Contents

CHAPTER 1

Reconstruction

In these lecture notes we want to present an introduction to (some of) the analytical aspects of regularity structures, with an emphasis on how to construct (some of) the most relevant objects.

1.1. Distributions

These lectures will concern the space $\mathscr{D}'(\mathbb{R}^d)$ of *distributions* or *generalised functions*. We consider the space $\mathscr{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support on $\mathbb{R}^d$.

A *distribution* on $\mathbb{R}^d$ is a linear functional $T : C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$ such that for every compact set $K \subset \mathbb{R}^d$ there is $r = r_K \in \mathbb{N}$

$$|T(\varphi)| \lesssim \|\varphi\|_{C^r} := \max_{|k|\leqslant r} \|\partial^k \varphi\|_\infty, \qquad \forall\, \varphi \in C_0^\infty(K) \tag{1.1.1}$$

where throughout these lecture notes $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leqslant C g$. If one can find a $r \in \mathbb{N}$ such that (1.1.1) holds for all compact set $K \subset \mathbb{R}^d$ then we say that T has *order* r.

Every locally integrable (in particular continuous) function $f : \mathbb{R}^d \to \mathbb{R}$ defines a distribution by integration:

$$f(\varphi) := \int_{\mathbb{R}^d} f(x)\, \varphi(x)\, \mathrm{d}x, \qquad \varphi \in \mathscr{D}(\mathbb{R}^d).$$

A famous example of distribution from quantum mechanics is the Dirac measure δ_x at $x \in \mathbb{R}^d$

$$\delta_x(\varphi) = \varphi(x), \qquad \varphi \in C^\infty(\mathbb{R}^d).$$

One can also differentiate any distribution $T \in \mathscr{D}'(\mathbb{R}^d)$ and obtain a new distribution: for $k \in \mathbb{N}^d$

$$\partial^k T(\varphi) := (-1)^{k_1+\cdots+k_d}\, T(\partial^k \varphi).$$

Distributions form a linear space. If $\varphi \in C^\infty(\mathbb{R}^d)$ and $T \in \mathscr{D}'(\mathbb{R}^d)$ then it is possible to define canonically the product $\varphi \cdot T = T \cdot \varphi$ as

$$\varphi \cdot T(\psi) = T \cdot \varphi(\psi) := T(\varphi\psi), \qquad \forall\, \psi \in C_c^\infty(\mathbb{R}^d).$$

However, if $T, T' \in \mathscr{D}'(\mathbb{R}^d)$, in general there is no canonical way of defining $T \cdot T'$.

One may use some form of regularisation of T, T' or both. Then, the result could heavily depend on the regularisation and thus be neither unique nor canonical. For example, there does not seem to exist a reasonable way to define the square $(\delta_x)^2$ of the Dirac function.

Regularity structures give a framework to define products of *certain* distributions, and to prove well-posedness of some PDEs where such distributions appear.

1.2. The main question of this chapter

For every $x \in \mathbb{R}^d$ we fix a distribution $F_x \in \mathscr{D}'(\mathbb{R}^d)$ and we call the family $(F_x)_{x\in\mathbb{R}^d}$ a germ if for all $\psi \in \mathscr{D}$, the map $x \mapsto F_x(\psi)$ is measurable.

Problem: Can we find a distribution $f \in \mathscr{D}'(\mathbb{R}^d)$ which is locally well approximated by $(F_x)_{x\in\mathbb{R}^d}$?

1.2.1. Taylor expansions. For example, let us fix $f \in C^\infty(\mathbb{R}^d)$, and let us define for a fixed $\gamma > 0$

$$F_x(y) := \sum_{|k|<\gamma} \partial^k f(x)\, \frac{(y-x)^k}{k!}, \qquad x, y \in \mathbb{R}^d\,. \tag{1.2.1}$$

Then the classical Taylor theorem says that there exists a function $R(x,y)$ such that

$$f(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma \tag{1.2.2}$$

uniformly for x, y on compact sets of $\mathbb{R}^d$. By (1.2.2) we say that the distribution defined by f is *locally well approximated* by the germ $(F_x)_{x\in\mathbb{R}^d}$ formed by its Taylor polynomials.

1.2.2. Scaling. Let us introduce now the fundamental tool of *scaling*: for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$, $\lambda > 0$ and $y \in \mathbb{R}^d$ we set

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d}\, \varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d\,. \tag{1.2.3}$$

When $y = 0$ we write $\varphi^\lambda = \varphi_0^\lambda$,

Then the local approximation property (1.2.2) implies

Proposition 1.2.1. *Let $f \in C^\infty(\mathbb{R}^d)$, $\gamma > 0$ and F_x be defined by* (1.2.1). *Then*

$$\left|(f - F_y)(\varphi_y^\lambda)\right| \lesssim \lambda^\gamma, \tag{1.2.4}$$

uniformly for y in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$ and $\varphi \in \mathscr{D}(B(0,1))$ with $\int |\varphi| \leqslant 1$.

PROOF. By (1.2.2) we have $f - F_y = R(y, \cdot)$ and $|R(y,w)| \lesssim |w-y|^\gamma$. Since φ_y^λ is supported by $B(y,\lambda)$ with $\int |\varphi_y^\lambda| = \int |\varphi|$,

$$\left|(f - F_y)(\varphi_y^\lambda)\right| = \left|\int_{\mathbb{R}^d} R(y,w)\, \varphi_y^\lambda(w)\, \mathrm{d}w\right|$$
$$\lesssim \sup_{w \in B(y,\lambda)} |w-y|^\gamma \int |\varphi_y^\lambda| \leqslant \lambda^\gamma$$

uniformly for y in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$ and $\varphi \in \mathscr{D}(B(0,1))$ with $\int |\varphi| \leqslant 1$. □

In this context we have another simple formula, which does not seem so well known.

Proposition 1.2.2. *Let $f \in C^\infty(\mathbb{R}^d)$, $\gamma > 0$ and F_x be defined by* (1.2.1)*. Then*

$$\left|(F_z - F_y)(\varphi_y^\lambda)\right| \lesssim (|y-z| + \lambda)^\gamma, \tag{1.2.5}$$

uniformly for y, z in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$ and $\varphi \in \mathscr{D}(B(0,1))$ with $\int |\varphi| \leqslant 1$.

PROOF. Let us note that we can Taylor expand also the derivatives of f for $|k| < \gamma$

$$\partial^k f(y) = \sum_{|\ell| < \gamma - |k|} \partial^{k+\ell} f(z)\, \frac{(y-z)^\ell}{\ell!} + R^k(y,z), \qquad |R^k(y,z)| \lesssim |y-z|^{\gamma - |k|},$$

uniformly for x, y on compact sets of $\mathbb{R}^d$. Then we can write

$$\begin{aligned} F_y(w) &= \sum_{|k|<\gamma} \partial^k f(y)\, \frac{(w-y)^k}{k!} \\ &= \sum_{|k|<\gamma} \left(\sum_{|\ell| < \gamma - |k|} \partial^{k+\ell} f(z)\, \frac{(y-z)^\ell}{\ell!} + R^k(y,z) \right) \frac{(w-y)^k}{k!} \\ &= F_z(w) + \sum_{|k|<\gamma} R^k(y,z)\, \frac{(w-y)^k}{k!}. \end{aligned}$$

Therefore we obtain the expression

$$F_z(w) - F_y(w) = - \sum_{|k|<\gamma} R^k(y,z)\, \frac{(w-y)^k}{k!}. \tag{1.2.6}$$

In particular

$$
\begin{aligned}
|F_z(w) - F_y(w)| &\leqslant \sum_{|k|<\gamma} |R^k(y,z)| \frac{|w-y|^k}{k!} \\
&\lesssim \sum_{|k|<\gamma} |y-z|^{\gamma-|k|} |w-y|^k \lesssim (|y-z| + |w-y|)^\gamma
\end{aligned}
$$

since $a^t b^s \leqslant (a+b)^t (a+b)^s$ for $a, b, t, s \geqslant 0$. Now by (1.2.3), for all $\varphi \in \mathscr{D}(B(0,1))$ with $\int |\varphi| \leqslant 1$

$$
\begin{aligned}
\left| \int_{\mathbb{R}^d} (F_z(w) - F_y(w))\, \varphi_y^\lambda(w)\, \mathrm{d}w \right| &\lesssim \sup_{w \in B(y,\lambda)} (|y-z| + |w-y|)^\gamma \int |\varphi_y^\lambda| \\
&\leqslant (|y-z| + \lambda)^\gamma.
\end{aligned}
$$

We have obtained (1.2.5). □

1.3. Reconstruction

We define throughout the paper

$$
\varepsilon_n := 2^{-n}, \qquad n \in \mathbb{N}.
$$

We have seen in (1.2.4) that for the germ $(F_y)_{y \in \mathbb{R}^d}$ related to a Taylor expansion of order $\gamma > 0$

$$
\left| (f - F_y)(\varphi_y^{\varepsilon_n}) \right| \lesssim \varepsilon_n^\gamma,
$$

uniformly for y in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$ and $\varphi \in \mathscr{D}(B(0,1))$ with $\int |\varphi| \leqslant 1$. This property does not rely explicitly on the smoothness of f, and seems to be a promising way of expressing the fact that $(F_y)_{y \in \mathbb{R}^d}$ locally approximates well (at order $\gamma > 0$) the distribution f.

This motivates the following:

Definition 1.3.1. *Let $(F_y)_{y \in \mathbb{R}^d} \subseteq \mathscr{D}'(\mathbb{R}^d)$ a family of distributions. We say that $f \in \mathscr{D}'(\mathbb{R}^d)$ is a reconstruction of $(F_y)_{y \in \mathbb{R}^d}$ if there exists $\gamma > 0$ such that for all $\varphi \in \mathscr{D}$*

$$
\left| (f - F_y)(\varphi_y^{\varepsilon_n}) \right| \lesssim \varepsilon_n^\gamma, \tag{1.3.1}
$$

uniformly for y in compact sets of $\mathbb{R}^d$ and $n \in \mathbb{N}$.

We are going to see below sufficient conditions for a family $(F_y)_{y \in \mathbb{R}^d} \subseteq \mathscr{D}'(\mathbb{R}^d)$ of distributions to admit a reconstruction. A first important remark is that, with this definition, there is at most one reconstruction for a given $(F_y)_{y \in \mathbb{R}^d}$.

We are going to use a number of times the following formula: for all $T \in \mathscr{D}'$ and $\varphi, g \in \mathscr{D}$

$$T(\varphi * g) = \int_{\mathbb{R}^d} T(\varphi(\cdot - y))\, g(y)\, \mathrm{d}y\,.$$

With the notation $\varphi_y(x) := \varphi(x-y) = \varphi_y^1(x)$, recall (1.2.3), we obtain the basic formula

$$T(\varphi * g) = \int_{\mathbb{R}^d} T(\varphi_y)\, g(y)\, \mathrm{d}y\,, \tag{1.3.2}$$

Lemma 1.3.2 (Uniqueness)**.** *Given any $(F_x)_{x\in\mathbb{R}^d} \subseteq \mathscr{D}'(\mathbb{R}^d)$ and $\gamma > 0$, there is at most one reconstruction of $(F_x)_{x\in\mathbb{R}^d}$ in the sense of Definition 1.3.1.*

PROOF. We fix a test function $\varphi \in \mathscr{D}$ with $\int \varphi = 1$, and two distributions $f, g \in \mathscr{D}'$ which satisfy, uniformly for y in compact sets,

$$\lim_{n\to\infty} |(f - F_y)(\varphi_y^{\varepsilon_n})| = \lim_{n\to\infty} |(g - F_y)(\varphi_y^{\varepsilon_n})| = 0\,. \tag{1.3.3}$$

We set $T := f - g$. For any $\psi \in \mathscr{D}$ we have $T(\psi) = \lim_{n\to\infty} T(\psi * \varphi^{\varepsilon_n})$. If K is any compact set which contains the support of ψ we have by (1.3.2)

$$|T(\psi * \varphi^{\varepsilon_n})| = \left| \int_{\mathbb{R}^d} T(\varphi_y^{\varepsilon_n})\, \psi(y)\, \mathrm{d}y \right| \leqslant \|\psi\|_{L^1} \sup_{y\in K} |T(\varphi_y^{\varepsilon_n})|\,.$$

It remains to show that $\lim_{n\to\infty} T(\varphi_y^{\varepsilon_n}) = 0$ uniformly for $y \in K$, for which it is enough to observe that

$$|T(\varphi_y^{\varepsilon_n})| = |f(\varphi_y^{\varepsilon_n}) - g(\varphi_y^{\varepsilon_n})| \leqslant |(f - F_y)(\varphi_y^{\varepsilon_n})| + |(g - F_y)(\varphi_y^{\varepsilon_n})|$$

and these terms vanish as $n \to \infty$ uniformly for y in compact sets, by (1.3.3). □

1.4. Coherence

We have seen in (1.2.5) that for the germ related to a Taylor expansion we have for any $\gamma > 0$

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_N})\right| \lesssim (|y - z| + \varepsilon_N)^\gamma, \qquad \left|(f - F_y)(\varphi_y^{\varepsilon_N})\right| \lesssim \varepsilon_N^\gamma,$$

uniformly for y, z in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$ and $\varphi \in \mathscr{D}(B(0,1))$ with $\int |\varphi| \leqslant 1$.

However the first estimate implicitly relies on the information that the distribution $F_z - F_y$ is a locally bounded function: suppose indeed that this is not the case; then we expect that the quantity $(F_z - F_y)(\varphi_y^{\varepsilon_N})$ does not necessarily remain bounded as $n \to \infty$; this is the case for example if $F_z - F_y$ is a Dirac mass at y, where

$$(F_z - F_y)(\varphi_y^{\varepsilon_N}) = \frac{1}{\varepsilon_N^d}\, \varphi(0). \tag{1.4.1}$$

Therefore, if we want to consider more general families $(F_y)_{y\in\mathbb{R}^d}$ of genuine distributions, we expect (1.2.5) to be too strong a requirement.

Formula (1.4.1) suggests that a weaker version of (1.2.5), which could be convenient in this context, may be obtained by allowing a multiplicative factor ε_N^α with $\alpha \leqslant 0$ in (1.2.5):

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_N})\right| \lesssim \varepsilon_N^\alpha (|y-z| + \varepsilon_N)^\gamma. \tag{1.4.2}$$

However, it turns out that (1.4.2) may not be strong enough to obtain (1.3.1): the multiplicative factor ε_N^α, which explodes as $n \to \infty$ if $\alpha < 0$, makes a better control on the factor $(|y-z| + \varepsilon_N)$ necessary, as can be seen from the proof of Theorem 1.5.1 below. It turns out that a sufficient condition for the existence of a (unique) reconstruction is

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_N})\right| \lesssim \varepsilon_N^\alpha (|y-z| + \varepsilon_N)^{\gamma-\alpha},$$

uniformly for z, y in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$, We call this property *coherence*, see below.

Definition 1.4.1. *We say that a germ $(F_z)_{z\in\mathbb{R}^d} \subset \mathscr{D}'$ is (α,γ)-*coherent *for $\gamma \in \mathbb{R}$, and $\alpha \leqslant \gamma \wedge 0$, if there exists $\varphi \in \mathscr{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$, such that*

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\alpha (|y-z| + \varepsilon_n)^{\gamma-\alpha}, \tag{1.4.3}$$

uniformly for z, y in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$.

We denote by $\mathscr{G}^{\alpha,\gamma}$ the set of (α,γ)-coherent germs.

Remark 1.4.2.

- Measurability of the map $x \mapsto F_x(\psi)$ is a technical assumption, which is needed in the definition of suitable approximations to the reconstruction of $(F_z)_{z\in\mathbb{R}^d}$.
- It is a non obvious (but true) fact, see [**2**, Proposition 13.1], that relation (3.3.1) actually holds uniformly over $\varphi \in \mathscr{D}(B(0,1))$ with bounded $\|\varphi\|_{C^{r_\alpha}}$. More precisely:

$$|(F_z - F_y)(\varphi_y^{\varepsilon_n})| \lesssim \|\varphi\|_{C^{r_\alpha}}\, \varepsilon_n^\alpha \,(|y-z| + \varepsilon_n)^{\gamma-\alpha}, \tag{1.4.4}$$

  uniformly for x, y, z in compact sets, $n \in \mathbb{N}$ and $\varphi \in \mathscr{D}(B(0,2))$, where $r_\alpha := \min\{k \in \mathbb{N} : k > -\alpha\}$.
- In particular, $\mathscr{G}^{\alpha,\gamma}$ is a vector space.

1.5. Hairer's Reconstruction Theorem (without regularity structures)

We define the following family of test functions:

$$\mathscr{B}_r := \{\psi \in \mathscr{D}(B(0,1)) : \ \|\psi\|_{C^r} \leqslant 1\}. \tag{1.5.1}$$

THEOREM 1.5.1 (Reconstruction Theorem). *Suppose that* $(F_z)_{z\in\mathbb{R}^d} \subset \mathscr{D}'$ *is a* (α,γ)-coherent germ *in the sense of Definition 1.4.1 with* $\gamma > 0$, *namely there exist* $\gamma > 0$, $\alpha \leqslant \gamma$ *and a* $\varphi \in \mathscr{D}(\mathbb{R}^d)$ *with* $\int \varphi \neq 0$, *such that*

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\alpha}(|x-y| + \varepsilon_n)^{\gamma-\alpha},$$

uniformly for x, y *in compact sets of* $\mathbb{R}^d$, $n \in \mathbb{N}$. *Then there exists a unique* $\mathscr{R}F \in \mathscr{D}'(\mathbb{R}^d)$ *such that*

$$|(\mathscr{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\gamma} \tag{1.5.2}$$

uniformly for x *in compact sets of* $\mathbb{R}^d$, $n \in \mathbb{N}$, $\psi \in \mathscr{B}_r$, *see* (1.5.1), *for any fixed integer* $r > -\alpha$.

- This result was stated and proved by Martin Hairer in [**5**, Thm. 3.10] for a subclass of germs related to regularity structures. He used wavelets.
- Later Otto-Weber [**7**] proposed an approach based on a semigroup. This corresponds to a special choice of the test functions φ, ψ. See also [**6**].
- The above statement is a slight improvement of [**2**, Thm. 5.1]. It is more general and requires no knowledge of regularity structures. The improvement is due to [**8**] and concerns the fact that it is not necessary to impose a homogeneity condition on the germ (see below).
- This result can be seen as a generalisation of the Sewing Lemma in rough paths [**4, 3**].
- The construction is completely local: constants and even the exponent α can depend on the compact set.
- We also cover the case $\gamma \leqslant 0$ (see below).
- There is clearly an analogy between the Reconstruction Theorem and the Sewing Lemma: see [**1**, section 5] for a discussion.

Example 1.5.2. *Let* $A \subset \mathbb{R}$ *be a (locally) finite set such that* $\alpha := \inf A \in \mathbb{R}$. *Let* $F = (F_x)_{x\in\mathbb{R}^d}$ *be a germ such that, for some* $\gamma \geqslant \alpha$ *and a* $\varphi \in \mathscr{D}(\mathbb{R}^d)$ *with* $\int \varphi \neq 0$, *we have*

$$|(F_z - F_y)(\varphi_y^{\varepsilon})| \lesssim \sum_{a\in A:\ a<\gamma} \varepsilon^a \, |z-y|^{\gamma-a}, \tag{1.5.3}$$

uniformly for z, y *in compact sets and for* $\varepsilon \in (0,1]$.

Then the germ F *is* (α,γ)*-coherent, since*

$$\varepsilon^a \, |z-y|^{\gamma-a} = \varepsilon^{\alpha} \, \varepsilon^{a-\alpha} \, |z-y|^{\gamma-a} \leqslant \varepsilon^{\alpha} \, (\varepsilon + |z-y|)^{\gamma-\alpha}.$$

For example we saw in (1.2.5) *that the Taylor expansions* (1.2.1) *satisfy* (1.5.3) *with* $A = \mathbb{N}$ *and* $\alpha = 0$.

Remark 1.5.3. If $(F_z)_{z\in\mathbb{R}^d} \subset \mathscr{D}'$ is a (α,γ)-coherent germ and $\alpha > 0$, then the map $z \mapsto F_z$ is constant, so that we implicitly assume from now on that $\alpha \leqslant 0$. In order to prove the claim, we apply the triangular inequality

$$|(F_y - F_x)(\varphi_z^{\varepsilon_n})| \leqslant |(F_y - F_z)(\varphi_z^{\varepsilon_n})| + |(F_z - F_x)(\varphi_z^{\varepsilon_n})| \to 0$$

as $n \to +\infty$ (uniformly for x,y,z in compact sets) by the coherence assumption. Then we obtain for all $\psi \in \mathscr{D}$ by (1.3.2)

$$\begin{aligned}(F_y - F_x)(\psi) &= \lim_{n\to+\infty} (F_y - F_x)(\psi * \varphi^{\varepsilon_n}) \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^d} (F_y - F_x)(\varphi_z^{\varepsilon_n})\, \psi(z)\, \mathrm{d}z = 0.\end{aligned}$$

1.6. Sketch of the proof

In this section we give a sketch of the proof of Theorem 1.5.1.

