(-\infty,L]} \mathrm{e}^{-y} \tilde{K}_{F,x}(L,\zeta,b,y) h_2(L-y) dy$ when ζ, then b then L go to infinity. We introduce some notation.

Let P be some probability measure. Recall that $\mathcal{C}$ is the family of non-nested disjoint loops in the unit disk $\mathbb{U}$, and ψ is the conformal map from $\mathbb{U}$ to the interior of the loop that surrounds 0, such that $\psi(0) = 0$ and $\psi'(0) > 0$. Under P, we (re-)introduce $(\hat{\mathcal{C}}, \hat{\psi})$ having Radon-Nikodym derivative $\Phi(\beta_c)^{-1}(\psi'(0))^{\beta_c}$ with respect to the law of $(\mathcal{C}, \psi)$. Let $(\hat{\mathcal{C}}_k, \hat{\psi}_k)_{k\geq 0}$ be independent and distributed as $(\hat{\mathcal{C}}, \hat{\psi})$. The loop in $\hat{\mathcal{C}}_k$ that surrounds 0 is denoted by $\hat{\mathcal{C}}_k(0)$. We are going to reconstruct loops seen from 0, layer by layer, backwards in time. Let $\hat{\mathcal{O}}_1 := \hat{\mathcal{C}}_1$. Consider $\hat{\mathcal{C}}_2$, and map $\hat{\mathcal{O}}_1$ into the interior of $\hat{\mathcal{C}}_2(0)$ with the map $\hat{\psi}_2$. In each brother loop of $\hat{\mathcal{C}}_2(0)$ in $\hat{\mathcal{C}}_2$, draw an independent copy of Γ up to the first generation. Call $\hat{\mathcal{O}}_2$ the result. Step 2 is complete. At step k, map $\hat{\mathcal{O}}_{k-1}$ into $\hat{\mathcal{C}}_k(0)$ with the map $\hat{\psi}_k$. Fill each brother loop of $\hat{\mathcal{C}}_k(0)$ in $\hat{\mathcal{C}}_k$ with an independent copy of Γ up to generation $k-1$. Call $\hat{\mathcal{O}}_k$ the result. It ends the iteration. We let $\hat{\mathcal{O}}$ be the family $(\hat{\mathcal{O}}_k,\, k \geq 1)$, see Figure 3. For any integer $k \geq 1$, we denote by $\hat{\mathbf{S}}_k$ (resp. $\mathbf{S}_k^z$) the conformal radius seen from 0 (resp. from z) of the outermost loop that surrounds 0 (resp. z) in $\hat{\mathcal{O}}_k$. We define $\hat{V}_k := (\beta_c+2)\ln \hat{\mathbf{S}}_k - kv_c$ and $V_k^z := (\beta_c+2)\ln \mathbf{S}_k^z - kv_c$. We naturally associate a filtration $(\hat{\mathcal{G}}_k, k \in [0,\infty])$.

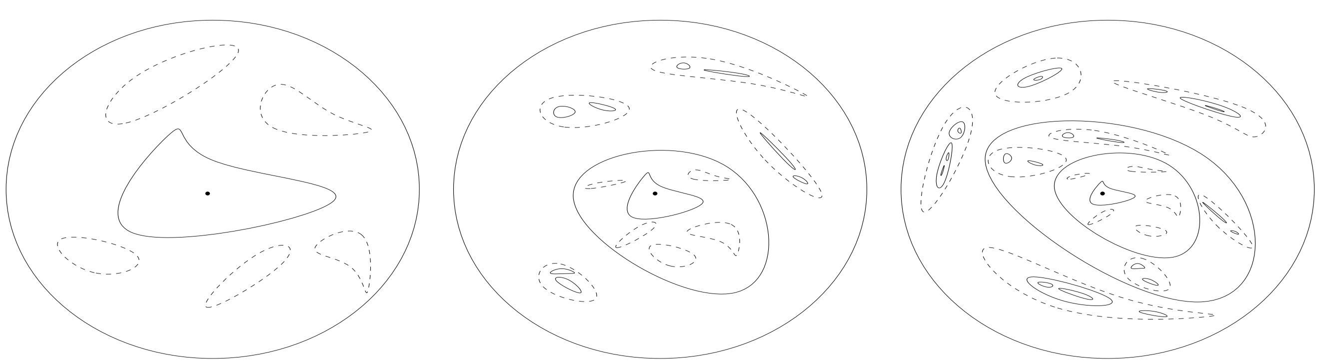

Figure 3: Construction of $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_2$ and $\hat{\mathcal{O}}_3$.

We define the descending ladder time and height processes $(\hat{T}_\ell, \hat{H}_\ell)_{\ell\geq 0}$ by $\hat{T}_0 = \hat{H}_0 := 0$, and for $\ell \geq 1$,

$$\begin{aligned}\hat{T}_\ell &:= \min\{i \geq \hat{T}_{\ell-1} : \hat{V}_i < \hat{H}_{\ell-1}\},\\ \hat{H}_\ell &:= \hat{V}_{\hat{T}_\ell}.\end{aligned}$$

For any integer $\ell \geq 0$, we define a probability measure $\mathrm{P}^{(\ell)}$ under which the random walk $(\hat{V}_k)_{k\geq 0}$ has the following distribution: Until the ℓ-th descending ladder time $\hat{T}_\ell$, $\hat{V}_k$ has the same distribution as under P. After that time, the random walk has the distribution of the random walk conditioned to stay above 0, translated by $\hat{H}_\ell$. In other words, $\mathrm{P}^{(\ell)}$ is such that, for any $k \geq 1$, (recall that h_2 is defined in (5.5)),

$$\frac{d\mathrm{P}^{(\ell)}}{d\mathrm{P}}\bigg|_{\hat{\mathcal{G}}_k} := \mathbf{1}_{\{\hat{T}_\ell \geq k\}} + \mathbf{1}_{\{\hat{T}_\ell < k\}} h_2(\hat{\mathbf{S}}_k + \hat{H}_\ell).$$

For any $z \in \mathbb{U}$ and any integer $j \geq 0$, we define

$$\Delta_j^z := (\beta_c + 2) \sum_{i \geq j+1} \ln \frac{|(\hat{\psi}_i)'(\hat{\psi}_{i-1} \circ \ldots \circ \hat{\psi}_{j+1}(z))|}{(\hat{\psi}_i)'(0)}$$

then

$$\mathscr{I} := \int_{\mathbb{U}} (\mathbf{S}_0^z)^{\beta_c+\delta(\beta_c+2)} \mathrm{e}^{(1+\delta)\Delta_0^z} dz + \sum_{k\geq 1} \frac{1}{(\hat{\mathbf{S}}_k)^{(1+\delta)(\beta_c+2)}} \int_{\mathbb{U}\setminus\hat{\psi}_k(\mathbb{U})} (\mathbf{S}_k^z)^{\beta_c+\delta(\beta_c+2)} \mathrm{e}^{(1+\delta)\Delta_k^z} dz.$$

We are now able to construct the decoration $\mathcal{Q}$. The measure $\mathcal{M}$ is given by

$$\int_{\Omega_{\mathbb{U}}^{\mathbb{N}}} f(\mathcal{Q}) \mathcal{M}(d\mathcal{Q}) = \sum_{\ell\geq 0} \mathrm{E}^{(\ell)} \left[\frac{f\left(\hat{\mathcal{O}}\right)}{\mathscr{I}} \right] \tag{5.6}$$

for any nonnegative measurable function f on the space $\Omega_{\mathbb{U}}^{\mathbb{N}}$.

Proposition 5.2 *We have*

$$\lim_{L\to\infty} \lim_{b\to\infty} \lim_{\zeta\to\infty} \int_{(-\infty,L]} \mathrm{e}^{-y} K_{F,x}(L,\zeta,b,y) h_2(L-y) dy = \int_{[x,+\infty)\times\Omega_{\mathbb{U}}^{\mathbb{N}}} F(y,\mathcal{Q}) \mathrm{e}^{-y} dy \mathcal{M}(d\mathcal{Q}).$$

Let

$$C_{F,x} := \frac{C_1 C_2 \sqrt{\pi}}{\sigma\sqrt{2}} \int_{[x,+\infty)\times\Omega_{\mathbb{U}}^{\mathbb{N}}} F(y,\mathcal{Q}) \mathrm{e}^{-y} dy \mathcal{M}(d\mathcal{Q}). \tag{5.7}$$

Corollary 5.3 *Let $x \in \mathbf{R}$, $\varepsilon > 0$ and $B \geq 1$. Recall (5.4). There exists a constant $c(\varepsilon, x, B) > 0$ such that for any bounded simply connected domain $\mathscr{D}$,*

$$\limsup_{n\to\infty} \left| \mathbb{E}_{\mathscr{D}} \left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n} \right) \mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x\}} \right] - D_0^{(0)} C_{F,x} \right| \leq \varepsilon D_0^{(0)} + cW_0 + C_{F,x}(D_0^{(0)} - D_{0,B}^{(0)}).$$

