Semimartingale Decompositions under a Continuous Expansion of the Filtration

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Based on joint work with Léo Neufcourt

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Historical Perspective

- K. Itô proposed in 1976 a type of filtration expansion now known as **initial expansion**
- Itô wanted to be able to treat B_1 as a constant, where B is standard Brownian motion. He wanted to write

$$\int_0^t \mathbf{B_1} H_s dB_s = \mathbf{B_1} \int_0^t H_s dB_s \tag{1}$$

  for a predictable integrand H.
- Itô created a larger filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B_1)$ and using semimartingale theory he justified the truth of (1)

- Itô's idea was reformulated in the following way: When can we enlarge a filtration in such a way so that martingales need not stay martingales, but at least remain semimartingales?
- Put more concisely, we want to expand the filtration in such a way that semimartingales remain semimartingales
- J. Jacod proved a useful result in this direction which is now known as **Jacod's Criterion:**
- **Theorem: Jacod's Criterion (1987):** Let L be a random variable with values in a standard Borel space $(E, \mathcal{E})$, and let $Q_t(\omega, dx)$ denote the regular conditional distribution of L given $\mathcal{F}_t$, each $t \geq 0$. Suppose that for each t there exists a positive σ-finite measure η_t on $(E, \mathcal{E})$ such that $Q_t(\omega, dx) \ll \eta_t(dx)$ a.s. Then every $\mathbb{F}$ semimartingale X is also an $\mathcal{F} \vee \sigma(L)$ semimartingale

A Second Approach

- A second approach began with the thesis of Martin Barlow in 1978.
- Instead of an initial enlargement à la Itô, he sought a gradual enlargement in order to render a nonnegative random variable L into a stopping time
- The obvious way to do this is to replace the filtration $\mathbb{F}$ by defining $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L \wedge t)$, and then L is a $\mathbb{G}$ stopping time
- The trick is to find conditions on L such that $\mathbb{F}$ semimartingales remain $\mathbb{G}$ semimartingales

New idea

- Younes Kchia (and the speaker) proposed to expand a filtration $\mathbb{F}$ continuously as time evolves, for example by continuously adding bits of information arising from another stochastic process
- **As always, the trick is to do this in a way that preserves the semimartingale property**
- The procedure is complicated, and we will give an outline of it
- **The new contribution presented in this talk however is to give a way to describe the new semimartingale decomposition as best as is possible**
- In particular, we are interested in the path properties of the new finite variation term under the expanded filtration
- First however we will try to explain why these types of results might be interesting or in extreme cases even useful

First Motivation is Trucking Networks in the US

- A primary means of shipping freight in the US is by truck. Trucks carry the largest share of freight by value, tons, and ton-miles for shipments less than 750 miles
- Rail is dominant for distances between 750 and 2000 miles (1200 and 3200 kilometers)
- Within a given state, trucking accounts for 81% of shipping
- One of the national shipping companies is Yellow Freight

- The basic problems related to a national truck routing framework can be summarized as follows:
- We have a set of indivisible and reusable resources
- Tasks arrive over time according to a probability distribution
- At each decision epoch, we have to decide what tasks to cover
- Resources generally have complex characteristics determining the feasibility of assigning a resource to a task
- We are interested in maximizing efficiency over a finite time horizon

- When making resource allocation decisions, one wants first to secure a "good" first period profit and also make sure that the allocation of trucks are going to be favorable in the second time period.
- Using dynamic programming, one can construct a function that gives the expected worth of a truck at a certain location at a certain time period.
- **As the trucks move, however, information is constantly received by management via GPS Tracking as well as communication from both the drivers and customers.**
- It is reasonable to model this information flow as continuous. If the underlying filtration is generated by truck positions and customer orders according to each "decision epoch," then this extra information, corrupted perhaps by noise, can be modeled as a continuous enlargement of the underlying filtration.