We fix a (α,γ)-coherent germ $(F_z)_{z\in\mathbb{R}^d} \subset \mathscr{D}'$, i.e. we suppose that there exist $\gamma > 0, \alpha \leqslant 0$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$, such that

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^{\alpha}(|y-z| + \varepsilon_n)^{\gamma-\alpha}, \tag{1.6.1}$$

uniformly for z, y in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$. We find in an elementary way a related $\hat{\varphi} \in \mathscr{D}(B(0,1))$ such that

$$\int_{\mathbb{R}^d} \hat{\varphi}(y)\, \mathrm{d}y = 1\,, \quad \int_{\mathbb{R}^d} y^k\, \hat{\varphi}(y)\, \mathrm{d}y = 0\,, \quad \forall\, k \in \mathbb{N}^d:\ 1 \leqslant |k| \leqslant r-1\,, \tag{1.6.2}$$

for a given $r > -\alpha$, and (1.6.1) holds with φ replaced by $\hat{\varphi}$, see [**2**, Lemma 8.3]. Then we define

$$\rho := \hat{\varphi}^2 * \hat{\varphi} \qquad \text{and} \qquad \varepsilon_n := 2^{-n}, \quad n \in \mathbb{N}\,, \tag{1.6.3}$$

where we recall that $\psi^{\varepsilon_N} = \psi_0^{\varepsilon_N}$ is a scaling of ψ as in (1.2.3). Note that $\int \rho = \int \hat{\varphi}^2 \int \hat{\varphi} = 1$. This peculiar choice of ρ ensures that *the difference* $\rho^{\frac{1}{2}} - \rho$ *is a convolution*:

$$\rho^{\frac{1}{2}} - \rho = \hat{\varphi} * \check{\varphi}\,, \qquad \text{where we define} \qquad \check{\varphi} := \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2\,. \tag{1.6.4}$$

By (1.6.2),

$$\int_{\mathbb{R}^d} y^k\, \check{\varphi}(y)\, \mathrm{d}y = 0\,, \quad \forall\, k \in \mathbb{N}^d:\ 0 \leqslant |k| \leqslant r-1\,. \tag{1.6.5}$$

This will be used below to subtract suitable Taylor polynomials. Moreover it follows that

$$\rho^{\varepsilon_{n+1}} - \rho^{\varepsilon_n} = (\rho^{\frac{1}{2}} - \rho)^{\varepsilon_n} = \hat{\varphi}^{\varepsilon_n} * \check{\varphi}^{\varepsilon_n}\,. \tag{1.6.6}$$

With these definitions, we can now set

$$f_n(\psi) := \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n})\, \psi(z)\, \mathrm{d}z\,, \qquad \psi \in \mathscr{D}\,, \tag{1.6.7}$$

recall (1.3.2). We study the function

$$f_{x,n}(z) := f_n - F_x(\rho_z^{\varepsilon_n}) = (F_z - F_x)(\rho_z^{\varepsilon_n}), \qquad x, z \in \mathbb{R}^d. \tag{1.6.8}$$

We write $f_{x,n}$ as a telescoping sum:

$$\begin{aligned} & f_{x,k+1}(z) - f_{x,k}(z) = (F_z - F_x)(\rho_z^{\varepsilon_{k+1}} - \rho_z^{\varepsilon_k}) \\ & = (F_z - F_x)(\hat{\varphi}^{\varepsilon_n} * \check{\varphi}^{\varepsilon_n}) = \int_{\mathbb{R}^d} (F_z - F_x)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y-z)\, \mathrm{d}y \\ & = \underbrace{\int_{\mathbb{R}^d} (F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y-z)\, \mathrm{d}y}_{g'_{x,k}(z)} + \underbrace{\int_{\mathbb{R}^d} (F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y-z)\, \mathrm{d}y}_{g''_k(z)}, \end{aligned} \tag{1.6.9}$$

where again we use (1.3.2). We have first, for all $z \in \mathbb{R}^d$

$$|g''_k(z)| \leqslant \|\check{\varphi}^{\varepsilon_k}\|_{L^1} \sup_{|y-z| \leqslant \varepsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \lesssim \varepsilon_k^{\alpha}\, \varepsilon_k^{\gamma-\alpha} = \varepsilon_k^{\gamma},$$

since $\|\check{\varphi}^{\varepsilon_k}\|_{L^1} = \|\check{\varphi}\|_{L^1}$. Then we obtain for all $\psi \in \mathscr{D}$

$$\left| \int_{\mathbb{R}^d} g''_k(z)\, \psi(z)\, \mathrm{d}z \right| \lesssim \varepsilon_k^{\gamma} \|\psi\|_{L^1}. \tag{1.6.10}$$

Now we want to estimate

$$\int_{\mathbb{R}^d} g'_{x,k}(z) \psi(z)\, \mathrm{d}z = \int_{\mathbb{R}^d} (F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})\, (\check{\varphi}^{\varepsilon_k} * \psi)(y)\, \mathrm{d}y. \tag{1.6.11}$$

If K is the support of ψ and $\bar{K}_1$ is the subset of $\mathbb{R}^d$ which has distance $\leqslant 1$ from K, we obtain that $\check{\varphi}^{\varepsilon} * \psi$ has support in $\bar{K}_1$. Then by the coherence condition

$$\left| \int_{\mathbb{R}^d} g'_{x,k}(z) \psi(z)\, \mathrm{d}z \right| \leqslant \sup_{y \in \bar{K}_1} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})|\, \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1} \lesssim \varepsilon_k^{\alpha} \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1}.$$

Note now that by (1.6.5)

$$(\check{\varphi}^{\varepsilon} * \psi)(y) = \int_{\mathbb{R}^d} \check{\varphi}^{\varepsilon}(y-z) \left\{ \psi(z) - p_y(z) \right\} \mathrm{d}z,$$

where $p_y(\cdot) := \sum_{|k| \leqslant r-1} \frac{\partial^k \psi(y)}{k!} (\cdot - y)^k$ the Taylor polynomial of ψ of order $r-1$ based at y; therefore

$$|(\check{\varphi}^{\varepsilon} * \psi)(y)| \leqslant \|\psi\|_{C^r} \int_{\mathbb{R}^d} |\check{\varphi}^{\varepsilon}(y-z)|\, |z-y|^r\, \mathrm{d}z \leqslant \|\psi\|_{C^r} \|\check{\varphi}\|_{L^1} \varepsilon^r, \quad y \in \mathbb{R}^d.$$

We obtain

$$\left| \int_{\mathbb{R}^d} g'_{x,k}(z)\, \psi(z)\, \mathrm{d}z \right| \lesssim \varepsilon_k^{\alpha+r} \|\psi\|_{C^r}. \tag{1.6.12}$$

In particular we obtain by (1.6.10)-(1.6.12), since $\gamma > 0$ and $\alpha + r > 0$, that

$$f_{x,n}(\psi) = f_{x,0}(\psi) + \sum_{k=0}^{n-1} \left[g'_{x,k}(\psi) + g''_k(\psi) \right]$$

converges as $n \to +\infty$ to a distribution of order r. Note now that $F_x(\rho^{\varepsilon_n}_\cdot)$ converges to F_x in $\mathscr{D}'$, since by (1.3.2)

$$\int_{\mathbb{R}^d} F_x(\rho^{\varepsilon_n}_z)\, \psi(z)\, \mathrm{d}z = F_x(\rho^{\varepsilon_n} * \psi) \to F_x(\psi)\,, \qquad \forall\, \psi \in \mathscr{D}.$$

We obtain by (1.6.8) that f_n converges to a distribution $\mathscr{R}F$ in $\mathscr{D}'$. Moreover, since for all $n \geqslant \ell$ we have

$$f_{x,n}(\psi) = f_{x,\ell}(\psi) + \sum_{k=\ell}^{n-1} \left[g'_{x,k}(\psi) + g''_k(\psi) \right], \tag{1.6.13}$$

letting $n \to +\infty$ we obtain that for all $x \in \mathbb{R}^d$, $\psi \in \mathscr{D}$ and $\ell \in \mathbb{N}$

$$\mathscr{R}F(\psi) = F_x(\psi) + f_{x,\ell}(\psi) + \sum_{k=\ell}^{\infty} \left[g'_{x,k}(\psi) + g''_k(\psi) \right]. \tag{1.6.14}$$

Formula (1.6.14) is due to [**8**].

We want now to prove the reconstruction bound (1.5.2). We recall the following result, proved in [**2**, Lemma 9.3]: let $\lambda, \varepsilon > 0$ and $G : \mathbb{R}^d \to \mathbb{R}$ a measurable function; then for all $x \in \mathbb{R}^d$ and $\psi \in \mathscr{B}_r$, see (1.5.1),

$$\left| \int_{\mathbb{R}^d} G(y)\, (\check{\varphi}^{\varepsilon} * \psi_x^{\varepsilon_N})(y)\, \mathrm{d}y \right| \leqslant 4^d\, \|\check{\varphi}\|_{L^1}\, \min\left\{ \varepsilon/\lambda, 1 \right\}^r \sup_{B(x,\lambda+\varepsilon)} |G|\,. \tag{1.6.15}$$

By (1.6.11) and (1.6.15)

$$\left| \int_{\mathbb{R}^d} g'_{x,k}(z)\, \psi_x^{\varepsilon_N}(z)\, \mathrm{d}z \right| \leqslant 4^d\, \|\check{\varphi}\|_{L^1}\, \min\left\{ \varepsilon_k/\lambda, 1 \right\}^r \sup_{y \in B(x,\lambda+\varepsilon_k)} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})|\,.$$

For $y \in B(x, \lambda + \varepsilon_k)$, by (1.6.1) with φ replaced by $\hat{\varphi}$, we have

$$|(F_x - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \lesssim \varepsilon_k^{\alpha} (|x-y| + \varepsilon_k)^{\gamma-\alpha} \lesssim \max\{\varepsilon_k, \lambda\}^{\gamma-\alpha}\, \varepsilon_k^{\alpha}\,.$$

We have obtained

$$\left| \int_{\mathbb{R}^d} g'_{x,k}(z)\, \psi_x^{\varepsilon_N}(z)\, \mathrm{d}z \right| \lesssim \begin{cases} \lambda^{\gamma-\alpha-r}\, \varepsilon_k^{\alpha+r} & \text{if } \varepsilon_k < \lambda \\ \varepsilon_k^{\gamma} & \text{if } \varepsilon_k \geqslant \lambda \end{cases}. \tag{1.6.16}$$

We now fix $\lambda = \varepsilon_\ell$. We want to estimate $\tilde{f}_x := \mathscr{R}F - F_x$, and in particular $\tilde{f}_x(\psi_x^{\varepsilon_\ell})$. We write

$$|\tilde{f}_x(\psi_x^{\varepsilon_\ell})| \leqslant |f_{x,\ell}(\psi_x^{\varepsilon_\ell})| + |(\tilde{f}_x - f_{x,\ell})(\psi_x^{\varepsilon_\ell})|.$$

First by (1.6.3)

$$f_{x,\ell}(\psi_x^{\varepsilon_\ell}) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (F_z - F_x)(\hat{\varphi}_y^{\varepsilon_\ell})\,\hat{\varphi}^{2\varepsilon_\ell}(y-z)\,\psi_x^{\varepsilon_\ell}(z)\,\mathrm{d}y\,\mathrm{d}z\,,$$

so that

$$|f_{x,\ell}(\psi_x^{\varepsilon_\ell})| \leqslant \|\hat{\varphi}^{2\varepsilon_\ell}\|_{L^1}\,\|\psi_x^{\varepsilon_\ell}\|_{L^1} \sup_{z\in B(x,\varepsilon_\ell),|y-z|\leqslant\varepsilon_\ell} |(F_z - F_x)(\hat{\varphi}_y^{\varepsilon_\ell})|.$$

Now we write $|(F_z - F_x)(\hat{\varphi}_y^{\varepsilon_\ell})| \leqslant |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_\ell})| + |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_\ell})|$ and

$$\sup_{z\in B(x,\varepsilon_\ell),|y-z|\leqslant\varepsilon_\ell} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_\ell})| \lesssim \varepsilon_\ell^\alpha\,\varepsilon_\ell^{\gamma-\alpha} \leqslant \varepsilon_\ell^\gamma\,,$$

$$\sup_{z\in B(x,\varepsilon_\ell),|y-z|\leqslant\varepsilon_\ell} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_\ell})| \lesssim \varepsilon_\ell^\alpha\,(\varepsilon_\ell + 2\varepsilon_\ell)^{\gamma-\alpha} \lesssim \varepsilon_\ell^\gamma\,,$$

so that we obtain

$$|f_{x,\ell}(\psi_x^{\varepsilon_\ell})| \lesssim \varepsilon_\ell^\gamma\,, \tag{1.6.17}$$

and this argument holds for any $\gamma \in \mathbb{R}$. Now by (1.6.14)

$$(\tilde{f}_x - f_{x,\ell})(\psi_x^{\varepsilon_\ell}) = \sum_{k=\ell}^{\infty}\left[g'_{x,k}(\psi) + g''_k(\psi)\right],$$

and by (1.6.10)-(1.6.16),

$$\begin{aligned}
|(\tilde{f}_x - f_{x,\ell})(\psi_x^{\varepsilon_\ell})| &\leqslant \sum_{k\geqslant\ell}\left[|g'_{x,k}(\psi_x^{\varepsilon_\ell})| + |g''_k(\psi_x^{\varepsilon_\ell})|\right] \\
&\lesssim \sum_{k\geqslant\ell}\left[\varepsilon_\ell^{\gamma-\alpha-r}\,\varepsilon_k^{\alpha+r} + \varepsilon_k^\gamma\right] \\
&\leqslant \frac{\varepsilon_\ell^{\gamma-\alpha-r}\,\varepsilon_\ell^{\alpha+r}}{1-2^{-(\alpha+r)}} + \frac{\varepsilon_\ell^\gamma}{1-2^{-\gamma}} \lesssim \varepsilon_\ell^\gamma\,,
\end{aligned}$$

since $\gamma > 0$ and $\alpha + r > 0$. The proof is complete.

1.7. The Reconstruction Theorem for $\gamma \leqslant 0$.

In Theorem 1.5.1 we have proved the existence and the uniqueness of the reconstruction of a (α,γ)-coherent germ in the case of $\gamma > 0$. If $\gamma \leqslant 0$ then we have a weaker result.

THEOREM 1.7.1. *Suppose that for a given $F : \mathbb{R}^d \to \mathscr{D}'(\mathbb{R}^d)$ there exist $\gamma \leqslant 0$ and $\alpha \leqslant \gamma$, such that for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$*

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\alpha(|x-y| + \varepsilon_n)^{\gamma-\alpha},$$

uniformly for x, y in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$. Then there exists a (non-unique) $\mathscr{R}F \in \mathscr{D}'(\mathbb{R}^d)$ such that

$$|(\mathscr{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \begin{cases} \varepsilon_n^\gamma & \text{if } \gamma < 0 \\ 1+n & \text{if } \gamma = 0 \end{cases}. \tag{1.7.1}$$

uniformly for x in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$, $\{\psi \in \mathscr{D}(B(0,1)) : \|\psi\|_{C^r} \leqslant 1\}$ with a fixed $r > -\alpha$.

PROOF. If one checks the proof of the case $\gamma > 0$, one sees that the convergence of the different terms depends either on $\gamma > 0$ or on $\alpha + r > 0$. More precisely, the estimate (1.6.10) on g_k'' is useful if $\gamma > 0$, while the estimate (1.6.12) on $g_{x,k}'$ is useful if $\alpha + r > 0$. If $\gamma \leqslant 0$, the estimate on g_k'' is simply not good enough.

On the other hand, for $\gamma \leqslant 0$ the reconstruction bound (1.7.1) is weaker, since ε_n^γ or n diverge as $n \to \infty$, and we do not state that there is a unique choice for $\mathscr{R}F$.

In fact, in order to prove the statement we can modify the approximating sequence f_n defined in (1.6.7), by eliminating the term g_k'' whose convergence is based on $\gamma > 0$. However, $g_{x,k}'$, given by (1.6.11) above, depends on $x \in \mathbb{R}^d$, while we want the approximating sequence $\bar{f}_n \in \mathscr{D}'$ to be independent of any base point.

We define, recalling (1.6.7) and (1.6.9),

$$\bar{f}_n := f_n - \sum_{k=0}^{n-1} g_k'',$$

$$\bar{f}_{x,n}(\psi) := \bar{f}_n(\psi) - F_x(\rho^{\varepsilon_n} * \psi) = f_{x,n}(\psi) - \sum_{k=0}^{n-1} g_k''(\psi).$$

Then, by (1.6.13), for all $n \geqslant \ell$,

$$\bar{f}_{x,n}(\psi) = f_{x,\ell}(\psi) + \sum_{k=\ell}^{n-1} g_{k,x}'(\psi) - \sum_{k=0}^{\ell-1} g_k''(\psi) = \bar{f}_{x,\ell}(\psi) + \sum_{k=\ell}^{n-1} g_{k,x}'(\psi). \tag{1.7.2}$$

By the estimate (1.6.12) on $g_{x,k}'$, we obtain that $\bar{f}_{x,n}$, and therefore $\bar{f}_n$, converge in $\mathscr{D}'$ and we can write for all $\psi \in \mathscr{D}$, $x \in \mathbb{R}^d$ and $\ell \in \mathbb{N}$

$$\mathscr{R}F(\psi) = \lim_n \bar{f}_n(\psi) = F_x(\psi) + \bar{f}_{x,\ell}(\psi) + \sum_{k=\ell}^{\infty} g_{k,x}'(\psi). \tag{1.7.3}$$

For the reconstruction bound (1.7.1), we want to estimate $\bar{f}_x := \mathscr{R}F - F_x$, and in particular $\bar{f}_x(\psi_x^{\varepsilon_\ell})$. We write

$$|\bar{f}_x(\psi_x^{\varepsilon_\ell})| \leqslant |\bar{f}_{x,\ell}(\psi_x^{\varepsilon_\ell})| + |(\bar{f}_x - \bar{f}_{x,\ell})(\psi_x^{\varepsilon_\ell})|.$$

Now, by (1.6.16) and (1.7.3), if $\gamma \leqslant 0$

$$\begin{aligned} |(\bar{f}_x - \bar{f}_{x,\ell})(\psi_x^{\varepsilon_\ell})| &\leqslant \sum_{k \geqslant \ell} |g'_{x,k}(\psi_x^{\varepsilon_\ell})| \\ &\lesssim \sum_{k \geqslant \ell} \varepsilon_\ell^{\gamma-\alpha-r}\, \varepsilon_k^{\alpha+r} \lesssim \varepsilon_\ell^{\gamma-\alpha-r}\, \varepsilon_\ell^{\alpha+r} = \varepsilon_\ell^{\gamma}, \end{aligned}$$

since $\alpha + r > 0$. By (1.6.17) and by (1.6.16), if $\gamma < 0$

$$\begin{aligned} |\bar{f}_{x,\ell}(\psi_x^{\varepsilon_\ell})| &\leqslant |f_{x,\ell}(\psi_x^{\varepsilon_\ell})| + \sum_{k=0}^{\ell-1} \left| g''_k(\psi_x^{\varepsilon_\ell}) \right| \\ &\lesssim \varepsilon_n^{\gamma} + \sum_{k=0}^{\ell-1} 2^{|\gamma| k} \lesssim 2^{|\gamma| \ell}. \end{aligned}$$

In the case $\gamma = 0$ we have rather

$$|\bar{f}_{x,\ell}(\psi_x^{\varepsilon_\ell})| \leqslant |f_{x,\ell}(\psi_x^{\varepsilon_\ell})| + \sum_{k=0}^{\ell-1} \left| g''_k(\psi_x^{\varepsilon_\ell}) \right| \lesssim 1 + \ell\,.$$

The proof is complete. □

1.8. Homogeneity

Definition 1.8.1. *Let F be a germ. We say that F satisfies a homogeneity bound with exponent $\bar{\alpha} \in \mathbb{R}$ if*

$$|F_x(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\bar{\alpha}}\,,$$

uniformly for x in compact sets, $n \in \mathbb{N}$ and $\psi \in \mathscr{B}_{r_{\bar{\alpha}}}$ with $r_{\bar{\alpha}} = \min\{n \in \mathbb{N} : n > -\bar{\alpha}\}$, see (1.5.1)*.*

We recall the following result, which is proved in [**2**, Lemma 4.12].

Lemma 1.8.2 (Homogeneity)**.** *Let $F = (F_x)_{x \in \mathbb{R}^d}$ be a (α, γ)-coherent germ. For any compact set $K \subseteq \mathbb{R}^d$, there is a real number $\bar{\alpha}_K < \gamma$ such that*

$$|F_x(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\bar{\alpha}_K} \quad \textit{uniformly for } x \in K \textit{ and } n \in \mathbb{N}\,, \tag{1.8.1}$$

with φ as in Definition 1.4.1.

Therefore coherence of a germ implies a local form of homogeneity of the same germ. However in Definition 1.8.1 we require the coefficient $\bar{\alpha}$ to be uniform over the compact set K.