Proof. Let $x \in \mathbf{R}$. Recall that $\tilde{K}(L, \zeta, b, y) := \frac{C_1 C_2 \sqrt{\pi}}{\sigma\sqrt{2}} K(L, \zeta, b, y)$. Write

$$C(L) := \lim_{b\to\infty} \lim_{\zeta\to\infty} \int_{(-\infty, L]} \mathrm{e}^{-y} \tilde{K}_{F,x}(L, \zeta, b, y) h_2(L-y) dy,$$

hence $C_{F,x} = \lim_{L\to\infty} C(L)$ by Proposition 5.2. Let $\varepsilon > 0$ and $B \geq 1$. Let L be as in Proposition 5.1, large enough so that $|C_{F,x} - C(L)| < \varepsilon$. Let b, ζ be as in Proposition 5.1, taken large enough so that

$$\left| \int_{(-\infty, L]} \mathrm{e}^{-y} \tilde{K}_{F,x}(L, \zeta, b, y) h_2(L-y) dy - C(L) \right| \leq \varepsilon.$$

By Proposition 5.1, there exists $c(\varepsilon, x, L, B)$ such that, for $n \to \infty$,

$$\begin{aligned} &\left| \mathbb{E}_{\mathscr{D}} \left[F\left(\mathcal{Q}_k^{\mathfrak{g}_n}, \mathfrak{G}_n - a_n \right) \mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x\}} \right] - D_0^{(0)} \int_{(-\infty, L]} \mathrm{e}^{-y} \tilde{K}_{F,x}(L, \zeta, b, y) h_2(L-y) dy \right| \\ \leq\ & (\varepsilon D_0^{(0)} + D_0^{(0)} - D_{0,B}^{(0)}) \int_{(-\infty, L]} \mathrm{e}^{-y} \tilde{K}_{F,x}(L, \zeta, b, y) h_2(L-y) dy + \varepsilon D_0^{(0)} + cW_0. \end{aligned}$$

By our choice of L, b, ζ, we have $\int_{(-\infty, L]} \mathrm{e}^{-y} \tilde{K}_{F,x}(L, \zeta, b, y) h_2(L-y) dy \leq C_{F,x} + 2\varepsilon$, hence, for $n \to \infty$,

$$\begin{aligned} &\left| \mathbb{E}_{\mathscr{D}} \left[F\left(\mathcal{Q}_n^{\mathfrak{g}_n}, \mathfrak{G}_n - a_n \right) \mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x\}} \right] - D_0^{(0)} \int_{(-\infty, L]} \mathrm{e}^{-y} \tilde{K}_{F,x}(L, \zeta, b, y) h_2(L-y) dy \right| \\ \leq\ & (\varepsilon D_0^{(0)} + D_0^{(0)} - D_{0,B}^{(0)})(C_{F,x} + 3\varepsilon) + cW_0. \end{aligned}$$

We deduce that

$$\limsup_{n\to\infty} \left| \mathbb{E}_{\mathscr{D}} \left[F\left(\mathcal{Q}_n^{\mathfrak{g}_n}, \mathfrak{G}_n - a_n \right) \mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x\}} \right] - D_0^{(0)} C_{F,x} \right| \leq (\varepsilon D_0^{(0)} + D_0^{(0)} - D_{0,B}^{(0)})(C_{F,x} + 5\varepsilon) + cW_0.$$

It implies the lemma. □

This section is organized as follows. We prove Proposition 5.1 in Section 5.1, then Proposition 5.2 in Section 5.2. Finally, we show in Section 5.3 how we can deduce Theorem 1.4 from Corollary 5.3.

5.1 Proof of Proposition 5.1

We work under the probability measure $\hat{\mathbb{P}}_{\mathscr{D}}$. Recall that $\hat{\rho}_i := \sup_{|z|<1} |\hat{\psi}_i(z)|$. For any $i \geq 0$, we define $\hat{\Psi}_i$ as the conformal bijection mapping $\mathbb{U}$ to $\mathbf{int}(\hat{\mathbf{A}}_i)$, with $\hat{\Psi}_i(0) = \hat{\xi}$, and $\hat{\Psi}_i'(0) > 0$. We see that $\hat{\Psi}_i = \hat{\psi}_0 \circ \ldots \circ \hat{\psi}_i$.

Define, for any $z \in \mathbb{U}$, and $n \geq \zeta \geq 1$,

$$\begin{aligned} \tilde{U}^z_{\zeta,n} &:= U_n^{\hat{\Psi}_{n-\zeta}(z)} - (\beta_c + 2)\ln|\hat{\Psi}'_{n-\zeta}(z)| + (n-\zeta)v_c \\ &= U_n^{\hat{\Psi}_{n-\zeta}(z)} - \hat{U}_{n-\zeta} - (\beta_c+2)\ln\frac{|\hat{\Psi}'_{n-\zeta}(z)|}{\hat{\Psi}'_{n-\zeta}(0)}. \end{aligned} \tag{5.8}$$

Observe that $(\tilde{U}^z_{\zeta,n})_{z\in\mathbb{U}}$ is distributed as $(U^z_\zeta)_{z\in\mathbb{U}}$ under $\hat{\mathbb{P}}^0_{\mathbb{U}}$. Similarly, let

$$\tilde{\mathbf{R}}^z_{\zeta,n} := \frac{\mathbf{R}_n^{\hat{\Psi}_{n-\zeta}(z)}}{|\Psi'_{n-\zeta}(z)|}; \qquad \tilde{\nu}^z_{\zeta,n} := \Phi(\beta_c)^{-\zeta}(\tilde{\mathbf{R}}^z_{\zeta,n})^{\beta_c} \mathrm{e}^{\delta \tilde{U}^z_{\zeta,n}}.$$

We see that $(\tilde{\mathbf{R}}^z_{\zeta,n})_{z\in\mathbb{U}}$, $(\tilde{\nu}^z_{\zeta,n})_{z\in\mathbb{U}}$ are distributed as $(\mathbf{R}^z_{\zeta,n})_{z\in\mathbb{U}}$, $(\nu^z_{\zeta,n})_{z\in\mathbb{U}}$ under $\hat{\mathbb{P}}^0_{\mathbb{U}}$. Mapping $\mathbf{int}(\hat{\mathbf{A}}_{n-\zeta})$ into $\mathbb{U}$ with the map $\hat{\Psi}^{-1}_{n-\zeta}$, we check that

$$\int_{\mathbf{int}(\hat{\mathbf{A}}_{n-b})} \nu^z_n dz = \mathrm{e}^{(1+\delta)\hat{U}_{n-\zeta}} \int_{\hat{\psi}_{n-\zeta+1}\circ\ldots\circ\hat{\psi}_{n-b}(\mathbb{U})} \tilde{\nu}^z_{\zeta,n} \left(\frac{|\hat{\Psi}'_{n-\zeta}(z)|}{\hat{\Psi}'_{n-\zeta}(0)}\right)^{(1+\delta)(\beta_c+2)} dz.$$

By Koebe's distortion Theorem, we have, $||\hat{\Psi}'_{n-\zeta}(z)| - \hat{\Psi}'_{n-\zeta}(0)| \leq c\hat{\Psi}'_{n-\zeta}(0)|z|$ if $|z| < 1/2$. We deduce the following lemma.

Lemma 5.4 *Let $\varepsilon > 0$. There exists d (which does not depend on $\mathscr{D}$) such that for any $\zeta - b \geq d$ and any $n \geq \zeta$, we have $\hat{\mathbb{P}}_{\mathscr{D}}$-almost surely on the event $\{\prod_{i=n-\zeta}^{n-b} \hat{\rho}_i \leq \mathrm{e}^{-(\zeta-b)^{1/2}}\}$,*

$$\int_{\mathbf{int}(\hat{\mathbf{A}}_{n-b})} \nu^z_n dz \geq (1-\varepsilon)\mathrm{e}^{(1+\delta)\hat{U}_{n-\zeta}} \int_{\hat{\psi}_{n-\zeta+1}\circ\ldots\circ\hat{\psi}_{n-b}(\mathbb{U})} \tilde{\nu}^z_{\zeta,n} dz$$

and

$$\int_{\mathbf{int}(\hat{\mathbf{A}}_{n-b})} \nu^z_n dz \leq (1+\varepsilon)\mathrm{e}^{(1+\delta)\hat{U}_{n-\zeta}} \int_{\hat{\psi}_{n-\zeta+1}\circ\ldots\circ\hat{\psi}_{n-b}(\mathbb{U})} \tilde{\nu}^z_{\zeta,n} dz.$$

Recall the definition of $\tilde{K}$ and h_2 in (5.3) and (5.5) . We first prove the lower bound of Proposition 5.1.