Trucking Route Networks

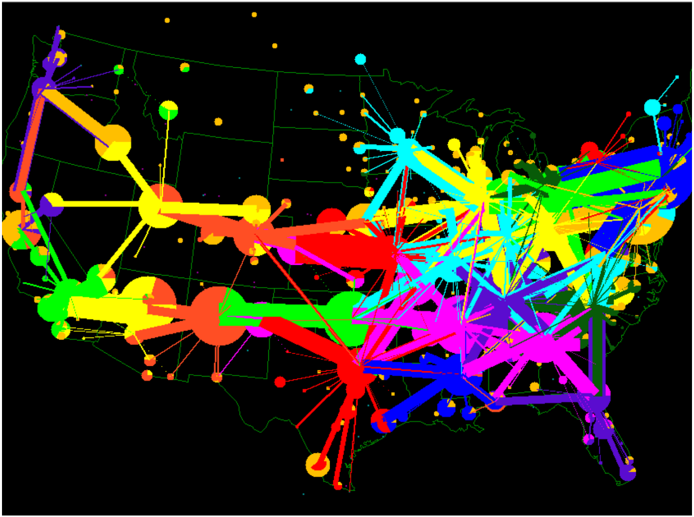

Figure: Yellow Freight Routing Map; H. Topaloglu

- The goal is to maximize the sum of the immediate profits, as well as the expected worth of a truck in the next time period. In this way, the functions estimate the impact of a decision in the current time period on the future time periods. Once the system is translated into profits in this way, we can construct a mathematical model along the lines of

$$X_t = M_t + A_t \text{ is a semimartingale modeling the profits at time } t \geq 0 \quad (2)$$

  where the profits include both profits and losses.
- The finite variation term $(A_t)_{t \geq 0}$ of (2) above can be modeled as $A_t = \int_0^t h_s ds$, denoting the cumulative profits (or losses) from time 0 to time $t > 0$.

- If one accepts this model, then we are in a framework where we can consider filtration enlargement.
- In this case the filtration enlargement process would come from information entering into the system as the semimartingale X of (2) evolves with time.
- Such information can take the form of order updates or cancelations, unforeseen supply chain issues, truck breakdowns and accidents, police interventions, unreliable drivers, and the like.

Second Motivation: High Energy Particle Collisions

- Michigan State University is building a new Facility for Rare Isotope Beams.
- High energy particles will be destroyed, allowing physicists to follow the reactions, by observing the light fragments (π, p, n, d, He) that are emitted.
- One then hopes to infer the detailed structure of the nuclei.
- This is related to nuclear liquid-gas phase transition.

- According to current theory, these collisions typically behave as follows: When two nuclei collide at high energy, a strong non-linear shock wave is formed.
- High density, pressure and temperature are formed in the region of the collision.
- The system expands due to its large internal pressure until the collisions between the particles cease to occur. Then the hydrodynamic description loses its validity and this is when one sees the light fragments.
- The light fragments are the data

- Fragment formation is a topic of “great current interest" in connection with the discovery of the nuclear liquid-gas phase transition.
- Were we to model the behavior of the particles as systems of stochastic differential equations (for example), then we would want to enlarge the underlying filtration with the evolving entropy and the viscous effects, which the physical theories show present a serious effect on the evolution of the particles.
- This would represent our filtration enlargement and subsequent change in the evolution of the particles.

Third Motivation: Insider Trading

- A model of a specialized kind of insider trading is the original motivation for our study.
- In the US the legality of insider trading is a complicated subject filled with nuance.
- There are types of trades which are (at least currently) perfectly legal but are most accurately described as insider trading.
- A dramatic example is provided by the high frequency traders known as co-locators.

- These companies place their trading machines next to the computers that process the trades of the stock exchange.
- They rent these co-locations directly from the stock exchanges themselves, and the fiber optics are carefully measured so that no one co-locator has a physical advantage over another. **Trades take place at intervals of 0.007 seconds.**
- These co-locators use various specialized orders (such as "immediate or cancel" orders) effectively to "see" the limit order book in the immediate future, and thus determine if a given stock is likely to go up or to go down in the immediate future.
- **With an enlarged filtration we get a new semimartingale decomposition and that affects the collection of risk neutral measures since we need to remove the drift with a Giranov transformation (if we can), and the point is that the drift has changed**
- The techniques presented today give a path to the mathematical modeling of such insider trading