If a germ satisfies a homogeneity bound with exponent $\bar{\alpha} \in \mathbb{R}$, then it satisfies a homogeneity bound with exponent $\bar{\alpha}'$ for all $\bar{\alpha}' \leqslant \bar{\alpha}$. Therefore the set of $\bar{\alpha} \in \mathbb{R}$ such that a fixed germ satisfies a homogeneity bound with exponent $\bar{\alpha}$ takes the form $]-\infty, b]$ or $]-\infty, b[$; in particular the exponent which appears in the proof of Lemma 1.8.2 is not necessarily optimal.

Definition 1.8.3. *We denote by $\mathscr{G}^{\bar{\alpha};\alpha,\gamma}$ the set of (α,γ)-coherent germs which satisfy a homogeneity bound with exponent $\bar{\alpha}$.*

1.9. Negative Hölder (Besov) spaces

Given $\alpha \in]-\infty, 0[$, we define $\mathscr{C}^\alpha = \mathscr{C}^\alpha(\mathbb{R}^d)$ as the space of distributions $T \in \mathscr{D}'$ such that

$$\frac{|T(\psi_x^{\varepsilon_n})|}{\|\psi\|_{C^{r_\alpha}}} \lesssim \varepsilon_n^\alpha, \tag{1.9.1}$$

uniformly for x in compact sets, $\psi \in \mathscr{B}_{r_\alpha} \setminus \{0\}$ and $n \in \mathbb{N}$, where we define r_α as the smallest integer $r \in \mathbb{N}$ such that $r > -\alpha$. For any distribution $T \in \mathscr{D}'$ and $\alpha < 0$, we define $\|T\|_{\mathscr{C}^\alpha(K)}$ as the best constant in (1.9.1):

$$\|T\|_{\mathscr{C}^\alpha(K)} := \sup_{z \in K,\, n \in \mathbb{N},\, \psi \in \mathscr{B}_{r_\alpha}} \frac{|T(\psi_x^{\varepsilon_n})|}{\varepsilon_n^\alpha \|\psi\|_{C^{r_\alpha}}}. \tag{1.9.2}$$

Then $T \in \mathscr{C}^\alpha$ if and only if $\|T\|_{\mathscr{C}^\alpha(K)} < \infty$, for all compact sets $K \subseteq \mathbb{R}^d$.

We want now to show that a coherent germ which satisfies a homogeneity bound with exponent $\bar{\alpha} < 0$ has a reconstruction (unique or not) which belongs to the Besov space $\mathscr{C}^{\bar{\alpha}}$, and the map $F \mapsto \mathscr{R}F$ is linear continuous.

We introduce the semi-norms

$$|||F|||^{\mathrm{coh}}_{K,\varphi,\alpha,\gamma} := \sup_{y,z \in K,\ n \in \mathbb{N}} \frac{|(F_z - F_y)(\varphi_y^{\varepsilon_n})|}{\varepsilon_n^\alpha\, (|z-y| + \varepsilon_n)^{\gamma-\alpha}}, \tag{1.9.3}$$

$$|||F|||^{\mathrm{hom}}_{K,\varphi,\bar{\alpha}} := \sup_{x \in K,\ n \in \mathbb{N}} \frac{|F_x(\varphi_x^{\varepsilon_n})|}{\varepsilon_n^{\bar{\alpha}}}, \tag{1.9.4}$$

where φ is as in Definition 1.4.1. We can now state the following result.

THEOREM 1.9.1 (Reconstruction Theorem and Hölder spaces). *Let $\alpha \leqslant \gamma$ and $\gamma \neq 0$. Let $(F_x)_{x \in \mathbb{R}^d}$ be a (α,γ)-coherent germ with local homogeneity bound $\bar{\alpha} \leqslant \gamma$, namely $F \in \mathscr{G}^{\bar{\alpha};\alpha,\gamma}$. If $\bar{\alpha} > 0$, then $\mathscr{R}F = 0$. If $\bar{\alpha} < 0$, then $\mathscr{R}F$ belongs to $\mathscr{C}^{\bar{\alpha}}$ and for every compact set $K \subseteq \mathbb{R}^d$*

$$\|\mathscr{R}F\|_{\mathscr{C}^{\bar{\alpha}}(K)} \leqslant \mathfrak{C}\left(|||F|||^{\mathrm{coh}}_{\bar{K}_4,\varphi,\alpha,\gamma} + |||F|||^{\mathrm{hom}}_{\bar{K}_2,\varphi,\bar{\alpha}} \right), \tag{1.9.5}$$

where φ is the test function in the coherence condition (1.4.3) *and $\mathfrak{C} = \mathfrak{C}_{\alpha,\gamma,\bar{\alpha},d,\varphi} < \infty$ is a constant which depends neither on F nor on K.*

PROOF. We fix a compact set $K \subset \mathbb{R}^d$ and $y \in K$. When $\bar{\alpha} > 0$ then $\gamma > 0$ and $\mathscr{R}F = 0$ satisfies
$$\left|J_y(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^{\bar{\alpha}},$$
and we have uniqueness of the reconstruction by Lemma 1.3.2. Henceforth we fix $\bar{\alpha} < 0$. Let φ be the test function in the coherence condition (1.4.3). Let $f = \mathscr{R}F$ be a reconstruction of F. Fix a compact set K: we want to show that
$$\sup_{x \in \bar{K}_2,\ N \in \mathbb{N}} \frac{|f(\varphi_x^{\varepsilon_N})|}{\varepsilon_N^{\bar{\alpha}}} \leqslant \mathfrak{C}' \left(|||F|||_{\bar{K}_4,\varphi,\alpha,\gamma}^{\mathrm{coh}} + |||F|||_{\bar{K}_2,\varphi,\bar{\alpha}}^{\mathrm{hom}} \right) \tag{1.9.6}$$
for some $\mathfrak{C}' = \mathfrak{C}'_{\alpha,\gamma,\bar{\alpha},d,\varphi} < \infty$. Set $\bar{r} := \min\{r \in \mathbb{N} : \ r > \max\{-\alpha, -\bar{\alpha}\}\}$. Then we have, uniformly for $x \in \bar{K}_2$ and $N \in \mathbb{N}$,
$$\begin{aligned} |(f - F_x)(\varphi_x^{\varepsilon_N})| &= c^{-1} \, |(f - F_x)(\psi_x^{\beta^{-1}\varepsilon_N})| \\ &\leqslant \mathfrak{c}' \, |||F|||_{\bar{K}_4,\varphi,\alpha,\gamma}^{\mathrm{coh}} \cdot \begin{cases} \varepsilon_N^{\gamma} & \text{if } \gamma \neq 0 \\ (1 + |\log \varepsilon_N|) & \text{if } \gamma = 0 \end{cases} \end{aligned}$$
for a suitable $\mathfrak{c}' = \mathfrak{c}'_{\alpha,\gamma,\bar{\alpha},d,\varphi}$. Since $\bar{\alpha} \leqslant \gamma \neq 0$, we bound $\varepsilon_N^{\gamma} \leqslant \varepsilon_N^{\bar{\alpha}}$, for all $n \in \mathbb{N}$. Recalling (1.9.4), by the triangle inequality we obtain
$$\begin{aligned} \sup_{x \in \bar{K}_2,\ N \in \mathbb{N}} \frac{|f(\varphi_x^{\varepsilon_N})|}{\varepsilon_N^{\bar{\alpha}}} &\leqslant \sup_{x \in \bar{K}_2,\ N \in \mathbb{N}} \frac{|(f - F_x)(\varphi_x^{\varepsilon_N})| + |F_x(\varphi_x^{\varepsilon_N})|}{\varepsilon_N^{\bar{\alpha}}} \\ &\leqslant (1 + c_{\bar{\alpha}}) \, \mathfrak{c}' \, |||F|||_{\bar{K}_4,\varphi,\alpha,\gamma}^{\mathrm{coh}} + |||F|||_{\bar{K}_2,\varphi,\bar{\alpha}}^{\mathrm{hom}}, \end{aligned}$$
which completes the proof of (1.9.6). □

Also in this setting, we can show that (1.9.1) holds for one ψ if and only if it holds for all ψ: see [**2**, Theorem 12.4].

1.10. Singular product

Let $f \in \mathscr{C}^{\alpha}$ with $\alpha > 0$ and $F_y(w) = \sum_{|k|<\alpha} \partial^k f(y) \frac{(w-y)^k}{k!}$. Let also $g \in \mathscr{C}^{\beta}$ with $\beta \leqslant 0$. We define the germ $P = (P_x := g \cdot F_x)_{x \in \mathbb{R}^d}$ as
$$P_x(\varphi) = (g \cdot F_x)(\varphi) := g(\varphi F_x), \qquad \varphi \in \mathscr{D}.$$
Note that this makes sense and defines a distribution in $\mathscr{D}'$ since $\varphi F_x \in \mathscr{D}$ for all $\varphi \in \mathscr{D}$.

THEOREM 1.10.1. *If $f \in \mathscr{C}^{\alpha}$ and $g \in \mathscr{C}^{\beta}$, with $\alpha > 0$ and $\beta \leqslant 0$, then the germ $P = (P_x)_{x \in \mathbb{R}^d}$ is $(\beta, \alpha + \beta)$-coherent and satisfies a homogeneity bound with exponent β,*
$$\left|(P_z - P_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^{\beta}(|y - z| + \varepsilon_n)^{\alpha}, \qquad \left|P_y(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^{\beta},$$
uniformly over z, y in compact sets, $n \in \mathbb{N}$ and $\varphi \in \mathscr{B}_r$, with $r > -\beta$.

PROOF. Since $g \in \mathscr{C}^\beta$ we have for all $\varepsilon \in (0,1]$, $\psi \in \mathscr{D}(B(0,1))$ and $y \in K$

$$\left|g(\psi_y^\varepsilon)\right| \leqslant \|g\|_{\mathscr{C}^\beta(K)} \|\psi\|_{C^r} \varepsilon^\beta . \tag{1.10.1}$$

Fix now any $\varphi \in \mathscr{D}(B(0,1))$ with $\int \varphi \neq 0$ and $\|\varphi\|_{C^r} \leqslant 1$. By (1.2.6), for any $y,z \in K$ (and γ replaced by α)

$$(P_z - P_y)(\varphi_y^\varepsilon) = - \sum_{0 \leqslant |k| < \alpha} g\left((\cdot - y)^k \varphi_y^\varepsilon\right) \frac{R^k(y,z)}{k!}$$

where $|R^k(y,z)| \lesssim \|f\|_{\mathscr{C}^\alpha(K)} |z-y|^{\alpha - |k|}$. We have for fixed $y \in \mathbb{R}^d$, $k \in \mathbb{N}^d$ and $\varepsilon > 0$

$$(w-y)^k \varphi_y^\varepsilon(w) = \varepsilon^{|k|} \psi_y^\varepsilon(w) , \quad \text{where} \quad \psi(w) := w^k \varphi(w) .$$

Then $\psi \in \mathscr{D}(B(0,1))$ and $\|\psi\|_{C^r} \lesssim \|\varphi\|_{C^r} \leqslant 1$, hence it follows by (1.10.1) that

$$|g\left((\cdot - y)^k \varphi_y^\varepsilon\right)| = \varepsilon^{|k|} g\left(\psi_y^\varepsilon\right) \lesssim \|g\|_{\mathscr{C}^\beta(K)} \varepsilon^{\beta + |k|} . \tag{1.10.2}$$

We thus obtain, uniformly for $z,y \in K$ and $\varepsilon \in (0,1]$,

$$\begin{aligned}
|(P_z - P_y)(\varphi_y^\varepsilon)| &\lesssim \|f\|_{\mathscr{C}^\alpha(K)} \|g\|_{\mathscr{C}^\beta(K)} \sum_{0 \leqslant |k| < \alpha} \varepsilon^{\beta + |k|} |z-y|^{\alpha - |k|} \\
&\lesssim \|f\|_{\mathscr{C}^\alpha(K)} \|g\|_{\mathscr{C}^\beta(K)} \varepsilon^\beta (|z-y| + \varepsilon)^\alpha ,
\end{aligned}$$

which completes the proof of coherence. We next prove homogeneity. By (1.10.2), we obtain

$$\begin{aligned}
|P_x(\varphi_x^\varepsilon)| &\leqslant \sum_{0 \leqslant |k| < \gamma} \left| g\left((\cdot - x)^k \varphi_x^\varepsilon\right) \right| \left| \frac{\partial^k f(x)}{k!} \right| \\
&\lesssim \|g\|_{\mathscr{C}^\beta(K)} \sum_{0 \leqslant |k| < \gamma} \varepsilon^{\beta + |k|} \left| \frac{\partial^k f(x)}{k!} \right| \\
&\lesssim \|f\|_{\mathscr{C}^\alpha(K)} \|g\|_{\mathscr{C}^\beta(K)} \sum_{0 \leqslant |k| < \gamma} \varepsilon^{\beta + |k|} \\
&\lesssim \|f\|_{\mathscr{C}^\alpha(K)} \|g\|_{\mathscr{C}^\beta(K)} \varepsilon^\beta ,
\end{aligned}$$

uniformly for x in compact sets and $\varepsilon \in (0,1]$. This completes the proof. $\square$

If $\alpha + \beta > 0$ the (unique) distribution $\mathscr{R}P$ can be used to construct a *canonical product* of f and g. Moreover $\mathscr{R}P \in \mathscr{C}^\beta$.

If $\alpha + \beta \leqslant 0$, the (non-unique) distribution $\mathscr{R}P$ can be used to construct a *non-canonical product* of f and g. Moreover $\mathscr{R}P \in \mathscr{C}^\beta$.

1.11. A special case

Let us assume that $F_x \in C(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$ and moreover that the map $\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto F_x(y)$ is continuous. We recall that in Section 1.6 we proved that for all $\psi \in \mathscr{D}$

$$\mathscr{R}(F)(\psi) = \lim_{n \to +\infty} \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n})\, \psi(z)\, \mathrm{d}z\,.$$

Now if $(x,y) \mapsto F_x(y)$ is continuous, we obtain by dominated convergence that

$$\mathscr{R}(F)(\psi) = \int_{\mathbb{R}^d} F_z(z)\, \psi(z)\, \mathrm{d}z\,,$$

namely $\mathscr{R}(F)$ is also a continuous function and coincides with $z \mapsto F_z(z)$.

For an example one can consider the germ F defined by the Taylor expansion of a smooth function f, see Section 1.2.1. In this case it is clear that $\mathscr{R}(F) = f$ is a function and $f(x) = F_x(x)$, $x \in \mathbb{R}^d$.

1.12. Recent developments

- *Reconstruction Theorem for Germs of Distributions on Smooth Manifolds*
  by Paolo Rinaldi and Federico Sclavi
- *On a Microlocal Version of Young's Product Theorem*
  by Claudio Dappiaggi, Paolo Rinaldi and Federico Sclavi
- *Besov Reconstruction*
  by Lucas Broux and David Lee
- *Reconstruction theorem in quasinormed spaces*
  by Pavel Zorin-Kranich
- *A stochastic reconstruction theorem*
  by Hannes Kern

CHAPTER 2

Models and modelled distributions

In the previous chapter we have introduced the notion of coherent germs and the operation of reconstruction. In this chapter we define a special class of germs which arise in regularity structures.

2.1. Pre-models and modelled distributions

We are going to study germs which can be written as suitable linear combinations of a fixed finite family of germs. First we introduce the notion of *pre-models*:

Definition 2.1.1. *A* pre-model *is a pair* (Π, Γ) *where*

(1) $(\Pi_x^i)_{i\in I, x\in\mathbb{R}^d}$ *is a family of germs, with I a finite index set*

(2) $\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto (\Gamma_{xy}^{ij})_{i,j\in I}$ *is a matrix-valued function such that*

$$\Pi_y^j = \sum_{i\in I} \Pi_x^i \Gamma_{xy}^{ij}, \qquad j \in I,\ x, y \in \mathbb{R}^d, \tag{2.1.1}$$

and we suppose that

(3) there exist $(\alpha_i)_{i\in I} \subset \mathbb{R}$ *and a* $\varphi \in \mathscr{D}(\mathbb{R}^d)$ *with* $\int \varphi \neq 0$ *such that*

$$|\Pi_x^i(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\alpha_i},$$

uniformly over x in compact sets of $\mathbb{R}^d$*,* $n \in \mathbb{N}$*.*

We denote $\bar{\alpha} := \min_{i\in I} \alpha_i$*.*

Example 2.1.2. *For a fixed* $\gamma > 0$*, the family of classical monomials*

$$\Pi_y^j(w) = \frac{(w-y)^j}{j!}, \qquad j \in \mathbb{N}^d, \quad y, w \in \mathbb{R}^d, \quad j \in I := \{i \in \mathbb{N}^d : |i| \leqslant \gamma\},$$

with $\alpha_i = |i|$*, any* $\varphi \in \mathscr{D}$ *and*

$$\Gamma_{xy}^{ij} = \mathbb{1}_{(i\leqslant j)} \frac{(x-y)^{j-i}}{(j-i)!}, \qquad i \in \mathbb{N}^d,$$

forms a pre-model. Note that for $j \in \mathbb{N}^d$*,* $w \in \mathbb{R}^d$*, we use the notation*

$$|j| := \sum_{k=1}^d j_k, \qquad w^j := \prod_{k=1}^d w_k^{j_k}, \qquad j! := \prod_{k=1}^d j_k!$$

with the convention $0^0 := 1$.

Now we can define the notion of *modelled distribution*.

Definition 2.1.3. *Let* (Π,Γ) *be a pre-model, and let* $\gamma \in \mathbb{R}$. *If* $f : \mathbb{R}^d \to \mathbb{R}^I$ *is measurable locally bounded and satisfies*

(1) $f^i \equiv 0$ *whenever* $\alpha_i \geqslant \gamma$,
(2) for all $i \in I$ *with* $\alpha_i < \gamma$,

$$\left| f_x^i \right| \lesssim 1, \qquad \left| f_x^i - \sum_{j\in I} \Gamma_{xy}^{ij} f_y^j \right| \lesssim |x-y|^{\gamma-\alpha_i},$$

uniformly for x,y *in compact subsets of* $\mathbb{R}^d$,

then we call f *a* distribution modelled *by* (Π,Γ), *or simply a* modelled distribution, *and we write* $f \in \mathscr{D}^\gamma_{(\Pi,\Gamma)}$.

Given a pre-model (Π,Γ) and a modelled distribution $f \in \mathscr{D}^\gamma_{(\Pi,\Gamma)}$, we define the germ

$$\langle \Pi, f\rangle_x := \sum_{i\in I} \Pi_x^i f_x^i, \qquad x \in \mathbb{R}^d. \tag{2.1.2}$$

We want to show that $\langle \Pi, f\rangle$ is $(\min_I \bar{\alpha}, \gamma)$-coherent (note that if $\gamma \leqslant \min_I \bar{\alpha}$ then f and $\langle \Pi, f\rangle$ are null). Using the reexpansion property (2.1.1) we have

$$\sum_{i\in I} \left(\Pi_z^i f_z^i - \Pi_y^i f_y^i \right) = -\sum_{i\in I} \Pi_y^i \left(f_y^i - \sum_{j\in I} \Gamma_{yz}^{ij} f_z^j \right).$$

Therefore

$$(\langle \Pi, f\rangle_z - \langle \Pi, f\rangle_y)(\varphi_y^\varepsilon) = -\sum_{i\in I} \Pi_y^i(\varphi_y^\varepsilon) \left(f_y^i - \sum_{j\in I} \Gamma_{yz}^{ij} f_z^j \right),$$

namely

$$\left| (\langle \Pi, f\rangle_z - \langle \Pi, f\rangle_y)(\varphi_y^\varepsilon) \right| \lesssim \sum_{i\in I} \varepsilon^{\alpha_i} |z-y|^{\gamma-\alpha_i} \lesssim \varepsilon^{\bar{\alpha}} (\varepsilon + |z-y|)^{\gamma-\bar{\alpha}},$$

uniformly for y,z in compact sets. Moreover

$$\left| \langle \Pi, f\rangle_y(\varphi_y^\varepsilon) \right| \leqslant \sum_{i\in I} f_y^i |\Pi_y^i(\varphi_y^\varepsilon)| \lesssim \sum_{i\in I} \varepsilon^{\alpha_i} \lesssim \varepsilon^{\bar{\alpha}},$$

uniformly over y in compact subsets of $\mathbb{R}^d$. In other words we have proved that

Proposition 2.1.4. *If* (Π,Γ) *is a pre-model and* $f \in \mathscr{D}^\gamma_{(\Pi,\Gamma)}$, *then* $\langle \Pi, f\rangle$ *is a* $(\bar{\alpha},\gamma)$*-coherent germs with uniform homogeneity bound with exponent* $\bar{\alpha}$. *In other words,* $\langle \Pi, f\rangle$ *belongs to* $\mathscr{G}^{\bar{\alpha};\bar{\alpha},\gamma}$.