Lemma 5.5 *Let $B \geq 1$, $L \geq 0$ and $x \leq L$. For any $\varepsilon > 0$, there exists $c(\varepsilon, L, x, B) > 0$ and $b_0, d > 0$ such that for any bounded simply connected domain $\mathscr{D}$, for any $b \geq b_0$ and $\zeta \geq d + b$,*

$$\begin{aligned} &\liminf_{n\to\infty} \mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x, \max_{\mathscr{D}}|z-\mathfrak{g}_n| \leq B\mathbf{R}_0^{\mathfrak{g}_n}\}}\right] \\ \geq\ &(1-\varepsilon)D_{0,B}^{(0)}\int_{(-\infty,L]} \mathrm{e}^{-y}\tilde{K}_{F,x}(L,\zeta,b,y)h_2(L-y)dy - \left(\varepsilon D_0^{(0)} + cW_0\right). \end{aligned}$$

Proof. Fix $\varepsilon > 0$, $B \geq 1$, $L \geq 0$ and $x \leq L$. Recall the definition of $\mathcal{Z}_n^{x,L}$ in (4.5) . We observe that

$$\mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x, \max_{\mathscr{D}}|z-\mathfrak{g}_n| \leq B\mathbf{R}_0^{\mathfrak{g}_n}\}}\right] \geq \mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{g}_n \in \mathcal{Z}_n^{x,L,B}\}}\right].$$

By Lemma 4.1, we have that

$$\begin{aligned} \mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{g}_n \in \mathcal{Z}_n^{x,L,B}\}}\right] &= \mathbb{E}_{\mathscr{D}}\left[\int_{\mathscr{D}_n} F\left(U_n^z - a_n, \mathcal{Q}_n^z\right)\mathbf{1}_{\{z \in \mathcal{Z}_n^{x,L,B}\}}\frac{\nu_n^z}{\int_{\mathscr{D}}\nu_n^{z'}dz'}dz\right] \\ &= W_0\hat{\mathbb{E}}_{\mathscr{D}}\left[\frac{\mathrm{e}^{\delta\hat{U}_n}}{\int_{\mathscr{D}_n}\nu_n^z dz}F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}_{\{\hat{\xi} \in \mathcal{Z}_n^{x,L,B}\}}\right]. \end{aligned}$$

In view of the definition of ν in (1.4) and of Lemma 2.4, we see that $\int_{\mathbf{int}(\hat{\mathbf{A}}_n)}\nu_n^z dz \geq c\mathrm{e}^{(1+\delta)\hat{U}_n}$, hence, choosing $\eta \in (0, \varepsilon)$ small enough,

$$\begin{aligned} &\hat{\mathbb{E}}_{\mathscr{D}}\left[\frac{\mathrm{e}^{\delta\hat{U}_n}}{\int_{\mathscr{D}_n}\nu_n^z dz}F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}_{\{\hat{\xi} \in \mathcal{Z}_n^{x,L,B}\}}\right] \\ \geq\ &(1-\varepsilon)\hat{\mathbb{E}}_{\mathscr{D}}\left[\frac{\mathrm{e}^{\delta\hat{U}_n}}{\int_{\mathbf{int}(\hat{A}_{n-b})}\nu_n^z dz}F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}\left\{\hat{\xi} \in \mathcal{Z}_n^{x,L,B}, \int_{z \notin \mathbf{int}(\hat{\mathbf{A}}_{n-b})}\nu_n^z dz \leq \eta\mathrm{e}^{(1+\delta)a_n}\right\}\right]. \end{aligned}$$

We want to remove now the event $\{\int_{z \notin \mathbf{int}(\hat{\mathbf{A}}_{n-b})}\nu_n^z dz \leq \eta\mathrm{e}^{(1+\delta)a_n}\}$. Observe that

$$\begin{aligned} &\hat{\mathbb{E}}_{\mathscr{D}}\left[\frac{\mathrm{e}^{\delta\hat{U}_n}}{\int_{\mathbf{int}(\hat{A}_{n-b})}\nu_n^z dz}\mathbf{1}\left\{\hat{\xi} \in \mathcal{Z}_n^{x,L,B}, \int_{z \notin \mathbf{int}(\hat{\mathbf{A}}_{n-b})}\nu_n^z dz > \eta\mathrm{e}^{(1+\delta)a_n}\right\}\right] \\ \leq\ &c\mathrm{e}^{-a_n - x}\hat{\mathbb{P}}_{\mathscr{D}}\left(\hat{\xi} \in \mathcal{Z}_n^{x,L,B}, \int_{z \notin \mathbf{int}(\hat{\mathbf{A}}_{n-b})}\nu_n^z dz > \eta\mathrm{e}^{(1+\delta)a_n}\right) \end{aligned}$$

which is less than $\frac{1}{W_0}\left(\eta D_0^{(0)} + c\frac{1+\eta}{\eta}W_0\right)$ for b large enough, by Lemma 4.9 and 4.12. It yields that, for some $c = c(\varepsilon, L, x, B)$,

$$\begin{aligned} &\mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x, \max_{\mathscr{D}}|z-\mathfrak{g}_n| \leq B\mathbf{R}_0^{\mathfrak{g}_n}\}}\right] \\ \geq\ &(1-\varepsilon)W_0\hat{\mathbb{E}}_{\mathscr{D}}\left[\frac{\mathrm{e}^{\delta\hat{U}_n}}{\int_{\mathbf{int}(\hat{A}_{n-b})}\nu_n^z dz}F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}_{\{\hat{\xi} \in \mathcal{Z}_n^{x,L,B}\}}\right] - \left(\varepsilon D_0^{(0)} + cW_0\right). \end{aligned} \tag{5.9}$$

Finally, we see that

$$\begin{aligned}
& \hat{\mathbb{E}}_{\mathscr{D}}\left[\frac{\mathrm{e}^{\delta \hat{U}_n}}{\int_{\mathbf{int}(\hat{A}_{n-b})}\nu_n^z dz} F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}_{\{\hat{\xi}\in \mathcal{Z}_n^{x,L,B}\}}\right] \\
\geq\ & \hat{\mathbb{E}}_{\mathscr{D}}\left[\frac{\mathrm{e}^{\delta \hat{U}_n}}{\int_{\mathbf{int}(\hat{A}_{n-b})}\nu_n^z dz} F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}\left\{\hat{\xi}\in \mathcal{Z}_n^{x,L,B}, \prod_{i=n-\zeta}^{n-b}\hat{\rho}_i \leq \mathrm{e}^{-(\zeta-b)^{1/2}}\right\}\right] \\
\geq\ & (1-\varepsilon)\hat{\mathbb{E}}_{\mathscr{D}}\left[\frac{\mathrm{e}^{-\hat{U}_{n-\zeta}}\mathrm{e}^{\delta(\hat{U}_n-\hat{U}_{n-\zeta})}}{\int_{\hat{\psi}_{n-\zeta+1}\circ\ldots\circ\hat{\psi}_{n-b}(\mathbb{U})}\tilde{\nu}_{\zeta,n}^z dz} F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}\left\{\hat{\xi}\in \mathcal{Z}_n^{x,L,B}, \prod_{i=n-\zeta}^{n-b}\hat{\rho}_i \leq \mathrm{e}^{-(\zeta-b)^{1/2}}\right\}\right]
\end{aligned}$$

by Lemma 5.4, for $\zeta - b$ large enough. We apply now the Markov property at time $n-\zeta$. We check that the expectation of the right-hand side actually is

(5.10)
$$\hat{\mathbb{E}}_{\mathscr{D}}\left[\mathrm{e}^{-\hat{U}_{n-\zeta}}K_{F,x}\left(L,\zeta,b,\hat{U}_{n-\zeta}-a_n\right)\mathbf{1}\left\{\max_{\ell\leq n-\zeta}\hat{U}_\ell \leq 0, \max_{\ell\in[n/2,n-\zeta]}\hat{U}_\ell \leq a_n + L, \max_{\mathscr{D}}|z-\hat{\xi}| \leq B\hat{\mathbf{R}}_0\right\}\right]$$

where $K_{F,x}(L,\zeta,b,y)$ is defined in (5.2). We rewrite (5.10) as

$$\begin{aligned}
& n^{3/2}\hat{\mathbb{E}}_{\mathscr{D}}\Big[\mathrm{e}^{-(\hat{U}_{n-\zeta}-a_n)}K_{F,x}\left(L,\zeta,b,\hat{U}_{n-\zeta}-a_n\right) \\
& \mathbf{1}\left\{\max_{\ell\leq n-\zeta}\hat{U}_\ell \leq 0, \max_{\ell\in[n/2,n-\zeta]}\hat{U}_\ell \leq a_n + L, \max_{\mathscr{D}}|z-\hat{\xi}| \leq B\hat{\mathbf{R}}_0\right\}\Big].
\end{aligned}$$

By Lemma 6.1, we know that the limit in $n\to\infty$ of the latter is

$$\begin{aligned}
& \frac{C_1C_2\sqrt{\pi}}{\sigma\sqrt{2}}\hat{\mathbb{E}}_{\mathscr{D}}[h_1(-\hat{U}_0)\mathbf{1}_{\{\max_{\mathscr{D}}|z-\hat{\xi}|\leq B\hat{\mathbf{R}}_0\}}]\int_{(-\infty,L]}\mathrm{e}^{-y}K_{F,x}(L,\zeta,b,y)h_2(L-y)dy \\
=\ & \frac{C_1C_2\sqrt{\pi}}{\sigma\sqrt{2}}\frac{D_{0,B}^{(0)}}{W_0}\int_{(-\infty,L]}\mathrm{e}^{-y}K_{F,x}(L,\zeta,b,y)h_2(L-y)dy.
\end{aligned}$$

Going back to (5.9), it completes the proof. □

We look now at the upper bound of Proposition 5.1.