The idea of a Continuous Expansion of a Filtration

1. We want to expand the filtration not just initially or with a random time, but dynamically with a stochastic process
2. The results of **Y. Kchia & M. Larsson** (also **N. El Karoui & Y. Jiao**) allows us to expand with a not necessarily adapted marked point process
3. This means to expand with a process X we can approximate it with a sequence of point processes, as time discretizations tend to zero in mesh size
4. With each point process approximation we expand the filtration, getting a sequence of larger and larger filtrations
5. We need the new filtrations to converge (we use the theory of **F. Antonelli, A. Kohatsu-Higa, F. Coquet, J. Mémin, and L. Słominski**)
6. We need the semimartingale decompositions to converge, so that we have a semimartingale in the limiting filtration (to do this we use and improve an old result of **M. Barlow-P^2**)

A Little More Detail

- **Step 1** is to approximate X with a sequence $(X^n)_{n\geq 1}$ of càdlàg processes that are marked point processes with possibly unordered jumps, and then expand with X^n to get a larger filtration $\mathbb{G}^n$
- **Step 2:** We choose the approximations X^n in such a way that we know that if M is an $\mathbb{F}$ semimartingale, then it is also a $\mathbb{G}^n$ semimartingale, and we can calculate N^n and A^n of its $\mathbb{G}^n$ Doob-Meyer decomposition:

$$M_t^n = N_t^n + A_t^n \tag{3}$$

- **We need some sort of a control on N^n and A^n in (3) as n increases to ∞,** to get a convergence of of the components of M^n, which is the $\mathbb{F}$ semimartingale M after the expansion

Weak convergence of filtrations-Step 2

- We combine the preceding with the (somewhat obscure) theory of the convergence of filtrations
- **Lemma:** A sequence of σ-fields $\mathcal{A}^n$ converges weakly to a σ-field $\mathcal{A}$ if and only if $E(Z \mid \mathcal{A}^n)$ converges in probability to Z for any integrable and $\mathcal{A}$ measurable random variable Z.
- **Lemma:** A sequence of filtrations $\mathbb{F}^n$ converges weakly to a filtration $\mathbb{F}$ if and only if $E(Z \mid \mathcal{F}_t^n)_{t\geq 0}$ converges in probability under the Skorohod J_1 topology to $E(Z \mid \mathcal{F}_t)_{t\geq 0}$, for any integrable, $\mathcal{F}_T$ measurable random variable Z.

- **Coquet, Mémin and Słominsky** provide a characterization of weak convergence of filtrations when the limiting filtration is the natural filtration of some càdlàg process X
- **Theorem:** Let $(\mathbb{F}^m)_{m\geq 1}$ be a sequence of filtrations. Let $\mathbb{F}$ be a filtration such that for all $t \in [0, T]$, $\mathcal{F}_t^m \xrightarrow{w} \mathcal{F}_t$. Define the filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0\leq t\leq T}$, where $\tilde{\mathcal{F}}_t = \bigvee_m \mathcal{F}_t^m$. Let X be an $\mathbb{F}$ adapted càdlàg process such that X is an $\tilde{\mathbb{F}}$ semimartingale. Then X is an $\mathbb{F}$ semimartingale.
- We next go to Step 3 and extend an old theorem of Barlow & P^2.

Convergence of Semimartingale Decompositions-Step 3

- A new version of the old result of **Barlow-P**^{2}
- **Theorem:** Let $(\mathbb{G}^n)_{n\geq 1}$ be a sequence of right-continuous filtrations and let $\mathbb{G}$ be a filtration such that $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$ for all t. Let $(X^n)_{n\geq 1}$ be a sequence of $\mathbb{G}^n$ semimartingales with canonical decomposition $X^n = X_0^n + M^n + A^n$. Assume there exists $K > 0$ such that for all n,

$$E(\int_0^T |dA_s^n|) \leq K \qquad \text{and} \qquad E(\sup_{0\leq s\leq T} |M_s^n|) \leq K$$

Then the following holds.