2.2. A special case

We have seen in Section 1.11 that under certain sufficient conditions on the coherent germ $(F_x)_{x\in\mathbb{R}^d}$, the reconstruction $\mathscr{R}F$ is a function and has an explicit form. An important example of this setting, where moreover $\mathscr{R}F$ is a (locally) Hölder-continuous function, is the following:

Example 2.2.1. Suppose we have a pre-model (Π,Γ) and a modelled distribution $f\in\mathscr{D}^\gamma_{(\Pi,\Gamma)}$ as in Section 2.1. We suppose that for all $i\in I$

$$\|\Pi^i_x\|_{C^\beta(\mathbb{R}^d)}<+\infty$$

uniformly for x in compact subsets of $\mathbb{R}^d$, where $\beta\in]0,1[$, namely Π^i_x is Hölder-continuous (locally uniformly in x). Then we can write unambiguously $y\mapsto\Pi^i_x(y)$ and

$$y\mapsto F_x(y):=\sum_{i\in I}f^i_x\,\Pi^i_x(y).$$

Now by the reexpansion property (2.1.1)

$$F_x(y)-F_{x'}(y)=-\sum_{i\in I}\Pi^i_x(y)\left(f^i_x-\sum_{j\in I}\Gamma^{ij}_{xx'}f^j_{x'}\right).$$

Then

$$\begin{aligned}|F_x(y)-F_{x'}(y')|&\leqslant|F_x(y)-F_{x'}(y)|+|F_{x'}(y')-F_{x'}(y')|\\&\lesssim\sum_{i\in I}|\Pi^i_x(y)|\left|x-x'\right|^{\gamma-\alpha_i}+|y-y'|^\beta\end{aligned}$$

which shows that $(x,y)\mapsto F_x(y)$ is continuous. Therefore, in this case the reconstruction of F is equal to $x\mapsto F_x(x)$. Moreover setting $y=x$ and $y'=x'$ we obtain

$$|F_x(x)-F_{x'}(x')|\lesssim\sum_{i\in I}|\Pi^i_x(x)|\left|x-x'\right|^{\gamma-\alpha_i}+|x-x'|^\beta,$$

namely the reconstruction of $F=\langle\Pi,f\rangle$ is even locally Hölder-continuous.

2.3. Models

We now define the notion of a *model*.

Definition 2.3.1. *A* model *is a pre-model* (Π,Γ) *as in Definition 2.1.1, such that moreover*

(1) $\Gamma^{ii}_{xy}=1$ *for all* $i\in I$,
(2) $\Gamma^{ij}_{xy}=0$ *if* $\alpha_i\geqslant\alpha_j$ *and* $i\neq j$,
(3) $|\Gamma^{ij}_{xy}|\lesssim|x-y|^{\alpha_j-\alpha_i}$ *if* $\alpha_i<\alpha_j$.

If (Π,Γ) is a model, then spaces $\mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$ of modelled distributions satisfy the following properties.

Lemma 2.3.2. *Let (Π,Γ) be a model as in Definition 2.3.1. Then*

(1) If $\gamma > \bar{\alpha} = \min_I \alpha$, the space $\mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$ is not reduced to the null vector.

(2) For $\gamma' > \gamma$ the natural projection

$$\mathscr{D}^{\gamma'}_{(\Pi,\Gamma)} \ni (f^i)_{i\in I} \mapsto (\mathbb{1}_{(\alpha_i<\gamma)} f^i)_{i\in I}$$

maps $\mathscr{D}^{\gamma'}_{(\Pi,\Gamma)}$ to $\mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$.

PROOF. For the first assertion, we consider an element Π^i_x of minimal homogeneity $\bar{\alpha} = \min_I \alpha$. In this case by the properties (1)-(2) in Definition 2.3.1 we see that $\Gamma^{ij}_{xy} = \delta_{ij}$ for all $j \in I$, where δ is the Kronecker symbol, and the function $f^j_x = \delta_{ij}$ is a modelled distribution for any $\gamma > \bar{\alpha} = \min_I \alpha$.

Let us prove now the second assertion. We write for i such that $\alpha_i < \gamma$

$$\begin{aligned}
\left| f^i_x - \sum_{\alpha_j<\gamma} \Gamma^{ij}_{xy} f^j_y \right| &= \left| f^i_x - \sum_{\alpha_j<\gamma'} \Gamma^{ij}_{xy} f^j_y \right| + \sum_{\gamma\leqslant\alpha_j<\gamma'} \left|\Gamma^{ij}_{xy} f^j_y\right| \\
&\lesssim |x-y|^{\gamma'-\alpha_i} + \sum_{\gamma\leqslant\alpha_j<\gamma'} |x-y|^{\alpha_j-\alpha_i} \\
&\lesssim |x-y|^{\gamma-\alpha_i},
\end{aligned}$$

uniformly for x,y in compact subsets of $\mathbb{R}^d$, by the property (3) in Definition 2.3.1. $\square$

We also have another instructive remark. Suppose that (Π,Γ) is a model. Then for every $j \in I$, the germ $(\Pi^j_x)_{x\in\mathbb{R}^d}$ is $(\bar{\alpha} = \min_I \alpha, \alpha_j)$-coherent. Indeed

$$\Pi^j_y - \Pi^j_x = \sum_{i\in I} \Pi^i_x \Gamma^{ij}_{xy} - \Pi^j_x = \sum_{i\neq j} \Pi^i_x \Gamma^{ij}_{xy},$$

so that

$$\begin{aligned}
|(\Pi^j_y - \Pi^j_x)(\varphi^{\varepsilon_n}_x)| &\leqslant \sum_{\alpha_i<\alpha_j} |\Pi^i_x(\varphi^{\varepsilon_n}_x)||x-y|^{\alpha_j-\alpha_i} \\
&\lesssim \sum_{\alpha_i<\alpha_j} \varepsilon_n^{\alpha_i}|x-y|^{\alpha_j-\alpha_i} \\
&\lesssim \varepsilon_n^{\bar{\alpha}}(|x-y|+\varepsilon_n)^{\alpha_j-\bar{\alpha}}.
\end{aligned}$$

Moreover, by property (3) in Definition 2.1.1, this germ satisfies a homogeneity bound with exponent α_j. The same property is in fact a reconstruction bound for this germ, with $\mathscr{R}(\Pi^j) = 0$. If $\alpha_j > 0$ then the reconstruction is unique.

Note that we can write, as in notation (2.1.2), $\Pi^j = \langle \Pi, f \rangle$ with $f^i_x := \delta_{ij}$, with δ the Kronecker symbol. However in this setting f does not belong to $\mathscr{D}^{\alpha_j}_{(\Pi,\Gamma)}$, because it has a non-zero coordinate corresponding to an element of the basis with homogeneity equal to α_j, which is not allowed by property (1) of Definition 2.1.3.

Proposition 2.3.3. *Let (Π,Γ) be a model and $\gamma > \bar\alpha = \min_I \alpha$. The family $(\Pi^i,\Gamma^{ij})_{\alpha_i,\alpha_j \leqslant \gamma}$ is again a model.*

PROOF. The re-expansion property (2.1.1) for $\alpha_j \leqslant \gamma$ is

$$\Pi^j_y = \sum_{i\in I} \Pi^i_x \Gamma^{ij}_{xy}, \qquad x,y \in \mathbb{R}^d.$$

By the property (2) in Definition 2.3.1, this can be rewritten as

$$\Pi^j_y = \sum_{i\in I,\alpha_j\leqslant\alpha_i} \Pi^i_x \Gamma^{ij}_{xy}, \qquad x,y \in \mathbb{R}^d.$$

It is therefore easy to show that $(\Pi^i,\Gamma^{ij})_{\alpha_i,\alpha_j\leqslant\gamma}$ is a model. $\square$

2.4. Hölder functions as modelled distributions

We have see in Example 2.1.2 that the classical polynomial family

$$\Pi^i_y(w) = \frac{(w-y)^i}{i!}, \quad i \in \mathbb{N}^d,\ \alpha_i = |i| < \gamma,$$

$$\Gamma^{ij}_{xy} = \mathbb{1}_{(i\leqslant j)} \frac{(x-y)^{j-i}}{(j-i)!}, \quad i,j \in \mathbb{N}^d,$$

forms a pre-model and actually a model. It is an interesting exercise to check that modelled distributions with respect to this model are actually classical Hölder functions.

This model belongs to the class that we have considered in Section 2.2, namely the function $(x,y) \mapsto \Pi^i_x(y)$ is continuous for all i and $\|\Pi^i_x\|_{C^\beta(\mathbb{R}^d)} < +\infty$ uniformly for x in compact subsets of $\mathbb{R}^d$ for any $\beta \in]0,1[$. Therefore by the discussion in Section 2.2 we know that any modelled distribution $f \in \mathscr{D}^\gamma_{(\Pi,\Gamma)}$ gives rise to a $(0,\gamma)$-coherent germ $\langle \Pi, f\rangle$ and that the reconstruction of $\langle \Pi, f\rangle$ is a locally Hölder-continuous function.

Let us consider for simplicity the case $\gamma \notin \mathbb{N}$. Now, a modelled distribution $f \in \mathscr{D}^\gamma_{(\Pi,\Gamma)}$ satisfies

$$\left| f^i_x - \sum_{j\geqslant i,|j|<\gamma} \frac{(x-y)^{j-i}}{(j-i)!} f^j_y \right| \lesssim |x-y|^{\gamma-|i|}, \qquad \forall\ |i| < \gamma.$$

This is in fact a Taylor expansion of f^i at order $\lfloor \gamma - |i| \rfloor$ with a remainder of order $\gamma - |i|$, and this implies that f^i is of class $C^{\gamma-|i|}$ and

$$f^j = \partial_{j-i} f^i, \qquad \forall\ j \geqslant i.$$

In particular, for $i = 0$ we see that f^0 is of class C^γ and satisfies (1.2.2); in particular by Proposition 1.2.1 we have that f^0 is a reconstruction of $\langle \Pi, f \rangle$, and since $\gamma > 0$ it is the unique reconstruction. In other words we have seen that

$$f^0 = \mathscr{R}\langle \Pi, f \rangle \in C^\gamma, \qquad f^i = \partial_i f^0, \quad \forall |i| < \gamma.$$

2.5. Semi-norms

Back to the general case, for a fixed pre-model (Π, Γ) we can interpret, by analogy with the case of Hölder functions of the previous section, the space $\mathscr{D}^\gamma_{(\Pi,\Gamma)}$ of all distributions modelled by (Π, Γ) as the collection of *generalised derivatives* of $u := \mathscr{R}\langle \Pi, f \rangle$ with respect to the model (Π, Γ).

We can define a system of seminorms for $f \in \mathscr{D}^\gamma_{(\Pi,\Gamma)}$

$$[f]_{\mathscr{D}^\gamma_{(\Pi,\Gamma)},K} = \sup_{i \in I} \sup_{x,y \in K, x \neq y} \frac{\left| f^i_x - \sum_{j \in I} \Gamma^{ij}_{xy} f^j_y \right|}{|x-y|^{\gamma - \alpha_i}},$$

where K is a compact subset of $\mathbb{R}^d$.

This is rather original with respect to the standard situation in ODEs or PDEs, where one sets an equation in a fixed Banach space. Here the Banach (Fréchet) space depends on an external parameter, the model (Π, Γ). For SDEs and SPDEs, the model (Π, Γ) is actually *random*.

CHAPTER 3

Schauder estimates for coherent germs

In this chapter we discuss one of the most important operations on coherent germs: the convolution with a regularising integration kernel.

3.1. Integration kernels

Definition 3.1.1 (Regularising kernel)**.** *Fix a dimension $d \in \mathbb{N}$, an exponent $\beta \in (0,d)$ and an integer $r \in \mathbb{N}$. A measurable function $\mathsf{K} : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ is called β*-regularizing kernel up to degree $m \in \mathbb{N}$ *if the following conditions hold:*

- *the function $x \mapsto \mathsf{K}(x)$ is of class C^m on $\mathbb{R}^d \backslash \{0\}$;*
- *the following upper bound holds:*

$$\forall k \in \mathbb{N}^d \ \text{ with } \ |k| \leqslant m : \qquad |\partial_x^k \mathsf{K}(x)| \lesssim \frac{1}{|x|^{d-\beta+|k|}} \mathbb{1}_{\{|x|\leqslant 1\}} \quad \text{uniformly for } x \text{ in compact sets}\,. \tag{3.1.1}$$

In particular, note that for $k=0$ equation (3.1.1) reduces to

$$|\mathsf{K}(x)| \lesssim \frac{1}{|x|^{d-\beta}} \mathbb{1}_{\{|x|\leqslant 1\}}\,. \tag{3.1.2}$$

This shows that a β-regularizing kernel is locally integrable on $\mathbb{R}^d$.

3.1.1. Singular convolution. We want to consider the convolution $\mathsf{K} * f \in \mathscr{D}'$ between a kernel $\mathsf{K}(x-y)$ and a distribution $f \in \mathscr{D}'$. This is *formally* defined by

$$(\mathsf{K} * f)(x) := f(\mathsf{K}(x-\cdot)) = \int_{\mathbb{R}^d} \mathsf{K}(x-y)\, f(\mathrm{d}y)\,, \tag{3.1.3}$$

but we stress that in general $\mathsf{K} * f$ is ill-defined. Under suitable conditions, $\mathsf{K} * f$ can be defined as a distribution by duality: for any test function $\psi \in \mathscr{D}$ we set

$$(\mathsf{K} * f)(\psi) := f(\mathsf{K}^* \psi) \quad \text{where} \quad (\mathsf{K}^* \psi)(y) := \int_{\mathbb{R}^d} \psi(x)\, \mathsf{K}(x-y)\, \mathrm{d}x\,, \tag{3.1.4}$$

provided $f(\mathsf{K}^*\psi)$ makes sense, of course. We are going to study the convolution $\mathsf{K}^*\psi$ between the kernel K and a test function ψ, to ensure that $f(\mathsf{K}^*\psi)$ is well-defined.

We start with an elementary observation: if $\mathsf{K}(\cdot)$ is β-regularizing up to some degree r, then $(\mathsf{K}^*\psi)(\cdot)$ *is a well-defined compactly supported measurable function*, because $\mathsf{K}(x-y)$ is jointly measurable, locally integrable and compactly supported in the difference $|x-y|$. The delicate point is that $\mathsf{K}^*\psi$ *needs not be smooth*, hence we cannot hope to define $f(\mathsf{K}^*\psi)$ for arbitrary $(f,\psi) \in \mathscr{D}' \times \mathscr{D}$.

3.1.2. Partition of unity. Let us introduce the usual dyadic sequence

$$\varepsilon_n := 2^{-n}, \qquad n \in \mathbb{Z}.$$

We call *dyadic partition of unity* a family of functions $(\rho_n)_{n\in\mathbb{Z}}$ such that:

- $\rho_n(z)$ is supported in the annulus $\{\frac{1}{2}\varepsilon_n \leqslant |z| \leqslant 2\varepsilon_n\}$ and
$$\forall z \in \mathbb{R}^d \backslash \{0\} : \qquad \sum_{n\in\mathbb{Z}} \rho_n(z) = 1\,;$$
- for any given $k \in \mathbb{N}^d$, one has
$$\|\partial^k \rho_n\|_\infty \lesssim \varepsilon_n^{-|k|} \quad \text{uniformly in } n \in \mathbb{N}.$$

It is easy to build a dyadic partition of unity. Given any smooth function $\chi : \mathbb{R}^d \to [0,1]$ such that

$$\chi(z) \begin{cases} = 1 & \text{if } |z| \leqslant 1 \\ \in [0,1] & \text{if } 1 \leqslant |z| \leqslant 2 \\ = 0 & \text{if } |z| \geqslant 2 \end{cases},$$

we obtain a dyadic partition of unity $(\rho_n)_{n\in\mathbb{Z}}$ by setting

$$\rho_n(z) := \chi(\varepsilon_n^{-1} z) - \chi(\varepsilon_{n+1}^{-1} z).$$

Such a partition of unity is *scale invariant*, since $\rho_n(z) = \rho_0(\varepsilon_n^{-1} z)$. We set

$$\mathsf{K}(x) = \sum_{n=0}^{\infty} \mathsf{K}_n(x) \qquad \text{where} \qquad \mathsf{K}_n(x) := \rho_n(x)\,\mathsf{K}(x). \tag{3.1.5}$$

We stress that $\mathsf{K}_n(x)$ is supported in the annulus $\{\frac{1}{2}\varepsilon_n \leqslant |x| \leqslant 2\varepsilon_n\}$.

$$\begin{aligned} &\forall k \in \mathbb{N}^d \text{ with } |k| \leqslant m : \\ &\qquad |\partial^k \mathsf{K}_n(x)| \lesssim \frac{1}{|x|^{d-\beta-|k|}} \mathbb{1}_{\{\frac{1}{2}\varepsilon_n \leqslant |x| \leqslant 2\varepsilon_n\}} \\ &\qquad\qquad \lesssim \varepsilon_n^{\beta-d-|k|} \mathbb{1}_{\{\frac{1}{2}\varepsilon_n \leqslant |x| \leqslant 2\varepsilon_n\}} \\ &\qquad \text{uniformly for } n \in \mathbb{N}. \end{aligned} \tag{3.1.6}$$

Moreover we have for all $y \in \mathbb{R}^d$ and $|\ell| < |k|$

$$\int_{\mathbb{R}^d} x^\ell \, \partial^k \mathsf{K}_n(x-y) \, \mathrm{d}x = (-1)^{|k|} \int_{\mathbb{R}^d} \partial^k x^\ell \, \mathsf{K}_n(x-y) \, \mathrm{d}x = 0 \,, \tag{3.1.7}$$

because $\partial^k x^\ell = 0$ for $|\ell| < |k|$.

3.2. Convolution with distributions

We show now that $\mathsf{K}^* \psi$ in (3.1.4) is well-defined and differentiable.

Proposition 3.2.1. *Given a kernel* K *which is* β*-regularizing up to degree* $m \in \mathbb{N}$ *and a test function* $\psi \in \mathscr{D}$*, the convolution* $\mathsf{K}^* \psi$ *defined in* (3.1.4) *belongs to* C^m.

More precisely, recalling K_n *defined in* (3.1.5)*, we have the following bound:*

$$\begin{aligned} \forall r \in \{0,1,\ldots,m\} : \qquad & \|\mathsf{K}_n^* \psi\|_{C^r} \lesssim \|\psi\|_{C^r} \, \varepsilon_n^\beta \\ & \textit{uniformly for } n \in \mathbb{N} \textit{ and } \psi \in \mathscr{D}(B(0,1)) \,, \end{aligned} \tag{3.2.1}$$

hence the series $\mathsf{K}^* \psi = \sum_{n=0}^\infty \mathsf{K}_n^* \psi$ *converges in* C^m *(recall that* $\beta > 0$*).*

Proof. We recall that $\mathsf{K}(x-y) = \sum_{n=0}^\infty \mathsf{K}_n(x-y)$ for all $x, y \in \mathbb{R}^d$ with $x \neq y$, by (3.1.5). Then by dominated convergence, thanks to (3.1.2), for any $y \in \mathbb{R}^d$ we can write

$$(\mathsf{K}^* \psi)(y) = \sum_{n=0}^\infty (\mathsf{K}_n^* \psi)(y) \quad \text{where} \quad (\mathsf{K}_n^* \psi)(y) := \int_{\mathbb{R}^d} \psi(x) \, \mathsf{K}_n(x-y) \, \mathrm{d}x \,.$$

To prove (3.2.1), it is sufficient to show that

$$\begin{aligned} \forall k \in \mathbb{N}^d \text{ with } |k| \leqslant m : \qquad & \|\partial^k (\mathsf{K}_n^* \psi)\|_\infty \lesssim \|\psi\|_{C^{|k|}} \, \varepsilon_n^\beta \\ & \text{uniformly for } n \in \mathbb{N} \text{ and } \psi \in \mathscr{D}(B(0,1)) \,. \end{aligned} \tag{3.2.2}$$

By Definition 3.1.1, for any $n \in \mathbb{N}$ the function $y \mapsto \mathsf{K}_n(x-y)$ is of class C^r *on the whole* $\mathbb{R}^d$ (including $y = x$, because $\mathsf{K}_n(x-y)$ vanishes for $|y-x| \leqslant \frac{1}{2}\varepsilon_n$, see (3.1.5)). Exchanging derivatives and integral by dominated convergence, thanks to (3.1.1), we see that $\mathsf{K}_n^* \psi \in C^m$ and

$$\forall k \in \mathbb{N}^d \ \text{ with } |k| \leqslant r : \quad \partial^k (\mathsf{K}_n^* \psi)(y) = -\int_{\mathbb{R}^d} \psi(x) \, \partial^k \mathsf{K}_n(x-y) \, \mathrm{d}x \,. \tag{3.2.3}$$

We now estimate $\partial^k (\mathsf{K}_n^* \psi)(y)$ for fixed $n \in \mathbb{N}$ and $y \in \mathbb{R}^d$, $k \in \mathbb{N}^d$. Denote by $Q^{[y,k]}(\cdot)$ the Taylor polynomial of ψ of degree $|k|-1$ based at y, that is

$$Q^{[y,k]}(x) := \sum_{|\ell| \leqslant |k|-1} \frac{\partial^\ell \psi(y)}{\ell!} \, (x-y)^\ell \,,$$

where we agree that for $k=0$ we set $Q^{[y,0]}(x) \equiv 0$. Then we can bound

$$|\psi(x) - Q^{[y,k]}(x)| \lesssim \|\psi\|_{C^{|k|}} \, |y-x|^{|k|}. \tag{3.2.4}$$

Starting from (3.2.3), we decompose

$$\begin{aligned} \partial^k(\mathsf{K}_n^* \psi)(y) = &- \underbrace{\int_{\mathbb{R}^d} (\psi - Q^{[y,k]})(x) \, \partial^k \mathsf{K}_n(x-y) \, \mathrm{d}x}_{A_{n,k}(y)} \\ &- \underbrace{\int_{\mathbb{R}^d} Q^{[y,k]}(x) \, \partial^k \mathsf{K}_n(x-y) \, \mathrm{d}x}_{B_{n,k}(y)}. \end{aligned}$$

By (3.1.7) we have that $B_{n,k}(y) = 0$. By (3.2.4) and (3.1.6), for $|k| \leqslant m$, the first term is bounded by

$$|A_{n,k}(y)| \lesssim \|\psi\|_{C^{|k|}} \int_{|y-x| \leqslant \varepsilon_n} |y-x|^{|k|} \, |y-x|^{\beta - |k| - d} \, \mathrm{d}x \lesssim \|\psi\|_{C^{|k|}} \, \varepsilon_n^{\beta},$$

uniformly for y in compact sets and $n \in \mathbb{N}$. This completes the proof of (3.2.2). ☐

We obtain the following useful

Proposition 3.2.2. *Given a kernel K which is β-regularizing up to degree $m \in \mathbb{N}$ and a distribution $T \in \mathscr{D}'$ of order $r \leqslant m$, the distribution*

$$\mathscr{D} \ni \psi \mapsto \mathsf{K} * T(\psi) := T(\mathsf{K}^* \psi),$$

where $\mathsf{K}^* \psi \in C^m$ is as in Proposition 3.2.1, is well-defined in $\mathscr{D}'$ and has order r.