Lemma 5.6 *For some constant c, the following holds: let $\varepsilon > 0$, $x \in \mathbf{R}$. There exists $L_0 \geq \max(x,0)$ such that of any $L \geq L_0$, we can find $b_0, d > 0$ such that for any bounded simply connected domain $\mathscr{D}$, for any $b \geq b_0$ and $\zeta \geq d+b$,*

$$\begin{aligned}
& \limsup_{n\to\infty}\mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{G}_n\geq a_n+x\}}\right] \\
\leq\ & (1+\varepsilon)D_0^{(0)}\int_{(-\infty,L]}\mathrm{e}^{-y}\tilde{K}_{F,x}(L,\zeta,b,y)h_2(L-y)dy + (\varepsilon D_0^{(0)} + cW_0).
\end{aligned}$$

Proof. Fix $\varepsilon > 0$ and $x \in \mathbf{R}$. We have by Lemma 4.3 and 4.4,

$$\mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{G}_n \geq a_n + x\}}\right] \leq \mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{g}_n \in \mathcal{Z}_n^{x,L}\}}\right] + \varepsilon D_0^{(0)} + cW_0$$

for L large enough. Fix L from now on. By Lemma 4.1, we have

$$\mathbb{E}_{\mathscr{D}}\left[F\left(\mathfrak{G}_n - a_n, \mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{g}_n \in \mathcal{Z}_n^{x,L}\}}\right] = W_0 \hat{\mathbb{E}}_{\mathscr{D}}\left[F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}_{\{\hat{\xi} \in \mathcal{Z}_n^{x,L}\}}\frac{\mathrm{e}^{\delta \hat{U}_n}}{\int_{\mathscr{D}} \nu_n^z dz}\right].$$

Using the fact that $\int_{\mathbf{int}(\hat{\mathbf{A}}_n)} \nu_n^z dz \geq c(x)\mathrm{e}^{(1+\delta)a_n}$ which can be deduced from Lemma 2.4, we see that Lemma 4.8 implies that for b and $\zeta - b$ large enough,

$$\begin{aligned}
&\hat{\mathbb{E}}_{\mathscr{D}}\left[F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}_{\{\hat{\xi} \in \mathcal{Z}_n^{x,L}\}}\frac{\mathrm{e}^{\delta \hat{U}_n}}{\int_{\mathscr{D}} \nu_n^z dz}\right] \\
\leq\ & \hat{\mathbb{E}}_{\mathscr{D}}\left[F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}_{\{\hat{\xi} \in \mathcal{Z}_n^{x,L}, \prod_{i=n-\zeta}^{n-b} \hat{\rho}_i \leq \mathrm{e}^{-(\zeta-b)^{1/2}}\}}\frac{\mathrm{e}^{\delta \hat{U}_n}}{\int_{\mathscr{D}} \nu_n^z dz}\right] + \frac{\varepsilon}{W_0}(D_0^{(0)} + W_0).
\end{aligned}$$

The expectation in the right-hand side is less than

$$\hat{\mathbb{E}}_{\mathscr{D}}\left[F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}_{\{\hat{\xi} \in \mathcal{Z}_n^{x,L}, \prod_{i=n-\zeta}^{n-b} \hat{\rho}_i \leq \mathrm{e}^{-(\zeta-b)^{1/2}}\}}\frac{\mathrm{e}^{\delta \hat{U}_n}}{\int_{\mathbf{int}(\hat{\mathbf{A}}_{n-b})} \nu_n^z dz}\right].$$

In view of Lemma 5.4, we have for $\zeta - b$ large enough,

$$\begin{aligned}
&\hat{\mathbb{E}}_{\mathscr{D}}\left[F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}_{\{\hat{\xi} \in \mathcal{Z}_n^{x,L}, \prod_{i=n-\zeta}^{n-b} \hat{\rho}_i \leq \mathrm{e}^{-(\zeta-b)^{1/2}}\}}\frac{\mathrm{e}^{\delta \hat{U}_n}}{\int_{\mathbf{int}(\hat{\mathbf{A}}_{n-b})} \nu_n^z dz}\right] \\
\leq\ & (1+\varepsilon)\hat{\mathbb{E}}_{\mathscr{D}}\left[F\left(\hat{U}_n - a_n, \hat{\mathcal{Q}}_n\right)\mathbf{1}_{\{\hat{\xi} \in \mathcal{Z}_n^{x,L}, \prod_{i=n-\zeta}^{n-b} \hat{\rho}_i \leq \mathrm{e}^{-(\zeta-b)^{1/2}}\}}\frac{\mathrm{e}^{-\hat{U}_{n-\zeta}}\mathrm{e}^{\delta(\hat{U}_n - \hat{U}_{n-\zeta})}}{\int_{\hat{\psi}_{n-\zeta+1}\circ\ldots\circ\hat{\psi}_{n-b}(\mathbb{U})} \tilde{\nu}_{\zeta,n}^z dz}\right].
\end{aligned}$$

The rest of the proof is similar to the proof of the previous lemma. □

5.2 Proof of Proposition 5.2

We work under $\hat{\mathbb{P}}_{\mathbb{U}}^0$. We investigate the limit in $\zeta \to \infty$ then $b \to \infty$ then $L \to \infty$ of

$$\int_{(-\infty,L]} \mathrm{e}^{-y} K_{F,x}(L,\zeta,b,y) h_2(L-y) dy$$

where

$$K_{F,x}(L,\zeta,b,y) = \hat{\mathbb{E}}_{\mathbb{U}}^0\left[\frac{\mathrm{e}^{\delta \hat{U}_\zeta} F\left(y + \hat{U}_\zeta, \hat{\mathcal{Q}}_\zeta\right)}{\int_{\mathbf{int}(\hat{A}_{\zeta-b})} \nu_\zeta^z dz}\mathbf{1}_{\{\max_{\ell \leq \zeta} \hat{U}_\ell \leq L-y, \hat{U}_\zeta \geq x-y, \prod_{j=1}^{\zeta-b} \hat{\rho}_j \leq \mathrm{e}^{-(\zeta-b)^{1/2}}\}}\right].$$

First, we decompose

$$\int_{\mathbf{int}(\hat{\mathbf{A}}_{\zeta-b})} \nu_\zeta^z dz = \sum_{k=\zeta-b}^{\zeta-1} \int_{\mathbf{int}(\hat{\mathbf{A}}_k)\setminus\mathbf{int}(\hat{\mathbf{A}}_{k+1})} \nu_\zeta^z dz + \int_{\mathbf{int}(\hat{\mathbf{A}}_\zeta)} \nu_\zeta^z dz. \tag{5.11}$$

Recall that $\hat{\Psi}_j$ is the conformal map sending $\mathbb{U}$ to $\mathbf{int}(\hat{\mathbf{A}}_j)$, and $\hat{\psi}_{j+1} = \hat{\Psi}_j^{-1} \circ \hat{\Psi}_{j+1}$. For $k \in [\zeta - b, \zeta - 1]$, we get that

$$\int_{\mathbf{int}(\hat{\mathbf{A}}_k)\setminus\mathbf{int}(\hat{\mathbf{A}}_{k+1})} \nu_\zeta^z dz = \Phi(\beta_c)^{-\zeta(1+\delta)} \int_{\mathbb{U}\setminus\hat{\psi}_{k+1}(\mathbb{U})} (\mathbf{R}_\zeta^{\hat{\Psi}_k(z)})^{\beta_c+\delta(\beta_c+2)} |\hat{\Psi}'_k(z)|^2 dz. \tag{5.12}$$

For fixed ζ, define for any $1 \le i \le \zeta$, $\hat{\psi}_i^- := \hat{\psi}_{\zeta-i+1}$, then $\hat{\rho}_i^- := \sup_{|z|<1} |\hat{\psi}_i^-(z)|$. Therefore, $\hat{\Psi}_j$ is $\hat{\psi}_\zeta^- \circ \ldots \circ \hat{\psi}_{\zeta-j+1}^-$. Let $\hat{\mathbf{R}}_i^- := \mathbf{R}(\hat{\Psi}_{\zeta-i}^{-1}(\mathbf{int}(\hat{\mathbf{A}}_\zeta)), 0)$, and for any $z \in \mathbb{U}$, $\mathbf{R}_i^{-,z} := \mathbf{R}(\hat{\Psi}_{\zeta-i}^{-1}(\mathbf{int}(\mathbf{A}_\zeta^{\hat{\Psi}_{\zeta-i}(z)})), z)$. Finally, set $\hat{U}_i^- := (\beta_c + 2) \ln \hat{\mathbf{R}}_i^- - iv_c$ and $\hat{\mathcal{Q}}_i^- := \hat{\mathcal{Q}}_{\zeta-i+1}$.