(i) Assume there exists a $\mathbb{G}$ adapted process X such that $E(\sup_{0\leq s\leq T} |X_s^n - X_s|) \to 0$. Then X is a $\mathbb{G}$ special semimartingale.

(ii) Moreover, assume $\mathbb{G}$ is right-continuous and let $X = M + A$ be the canonical decomposition of X. Then M is a $\mathbb{G}$ martingale and $\int_0^T |dA_s|$ and $\sup_{0\leq s\leq T} |M_s|$ are integrable.

- We say that a semimartingale Y is an $\mathbb{L}$ **nicely integrable** semimartingale if $Y = N + A$ is its canonical decomposition in $\mathbb{L}$ and there exists a constant K such that

$$E\left(\int_0^T |dA_s|\right) \leq K, \quad \text{and} \quad E\left(\sup_{o\leq s\leq T} |N_s|\right) \leq K. \tag{4}$$

- For a given semimartingale X that we are using for our expansion, we approximate X with X^n, where

$$X_t^n = \sum_{i=0}^{n+1}(X_{t_n^i} - X_{t_n^{i-1}})1_{\{t\geq t_n^i\}} \tag{5}$$

A Generalized Jacod's Criterion

- **Generalized Jacod's criterion:** There exists a sequence $(\pi_n)_{n\geq 1} = (\{t_i^n\})_{n\geq 1}$ of subdivisions of $[0, T]$ whose mesh tends to zero and such that for each n, $(X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, \ldots, X_T - X_{t_n^n})$ satisfies Jacod's criterion, i.e. there exists a σ-finite measure η_n on $\mathcal{B}(\mathbb{R}^{n+2})$ such that $P\big((X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, \ldots, X_T - X_{t_n^n}) \in \cdot \mid \mathcal{F}_t\big)(\omega) \ll \eta_n(\cdot)$ a.s

- We let $\mathbb{G}^0$ (resp. $\mathbb{G}$) be the smallest (resp. the smallest right-continuous) filtration containing $\mathbb{F}$ and relative to which X is adapted.
- **Theorem:** Assume X and $\mathbb{F}$ satisfy the Generalized Jacod's Criterion, and that either X is quasi-left continuous, or the sequence of subdivisions $(\pi_n)_{n\geq 1}$ is refining and all fixed times of discontinuity of X belong to $\cup_n \pi_n$.

  Let M be a continuous $\mathbb{F}$ martingale such that $E(\sup_{s\leq T}|M_s|)\leq K$ and $E(\int_0^T |dA_s^{(n)}|)\leq K$ for some K and all n. Then

  (i) M is a $\mathbb{G}^0$ special semimartingale, and

  (ii) Moreover, if $\mathbb{F}$ is the natural filtration of some càdlàg process Z, then M is a $\mathbb{G}$ special semimartingale with canonical decomposition $M = N + A$ such that N is a $\mathbb{G}$ martingale and $\sup_{0\leq s\leq T}|N_s|$ and $\int_0^T |dA_s|$ are integrable.

New Results with Léo Neufcourt

- We want to find the finite variation term A or at least determine its properties.
- In particular, within the Brownian paradigm, we would like sufficient conditions so that $dA_t \ll dt$ a.s. Such results are necessary conditions for an absence of arbitrage within a Math Finance context
- The idea is simple (its analysis is more difficult): Suppose for each decomposition in our sequence we have $X_t^n = M_t^n + \int_0^t \alpha_s^n ds$ for a filtration $\mathbb{G}^n$
- We call the integrand processes $(\alpha_s^n)_{s\geq 0}$ the **information drifts**

- We look for conditions such that $\int_0^t \alpha_s^n ds$ converge to a process of the form $\int_0^t \alpha_s ds$
- An old example due to **Th. Jeulin** shows that this need not be the case; that is, the limiting process A_t can have paths that are a.s. singular with respect to Lebesgue measure, similar to the paths of a local time
- We say that a sequence of σ algebras $(\mathcal{H}^n)_{n\geq 1}$ **converges to a σ algebra $\mathcal{H}$ in L^p if**