3.3. Schauder estimate for coherent germs

3.3.1. Coherent germs. Fix two real numbers α, γ such that

$$\alpha \leqslant \gamma, \quad \gamma \neq 0.$$

Let $F = (F_x)_{x \in \mathbb{R}^d}$ be a (α, γ)-coherent germ, i.e. we have

$$\begin{gathered} |(F_z - F_y)(\varphi_y^{\varepsilon_n})| \lesssim \varepsilon_n^{\alpha} \, (|y-z| + \varepsilon_n)^{\gamma - \alpha} \\ \text{uniformly for } y, z \text{ in compact sets and } n \in \mathbb{N}, \end{gathered} \tag{3.3.1}$$

for some test function $\varphi \in \mathscr{D}$ with $\int \varphi \neq 0$.

We define r_α as the smallest integer larger than $-\alpha$

$$r_\alpha := \min\{k \in \mathbb{N} : \ k > -\alpha\}. \tag{3.3.2}$$

Since F is γ-coherent, by the Reconstruction Theorem 1.5.1-1.7.1 there is a distribution $\mathscr{R}F \in \mathscr{D}'$ such that

$$\begin{gathered} |(\mathscr{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \|\psi\|_{C^{r_\alpha}}\, \varepsilon_n^\gamma \\ \text{uniformly for } x \text{ in compact sets, } n \in \mathbb{N} \text{ and } \psi \in \mathscr{D}(B(0,1)) . \end{gathered} \tag{3.3.3}$$

If $\gamma > 0$ then $\mathscr{R}F$ is unique.

3.3.2. Singular convolution. Fix a kernel K which is β-regularizing up to degree r for some $\beta \in (0,d)$, see Definition 3.1.1. We now want to "lift the convolution with K on the space of coherent germs", i.e. to find a coherent germ $H = (H_x)_{x\in\mathbb{R}^d}$ with the property that

$$\mathscr{R}H = \mathsf{K} * \mathscr{R}F . \tag{3.3.4}$$

A simple solution of (3.3.4) is the constant germ $H_x \equiv \mathsf{K} * \mathscr{R}F$, which is trivially coherent, but typically it does not satisfy (3.3.5). The naive guess $H_x = \mathsf{K} * F_x$ needs not give a coherent germ, therefore we need to enrich it. To this purpose, we look for H_x of the following special form:

$$\forall x \in \mathbb{R}^d : \qquad H_x = \mathsf{K} * F_x + R_x \qquad \text{where } R_x(\cdot) \text{ is a } \textit{polynomial} . \tag{3.3.5}$$

Remarkably, this is possible with the following explicit solution:

$$H_x := \mathsf{K} * F_x + \underbrace{\sum_{|\ell|<\gamma+\beta} (\mathscr{R}F - F_x)\left(\partial^\ell \mathsf{K}(x-\cdot)\right) \mathbb{X}_x^\ell}_{R_x(\cdot)} , \tag{3.3.6}$$

where we denote

$$\mathbb{X}_x^\ell(\cdot) := \frac{(\,\cdot - x)^\ell}{\ell!} \tag{3.3.7}$$

to be the monomial germs, and where we agree that

$$R_x(\cdot) \equiv 0 \qquad \text{if} \qquad \gamma + \beta \leqslant 0.$$

Note that $R_x(\cdot)$ is a family of polynomials labelled by x, whose coefficients depend on F_x, on $\mathscr{R}F$ and on the derivatives $\partial^k \mathsf{K}$ for $|k| < \gamma + \beta$. Then we also assume that $\gamma + \beta \notin \mathbb{N}$ and we suppose that the integer m which appears in Definition 3.1.1 satisfies

$$m > \gamma + \beta + r_\alpha. \tag{3.3.8}$$

THEOREM 3.3.1 (Schauder estimate for coherent germs). *Fix a dimension $d \in \mathbb{N}$ and real numbers $\alpha, \gamma, \beta \in \mathbb{R}$ such that*

$$\alpha \leqslant \gamma, \qquad \gamma \neq 0, \qquad \beta > 0 ,$$

where we further assume for simplicity that

$$\{\alpha + \beta,\ \gamma + \beta\} \cap \mathbb{N} = \varnothing .$$

Consider the following ingredients:

- *$F = (F_x)_{x\in\mathbb{R}^d} \in \mathscr{G}^{\alpha,\gamma}$ is a (α,γ)-coherent germ;*
- *K is a β-regularizing kernel (see Definition 3.1.1) up to degree $r = r_\alpha$ given in (3.3.2).*

Then

(1) the germ $H = (H_x)_{x\in\mathbb{R}^d}$ in (3.3.6) is locally well-defined, i.e. $H_x(\varphi)$ is well-defined for all $\varphi \in \mathscr{D}(B(x,1))$.
(2) H is $((\alpha+\beta)\wedge 0, \gamma+\beta)$-coherent, namely $H \in \mathscr{G}^{(\alpha+\beta)\wedge 0,\gamma+\beta}$.
(3) H satisfies $\mathscr{R}H = \mathsf{K} * \mathscr{R}F$.

In other words, setting $\mathscr{K}F := H$, we have a linear operator satisfying

$$\mathscr{K} : \mathscr{G}^{\alpha,\gamma} \to \mathscr{G}^{(\alpha+\beta)\wedge 0,\gamma+\beta}, \qquad \mathscr{R}\circ\mathscr{K} = \mathsf{K} * \mathscr{R}.$$

Let us define the new germ

$$J_x := F_x - \mathscr{R}F\,,$$

which lets us rewrite (3.3.6) as

$$H_x = \mathsf{K} * \mathscr{R}F + L_x, \qquad \text{where} \qquad L_x := \mathsf{K} * J_x - R_x\,. \tag{3.3.9}$$

From (3.3.6), observe that

$$L_x = \mathsf{K} * J_x - \sum_{|\ell|<\gamma+\beta} J_x(\partial^\ell \mathsf{K}(x-\cdot))\,\mathbb{X}_x^\ell. \tag{3.3.10}$$

We are going to prove that L_x is $((\alpha+\beta)\wedge 0, \gamma+\beta)$-coherent, that is

$$\begin{aligned} &|(L_z - L_y)(\psi_y^{\varepsilon_n})| \lesssim \|\psi\|_{C^{r_\alpha}}\, \varepsilon_n^{(\alpha+\beta)\wedge 0}\,(|y-z|+\varepsilon_n)^{\gamma+\beta-(\alpha+\beta)\wedge 0}\,, \\ &\text{uniformly for } y,z \text{ in compact sets, } n\in\mathbb{N} \text{ and } \psi\in\mathscr{D}(B(0,1))\,. \end{aligned} \tag{3.3.11}$$

More explicitly:

$$|(L_z - L_y)(\psi_y^{\varepsilon_n})| \lesssim \|\psi\|_{C^{r_\alpha}} \times \begin{cases} \varepsilon_n^{\alpha+\beta}\,(|y-z|+\varepsilon_n)^{\gamma-\alpha} & \text{if } \alpha+\beta<0\,, \\ (|y-z|+\varepsilon_n)^{\gamma+\beta} & \text{if } \alpha+\beta>0\,. \end{cases}$$

Then we are going to prove that L has homogeneity bound with exponent $\gamma+\beta$, that is,

$$|L_x(\psi_x^{\varepsilon_n})| \lesssim \|\psi\|_{C^{r_\alpha}}\, \varepsilon_n^{\gamma+\beta} \qquad \begin{array}{l}\text{uniformly for } x \text{ in compact sets,}\\ n\in\mathbb{N} \text{ and } \psi\in\mathscr{D}(B(0,1)).\end{array}$$

Recalling (3.3.9), this implies that $\mathscr{R}H = \mathsf{K} * \mathscr{R}F$; indeed we recall that $h = \mathscr{R}H$ means precisely $|(h-H_x)(\psi_x^{\varepsilon_n})| \lesssim \|\psi\|_{C^{r_\alpha}}\, \varepsilon_n^{\gamma+\beta}$, as the coherence exponent of H_x is $\gamma+\beta$.

One of the tools in the proof of Theorem 3.3.1 is the following simple result.

Lemma 3.3.2. *Fix $\gamma \in \mathbb{R}$, $\beta > 0$ such that $\gamma + \beta > 0$ and a point $x \in \mathbb{R}^d$. Let $T \in \mathscr{D}'$ have order r_α and homogeneity bound γ at the point x, i.e. for some $r \in \mathbb{N}$ and $\mathsf{C}_x < \infty$*

$$\begin{aligned} &|T(\varphi_x^\varepsilon)| \leqslant \mathsf{C}_x \, \|\varphi\|_{C^{r_\alpha}} \, \varepsilon^\gamma \\ &\qquad \text{uniformly for } \varepsilon \in (0,1] \text{ and } \varphi \in \mathscr{D}(B(0,1)) \, . \end{aligned} \tag{3.3.12}$$

Let K be a β-regularizing kernel up to degree $m > \gamma + \beta + r_\alpha$.. Then for all $\ell \in \mathbb{N}_0^d$ with $|\ell| < \gamma + \beta$,

$$T(\partial^\ell \mathsf{K}(x - \cdot)) := \sum_n T(\partial^\ell \mathsf{K}_n(x - \cdot))$$

is well-defined and, writing $\partial^\ell \mathsf{K} = \sum_{n=0}^\infty \partial^\ell \mathsf{K}_n$ as in (3.1.5)*, we have*

$$\forall N \in \mathbb{N}: \qquad \left| T\left(\sum_{n=N}^\infty \partial^\ell \mathsf{K}_n(x - \cdot) \right) \right| \lesssim \mathsf{C}_x \, \varepsilon_N^{\gamma + \beta - |\ell|} \, . \tag{3.3.13}$$

Before proving Lemma 3.3.2 we need the following simple

Lemma 3.3.3. *We introduce the function*

$$\varphi^{[k,n]}(w) := (2\varepsilon_n)^d \, \partial^k \mathsf{K}_n(-2\varepsilon_n w) \, , \tag{3.3.14}$$

so that

$$\partial^k \mathsf{K}_n(x - \cdot) = \left(\varphi^{[k,n]} \right)_x^{2\varepsilon_n} . \tag{3.3.15}$$

Then

$$\operatorname{supp}\left(\varphi^{[k,n]} \right) \subset B(0,1), \qquad \forall \, |k| < \; \gamma + \beta, \tag{3.3.16}$$

$$\left\| \varphi^{[k,n]} \right\|_{\mathscr{C}^{r_\alpha}} \lesssim \varepsilon_n^{\beta - |k|}, \qquad \forall \, |k| < \; \gamma + \beta, \tag{3.3.17}$$

PROOF. Observe that (3.3.15) is straightforward from the definition of $\varphi^{[k,n]}$. One has $\operatorname{supp}\left(\partial^k \mathsf{K}_n(\cdot)\right) \subset B(0, 2\varepsilon_n)$ and thus one has as announced $\operatorname{supp}\left(\varphi^{[k,n]} \right) \subset B(0,1)$. Now, if $1 \leqslant |l| \leqslant r_\alpha$ then $\partial^l \varphi^{[k,n]} = (2\varepsilon_n)^{d + |l|} \, \partial^{k+l} \mathsf{K}_n(-2\varepsilon_n w)$. Thus from (3.1.6), one obtains (3.3.17). □

PROOF OF LEMMA 3.3.2. By (3.3.15) and by the homogeneity bound at x (3.3.12), using the properties (3.3.16) and (3.3.17) of $\varphi^{[\ell,n]}$ we can bound

$$|T(\partial^\ell \mathsf{K}_n(x - \cdot))| \leqslant \mathsf{C}_x \, \|\varphi^{[\ell,n]}\|_{C^{r_\alpha}} \, \varepsilon_n^\gamma \lesssim \mathsf{C}_x \, \varepsilon_n^{\gamma + \beta - |\ell|} .$$

Thus $T(\partial^\ell \mathsf{K}(x - \cdot)) := \sum_{n=0}^\infty T(\partial^\ell \mathsf{K}_n(x - \cdot))$ is well-defined in $\mathscr{D}'$ and moreover we obtain (3.3.13). □

3.4. Proof

In this section we prove Theorem 3.3.1.

Lemma 3.4.1. L_x *in* (3.3.10) *is a well-defined distribution.*

PROOF. By the discussion of the Reconstruction Theorem in section 3.3.1 we know that $J_x = F_x - \mathscr{R}F$ is a distribution of order r_α. Then by Proposition 3.2.2 the distribution $\mathsf{K} * J_x$ is well defined and has order r_α.

If we apply Lemma 3.3.2 to the distribution $T = J_x$ then we know that $T(\partial^\ell \mathsf{K}(x-\cdot)) \in \mathbb{R}$ is well-defined for all $\ell \in \mathbb{N}^d$ such that $|\ell| < \gamma + \beta$. Then L_x is a well-defined distribution. $\square$

Remark 3.4.2. We will write $(L_z - L_y)(\psi_y^\lambda)$ for $\lambda \in]0,1]$ as a sum of various terms and show that

$$\text{each term is } \lesssim \lambda^a(|y-z|+\lambda)^{\gamma+\eta-a} \quad \text{for a suitable } a \geqslant (\alpha+\eta) \wedge 0 .$$

This implies (3.3.11) because $a \mapsto \lambda^a(|y-z|+\lambda)^{\gamma+\eta-a}$ is decreasing (note that we can write $\lambda^a(|y-z|+\lambda)^{\gamma+\eta-a} = A^a B$ with $A = \frac{\lambda}{\lambda+|y-z|} \leqslant 1$).

We take a compact set $K \subseteq \mathbb{R}^d$ and fix $y, z \in K$ as well as $N \in \mathbb{N}$. We set

$$M_{y,z,N} := \min\{n \in \mathbb{N} : \ \varepsilon_n \leqslant |y-z| + \varepsilon_N\} ,$$

and note that $0 \leqslant M_{y,z,N} \leqslant N < \infty$. Then we decompose

$$\mathsf{K}(\cdot,\cdot) = \underbrace{\sum_{n=0}^{M_{y,z,N}-1} \mathsf{K}_n(\cdot,\cdot)}_{\mathsf{K}_{[0,M)}(\cdot,\cdot)} + \underbrace{\sum_{n=M_{y,z,N}}^{N-1} \mathsf{K}_n(\cdot,\cdot)}_{\mathsf{K}_{[M,N)}(\cdot,\cdot)} + \underbrace{\sum_{n=N}^{\infty} \mathsf{K}_n(\cdot,\cdot)}_{\mathsf{K}_{[N,\infty)}(\cdot,\cdot)} ,$$

where we stress that in this decomposition the sum is split at the points $M_{y,z,N}$ and N, for the fixed values of y,z,N, irrespective of the arguments of $\mathsf{K}(\cdot)$. We also define for $A \subset \mathbb{N}$, $x \in \mathbb{R}^d$, $\psi \in \mathscr{D}$

$$P_x^A(\psi) := \sum_{n \in A} \sum_{|\ell| < \gamma+\beta} \partial^\ell \mathsf{K}_n(x-\cdot) \int_{\mathbb{R}^d} \mathbb{X}_x^\ell(w)\, \psi(z)\, \mathrm{d}w, \tag{3.4.1}$$

so that in particular

$$L_x(\psi) = J_x\Big(\mathsf{K}^*\psi - P_x^{\mathbb{N}}(\psi)\Big), \qquad P_x^{\mathbb{N}} = P_x^{[0,M)} + P_x^{[M,N)} + P_x^{[N,\infty)}. \tag{3.4.2}$$

Then we bound for $\psi \in \mathscr{D}(B(0,1))$

$$
\begin{aligned}
\left|(L_z - L_y)(\psi_y^{\varepsilon_N})\right| &\leqslant \left|(J_z - J_y)\left(\mathsf{K}^* \psi_y^{\varepsilon_N}\right) - J_z\left(P_z^{\mathbb{N}}(\psi_y^{\varepsilon_N})\right) + J_y\left(P_y^{\mathbb{N}}(\psi_y^{\varepsilon_N})\right)\right| \\
&\leqslant \underbrace{\left|(J_z - J_y)\left(\mathsf{K}^*_{[N,\infty)} \psi_y^{\varepsilon_N}\right)\right|}_{A} + \underbrace{\left|(J_z - J_y)\left(\mathsf{K}^*_{[M,N)} \psi_y^{\varepsilon_N}\right)\right|}_{B} \\
&\quad + \underbrace{\left|J_z\left(P_z^{[M,\infty)}(\psi_y^{\varepsilon_N})\right)\right| + \left|J_y\left(P_y^{[M,\infty)}(\psi_y^{\varepsilon_N})\right)\right|}_{C} \\
&\quad + \underbrace{\left|(J_z - J_y)\left(\mathsf{K}_{[0,M)}(\psi_y^{\varepsilon_N}) - P_y^{[0,M)}(\psi_y^{\varepsilon_N})\right)\right|}_{D} \\
&\quad + \underbrace{\left|J_z\left(P_y^{[0,M)}(\psi_y^{\varepsilon_N}) - P_z^{[0,M)}(\psi_y^{\varepsilon_N})\right)\right|}_{E}.
\end{aligned}
$$

We are going to need the following technical result, which can be proved as Lemma 3.3.2.

Lemma 3.4.3. *Let $\zeta^{[n,N,y]} : \mathbb{R}^d \to \mathbb{R}$ for $n \geqslant N$ and $y \in \mathbb{R}^d$*

$$\zeta^{[n,N,y]}(w) := (3\varepsilon_N)^d \left(\mathsf{K}_n^* \psi_y^{\varepsilon_N}\right)(y + (3\varepsilon_N)w). \tag{3.4.3}$$

Then $\zeta^{[n,N,y]}$ is supported in $B(0,1)$, and

$$\left\|\zeta^{[n,N,y]}\right\|_{\mathscr{C}^{r\alpha}} \lesssim \|\psi\|_{\mathscr{C}^{r\alpha}} \varepsilon_n^{\beta}, \qquad n \geqslant N,\ \psi \in \mathscr{D}(B(0,1)), \tag{3.4.4}$$

uniformly over y in compacts. Let $\varphi^{[n,z]} : \mathbb{R}^d \to \mathbb{R}$

$$\varphi^{[n,z]}(w) := (3\varepsilon_n)^d \mathsf{K}_n(z - 3\varepsilon_n w). \tag{3.4.5}$$

Then $\varphi^{[n,z]}$ is supported in $B(0,1)$ for all $|z| \leqslant \varepsilon_n$ and

$$\left\|\varphi^{[n,z]}\right\|_{\mathscr{C}^{r\alpha}} \lesssim \varepsilon_n^{\beta}, \qquad \text{uniformly over } |z| \leqslant \varepsilon_n. \tag{3.4.6}$$

Let $\xi^{[k,n,z,t]} : \mathbb{R}^d \to \mathbb{R}$ for $k, n \in \mathbb{N}$, $z \in \mathbb{R}^d$, $t \in [0,1]$,

$$\xi^{[k,n,z,t]}(w) := (3\varepsilon_n)^d \frac{d^{\lceil\gamma+\beta\rceil-|k|}}{dt^{\lceil\gamma+\beta\rceil-|k|}} \partial^k \mathsf{K}_n((1-t)z - 3\varepsilon_n w). \tag{3.4.7}$$

Then $\xi^{[k,n,z,t]}$ is supported in $B(0,1)$ and

$$\left\|\xi^{[k,n,z,t]}\right\|_{\mathscr{C}^{r\alpha}} \lesssim |z|^{\lceil\gamma+\beta\rceil-|k|} \varepsilon_n^{\beta-\lceil\gamma+\beta\rceil} \tag{3.4.8}$$

uniformly over z in compacts, $|k| < \gamma+\beta$, $t \in [0,1]$, $n \in \mathbb{N}$.