We observe that $\mathbf{R}_\zeta^{\hat{\Psi}_k(z)} = |\hat{\Psi}'_k(z)| \mathbf{R}(\hat{\Psi}_k^{-1}(\mathbf{int}(\mathbf{A}_\zeta^{\hat{\Psi}_k(z)})), z) = |\hat{\Psi}'_k(z)| \mathbf{R}_{\zeta-k}^{-,z}$. We deduce that

$$\int_{\mathbf{int}(\hat{\mathbf{A}}_k)\setminus\mathbf{int}(\hat{\mathbf{A}}_{k+1})} \nu_\zeta^z dz = \Phi(\beta_c)^{-\zeta(1+\delta)} \int_{\mathbb{U}\setminus\hat{\psi}_{k+1}(\mathbb{U})} (\mathbf{R}_{\zeta-k}^{-,z})^{\beta_c+\delta(\beta_c+2)} |\hat{\Psi}'_k(z)|^{(1+\delta)(\beta_c+2)} dz.$$

Notice that

$$\ln \frac{|\hat{\Psi}'_k(z)|}{\hat{\Psi}'_k(0)} = \sum_{i=\zeta-k+1}^{\zeta} \ln \frac{|(\hat{\psi}_i^-)'(\hat{\psi}_{i-1}^- \circ \ldots \circ \hat{\psi}_{\zeta-k+1}^-(z))|}{(\hat{\psi}_i^-)'(0)}.$$

Set for any integer $j \le \zeta$, and any $z \in \mathbb{U}$,

$$\Delta_{j,\zeta}^{-,z} := (\beta_c + 2) \sum_{i=j+1}^{\zeta} \ln \frac{|(\hat{\psi}_i^-)'(\hat{\psi}_{i-1}^- \circ \ldots \circ \hat{\psi}_{j+1}^-(z))|}{(\hat{\psi}_i^-)'(0)}.$$

Then

$$|\hat{\Psi}'_k(z)|^{\beta_c+2} = (\hat{\Psi}'_k(0))^{\beta_c+2} \mathrm{e}^{\Delta_{\zeta-k,\zeta}^{-,z}}.$$

From (5.12), it entails that

$$\int_{\mathbf{int}(\hat{\mathbf{A}}_k)\setminus\mathbf{int}(\hat{\mathbf{A}}_{k+1})} \nu_\zeta^z dz = \frac{\mathrm{e}^{(1+\delta)\hat{U}_\zeta}}{(\hat{\mathbf{R}}_{\zeta-k}^-)^{(1+\delta)(\beta_c+2)}} \int_{\mathbb{U}\setminus\hat{\psi}_{k+1}(\mathbb{U})} (\mathbf{R}_{\zeta-k}^{-,z})^{\beta_c+\delta(\beta_c+2)} \mathrm{e}^{(1+\delta)\Delta_{\zeta-k,\zeta}^{-,z}} dz.$$

Going back to (5.11), it gives that

$$\mathrm{e}^{-(1+\delta)\hat{U}_\zeta} \int_{\mathbf{int}(\hat{\mathbf{A}}_{\zeta-b})} \nu_\zeta^z dz = \mathscr{I}_{\zeta,b}^-$$

where

$$\mathscr{I}^-_{\zeta,b} := \int_{\mathbb{U}} (\mathbf{R}_0^{-,z})^{\beta_c+\delta(\beta_c+2)} \mathrm{e}^{(1+\delta)\Delta^{-,z}_{0,\zeta}} dz + \sum_{k=1}^{b} \frac{1}{(\hat{\mathbf{R}}_k^-)^{(1+\delta)(\beta_c+2)}} \int_{\mathbb{U}\setminus\hat{\psi}_k^-(\mathbb{U})} (\mathbf{R}_k^{-,z})^{\beta_c+\delta(\beta_c+2)} \mathrm{e}^{(1+\delta)\Delta^{-,z}_{k,\zeta}} dz.$$

In view of (5.2), $K_{F,x}(L,\zeta,b,y)$ can be rewritten as

$$\begin{aligned} &K_{F,x}(L,\zeta,b,y) \\ =\ & \hat{\mathbb{E}}^0_{\mathbb{U}}\left[\frac{\mathrm{e}^{-\hat{U}_\zeta} F\left(y+\hat{U}_\zeta, \hat{\mathcal{Q}}_\zeta\right)}{\mathscr{I}^-_{\zeta,b}} \mathbf{1}\left\{ \max_{\ell\le\zeta} \hat{U}_\ell \le L-y, \hat{U}_\zeta \ge x-y, \prod_{j=1}^{\zeta-b} \hat{\rho}_j \le \mathrm{e}^{-(\zeta-b)^{1/2}} \right\}\right] \\ =\ & \hat{\mathbb{E}}^0_{\mathbb{U}}\left[\frac{\mathrm{e}^{-\hat{U}^-_\zeta} F\left(y+\hat{U}^-_\zeta, \hat{\mathcal{Q}}_{\zeta},\right)}{\mathscr{I}^-_{\zeta,b}} \mathbf{1}\left\{ \min_{\ell\le\zeta} \hat{U}^-_\ell \ge -L, y+\hat{U}^-_\zeta \in [x, L+\min_{i\le\zeta}\hat{U}^-_i], \prod_{j=b+1}^{\zeta} \hat{\rho}^-_j \le \mathrm{e}^{-(\zeta-b)^{1/2}} \right\}\right]. \end{aligned}$$

Observe that this is (recall the notation at the beginning of the section),

$$\mathrm{E}\left[\frac{\mathrm{e}^{-\hat{V}_\zeta} F\left(y+\hat{V}_\zeta, \mathcal{O}\right)}{\mathscr{I}_{\zeta,b}} \mathbf{1}\left\{ \min_{\ell\le\zeta} \hat{V}_\ell \ge -L, y+\hat{V}_\zeta \in [x, L+\min_{i\le\zeta}\hat{V}_i], \prod_{j=b+1}^{\zeta} \hat{\rho}_j \le \mathrm{e}^{-(\zeta-b)^{1/2}} \right\}\right],$$

where, under P, $\hat{\rho}_j := \sup_{|z|<1} |\hat{\psi}_j(z)|$,

$$\mathscr{I}_{\zeta,b} := \int_{\mathbb{U}} (\mathbf{S}_0^{z})^{\beta_c+\delta(\beta_c+2)} \mathrm{e}^{(1+\delta)\Delta^{z}_{0,\zeta}} dz + \sum_{k=1}^{b} \frac{1}{(\hat{\mathbf{S}}_k)^{(1+\delta)(\beta_c+2)}} \int_{\mathbb{U}\setminus\hat{\psi}_k(\mathbb{U})} (\mathbf{S}_k^{z})^{\beta_c+\delta(\beta_c+2)} \mathrm{e}^{(1+\delta)\Delta^{z}_{k,\zeta}} dz$$

and

$$\Delta^z_{j,\zeta} := (\beta_c+2)\sum_{i=j+1}^{\zeta} \ln \frac{|(\hat{\psi}_i)'(\hat{\psi}_{i-1}\circ\ldots\circ\hat{\psi}_{j+1}(z))|}{(\hat{\psi}_i)'(0)}.$$

It implies that $\int_{(-\infty,L]} \mathrm{e}^{-y} K_{F,x}(L,\zeta,b,y) h_2(L-y)dy$ is equal to (with a change of variables under the integral $y = y+\hat{V}_\zeta$),

$$\int_x^L \mathrm{e}^{-y}\mathrm{E}\left[\frac{F\left(y,\mathcal{O}\right)}{\mathscr{I}_{\zeta,b}} h_2(L+\hat{V}_\zeta - y) \mathbf{1}\left\{ \min_{\ell\le\zeta} \hat{V}_\ell \ge y-L, \prod_{j=b+1}^{\zeta} \hat{\rho}_j \le \mathrm{e}^{-(\zeta-b)^{1/2}} \right\}\right] dy. \tag{5.13}$$

Recall the definition of h_2 in (5.5). For any $a\ge 0$, define the probability measure $\mathrm{P}^{\uparrow}_a$ on $\hat{\mathcal{G}}_\infty$ such that for any $k\ge 1$,

$$\frac{d\mathrm{P}^{\uparrow,(a)}}{d\mathrm{P}}\Big|_{\hat{\mathcal{G}}_k} := \frac{h_2(a+\hat{V}_k)}{h_2(a)} \mathbf{1}_{\{\min_{i\le k}\hat{V}_i \ge -a\}}.$$