  For any $Z \in L^p(\mathcal{H}, P)$ we have $E(Z|\mathcal{H}^n) \to Z$ in L^p as $n \to \infty$

- We have an analogous notion for the convergence of filtrations in L^p

Convergence of the Information Drifts

- **Theorem:** Let $(\mathbb{G}^n)_{n\geq 1}$ be a non decreasing sequence of filtrations and suppose that M is a $\mathbb{G}^n$ semimartingale with decomposition $M = M^n + \int_0^t \alpha_s^n d[M,M]_s$ for every $n \geq 1$ for some $\mathbb{G}^n$ adapted process α^n. If $\mathbb{G}^n \to \mathbb{G}$ in L^2 and if

$$\sup_{n\geq 1} \int_0^T (\alpha_s^n)^2 d[M,M]_s < \infty \tag{6}$$

then M is also a $\mathbb{G}$ semimartingale with decomposition

$$M = \tilde{M} + \int \alpha_s d[M,M]_s \tag{7}$$

for a $\mathbb{G}$ adapted process α

- **Theorem:** Suppose a process X is continuous. Suppose for a sequence of finite partitions π^n with final element $\ell(n)+1$ we have that

$$(X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, \dots, X_{t^n_{\ell(n)+1}} - X_{t^n_{\ell(n)}})$$

satisfies the generalized Jacod's condition, and let α^n be the information drift of the n^{th} decomposition.

  (a) **If** $\sup_{n\geq 1} E \int_0^T |\alpha_s^n| d[M,M]_s < \infty$ **then** M **is a continuous** $\mathbb{G}$ **semimartingale;**

  (b) **If** $\sup_n E \int_0^T (\alpha_s^n)^2 d[M,M]_s < \infty$**, then** M **is a continuous** $\mathbb{G}$ **semimartingale decomposition**

$$M = \tilde{M} + \int \alpha_s d[M,M]_s$$

and $E \int_0^T (\alpha_s^n - \alpha_s)^2 d[M,M]_s \to 0$ as $n \to \infty$.

- **Remark:** The previous theorem has a version with X càdlàg and quasi-left continuous, such as for example a Hunt process. As examples we can take a class of Hunt processes of a special form: Let Z be a Lévy process and let X be the solution of an SDE of the form

$$X_t = X_0 + \int_0^t \sigma(X_{s-})dZ_s + \int_0^t b(X_s)ds \tag{8}$$

whjere σ and b are (for example) Lipschitz continuous. Then X is a Hunt process.
- For examples using this technique we first point out there are several interesting examples worked out by hand by **Jeulin, Yor**, long ago, and more recently **Corcuera, Imkeller, Kohatsu-Higa**, and **Jacod-P^2**, among others. We can reprove and sometimes improve them using these techniques
- **We also have some improvements of the old examples and a new example.**

A New Example

- In order to find expansions with a continuous anticipation satisfying the information drift property **we must consider the right speed of anticipation.**
- Let ϕ be a continuous time change, i.e. a non-decreasing stochastic process with continuous paths, independent from W, and let $X_t := W_{t \wedge \phi_t}$.
- The natural expansion of $\mathcal{F}$ with the process X is the filtration $\check{\mathcal{G}}$ given by $\check{\mathcal{G}}_t := \mathcal{F}_t \vee \sigma(X_s, s \leq t)$.
- It is equivalent and useful for applications to consider the right-continuous filtration $\mathcal{G}_t := \bigcap_{u>t}(\mathcal{F}_u \vee \sigma(X_s, s \leq u)), t \in I$.

- **Theorem:** For every $s \geq t$ define the random time $\tau(s,t) := \inf\{0 \leq u \leq s : u \vee \phi_u = t\}$ and suppose that it admits a $\mathcal{G}_s$-conditional density $u \mapsto f(u;s,t)$, with respect to Lebesgue measure, which is continuously differentiable in (s,t). Then the $\mathcal{G}$-information drift α of the Brownian motion W is given by

$$\alpha_s = \int_0^s W_{\phi_u} \partial_t f(u;s,t)\Big|_{t=s} du.$$

Thank You for Your Attention