Estimate of A. We analyze

$$(J_z - J_y)\left(\mathsf{K}^*_{[N,\infty)}\psi_y^{\varepsilon_N}\right) = \sum_{n=N}^{\infty}(J_z - J_y)\left(\mathsf{K}^*_n\psi_y^{\varepsilon_N}\right). \tag{3.4.9}$$

Note that we can write by (3.4.3)

$$\mathsf{K}^*_n\psi_y^{\varepsilon_N} = \left(\zeta^{[n,N,y]}\right)_y^{3\varepsilon_N}$$

where $\zeta^{[n,N,y]}$ is defined in (3.4.3). Then, by coherence (1.4.4) and (3.4.4), we can bound for $n \geqslant N$:

$$\begin{aligned}\left|(J_z - J_y)\left(\mathsf{K}^*_n\psi_y^{\varepsilon_N}\right)\right| &= \left|(J_z - J_y)\left(\left(\zeta^{[n,N,y]}\right)_y^{3\varepsilon_N}\right)\right| \\ &\lesssim \left\|\zeta^{[n,N,y]}\right\|_{C^{r_\alpha}}(3\varepsilon_N)^{\alpha}\left(|y-z| + 3\varepsilon_N\right)^{\gamma-\alpha} \\ &\lesssim \|\psi\|_{C^{r_\alpha}}\varepsilon_n^{\beta}\varepsilon_N^{\alpha}\left(|y-z| + \varepsilon_N\right)^{\gamma-\alpha}.\end{aligned}$$

Plugging this bound into (3.4.9) we finally obtain since $\beta > 0$

$$\left|\mathsf{K}_{[N,\infty)} * (J_z - J_y)\left(\psi_y^{\varepsilon_N}\right)\right| \lesssim \|\psi\|_{C^{r_\alpha}}\,\varepsilon_N^{\alpha+\beta}\left(|y-z| + \varepsilon_N\right)^{\gamma-\alpha},$$

which coincides with (3.3.11) for $\alpha + \beta \leqslant 0$, while for $\alpha + \beta > 0$ it is even better than (3.3.11), by Remark 3.4.2.

Estimate of B. Then we analyze

$$\begin{aligned}(J_z - J_y)\left(\mathsf{K}^*_{[M,N)}\psi_y^{\varepsilon_N}\right) &= \sum_{n=M_{y,z,N}}^{N-1}(J_z - J_y)\left(\mathsf{K}^*_n\psi_y^{\varepsilon_N}\right) \\ &= \sum_{n=M_{y,z,N}}^{N-1}\int_{\mathbb{R}^d}\psi_y^{\varepsilon_N}(x)\,(J_z - J_y)(\mathsf{K}_n(x-\cdot))\,\mathrm{d}x.\end{aligned} \tag{3.4.10}$$

Note now that one can write $\mathsf{K}_n(x-\cdot) = \left(\varphi^{[n,x-y]}\right)_y^{3\varepsilon_n}$ where $\varphi^{[n,z]}$ is defined in (3.4.5). Then, by coherence (1.4.4), and using the property (3.4.6) of $\varphi^{[n,x-y]}$ we can bound

$$\begin{aligned}|(J_z - J_y)(\mathsf{K}_n(x-\cdot))| &= \left|(J_z - J_y)\left(\left(\varphi^{[n,x-y]}\right)_y^{3\varepsilon_n}\right)\right| \\ &\lesssim \left\|\varphi^{[n,x-y]}\right\|_{C^{r_\alpha}}(3\varepsilon_n)^{\alpha}\left(|y-z| + 3\varepsilon_n\right)^{\gamma-\alpha} \\ &\lesssim \varepsilon_n^{\beta}(3\varepsilon_n)^{\alpha}\left(|y-z| + 3\varepsilon_n\right)^{\gamma-\alpha} \\ &\leqslant \varepsilon_n^{\beta}(3\varepsilon_n)^{\alpha}\left(4|y-z| + 3\varepsilon_N\right)^{\gamma-\alpha},\end{aligned}$$

where in the last inequality we used the fact that $\varepsilon_n \leqslant |y-z| + \varepsilon_N$ for $n \geqslant M_{y,z,N}$. We plug this bound into (3.4.10). Note that

$$\sum_{n=M_{y,z,N}}^{N} \varepsilon_n^{\alpha+\beta} \lesssim \begin{cases} \displaystyle\sum_{n=0}^{N} \varepsilon_n^{\alpha+\beta} \lesssim \varepsilon_N^{\alpha+\beta} & \text{if } \alpha+\beta < 0\,, \\ \displaystyle\sum_{n=M_{y,z,N}}^{\infty} \varepsilon_n^{\alpha+\beta} \lesssim (|y-z| + \varepsilon_N)^{\alpha+\beta} & \text{if } \alpha+\beta > 0\,. \end{cases}$$

Moreover $\int_{\mathbb{R}^d} |\psi_y^{\varepsilon_N}(w)|\, \mathrm{d}w = \int_{\mathbb{R}^d} |\psi(w)|\, \mathrm{d}w \lesssim \|\psi\|_\infty \leqslant \|\psi\|_{C^{r\alpha}}$ for any $\psi \in \mathscr{D}(B(0,1))$, hence

$$\frac{|(\mathsf{K}_{[M,N)} * (J_z - J_y))(\psi_y^{\varepsilon_N})|}{\|\psi\|_{C^{r\alpha}}} \lesssim \begin{cases} \varepsilon_N^{\alpha+\beta}\,(|y-z| + \varepsilon_N)^{\gamma-\alpha} & \text{if } \alpha+\beta < 0\,, \\ (|y-z| + \varepsilon_N)^{\gamma+\beta} & \text{if } \alpha+\beta > 0\,, \end{cases}$$

which coincides with (3.3.11).

Estimate of C. If $\gamma+\beta \leqslant 0$ then $C = 0$. Let us consider the case $\gamma+\beta > 0$. By (3.3.3) and Lemma 3.3.2, see in particular (3.3.13), we have

$$\sum_{n=M_{y,z,N}}^{\infty} \left| J_y \left(\partial^\ell \mathsf{K}_n(y-\cdot) \right) \right| \lesssim \varepsilon_{M_{y,z,N}}^{\gamma+\beta-|\ell|},$$

while

$$\int_{\mathbb{R}^d} \left| \mathbb{X}_y^\ell(w)\, \psi_y^{\varepsilon_N}(w) \right| \mathrm{d}w \lesssim \varepsilon_N^\ell.$$

Then

$$\left| J_y \left(P_y^{[M,\infty)}(\psi_y^{\varepsilon_N}) \right) \right| \lesssim \sum_{|\ell|<\gamma+\beta} \varepsilon_{M_{y,z,N}}^{\gamma+\beta-|\ell|} \varepsilon_N^{|\ell|} \lesssim (|y-z| + \varepsilon_N)^{\gamma+\beta}\,. \tag{3.4.11}$$

Similarly

$$\sum_{n=M_{y,z,N}}^{\infty} \left| J_z \left(\partial^\ell \mathsf{K}_n(z-\cdot) \right) \right| \lesssim \varepsilon_{M_{y,z,N}}^{\gamma+\beta-|\ell|},$$

$$\int_{\mathbb{R}^d} \left| \mathbb{X}_z^\ell(w)\, \psi_y^{\varepsilon_N}(w) \right| \mathrm{d}w \lesssim (|y-z| + \varepsilon_N)^\ell,$$

so that

$$\left| J_z \left(P_z^{[M,\infty)}(\psi_y^{\varepsilon_N}) \right) \right| \lesssim (|y-z| + \varepsilon_N)^{\gamma+\beta}\,. \tag{3.4.12}$$

Note that both (3.4.11) and (3.4.12) are better than (3.3.11), by Remark 3.4.2.

Estimate of D. We now focus now on

$$(J_z - J_y) \left(\mathsf{K}_{[0,M)}^* \psi_y^{\varepsilon_N} - P_y^{[0,M)}(\psi_y^{\varepsilon_N}) \right). \tag{3.4.13}$$

We first assume that $\gamma+\beta>0$. Observe that one can write

$$\mathsf{K}_n(w-\cdot)-\sum_{|\ell|<\gamma+\beta}\partial^\ell\mathsf{K}_n(y-\cdot)\frac{(w-y)^\ell}{\ell!}=\int_0^1\frac{(1-t)^m}{m!}\left(\xi^{[0,n,w-y,t]}\right)_y^{3\varepsilon_n}\mathrm{d}t, \tag{3.4.14}$$

where $\xi^{[k,n,z,t]}$ is defined as in (3.4.7). Therefore:

$$\begin{aligned}(J_z-J_y)&\left(\mathsf{K}^*_{[0,M)}\psi_y^{\varepsilon_N}-P_y^{[0,M)}(\psi_y^{\varepsilon_N})\right)=\\&=\sum_{n=0}^{M_{y,z,N}-1}\int_0^1\frac{(1-t)^m}{m!}(J_z-J_y)\left(\left(\xi^{[0,n,w-y,t]}\right)_y^{3\varepsilon_n}\right)\mathrm{d}t\,.\end{aligned}$$

Applying the coherence bound (1.4.4), we can estimate

$$\begin{aligned}\left|(J_z-J_y)\left(\left(\xi^{[0,n,w-y,t]}\right)_y^{3\varepsilon_n}\right)\right|&\lesssim\left\|\xi^{[0,n,w-y,t]}\right\|_{C^{r_\alpha}}(3\varepsilon_n)^\alpha(|z-y|+\varepsilon_n)^{\gamma-\alpha}\\&\lesssim\left\|\xi^{[0,n,w-y,t]}\right\|_{C^{r_\alpha}}\varepsilon_n^\gamma,\end{aligned}$$

where in the last inequality we used the fact that for $n\leqslant M_{y,z,N}$, $(|z-y|+\varepsilon_n)^{\gamma-\alpha}\leqslant(2\varepsilon_n)^{\gamma-\alpha}$. If $|w-y|\leqslant\varepsilon_N\leqslant\varepsilon_n$, then from the property (3.4.8) of $\xi^{[0,n,w-y,t]}$ one obtains

$$\left\|\xi^{[0,n,w-y,t]}\right\|_{C^{r_\alpha}}\lesssim|y-w|^{m+1}\varepsilon_n^{\beta-m-1}\leqslant\varepsilon_N^{m+1}\varepsilon_n^{\beta-m-1},$$

uniformly for $n\leqslant N$ and $t\in[0,1]$. Collecting all those estimates,

$$\begin{aligned}\left|(J_z-J_y)\left(\mathsf{K}^*_{[0,M)}\psi_y^{\varepsilon_N}-\mathsf{K}_{[0,M)}(\psi_y^{\varepsilon_N})\right)\right|&\lesssim\varepsilon_N^{m+1}\sum_{n=0}^{M_{y,z,N}-1}\varepsilon_n^{\gamma+\beta-m-1}\\&\lesssim\varepsilon_N^{m+1}(|z-y|+\varepsilon_N)^{\gamma+\beta-m-1}\leqslant(|z-y|+\varepsilon_N)^{\gamma+\beta},\end{aligned}$$

which, recalling (3.4.13), is better than (3.3.11) by Remark 3.4.2.

We next assume that $\gamma+\beta<0$. In this case we have $P_y^{[0,M)}\equiv0$ in (3.4.13). Then, recall from (3.4.5) that one can write

$$\mathsf{K}_n(w-\cdot)=\left(\varphi^{[n,w-y]}\right)_y^{3\varepsilon_n}.$$

Thus, from the coherence bound (1.4.4), and the property (3.4.6) of $\varphi^{[n,w-y]}$ one can estimate (recall that $\varepsilon_N \leqslant \varepsilon_n$ and $\beta > 0$)

$$
\begin{aligned}
\left|(J_z - J_y)\left(\mathsf{K}_n^* \psi_y^{\varepsilon_N}\right)\right| &\lesssim \sup_{|w-y|\leqslant\varepsilon_N} \left|(J_z - J_y)\left(\left(\varphi^{[n,w-y]}\right)_y^{3\varepsilon_n}\right)\right| \\
&\lesssim \sup_{|w-y|\leqslant\varepsilon_N} \left\|\varphi^{[n,w,y]}\right\|_{C^{r\alpha}} (3\varepsilon_n)^{\alpha} (|z-y|+\varepsilon_n)^{\gamma-\alpha} \\
&\lesssim \varepsilon_n^{\beta} (3\varepsilon_n)^{\alpha} (|z-y|+\varepsilon_n)^{\gamma-\alpha}.
\end{aligned}
$$

For $n \leqslant M_{y,z,N}$ we have $(|z-y|+\varepsilon_n)^{\gamma-\alpha} \leqslant (2\varepsilon_n)^{\gamma-\alpha}$, hence

$$
\left|(J_z - J_y)\left(\mathsf{K}_{[0,M)}^* \psi_y^{\varepsilon_N}\right)\right| \lesssim \sum_{n=0}^{M_{y,z,N}-1} \varepsilon_n^{\gamma+\beta} \lesssim (|z-y|+\varepsilon_N)^{\gamma+\beta}
$$

which, recalling (3.4.13), is better than (3.3.11) by Remark 3.4.2.

Estimate of E. We have

$$
P_z^n(\psi_y^{\varepsilon_N}) - P_y^n(\psi_y^{\varepsilon_N}) = -\sum_{|k|<\gamma+\beta} R^k(y,z,\cdot) \int_{\mathbb{R}^d} \frac{(w-y)^k}{k!}\, \psi_y^{\varepsilon_N}(w)\, \mathrm{d}w,
$$

see [**2**, formula (4.7)], where

$$
\begin{aligned}
R^k(y,z,\zeta) &:= \partial_x^k \mathsf{K}_n(y,\zeta) - \sum_{|\ell|<\gamma+\beta-|k|} \partial_x^{k+\ell} \mathsf{K}_n(z,\zeta)\, \frac{(y-z)^{\ell}}{\ell!} \\
&= \int_0^1 \frac{(1-t)^{m-|k|}}{(m-|k|)!} \left(\xi^{[k,n,y-z,t]}\right)_z^{3\varepsilon_n} (\zeta)\, \mathrm{d}t,
\end{aligned}
$$

where $\xi^{[k,n,z,t]}$ is the function defined in (3.4.7). Then

$$
\begin{aligned}
&J_z\left(P_y^{[0,M)}\left(\psi_y^{\varepsilon_N}\right) - P_z^{[0,M)}\left(\psi_y^{\varepsilon_N}\right)\right) = \\
&= -\sum_{|k|<\gamma+\beta} \sum_{n=0}^{M_{y,z,N}-1} \int_0^1 \frac{(1-t)^{m-|k|}}{(m-|k|)!} J_z(\left(\xi^{[k,n,y-z,t]}\right)_z^{3\varepsilon_n})\, \mathrm{d}t\, \mathbb{X}_y^k(\psi_y^{\varepsilon_N}).
\end{aligned}
$$

Applying the coherence bound (1.4.4), and the property (3.4.8) of $\xi^{[k,n,y-z,t]}$ (observe that because $n \leqslant M_{y,z,\lambda}$, one has indeed $|y-z| \leqslant \varepsilon_n$), we can estimate

$$
\begin{aligned}
\left|J_z\left(\left(\xi^{[k,n,y-z,t]}\right)_z^{3\varepsilon_n}\right)\right| &\lesssim \|\xi^{[k,n,y-z,t]}\|_{C^{r\alpha}} (3\varepsilon_n)^{\alpha} (|z-y|+\varepsilon_n)^{\gamma-\alpha} \\
&\lesssim |y-z|^{m+1-|k|} \varepsilon_n^{\beta-m-1} (3\varepsilon_n)^{\alpha} (|z-y|+\varepsilon_n)^{\gamma-\alpha},
\end{aligned}
$$

Recalling that $|w-y| \leqslant \varepsilon_N$ and that $(|z-y|+\varepsilon_n)^{\gamma-\alpha} \leqslant (2\varepsilon_n)^{\gamma-\alpha}$, we bound

$$\begin{aligned}\left|J_z\left(P_y^{[0,M)}\left(\psi_y^{\varepsilon_N}\right)-P_z^{[0,M)}\left(\psi_y^{\varepsilon_N}\right)\right)\right| &\lesssim \varepsilon_N^k |y-z|^{m+1-|k|} \sum_{n=0}^{M_{y,z,N}-1} \varepsilon_n^{\gamma+\beta-m-1} \\ &\lesssim (|y-z|+\varepsilon_N)^{m+1} \sum_{n=0}^{M_{y,z,N}-1} \varepsilon_n^{\gamma+\beta-m-1} \\ &\lesssim (|y-z|+\varepsilon_N)^{\gamma+\beta}.\end{aligned}$$

which, recalling (3.4.13), is better than (3.3.11) by Remark 3.4.2.

L has homogeneity $\gamma+\beta$. Finally we prove that

$$|L_x(\psi_x^{\varepsilon_N})| \lesssim \varepsilon_N^{\gamma+\beta}$$

uniformly for $x \in K$ and $n \in \mathbb{N}$. This is a consequence of the following

Lemma 3.4.4. *Fix $\gamma \in \mathbb{R}$, $\beta > 0$ and a point $x \in \mathbb{R}^d$. Let $T \in \mathscr{D}'$ have order r_α and homogeneity bound γ at the point x, i.e. for some $r \in \mathbb{N}$ and $\mathsf{C}_x < \infty$*

$$\begin{gathered}|T(\varphi_x^\varepsilon)| \leqslant \mathsf{C}_x \, \|\varphi\|_{C^{r_\alpha}} \, \varepsilon^\gamma \\ \text{uniformly for } \varepsilon \in (0,1] \text{ and } \varphi \in \mathscr{D}(B(0,1)).\end{gathered} \tag{3.4.15}$$

Let K be a β-regularizing kernel up to degree $m > \gamma+\beta+r_\alpha$. Then

$$\left|T\left(\mathsf{K}^*\psi_x^{\varepsilon_N} - P_x^{\mathbb{N}}(\psi_x^{\varepsilon_N})\right)\right| \lesssim C_x \varepsilon_N^{\gamma+\beta},$$

recall (3.4.1) *and* (3.4.2).

Proof. We consider the decomposition

$$\begin{aligned}T\left(\mathsf{K}^*\psi_x^{\varepsilon_N} - P_x^{\mathbb{N}}(\psi_x^{\varepsilon_N})\right) = &\underbrace{T\left(\mathsf{K}^*_{[N,+\infty)}\psi_x^{\varepsilon_N}\right)}_{F} - \underbrace{T\left(P_x^{[N,+\infty)}(\psi_x^{\varepsilon_N})\right)}_{G} \\ &+ \underbrace{T\left(\mathsf{K}^*_{[0,N)}\psi_x^{\varepsilon_N} - P_x^{[0,N)}(\psi_x^{\varepsilon_N})\right)}_{H}.\end{aligned}$$

We shall estimate F, G, H separately. We analyze first

$$F = \sum_{n=N}^{\infty} T(\mathsf{K}_n^*\psi_x^{\varepsilon_N}). \tag{3.4.16}$$

Recall from (3.4.3) that one can write $\mathsf{K}_n^*\psi_x^{\varepsilon_N} = (\zeta^{[n,N,x]})_x^{3\varepsilon_N}$. Then, by the homogeneity bound (3.3.3) for J, and using the property (3.4.4) of $\zeta^{[n,N,x]}$, we can bound for $n \geqslant N$:

$$\begin{aligned} |T(\mathsf{K}_n^* \psi_x^{\varepsilon_N})| &= \left| T\left(\left(\zeta^{[n,N,x]} \right)_x^{3\varepsilon_N} \right) \right| \\ &\lesssim \left\| \zeta^{[n,N,x]} \right\|_{C^{r\alpha}} (3\varepsilon_N)^\gamma \\ &\lesssim \|\psi\|_{\mathscr{C}^{r\alpha}} \varepsilon_n^\beta (3\varepsilon_N)^\gamma. \end{aligned}$$

Plugging this bound into (3.4.16) we finally obtain

$$|F| \lesssim \|\psi\|_{C^{r\alpha}} \varepsilon_N^{\gamma+\beta},$$

as required. The quantity G is treated in the same way as (3.4.11), so that:

$$|G| \lesssim \varepsilon_N^{\gamma+\beta}.$$

We are ready to control the contribution of H. As in the estimate of D above, we distinguish two cases. First assume that $\gamma+\beta > 0$, then we use (3.4.14) again. Therefore:

$$H = \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \int_0^1 \frac{(1-t)^m}{m!} T\left(\left(\xi^{[0,n,w-x,t]} \right)_x^{3\varepsilon_n} \right) \mathrm{d}t \, \psi_x^{\varepsilon_N}(w) \, \mathrm{d}w \, .$$

By the homogeneity bound (3.3.3) for J, and using the property (3.4.8) of $\xi^{[0,n,w-x,t]}$ (note that here $|x-w| \leqslant \varepsilon_N \leqslant \varepsilon_n$), we can bound

$$\left| T\left(\left(\xi^{[0,n,w-x,t]} \right)_x^{3\varepsilon_n} \right) \right| \lesssim \left\| \xi^{[0,n,w-x,t]} \right\|_{\mathscr{C}^{r\alpha}} (3\varepsilon_n)^\gamma \lesssim \varepsilon_N^{m+1} \varepsilon_n^{\gamma+\beta-m-1} \, .$$

And thus after summing the geometric series one obtains since $\gamma+\beta < m+1$

$$|H| \lesssim \varepsilon_N^{\gamma+\beta}.$$

Finally, we bound H in the case when $\gamma+\beta < 0$. In this case, $P_x^{[0,N)} \equiv 0$. Then, recall from (3.4.5) that one can write $\mathsf{K}_n(w-\cdot) = \left(\varphi^{[n,w-x]} \right)_x^{3\varepsilon_n}$, so that

$$H = \int_{\mathbb{R}^d} T\left(\left(\varphi^{[n,w-x]} \right)_x^{3\varepsilon_n} \right) \psi_x^{\varepsilon_N}(w) \, \mathrm{d}w.$$

Thus, from the homogeneity bound (3.3.3) for J, and the property (3.4.6) of $\varphi^{[n,w-x]}$ one can estimate (note that here $|w-x| \leqslant \varepsilon_N \leqslant \varepsilon_n$)

$$\left| T\left(\left(\varphi^{[n,w-x]} \right)_x^{3\varepsilon_n} \right) \right| \lesssim \left\| \varphi^{[n,w-x]} \right\|_{\mathscr{C}^{r\alpha}} (3\varepsilon_n)^\gamma \lesssim \varepsilon_n^{\beta+\gamma}.$$

And thus after summing the geometric series one obtains as announced $|H| \lesssim \varepsilon_N^{\gamma+\beta}$. The proof of Lemma 3.4.4 is complete. $\square$

Conclusion. We have shown that L is $((\alpha+\beta)\wedge 0, \gamma+\beta)$-coherent and that it has homogeneity bound with exponent $\gamma+\beta$. Then its $(\gamma+\beta)$-reconstruction is 0, and therefore the $(\gamma+\beta)$-reconstruction of H is $\mathsf{K} * \mathscr{R}F$.