We obtain that, for any $y \in [x, L]$,

$$
\begin{aligned}
& \mathrm{E}\left[\frac{F\left(\hat{V}_\zeta, \mathcal{O}\right)}{\mathscr{I}_{\zeta,b}} h_2(L+\hat{V}_\zeta - y)\mathbf{1}\left\{\min_{\ell\le\zeta}\hat{V}_\ell \ge y-L,\ \prod_{j=b+1}^{\zeta}\hat{\rho}_j \le \mathrm{e}^{-(\zeta-b)^{1/2}}\right\}\right] \\
= \;& h_2(L-y)\mathrm{E}^{\uparrow}_{L-y}\left[\frac{F(y,\mathcal{O})}{\mathscr{I}_{\zeta,b}}\mathbf{1}\left\{\prod_{j=b+1}^{\zeta}\hat{\rho}_j \le \mathrm{e}^{-(\zeta-b)^{1/2}}\right\}\right].
\end{aligned}
$$

We want to make ζ then b go to infinity in the expectation. Observe that

$$
\mathscr{I}_{\zeta,b} \ge \int_{\mathbb{U}} (\mathbf{S}_0^z)^{\beta_c+\delta(\beta_c+2)}\mathrm{e}^{(1+\delta)\Delta^z_{0,\zeta}}dz \ge c\int_{|z|<1/2}\mathrm{e}^{(1+\delta)\Delta^z_{0,\zeta}}dz.
$$

On the other hand, applying Koebe's distortion Theorem to the map $\hat{\psi}_\zeta\circ\ldots\circ\hat{\psi}_1$, we see that $\Delta^z_{0,\zeta}\ge c>0$ for any $|z|<1/2$. Therefore, $\frac{1}{\mathscr{I}_{\zeta,b}}$ is bounded from above by a constant, and we can apply dominated convergence. We get

$$
\lim_{b\to\infty}\lim_{\zeta\to\infty}\mathrm{E}^{\uparrow}_{L-y}\left[\frac{F(y,\mathcal{O})}{\mathscr{I}_{\zeta,b}}\mathbf{1}\left\{\prod_{j=b+1}^{\zeta}\hat{\rho}_j \le \mathrm{e}^{-(\zeta-b)^{1/2}}\right\}\right] = \mathrm{E}^{\uparrow}_{L-y}\left[\frac{F(y,\mathcal{O})}{\mathscr{I}}\right]
$$

where

$$
\mathscr{I} := \int_{\mathbb{U}}(\mathbf{S}_0^z)^{\beta_c+\delta(\beta_c+2)}\mathrm{e}^{(1+\delta)\Delta^z_0}dz + \sum_{k\ge1}\frac{1}{(\hat{\mathbf{S}}_k)^{(1+\delta)(\beta_c+2)}}\int_{\mathbb{U}\setminus\hat{\psi}_k(\mathbb{U})}(\mathbf{S}_k^z)^{\beta_c+\delta(\beta_c+2)}\mathrm{e}^{(1+\delta)\Delta^z_k}dz
$$

and

$$
\Delta^z_j := (\beta_c+2)\sum_{i\ge j+1}\ln\frac{|(\hat{\psi}_i)'(\hat{\psi}_{i-1}\circ\ldots\circ\hat{\psi}_{j+1}(z))|}{(\hat{\psi}_i)'(0)}.
$$

By equation (5.13), it proves that

$$
(5.14)\quad \lim_{b\to\infty}\lim_{\zeta\to\infty}\int_{(-\infty,L]}\mathrm{e}^{-y}K_{F,x}(L,\zeta,b,y)h_2(L-y)dy = \int_x^L \mathrm{e}^{-y}h_2(L-y)\mathrm{E}^{\uparrow}_{L-y}\left[\frac{F(y,\mathcal{O})}{\mathscr{I}}\right].
$$

Finally, we need to make $L\to\infty$. We use the following decomposition. We introduced the probability measure $\mathrm{P}^{(\ell)}$ at the beginning of the section.

Lemma 5.7 *Let $a\ge 0$. Then, for any integrable variable X, we have*

$$
h_2(a)\mathrm{E}^{\uparrow}_a[X] = \sum_{\ell\ge0}\mathrm{E}^{(\ell)}\left[X\mathbf{1}_{\{\min_{i\ge0}\hat{V}_i\ge -a\}}\right].
$$

Proof. Suppose that X is measurable with respect to $\hat{\mathcal{G}}_k$ and let $n \geq k$. Then

$$h_2(a)\mathrm{E}_a^{\uparrow}[X] = \mathrm{E}\left[X\mathbf{1}_{\{\min_{i\leq n}\hat{V}_i \geq -a\}} h_2(\hat{V}_n + a)\right].$$

We decompose the trajectory of the random walk with respect to the minimum on the time interval $[0,n]$. We get

$$\begin{aligned}\mathrm{E}\left[X\mathbf{1}_{\{\min_{i\leq n}\hat{V}_i \geq -a\}} h_2(\hat{V}_n + a)\right] &= \sum_{\ell\geq 0}\mathrm{E}\left[X\mathbf{1}_{\{\hat{H}_\ell = \min_{i\leq n}\hat{V}_i \geq -a\}} h_2(\hat{V}_n + a)\right] \\ &= \sum_{\ell \geq 0}\mathrm{E}^{(\ell)}\left[X\mathbf{1}_{\{\hat{H}_\ell \geq -a, \hat{T}_\ell \leq n\}}\frac{h_2(\hat{V}_n + a)}{h_2(\hat{V}_n - \hat{H}_\ell)}\right].\end{aligned}$$

Using $\frac{h_2(\hat{V}_n + a)}{h_2(\hat{V}_n - \hat{H}_\ell)} \leq c + a$, we can justify dominated convergence. Letting $n \to \infty$, we see that

$$h_2(a)\mathrm{E}_a^{\uparrow}[X] = \sum_{\ell\geq 0}\mathrm{E}^{(\ell)}\left[X\mathbf{1}_{\{\hat{H}_\ell \geq -a\}}\right].$$

It completes the proof. □

Let us go back to (5.14). By Lemma 5.7, we have

$$h_2(L-y)\mathrm{E}_{L-y}^{\uparrow}\left[\frac{F(y,\mathcal{O})}{\mathscr{I}}\right] = \sum_{\ell\geq 0}\mathrm{E}^{(\ell)}\left[\frac{F(y,\mathcal{O})}{\mathscr{I}}\mathbf{1}_{\{\min_{i\geq 0}\hat{V}_i \geq y - L\}}\right].$$

By monotone convergence, it follows that

$$\lim_{L\to\infty}\int_x^L \mathrm{e}^{-y} h_2(L-y)\mathrm{E}_{y-L}^{\uparrow}\left[\frac{F(y,\mathcal{O})}{\mathscr{I}}\right] dy = \int_{[x,+\infty)} \mathrm{e}^{-y}\sum_{\ell\geq 0}\mathrm{E}^{(\ell)}\left[\frac{F(y,\mathcal{O})}{\mathscr{I}}\right] dy.$$

It completes the proof of Proposition 5.2.

5.3 Proof of Theorem 1.4

Let $x \in \mathbf{R}$ and $(A_i)_{i\leq I}$ disjoint deterministic open subsets of $\mathscr{D}$. Let $(\lambda_i)_{i\leq I}$ be real numbers. Following Section 6.3, we suppose that each A_i is the interior of a square which vertices have rational coordinates. Let $f : \mathbf{R}\times\Omega_{\mathscr{D}}^p \to \mathbf{R}_+$ be a continuous function with support included in $[x,+\infty)\times\Omega_{\mathscr{D}}^p$, where p is some integer. We extend the definition of f to $\mathbf{R}\times\Omega_{\mathscr{D}}^{\mathbb{N}}$ by looking at the projection onto $\mathbf{R}\times\Omega_{\mathscr{D}}^p$. It is enough to show that,

$$\lim_{n\to\infty}\mathrm{E}_{\mathscr{D}}\left[\prod_{i=1}^{I}\prod_{|\gamma|=k,\mathfrak{g}_{n,\gamma}\in A_i}\mathrm{e}^{-\lambda_i f(\mathfrak{G}_{n,\gamma} - a_n, \mathcal{Q}_n^{\mathfrak{g}_{n,\gamma}})} \mid \mathcal{F}_k\right] \xrightarrow[k\to\infty]{(d)} \mathrm{e}^{-\sum_{i=1}^{I} D_\infty(A_i)\tilde{C}_{F_i,x}}$$

where

$$\tilde{C}_{F_i,x} := C_2 \int_{\mathbf{R}\times\Omega_{\mathscr{D}}^{\mathbb{N}}} F_i(y,\mathcal{Q})\mathrm{e}^{-y} dy \mathcal{M}(d\mathcal{Q})$$

and $F_i(y,\mathcal{Q}) := 1-\mathrm{e}^{-\lambda_i f(y,\mathcal{Q})}$. Recall that c_- is the constant such that $h_1(x)\sim c_- x$ as $x\to\infty$. We have $c_- C_1 = \frac{\sigma\sqrt{2}}{\sqrt{\pi}}$ (for example see Lemma 2.1 in [4]). Therefore, with the notation of (5.7), we have

$$\tilde{C}_{F_i,x} = c_- C_{F_i,x}.$$

Moreover, using the fact that the loops are uniformly small, we can restrict to $\mathfrak{g}_{n,\gamma}$ for which the loop γ is included in A_i (hence we do not consider cases where the loop γ intersect the boundary of A_i). Finally, since the loop ensembles contained in disjoint γ at generation k are independent, we may restrict to the case $I=1$. To sum up, we will prove for a deterministic open subset A of $\mathscr{D}$,

$$\lim_{n\to\infty} \mathbb{E}_{\mathscr{D}}\left[\prod_{|\gamma|=k,\gamma\subset A} \mathrm{e}^{-f(\mathfrak{G}_{n+k,\gamma}-a_{n+k},\mathcal{Q}_{n+k}^{\mathfrak{g}_{n+k,\gamma}})} \mid \mathcal{F}_k\right] \xrightarrow[k\to\infty]{(d)} \mathrm{e}^{-c_- D_\infty(A) C_{F,x}} \tag{5.15}$$

where $F(y,\mathcal{Q}) := 1-\mathrm{e}^{-f(y,\mathcal{Q})}$. By Remark 4.2,

$$\mathbb{E}_{\mathscr{D}}\left[\prod_{|\gamma|=k,\gamma\subset A} \mathrm{e}^{-f(\mathfrak{G}_{n+k,\gamma}-a_{n+k},\mathcal{Q}_{n+k}^{\mathfrak{g}_{n+k,\gamma}})} \mid \mathcal{F}_k\right] = \prod_{|\gamma|=k,\gamma\subset A} \mathbb{E}_{\mathrm{e}^{-k\tilde{v}_c}\mathbf{int}(\gamma)}\left[\mathrm{e}^{-f(\mathfrak{G}_n-a_{n+k},\mathcal{Q}_n^{\mathfrak{g}_n})}\right].$$

For any loop γ at generation k, we have

$$\mathbb{E}_{\mathrm{e}^{-k\tilde{v}_c}\mathbf{int}(\gamma)}\left[\mathrm{e}^{-f(\mathfrak{G}_n-a_n,\mathcal{Q}_n^{\mathfrak{g}_n})}\right] = 1-\mathbb{E}_{\mathrm{e}^{-k\tilde{v}_c}\mathbf{int}(\gamma)}\left[F(\mathfrak{G}_n-a_n,\mathcal{Q}_n^{\mathfrak{g}_n})\right].$$

Let $\varepsilon>0$ and $B\geq 1$. By Corollary 5.3, for n large enough,

$$\begin{aligned} &\left|\mathbb{E}_{\mathrm{e}^{-k\tilde{v}_c}\mathbf{int}(\gamma)}\left[F\left(\mathfrak{G}_n-a_n,\mathcal{Q}_n^{\mathfrak{g}_n}\right)\mathbf{1}_{\{\mathfrak{G}_n\geq a_n+x\}}\right]-D_k^+(\gamma)C_{F,x}\right| \\ \leq\ & \varepsilon D_k^+(\gamma)+c(\varepsilon,x,B)W_k(\gamma)+C_{F,x}(D_k^+(\gamma)-D_{k,B}^+(\gamma)) \end{aligned}$$

with

$$\begin{aligned} W_k(\gamma) &:= \Phi(\beta_c)^{-k}\int_{\mathbf{int}(\gamma)} (\mathbf{R}_k^z)^{\beta_c} dz, \\ D_k^+(\gamma) &:= \Phi(\beta_c)^{-k}\int_{\mathbf{int}(\gamma)} h_1(-U_k^z)(\mathbf{R}_k^z)^{\beta_c} dz, \\ D_{k,B}^+(\gamma) &:= \Phi(\beta_c)^{-k}\int_{\mathbf{int}(\gamma)} h_1(-U_k^z)(\mathbf{R}_k^z)^{\beta_c}\mathbf{1}_{\{\max_{\mathbf{int}(\gamma)}|z'-z|\leq B\mathbf{R}_k^z\}} dz. \end{aligned}$$

Hence, for $n \to \infty$,

$$\begin{aligned}
&\prod_{|\gamma|=k,\gamma\subset A} \mathbb{E}_{e^{-k\tilde{v}_c}\mathbf{int}(\gamma)}\left[e^{-f(\mathfrak{G}_n-a_n,\mathcal{Q}_n^{\mathfrak{g}_n})}\right] \\
=& \prod_{|\gamma|=k,\gamma\subset A} (1-D_k^+(\gamma)C_{F,x}) + O\left(\varepsilon D_k^+ + c(\varepsilon,x,B)W_k + (D_k^+ - D_{k,B}^+)\right)
\end{aligned}$$

where $D_k^+ := \sum_{|\gamma|=k} D_k^+(\gamma)$ and $D_{k,B}^+ := \sum_{|\gamma|=k} D_{k,B}^+(\gamma)$. Using the fact that $a_{n+k} = a_n + o_n(1)$, we have as well, for $n \to \infty$,

$$\begin{aligned}
&\prod_{|\gamma|=k,\gamma\subset A} \mathbb{E}_{e^{-k\tilde{v}_c}\mathbf{int}(\gamma)}\left[e^{-f(\mathfrak{G}_n-a_{n+k},\mathcal{Q}_n^{\mathfrak{g}_n})}\right] \\
=& \prod_{|\gamma|=k,\gamma\subset A} (1-D_k^+(\gamma)C_{F,x}) + O(\varepsilon D_k^+ + c(\varepsilon,x,B)W_k + (D_k^+ - D_{k,B}^+)),
\end{aligned}$$

which is the same as

$$\begin{aligned}
&\mathbb{E}_{\mathscr{D}}\left[\prod_{|\gamma|=k,\gamma\subset A} e^{-f(\mathfrak{G}_{n+k,\gamma}-a_{n+k},\mathcal{Q}_{n+k}^{\mathfrak{g}_{n+k},\gamma})} \mid \mathcal{F}_k\right] \\
=& \prod_{|\gamma|=k,\gamma\subset A} (1-D_k^+(\gamma)C_{F,x}) + O(\varepsilon D_k^+ + c(\varepsilon,x,B)W_k + (D_k^+ - D_{k,B}^+)).
\end{aligned}$$

By Lemma 4.3, then equation (4.4) and Lemma 4.1, $\limsup_{k\to\infty} \mathbb{P}_{\mathscr{D}}(D_k^+ - D_{k,B}^+ > \varepsilon)$ goes to 0 as $B \to \infty$. Recall that $W_k \to 0$ a.s. when $k \to \infty$. We deduce that the limit in distribution as $k \to \infty$ of

$$\lim_{n\to\infty} \mathbb{E}_{\mathscr{D}}\left[\prod_{|\gamma|=k,\gamma\subset A} e^{-f(\mathfrak{G}_{n+k,\gamma}-a_{n+k},\mathcal{Q}_{n+k}^{\mathfrak{g}_{n+k},\gamma})} \mid \mathcal{F}_k\right]$$

is the limit in distribution as $k \to \infty$ (if it exists) of $\prod_{|\gamma|=k,\gamma\subset A}(1-D_k^+(\gamma)C_{F,x})$. Observe that $D_k^+ \sim c_- D_k$. Therefore, if we admit that $\max_{|\gamma|=k} D_k^+(\gamma) \to 0$ almost surely, then, almost surely,

$$\lim_{k\to\infty} \prod_{|\gamma|=k,\gamma\subset A} (1-D_k^+(\gamma)C_{F,x}) = e^{-c_- C_{F,x} D_\infty(A)}$$

which ends the proof of (5.15). It remains to prove that $\max_{|\gamma|=k} D_k^+(\gamma) \to 0$ indeed. We can suppose that $\max_{\ell\geq 0, z\in\mathscr{D}_\ell} U_\ell^z \leq \alpha$. By Lemma 4.1,

$$\begin{aligned}
\mathbb{E}_{\mathscr{D}}\left[\int_{\mathscr{D}_k} \phi(\beta_c)^{-k} h_1(-U_k^z)(\mathbf{R}_k^z)^{\beta_c} \mathbf{1}_{\{\max_{\ell\leq k} U_\ell^z \leq \alpha, U_k^z \geq -k^{1/9}\}} dz\right] &= W_0 \hat{\mathbb{E}}_{\mathscr{D}}\left[h_1(-\hat{U}_k)\mathbf{1}_{\{\max_{\ell\leq k}\hat{U}_\ell\leq\alpha,\hat{U}_k\geq -k^{1/9}\}}\right] \\
&\leq c(\alpha) W_0 k^{1/3} k^{-3/2}
\end{aligned}$$