CHAPTER 4

Multi-level Schauder estimates for modelled distributions

In this chapter we discuss one of the most important operations on modelled distributions: the convolution with a regularising integration kernel.

We fix a pre-model (Π,Γ) as in Definition 2.1.1 and we consider $f \in \mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$. We have seen in Theorem 3.3.1 how we can build a linear operator

$$\mathscr{K} : \mathscr{G}^{\alpha,\gamma} \to \mathscr{G}^{(\alpha+\beta)\wedge 0,\gamma+\beta}, \qquad \mathscr{R} \circ \mathscr{K} = \mathsf{K} * \mathscr{R}.$$

Now we want to address an analogous question for $F = \langle \Pi, f\rangle$. In other words, we want to show that it is possible to construct

(1) *another* pre-model $(\hat\Pi,\hat\Gamma)$, such that

(2) for every $f \in \mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$ there is a modelled distribution $\hat f \in \mathscr{D}^{\gamma+\beta}_{(\hat\Pi,\hat\Gamma)}$

such that

$$\mathscr{K}\langle \Pi, f\rangle = \langle \hat\Pi, \hat f\rangle.$$

4.1. The pre-model

We need an additional property for a pre-model (see Definition 2.1.1).

Definition 4.1.1. *A pre-model is* good *if there exists $r \in \mathbb{N}$ such that*

$$|\Pi^i_x(\varphi^{\varepsilon_n}_x)| \lesssim \varepsilon_n^{\alpha_i},$$

uniformly over x in compact subsets of $\mathbb{R}^d$, $n \in \mathbb{N}$ and $\varphi \in \mathscr{D}$ such that $\|\varphi\|_{C^r} \leqslant 1$.

Remark 4.1.2. A model (Definition 2.1.3) is *a fortiori* a good pre-model. Indeed, one can see that for a coherent and homogeneous germ, on can replace the single $\varphi \in \mathscr{D}$ by a generic $\psi \in \mathscr{B}_r$ for any $r > \max\{-\alpha, -\bar\alpha\}$.

We fix throughout this chapter an integration kernel K, which is supposed to β-regularising up to order $m > \max\{\gamma, \max_I \alpha\} + \beta + r$, where r is as in Definition 4.1.1.

We work from now on with a good pre-model (Π,Γ), and we want to construct a pre-model $(\hat\Pi,\hat\Gamma)$ with the property discussed at the beginning of this chapter.

We start discussing the family $(\hat\Pi^i_x)_{i\in\hat I, x\in\mathbb{R}^d}$. A reasonable guess would be to set $\hat I = I$ and $\hat\Pi^i_x = \mathsf{K} * \Pi^i_x$, recall (3.1.3). However we expect $\hat\Pi^i_x(\psi^{\varepsilon_n}_x)$

to be small as $n \to +\infty$, at least if the homogeneity $\alpha_i + \beta$ which is expected for $\hat{\Pi}^i$ is positive.

However $\mathsf{K} * \Pi^i_x(\psi^{\varepsilon_n}_x)$ has no reason to become small for large n. To this aim we can subtract a Taylor polynomial which can yield the desired behaviour. We are going to set for $i \in I$

$$\hat{\Pi}^i_x = \mathsf{K} * \Pi^i_x - \sum_{|k|<\alpha_i+\beta} \Pi^i_x\left(\partial^k \mathsf{K}(x-\cdot)\right)\mathbb{X}^k_x, \tag{4.1.1}$$

where we recall that $\mathbb{X}^k_x(w) := \frac{(w-x)^k}{k!}$.

Proposition 4.1.3. *The distribution $\hat{\Pi}^i_x$ in* (4.1.1) *for $i \in I$ is well defined, has order r and satisfies for all compact set $K \subset \mathbb{R}^d$*

$$\sup_{x\in K}\sup_{\ell\in\mathbb{N}}\sup_{\psi\in\mathscr{B}_r} \frac{|\hat{\Pi}^i_x(\psi^{\varepsilon_\ell}_x)|}{\varepsilon_\ell^{\alpha_i+\beta}} < +\infty. \tag{4.1.2}$$

PROOF. Since (Π, Γ) is a good pre-model, then Π^i_x is a distribution with order r. Then by Proposition 3.2.2 the distribution $\mathsf{K} * \Pi^i_x$ is well defined and has order r. By applying Lemma 3.3.2 to $T := \Pi^i_x$ and $\gamma = \alpha_i$, we obtain that $\Pi^i_x\left(\partial^k \mathsf{K}(x-\cdot)\right)$ is well defined for all $|k| < \alpha_i + \beta$.

Finally, (4.1.2) follows from Lemma 3.4.4. $\square$

We can therefore associate to $\hat{\Pi}^i$ the homogeneity $\alpha_i + \beta$. Then we construct a new basis by setting

$$\hat{I} := I \sqcup I_{\mathrm{Poly}}, \quad I_{\mathrm{Poly}} := \{k \in \mathbb{N}^d : |k| < \max\{\gamma, \max_I \alpha\} + \beta\},$$

$$\hat{\Pi}^k_x := \mathbb{X}^k_x, \qquad k \in I_{\mathrm{Poly}}.$$

recall (3.3.7); of course the homogeneity of $\hat{\Pi}^k_x$ is $|k|$.

Once this choice is made, it remains to construct $\hat{\Gamma}$ and $\hat{f}$. It turns out that there are very natural choices for these objects. Let us set for notational convenience

$$A^{i,\ell}_x := \mathbb{1}_{(|\ell|<\alpha_i+\beta)}\, \Pi^i_x\left(\partial^\ell \mathsf{K}(x-\cdot)\right), \qquad x \in \mathbb{R}^d,\ i \in I,\ \ell \in \mathbb{N}^d,$$

so that (4.1.1) becomes

$$\hat{\Pi}^i_x = \mathsf{K} * \Pi^i_x - \sum_{k\in I_{\mathrm{Poly}}} A^{i,k}_x\, \mathbb{X}^k_x.$$

We define now the coefficients $(\hat{\Gamma}^{ij}_{xy})_{i,j\in\hat{I}}$. These are straightforward when $i, j \in I$ or $i, j \in I_{\mathrm{Poly}}$ or $i \in I$ and $j \in I_{\mathrm{Poly}}$ (see (4.1.3) below for the precise values). The less simple case is that of $i \in I_{\text{Poly}}$ and $j \in I$, to which we turn now. By the definition of $(\hat{\Pi}^i_x)_{i\in\hat{I}}$ we find that for $j \in I$

$$\hat{\Pi}^j_y - \sum_{i\in I} \hat{\Pi}^i_x \Gamma^{ij}_{xy} = \sum_{k\in I_{\text{Poly}}} \left(-A^{j,k}_y \mathbb{X}^k_y + \sum_{i\in I} \Gamma^{ij}_{xy} A^{i,k}_x \mathbb{X}^k_x \right).$$

Since $\mathbb{X}^k_y = \sum_{\ell\leqslant k} \mathbb{X}^{k-\ell}_y(x)\,\mathbb{X}^\ell_x$, the left-hand side of the latter expression is equal to

$$\sum_{i\in I_{\text{Poly}}} \mathbb{X}^i_x \left(\sum_{k\in I} \Gamma^{kj}_{xy} A^{k,i}_x - \sum_{\ell\in\mathbb{N}^d} \mathbb{X}^\ell_y(x) A^{j,i+\ell}_y \right),$$

namely a linear combination of elements in I_{Poly}. Therefore we set for $j \in I$ and $i \in I_{\text{Poly}}$

$$\hat{\Gamma}^{ij}_{xy} := \sum_{k\in I} \Gamma^{kj}_{xy} A^{k,i}_x - \sum_{\ell\in\mathbb{N}^d} \mathbb{X}^\ell_y(x) A^{j,i+\ell}_y,$$

and to summarize

$$\hat{\Gamma}^{ij}_{xy} = \begin{cases} \Gamma^{ij}_{xy}, & \text{if } \ i,j \in I\,, \\[2ex] \mathbb{X}^{j-i}_y(x), & \text{if } \ i,j \in I_{\text{Poly}},\ i \leqslant j\,, \\[2ex] \displaystyle\sum_{k\in I} \Gamma^{kj}_{xy} A^{k,i}_x - \sum_{\ell\in\mathbb{N}^d} \mathbb{X}^\ell_y(x) A^{j,i+\ell}_y\,, & \text{if } \ i \in I_{\text{Poly}},\ j \in I\,, \\[2ex] 0 & \text{if } \ i \in I,\ j \in I_{\text{Poly}}\,. \end{cases} \tag{4.1.3}$$

Then we have the desired property for the pre-model $(\hat{\Pi}, \hat{\Gamma})$

$$\sum_{i\in\hat{I}} \hat{\Pi}^i_x \hat{\Gamma}^{ij}_{xy} = \hat{\Pi}^j_y, \qquad j \in \hat{I}.$$

4.2. The modelled distribution

For a modelled distribution $f : \mathbb{R}^d \to \mathbb{R}^I$ we define now a new function $\hat{f} : \mathbb{R}^d \to \mathbb{R}^{\hat{I}}$

$$\hat{f}^i_x := \begin{cases} f^i_x & \text{if} \quad i \in I\,, \\[2ex] \left(\mathscr{R}F - \displaystyle\sum_{\alpha_a \leqslant |i|-\beta} f^a_x \Pi^a_x \right)(\partial^i \mathsf{K}(x-\cdot)) & \text{if} \quad i \in \mathbb{N}^d,\ |i| < \gamma+\beta\,. \end{cases}$$

Remark 4.2.1. Note that we have

$$\mathscr{K}\langle f,\Pi\rangle = \langle \hat{f},\hat{\Pi}\rangle,$$

where $\mathscr{K}$ is the operator of Theorem 3.3.1. Indeed, observe that from the definitions and the notation (2.1.2)

$$\begin{aligned}\langle \hat{f},\hat{\Pi}\rangle &= \sum_{i\in I} f_x^i\left(\left(\mathsf{K}*\Pi_x^i\right) - \sum_{|k|<\alpha_i+\beta}\Pi_x^i\left(\partial^k\mathsf{K}(x-\cdot)\right)\mathbb{X}_x^k\right)\\ &\quad+\sum_{|i|<\gamma+\beta}\left(\mathscr{R}F - \sum_{\alpha_a\leqslant|i|-\beta} f_x^a\,\Pi_x^a\right)(\partial^i\mathsf{K}(x-\cdot))\,\mathbb{X}_x^i\\ &= \mathsf{K}*\left(\sum_{i\in I} f_x^i\Pi_x^i\right) + \sum_{|i|\leqslant\gamma+\beta}\left(\mathscr{R}F - \sum_{a\in I} f_x^a\,\Pi_x^a\right)\left(\partial^i\mathsf{K}(x-\cdot)\right)\mathbb{X}_x^i\\ &= \mathscr{K}\langle f,\Pi\rangle_x.\end{aligned}$$

In particular, we have already proved in Theorem 3.3.1 that $\mathscr{R}\langle\hat{f},\hat{\Pi}\rangle = \mathsf{K}*\mathscr{R}F$.

Proposition 4.2.2. *Assume that $\Pi_x^a\left(\partial^i\mathsf{K}(x-\cdot)\right) = 0$ for all $x\in\mathbb{R}^d$ and all $a\in\{1,\ldots,n\}$ such that $\alpha_a+\beta\in\mathbb{N}_0$. L: More precisely: if $\alpha_a+\beta\in\mathbb{N}_0$ then we assume $\Pi_x^a\left(\partial_x^{\alpha_a+\beta}K_x\right) = 0$*

Then g is a modelled distribution of order $\gamma+\beta$ with respect to $\hat{\Gamma}$.

PROOF. We want $\hat{f}$ to be a modelled distribution of order $\gamma+\beta$ with respect to $\hat{\Gamma}$: the condition is obvious for $i\in I$, since it is equivalent to the condition on f with respect to Γ. We have to check the correct bound for $i\in\mathbb{N}^d$, $|i|<\gamma+\beta$:

Introduce the quantity

$$N_{x,y} := \min\{n\in\mathbb{N}:\ \varepsilon_n\leqslant|y-x|\}$$

We recall the notation $J_x = F_x - \mathscr{R}F$, and we write the decomposition:

$$
\begin{aligned}
&\hat{f}^i_x - \sum_{j\in\hat{I}} \hat{\Gamma}^{ij}_{xy}\hat{f}^j_y = \\
&= -\underbrace{\sum_{n=0}^{N_{x,y}-1} J_y\left(\partial^i \mathsf{K}_n(x-\cdot) - \sum_{|k|<\gamma+\beta-|i|} \partial^{i+k}\mathsf{K}_n(y-\cdot)\,\mathbb{X}^k_y(x)\right)}_{A} \\
&\quad -\underbrace{\sum_{n=0}^{N_{x,y}-1}\sum_{\alpha_a\leqslant|i|-\beta} \Pi^a_x(\partial^i\mathsf{K}_n(x-\cdot))\left(f^a_x - \sum_{j\in I}\Gamma^{aj}_{xy}f^j_y\right)}_{B} \\
&\quad -\underbrace{\sum_{n=N_{x,y}}^{+\infty} J_x\left(\partial^i\mathsf{K}_n(x-\cdot)\right)}_{C} + \underbrace{\sum_{n=N_{x,y}}^{+\infty}\sum_{|k|<\gamma+\beta-|i|} J_y(\partial^{i+k}\mathsf{K}_n(y-\cdot))\,\mathbb{X}^k_y(x)}_{D} \\
&\quad +\underbrace{\sum_{n=N_{x,y}}^{+\infty}\sum_{\alpha_a>|i|-\beta} \Pi^a_x(\partial^i\mathsf{K}_n(x-\cdot))\left(f^a_x - \sum_{j\in I}\Gamma^{aj}_{xy}f^j_y\right)}_{E}.
\end{aligned}
$$

Now, with the multiscale techniques of the proof of Theorem 3.3.1, we shall prove that each of those terms is bounded by $|x-y|^{\gamma+\beta-|i|}$.

Estimate of A. In view of (3.4.14), we rewrite:

$$
A = \sum_{n=0}^{N_{x,y}-1}\int_0^1 \frac{(1-t)^{m-|i|}}{(m-|i|)!} J_y\left(\left(\xi^{[i,n,x-y,t]}\right)^{3\varepsilon_n}_y\right)\mathrm{d}t,
$$

where $\xi^{[i,n,z,t]}$ is the function defined in (3.4.7). Note that because $n \leqslant N_{x,y}$ we are in the regime $|y-x| \leqslant \varepsilon_n$ and thus from (3.4.8) and the reconstruction bound on F, see (3.3.3), one obtains:

$$
\begin{aligned}
\left|J_y\left(\left(\xi^{[i,n,x-y,t]}\right)^{3\varepsilon_n}_y\right)\right| &\lesssim \left\|\xi^{[i,n,x-y,t]}\right\|_{\mathscr{C}^r}(3\varepsilon_n)^\gamma \\
&\lesssim |y-x|^{\lceil\gamma+\beta\rceil-|i|}\varepsilon_n^{\beta-m-1}(3\varepsilon_n)^\gamma.
\end{aligned}
$$

Thus, summing a geometric series and since $\gamma+\beta < \lceil\gamma+\beta\rceil$:

$$
|A| \lesssim |y-x|^{\gamma+\beta-|i|}.
$$

Estimate of B. Notice that because of the assumption that $\Pi^a_x\left(\partial^i\mathsf{K}(x-\cdot)\right) = 0$ when $\alpha_a + \beta \in \mathbb{N}$, only the terms with $\alpha_a < |i| - \beta$ contribute to the sum. In view of (3.3.15), we rewrite

$$\Pi_x^a(\partial^i \mathsf{K}_n(x-\cdot)) = \Pi_x^a\left(\left(\varphi^{[i,n]}\right)_x^{3\varepsilon_n}\right),$$

where $\varphi^{[i,n]}$ is defined in (3.3.14). Thus from the property (3.3.17) of $\varphi^{[i,n]}$ and the fact that Π^a has homogeneity bound α_a, we obtain:

$$\begin{aligned}\left|\Pi_x^a(\partial^i \mathsf{K}_n(x-\cdot))\right\| &\lesssim \left\|\varphi^{[i,n]}\right\|_{\mathscr{C}^r}(3\varepsilon_n)^{\alpha_a}\\ &\lesssim \varepsilon_n^{\beta-|i|}(3\varepsilon_n)^{\alpha_a} \lesssim \varepsilon_n^{\beta+\alpha_a-|i|}.\end{aligned}$$

Now since f is a modelled distribution with respect to Γ one can bound B by:

$$|B| \lesssim \sum_{n=0}^{N_{x,y}-1} \sum_{\alpha_a<|i|-\beta} \varepsilon_n^{\beta+\alpha_a-|i|}|x-y|^{\gamma-\alpha_a}. \tag{4.2.1}$$

Summing the geometric sums yields as announced

$$|B| \lesssim |y-x|^{\gamma+\beta-|i|}.$$

Estimate of C. As just above, we rewrite

$$C = \sum_{n=N_{x,y}}^{+\infty} J_x\left(\left(\varphi^{[i,n]}\right)_x^{3\varepsilon_n}\right),$$

where $\varphi^{[i,n]}$ satisfies (3.3.16), (3.3.17), and thus from the reconstruction bound on F, see (3.3.3), one obtains:

$$\begin{aligned}\left|J_x\left(\left(\varphi^{[i,n]}\right)_x^{3\varepsilon_n}\right)\right| &\lesssim \left\|\varphi^{[i,n]}\right\|_{\mathscr{C}^r}(3\varepsilon_n)^{\gamma}\\ &\lesssim \varepsilon_n^{\beta-|i|}(3\varepsilon_n)^{\gamma} \lesssim \varepsilon_n^{\gamma+\beta-|i|}.\end{aligned}$$

Hence, summing a geometric series and since $\gamma+\beta > |i|$:

$$|C| \lesssim |y-x|^{\gamma+\beta-|i|}.$$

Estimate of D. Here we use the estimate proved just above:

$$\left|J_y(\partial^{i+k}\mathsf{K}_n(y-\cdot))\right| \lesssim \varepsilon_n^{\gamma+\beta-|i|-|k|}.$$

Thus by summing a geometric series, one obtains:

$$|D| \lesssim \sum_{n=N_{x,y}}^{+\infty} \sum_{|k|<\gamma+\beta-|i|} \varepsilon_n^{\gamma+\beta-|i|-|k|}|y-x|^k \lesssim |y-x|^{\gamma+\beta-|i|}.$$

Estimate of E. Finally, for the term E, the estimates are the same than for the term B, but are summed over different indices. Hence, similarly as for (4.2.1), we get:

$$|E| \lesssim \sum_{n=N_{x,y}}^{+\infty} \sum_{\alpha_a > |i|-\beta} \varepsilon_n^{\beta+\alpha_a-|i|} |x-y|^{\gamma-\alpha_a},$$

and summing the geometric series yields as announced:

$$|E| \lesssim |y-x|^{\gamma+\beta-|i|}.$$

This concludes the proof. □

4.3. Recursive properties

Recall that we have not imposed a group property on the reexpansion operators Γ. The following proposition however establishes that if Γ enjoys such a property, then so does $\hat{\Gamma}$.

Proposition 4.3.1. *The following assertions are equivalent:*

(1) For all $x,y,z \in \mathbb{R}^d$, $\Gamma_{x,y}\Gamma_{y,z} = \Gamma_{x,z}$.