by equation (6.2). Therefore, $\phi(\beta_c)^{-k}\int_{\mathscr{D}_k} h_1(-U_k^z)(\mathbf{R}_k^z)^{\beta_c}\mathbf{1}_{\{\max_{\ell\leq k}U_\ell^z\leq\alpha, U_k^z\geq -k^{1/9}\}}$ goes to 0 almost surely. Similarly, using Lemma 4.1 and Lemma 2.3, we see that

$$\phi(\beta_c)^{-k}\int_{\mathscr{D}_k} h_1(-U_k^z)(\mathbf{R}_k^z)^{\beta_c}\mathbf{1}_{\{\int_{\mathbf{int}(\mathbf{A}_k^z)}dz\geq k^c R_k^z\}}dz$$

goes to 0 almost surely, if c is a constant big enough. Finally, observe that

$$\begin{aligned}&\phi(\beta_c)^{-k}\int_{\mathbf{int}(\gamma)} h_1(-U_k^z)(\mathbf{R}_k^z)^{\beta_c}\mathbf{1}_{\{U_k^z\leq -k^{1/9},\int_{\mathbf{int}(\mathbf{A}_k^z)}dz\geq k^c R_k^z\}}dz\\ =&\int_{\mathbf{int}(\gamma)} h_1(-U_k^z)\mathrm{e}^{U_k^z}\frac{1}{(R_k^z)^2}\mathbf{1}_{\{U_k^z\leq -k^{1/9},\int_{\mathbf{int}(\mathbf{A}_k^z)}dz\leq k^c R_k^z\}}dz\\ \leq& c'\mathrm{e}^{-k^{1/9}/2}k^{2c}\end{aligned}$$

which goes to 0 indeed.

6 Appendix

6.1 $\mathcal{C}$ contains none or infinitely many loops

Suppose that $\mathcal{C}$ has positive probability to have, say, $k\geq 1$ loops. Then the law of $\mathcal{C}$ conditionally on having k loops is again conformally invariant. Let $\varepsilon>0$ such that the probability that all the k loops are contained in $\{|z|<1-\varepsilon\}$ has probability strictly greater than $1/2$. Now we can send $\{|z|<1-\varepsilon\}$ inside its complementary via a Moëbius transformation. We deduce that we can have the k loops contained in $\{|z|\geq 1-\varepsilon\}$ with probability strictly greater than $1/2$, which yields a contradiction.

6.2 $\mathbf{R}_1^0$ can take arbitrarily small values

Let $\varepsilon>0$ such that $\mathbf{R}_1^0\in(0,1-\varepsilon)$ with positive probability. On this event, there exists a point $|z|<1-\varepsilon$ with rational coordinates such that $\mathbf{R}_1^z\in(0,\eta)$, where η is some fixed positive number. Therefore, for any $\eta>0$, there exists $|z|<1-\varepsilon$ such that $\mathbf{R}_1^z\in(0,\eta)$ with positive probability. By conformal invariance, we have $\mathbf{R}_1^0\in(0,c(\varepsilon)\eta)$ with positive probability, this for any $\eta>0$. Hence, ess inf $\mathbf{R}_1^0=0$.

6.3 The construction of measures

Let $\mathscr{D}$ a bounded simply connected domain. Let M_n be the measure $W_n^{(\beta)}$, for $0<\beta<\beta_c$ or $D_n^{(\alpha)}$, for $\alpha\geq 0$. We know that, for any deterministic Borelian subset A, $M_n(A)$ converges in

L^1 and almost surely towards some random variable $M_\infty(A)$. Let $(A_p)_{p\geq 1}$ be the countable collection of interiors of squares which vertices have rational coordinates, and $(B_p)_{p\geq 1}$ their closures. Then $M_n(A_p)$, resp. $M_n(B_p)$, converges to $M_\infty(A_p)$, resp. $M_\infty(B_p)$, for any $p \geq 1$ with probability 1. Since $M_n(\mathscr{D})$ converges almost surely, the family of (random) measures M_n is tight almost surely and there exists a (random) measure μ on the closure of $\mathscr{D}$ which is a limit of a convergent subsequence. Let A be the interior of a square with rational coordinates. We have necessarily $\mu(A) \leq M_\infty(A)$. On the other hand, A can be written as a limit of an increasing sequence of sets $(C_k)_{k\geq 1}$ belonging to $(B_p)_{p\geq 1}$. Notice that $M_\infty(C_k) \leq \mu(C_k)$ for any $k \geq 1$. Therefore, $\mu(A) \geq \sup_{k\geq 1} M_\infty(C_k)$. By the L^1 convergence, we see that $\mathbb{E}_{\mathscr{D}}[M_\infty(A) - M_\infty(C_k)] = M_0(A) - M_0(C_k)$ which goes to 0 as $k \to \infty$. It implies that $\sup_{k\geq 1} M_\infty(C_k) = M_\infty(A)$ almost surely, therefore, $M_\infty(A) = \mu(A)$ for any A belonging to $(A_p)_{p\geq 1}$ on an event of probability 1. Since $(A_p)_{p\geq 1}$ generates the Borel σ-algebra, we conclude that the measures M_n converge indeed towards μ, which is such that $\mu(A_p) = M_\infty(A_p)$ for any $p \geq 1$ on an event of probability 1. From the L^1 convergence, we also prove that μ is a measure on $\mathscr{D}$ (it does not put mass on the boundary). We emphasize that we did not prove that $\mu(A) = M_\infty(A)$ for any Borelian set A (we do not even know whether $M_\infty(A)$ exists for any Borelian set A with probability 1). Still, with an abuse of notation, we will denote μ by M_∞.

6.4 One-dimensional estimates

We collect some results on one-dimensional random walks. They can be found for example in [3] and in Section 2 of [1], see Lemma 2.3 and equations (2.8) and (2.9) there. The following lemma is a particular case of Lemma 2.4 of [18]. Let $(S_n)_{n\geq 0}$ be under P a centered random walk with finite variance σ^2. Let $C_1, C_2 > 0$ the constants such that

$$\mathrm{P}\left(\min_{\ell\leq n} S_\ell \leq 0\right) \sim_{n\to\infty} \frac{C_1}{n^{1/2}}, \qquad \mathrm{P}\left(\max_{\ell\leq n} S_\ell \geq 0\right) \sim_{n\to\infty} \frac{C_2}{n^{1/2}}.$$

Let h_1, resp. h_2, be the renewal function associated to $(S_\ell)_{\ell\geq 0}$, resp. $(-S_\ell)_{\ell\geq 0}$.

Lemma 6.1 *Let* $(r_n)_{n\geq 0}$ *and* $(\lambda_n)_{n\geq 0}$ *be two sequences of numbers resp. in* $\mathbf{R}_+$ *and in* $(0,1)$ *and such that resp.* $\lim_{n\to\infty} \frac{r_n}{n^{1/2}} = 0$, *and* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$. *Let* $F : \mathbf{R}_+ \to \mathbf{R}$ *be a Riemann integrable function. We suppose that there exists a non-increasing function* $\overline{F} : \mathbf{R}_+ \to \mathbf{R}$ *such that* $|F(x)| \leq \overline{F}(x)$ *for any* $x \geq 0$ *and* $\int_{x\geq 0} x\overline{F}(x) < \infty$. *Let*

$\alpha \geq 0$. *Then, as* $n \to \infty$,

(6.1)

$$\mathrm{E}\left[F(S_n - y), \min_{k\in[0,n]} S_k \geq -\alpha, \min_{k\in[\lambda_n n,n]} S_k \geq y\right] \sim h_1(\alpha)\frac{C_1 C_2 \sqrt{\pi}}{\sigma\sqrt{2}} n^{-3/2} \int_{x\geq 0} F(x) h_2(x) dx$$

uniformly in $y \in [0, r_n]$.

Finally, we state the following inequalities. There exists a constant $c > 0$ such that for any $\alpha \geq 0$, $b \geq a \geq -\alpha$, and $n \geq 1$,

$$\mathrm{P}(S_n \in [a, b], \min_{j\leq n} S_j \geq -\alpha) \leq c(1 + \alpha)(1 + b - a)(1 + b)n^{-3/2}. \tag{6.2}$$

Let $0 < \lambda < 1$. There exists a constant $c = c(\lambda) > 0$ such that for any $\alpha \geq 0$,$b \geq a \geq 0$, $y \geq 0$ and $n \geq 1$,

$$\begin{aligned} &\mathrm{P}(S_n \in [y + a, y + b], \min_{j\leq n} S_j \geq -\alpha, \min_{\lambda n\leq j\leq n} S_j \geq y) \\ \leq\ & c(1 + \alpha)(1 + b - a)(1 + b)n^{-3/2}. \end{aligned} \tag{6.3}$$

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