(2) For all $x,y,z \in \mathbb{R}^d$, $\hat{\Gamma}_{xy}\hat{\Gamma}_{yz} = \hat{\Gamma}_{xz}$.

(Here the product is understood as the matrix product.)

PROOF. The implication (2) ⇒ (1) is straightforward. Now assume (1) and let us establish (2). We have to prove that for all $i, j \in \hat{I}$,

$$\sum_{k\in\hat{I}} \hat{\Gamma}_{xy}^{ik}\hat{\Gamma}_{yz}^{kj} = \hat{\Gamma}_{xy}^{ij}. \tag{4.3.1}$$

We distinguish the different possible cases for $i, j \in \hat{I}$. If $i, j \in I$, (4.3.1) is straighforward from the definition of $\hat{\Gamma}$ and (1). If $i, j \in I_{\text{Poly}}$, then (4.3.1) is also straighforward from Newton's binomial formula. In the case when $i \in I$, $j \in I_{\text{Poly}}$, the left-hand side and the right-hand side of (4.3.1) vanish.

It remains to tackle the case when $i \in I_{\text{Poly}}$, $j \in I$. In this case, we can calculate explicitly

$$\begin{aligned}
\sum_{k\in\hat{I}} \hat{\Gamma}_{xy}^{ik}\hat{\Gamma}_{yz}^{kj} &= \sum_{k\in I} \hat{\Gamma}_{xy}^{ik}\hat{\Gamma}_{yz}^{kj} + \sum_{k\in I_{\text{Poly}}} \hat{\Gamma}_{xy}^{ik}\hat{\Gamma}_{yz}^{kj} \\
&= \sum_{k\in I} \left(\sum_{a\in I} A_x^{a,i}\Gamma_{xy}^{ak} - \sum_{\ell\in I_{\text{Poly}}} \mathbb{X}_y^{\ell}(x) A_y^{k,i+\ell} \right) \Gamma_{yz}^{kj} \\
&\quad + \sum_{k\in I_{\text{Poly}}} \mathbb{X}_y^{k-i}(x) \left(\sum_{a\in I} A_y^{a,k}\Gamma_{yz}^{aj} - \sum_{\ell\in I_{\text{Poly}}} \mathbb{X}_z^{\ell}(y) A_z^{j,k+\ell} \right).
\end{aligned}$$

Using the fact that $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ in the first term:

$$\begin{aligned}\sum_{k\in\hat{I}} \hat{\Gamma}^{ik}_{xy}\hat{\Gamma}^{kj}_{yz} = \sum_{a\in I} A^{a,i}_x \Gamma^{aj}_{xz} - \sum_{k\in I}\sum_{\ell\in I_{\mathrm{Poly}}} \mathbb{X}^{\ell}_y(x) A^{k,i+\ell}_y \Gamma^{kj}_{yz} \\ + \sum_{k\in I_{\mathrm{Poly}}}\sum_{a\in I} \mathbb{X}^{k-i}_y(x) A^{a,k}_y \Gamma^{aj}_{yz} - \sum_{k\in I_{\mathrm{Poly}}} \mathbb{X}^{k-i}_y(x) \sum_{\ell\in I_{\mathrm{Poly}}} A^{j,k+\ell}_z \mathbb{X}^{\ell}_z(y).\end{aligned}$$

Observe that the second and third term cancel out, and from Newton's binomial formula in the last term, we obtain

$$\sum_{k\in\hat{I}} \hat{\Gamma}^{ik}_{xy}\hat{\Gamma}^{kj}_{yz} = \sum_{a\in I} A^{a,i}_x \Gamma^{aj}_{xz} - \sum_{a\in I_{\mathrm{Poly}}} \mathbb{X}^{a}_z(y) A^{j,i+a}_z = \hat{\Gamma}^{ij}_{xz}.$$

The proof is complete $\square$

CHAPTER 5

The Schauder estimates

5.1. A theory, a theorem

This talk is based on work in progress with L. Broux and F. Caravenna. A temptative title for this work could be

- *Hairer's Schauder estimates without Regularity Structures*

In this paper we have extracted a single result (the multilevel Schauder estimates) from a larger theory (Regularity Structures).

We present the former in a simpler and more general version, without reference to the latter.

5.2. What we did yesterday

We defined the notion of coherent germs: $(F_x)_{x\in\mathbb{R}^d} \subset \mathscr{D}'(\mathbb{R}^d)$ such that

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_N})\right| \lesssim \lambda^{\alpha}(|y-z|+\lambda)^{\gamma-\alpha},$$

where for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$, $\lambda > 0$ and $y \in \mathbb{R}^d$

$$\varphi_y^{\varepsilon_N}(w) := \frac{1}{\lambda^d}\,\varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d\,.$$

Here $\gamma, \alpha \in \mathbb{R}$ and $\alpha \leqslant \gamma$.

We stated the Reconstruction Theorem: there exists $\mathscr{R}F \in \mathscr{D}'(\mathbb{R}^d)$ such that

$$|(\mathscr{R}F - F_x)(\psi_x^{\varepsilon_N})| \lesssim \lambda^{\gamma}$$

(with a log-correction for $\gamma = 0$) and $\mathscr{R}F$ is unique if $\gamma > 0$.

5.3. Models

Then we realised that the space of germs is too large.

The idea in regularity structures (and rough paths) is to find a suitable subspace of germs which can contain the solution to the equation of interest.

The space is defined in the following way: one fixes a finite family $(\Pi_x^1, \ldots, \Pi_x^N)$ of germs, such that for all $x \in \mathbb{R}^d$ Π_x^i has homogeneity $\alpha_i \in \mathbb{R}$, i.e.

$$|\Pi_x^i(\varphi_x^{\varepsilon_N})| \lesssim \lambda^{\alpha_i},$$

and there exists a matrix-valued function $(x,y) \mapsto (\Gamma^{ij}_{xy})_{i,j\in I}$ such that

$$\Pi^j_y = \sum_{i\in I} \Pi^i_x \Gamma^{ij}_{xy}, \qquad j \in I,\ x,y \in \mathbb{R}^d.$$

If the pair (Π,Γ) satisfies additional properties, then in regularity structures it is called a model. We also use this terminology.

5.4. Modelled distributions

For a fixed model (Π,Γ), we call any function $f : \mathbb{R}^d \to \mathbb{R}^I$ such that

$$\left| f^i_x - \sum_{j\in I} \Gamma^{ij}_{xy} f^j_y \right| \lesssim |x-y|^{\gamma-\alpha_i}, \qquad \forall\ i \in I$$

with $\gamma > \max_i \alpha_i$ a modelled distribution. We saw yesterday that the germ

$$F_x := \sum_{i\in I} \Pi^i_x f^i_x$$

turns out to be $(\bar{\alpha},\gamma)$-coherent with $\bar{\alpha} = \min_{i\in I} \alpha_i$.

Then $u := \mathscr{R}F$ is locally well approximated by F_x (say $\gamma > 0$).

We introduce a new notation:

$$\mathscr{R}_\Pi\, f := \mathscr{R} \sum_i \Pi^i f^i\,.$$

5.5. Fixed points

In the space $\mathscr{D}^\gamma(\Pi,\Gamma)$ we want to set an equation in the form of a fixed point

$$f = \hat{I}(f), \qquad \hat{I} : \mathscr{D}^\gamma \to \mathscr{D}^\gamma$$

with $\hat{I}$ a non-linear map.

Then one defines $u := \mathscr{R}_\Pi\, f$ (uniquely defined for $\gamma > 0$).

The point is that one would like u to be solution to a fixed point problem

$$u = \tilde{I}(u)\,,$$

where we would assume the following commutation

$$\mathscr{R} \circ \hat{I} = \tilde{I} \circ \mathscr{R}.$$

However $\tilde{I}$ is in general ill-defined and the space $\mathscr{R}\mathscr{D}^\gamma$ is not a Banach (Fréchet) space.

5.6. Operations on Modelled distributions

Therefore it is very important to define several operations on modelled distributions.

For an equation of the form (as for Φ^4_d)

$$u = G * (P(u) + \xi)$$

where G is the heat kernel and P is a polynomial, we can guess that two main ingredients are needed:

- an integration of modelled distributions with respect to a kernel
- a product of modelled distributions.

The integration with respect to a kernel is the Schauder estimate that we want to discuss now.

5.7. The Integration kernel

We fix an even measurable function $K : \mathbb{R}^d \to \mathbb{R}$ such that for some $\beta > 0$

$$|\partial^k K(x)| \lesssim \frac{1}{|x|^{d-\beta+|k|}} \mathbb{1}_{\{|x|\leqslant 1\}}, \qquad \forall\, |k| \leqslant N_{\gamma,\min_i \alpha_i},\ x \in \mathbb{R}^d.$$

Note the possible singularity at $x = 0$.

For the intuition, one can think of the special situation

$$K = \sum_{n=0}^{+\infty} \varepsilon_n^\beta L^{\varepsilon_n}, \qquad L^\varepsilon(x) = \frac{1}{\varepsilon^d} L\left(\frac{x}{\varepsilon}\right), \qquad \varepsilon_n := \varepsilon_n,$$

where $L \in \mathscr{D}(\mathbb{R}^d)$.

5.8. Integration

Suppose we have

- a model (Π, Γ)
- a modelled distribution $f \in \mathscr{D}^\gamma(\Pi, \Gamma)$ with $\gamma > 0$
- $u := \mathscr{R}_\Pi\ f$.

We want to define

- $K * u \in \mathscr{D}'(\mathbb{R}^d)$
- a new model $(\bar\Pi, \bar\Gamma)$ and a new modelled distribution $g \in \mathscr{D}^{\gamma+\beta}(\bar\Pi, \bar\Gamma)$ such that

$$K * u = \mathscr{R}_{\bar\Pi}\, g$$

namely $K * u = \mathscr{R} G$ where $G_x = \sum_k \bar\Pi_x^k g_x^k$.

Then g is the family of generalised derivatives of $K * u$ with respect to the model $(\bar{\Pi}, \bar{\Gamma})$.

In other words, if we set $\mathscr{K} : \mathscr{D}^{\gamma}(\Pi, \Gamma) \to \mathscr{D}^{\gamma+\beta}(\bar{\Pi}, \bar{\Gamma})$

$$\mathscr{K} f = g$$

then we have

$$\mathscr{R}_{\bar{\Pi}} \mathscr{K} = K * \mathscr{R}_{\Pi} .$$

Moreover the map $\mathscr{K} : \mathscr{D}^{\gamma}(\Pi, \Gamma) \to \mathscr{D}^{\gamma+\beta}(\bar{\Pi}, \bar{\Gamma})$ is linear and continuous.

5.9. The model $(\bar{\Pi}, \bar{\Gamma})$

The new basis $(\bar{\Pi}_x^k)_k$ is supposed to contain at least germs looking like $\bar{\Pi}_x^i := K * \Pi_x^i$.

However the elements of the basis must have a homogeneity property, which here should read (for $\alpha_i + \beta \neq 0$)

$$\left|\bar{\Pi}_x^i(\varphi_x^{\varepsilon_N})\right| \lesssim \lambda^{\alpha_i+\beta}.$$

This is however a very non-trivial constraint. If $\alpha_i + \beta > 0$, this means that $\bar{\Pi}_x^i(\varphi_x^{\varepsilon_N})$ has to be small for λ small.

However $K * \Pi_x^i(\varphi_x^{\varepsilon_N})$ has no reason to be small.

We must accept that $\bar{\Pi}_x^i(\varphi_x^{\varepsilon_N})$ need to be modified.

A natural choice is to allow $\bar{\Pi}_x^i$ to be equal to $K * \Pi_x^i$ up to a polynomial $P_x(\cdot)$.

5.10. The basis $\bar{\Pi}$

We want

$$\bar{\Pi}_x^i(\varphi) = K * \Pi_x^i(\varphi) - P_x(\varphi), \qquad \left|\bar{\Pi}_x^i(\varphi_x^{\varepsilon_N})\right| \lesssim \lambda^{\alpha_i+\beta} \qquad \text{where } P_x \text{ is a polynomial.}$$

If K and $\Pi_x^i(\cdot)$ are smooth, then we have essentially no choice but to subtract a Taylor polynomial of $K * \Pi_x^i$ centered at x:

$$\bar{\Pi}_x^i(\zeta) = K * \Pi_x^i(\zeta) - \sum_{|k| < \alpha_i + \beta} \left(\partial^k K * \Pi_x^i\right)(x) \, \frac{(\zeta - x)^k}{k!}.$$

In the general case, this must be defined as a distribution

$$\bar{\Pi}_x^i(\varphi) = K * \Pi_x^i(\varphi) - \sum_{|k| < \alpha_i + \beta} \left(\partial^k K * \Pi_x^i\right)(x) \int_{\mathbb{R}^d} \frac{(\zeta - x)^k}{k!} \, \varphi(\zeta) \, \mathrm{d}\zeta \, .$$

5.11. Local Schauder for distributions

Lemma 5.11.1. *Let $T \in \mathscr{D}'(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ fixed. Let $\bar{\alpha} \in \mathbb{R}$ and $r \in \mathbb{N}$ with $r > -\bar{\alpha}$. If*

$$|T(\varphi_x^\varepsilon)| \lesssim \|\varphi\|_{C^r} \varepsilon^{\bar{\alpha}}, \qquad \varepsilon \in (0,1],\ \varphi \in \mathscr{D}(B(0,1)),$$

then the distribution

$$\bar{T}(\varphi) := K * T(\varphi) - \sum_{|k|<\bar{\alpha}+\beta} \left(\partial^k K * T\right)(x) \int_{\mathbb{R}^d} \frac{(\zeta - x)^k}{k!} \varphi(\zeta)\, \mathrm{d}\zeta\,,$$

is well defined and we have

$$|\bar{T}(\varphi_x^\varepsilon)| \lesssim \varepsilon^{\bar{\alpha}+\beta}.$$

Note that for $\bar{\alpha} + \beta < 0$ we have $\bar{T}(\varphi) := K * T(\varphi)$.

5.12. The basis $\bar{\Pi}$

Therefore we have our first elements of the basis $\bar{\Pi}$

$$\bar{\Pi}_x^i(\varphi) = K * \Pi_x^i(\varphi) - \sum_{|k|<\alpha_i+\beta} \left(\partial^k K * \Pi_x^i\right)(x) \int_{\mathbb{R}^d} \frac{(\zeta - x)^k}{k!} \varphi(\zeta)\, \mathrm{d}\zeta\,.$$

and $\bar{\Pi}_x^i$ has homogeneity $\alpha_i + \beta$.

Is this enough?

We also have to build $\bar{\Gamma}$, for which we can necessarily start from Γ.

We have clearly

$$K * \Pi_y^j = \sum_{i \in I} (K * \Pi_x^i)\, \Gamma_{xy}^{ij}.$$

However what about the polynomial terms?

5.13. The matrix $\bar{\Gamma}$

$$\bar{\Pi}_x^i(\varphi) = K * \Pi_x^i(\varphi) - \sum_{|k|<\alpha_i+\beta} \left(\partial^k K * \Pi_x^i\right)(x) \int_{\mathbb{R}^d} \frac{(\zeta - x)^k}{k!} \varphi(\zeta)\, \mathrm{d}\zeta\,.$$

A computation shows

$$\begin{aligned} \bar{\Pi}_y^j - \sum_{i \in I} \bar{\Pi}_x^i \Gamma_{xy}^{ij} = &- \sum_{|k|<\alpha_j+\beta} \left(\partial^k K * \Pi_y^j\right)(y) \sum_{\ell \leqslant k} \frac{(x-y)^{k-\ell}}{(k-\ell)!} \frac{(\cdot - x)^\ell}{\ell!} \\ &+ \sum_{i \in I} \Gamma_{xy}^{ij} \sum_{|\ell|<\alpha_i+\beta} \left(\partial^\ell K * \Pi_x^i\right)(x) \frac{(\cdot - x)^\ell}{\ell!}. \end{aligned}$$

We must take care of the right hand side, which is a polynomial in $(\cdot)$ of degree at most $\gamma + \beta$, since $\alpha_i < \gamma$.

That means that we have to add to our basis $\bar{\Pi}$ the monomials of degree up to $\gamma+\beta$.

5.14. The model $(\bar{\Pi},\bar{\Gamma})$

We define $\hat{I} = I \sqcup \{i \in \mathbb{N}^d : |i| < \gamma+\beta\}$ and

$$\bar{\Pi}^i_x = \begin{cases} K * \Pi^i_x - \displaystyle\sum_{|k|<\alpha_i+\beta} \left(\partial^k K * \Pi^i_x\right)(x)\frac{(\cdot - x)^k}{k!}\,, & \text{if} \quad i \in I\,, \\[2ex] \dfrac{(\cdot - x)^i}{i!}, & \text{if} \quad i \in \mathbb{N}^d,\ |i| < \gamma+\beta\,. \end{cases}$$

The homogeneity is of course α_i if $i \in I$ and $|i|$ if $i \in \mathbb{N}^d$.
Now we have to define the corresponding $\bar{\Gamma}$.

5.15. The model $(\bar{\Pi},\bar{\Gamma})$

$$\bar{\Gamma}^{ij}_{xy} = \begin{cases} \Gamma^{ij}_{xy}, & \text{if} \quad i,j \in I\,, \\[2ex] \dfrac{(x-y)^{j-i}}{(j-i)!}, & \text{if} \quad i,j \in \mathbb{N}^d,\ i \leqslant j,\ \max\{|i|,|j|\} < \gamma+\beta\,, \\[2ex] \displaystyle\sum_{a:\,\alpha_a>|i|-\beta} (\partial^i K * \Pi^a_x)(x)\,\Gamma^{aj}_{xy} - \sum_{|k|<\alpha_j+\beta-|i|} (\partial^{i+k} K * \Pi^j_y)(y)\frac{(x-y)^k}{k!}\,, & \\ \qquad \text{if} \quad i \in \mathbb{N}^d,\ |i| < \alpha_j+\beta,\ j \in I\,, & \\[2ex] 0 & \text{otherwise.} \end{cases}$$

5.16. The modelled distribution g

We wanted to define a new model $(\bar{\Pi},\bar{\Gamma})$ and a new modelled distribution $g \in \mathscr{D}^{\gamma+\beta}(\bar{\Pi},\bar{\Gamma})$ such that

$$K * u = \mathscr{R}_{\bar{\Pi}}\, g$$

namely $K * u = \mathscr{R}G$ where $G_x = \sum_k \bar{\Pi}^k_x g^k_x$.
It remains to construct g and show that it has the desider properties. The definition is

$$g_x^i := \begin{cases} f_x^i & \text{if} \quad i \in I\,, \\ \left(\mathscr{R}_\Pi f - \sum\limits_{\alpha_a \leqslant |i|-\beta} f_x^a \, \Pi_x^a \right) * \partial^i K(x) & \text{if} \quad i \in \mathbb{N}^d,\ |i| < \gamma + \beta\,. \end{cases}$$

5.17. Multilevel Schauder estimates

THEOREM 5.17.1 (Hairer 14, Broux-Caravenna-Z. 21+). *With the above definitions*

- $g \in \mathscr{D}^{\gamma+\beta}(\bar{\Pi}, \bar{\Gamma})$
- *the map* $\mathscr{K} : \mathscr{D}^\gamma(\Pi, \Gamma) \to \mathscr{D}^{\gamma+\beta}(\bar{\Pi}, \bar{\Gamma})$ *defined by* $\mathscr{K} f = g$ *is linear continuous*
- *we have the commutation relation*

$$\mathscr{R}_{\bar{\Pi}} \mathscr{K} = K * \mathscr{R}_\Pi\,.$$

CHAPTER 6

Products and equations

6.1. Brief and very incomplete history

In the '60, '70 and '80 a huge activity on constructive quantum field theory.

In the '90 and '00: rough paths approach to stochastic analysis: T. Lyons, M. Gubinelli et al.: continuity of the solution map.

In the '10: application of rough path ideas to SPDEs: M. Hairer, Gubinelli-Imkeller-Perkowski.

The result is a robust and non-perturbative construction of Euclidean QFT models via stochastic quantization (Parisi '80).

6.2. Recap

We consider equations of the form (as for Φ^4_d)

$$u = G * (P(u) + \xi) = \tilde{\mathcal{I}}(u)$$

where G is the heat kernel and P is a polynomial.

This equation is singular because u may be a distribution and $P(u)$ would be ill-defined.

Indeed, if ξ is not smooth then in general the fixed point $u = \tilde{\mathcal{I}}(u)$ lacks a rigorous treatment.

In other words, we do not have a proper Banach space B which could contain the solution u, nor a good definition of the map $\tilde{\mathcal{I}}$.

Indeed $\tilde{\mathcal{I}}$ should contain the famous renormalised non-linearities like $: u^3 :$ which are hard to control analytically.

6.3. Recap

Martin's idea is to express the solution u in terms of an explicit (random) family $(\Pi^i_x)_{i \in I, x \in \mathbb{R}^d}$ of distributions.

The idea is to lift the equation to the space $\mathscr{D}^\gamma(\Pi, \Gamma)$ of modelled distributions, where (Π, Γ) is a model.

(Π, Γ) is an enhancement of the noise ξ. Its construction is the only probabilistic argument.

Via the Reconstruction Theorem, we represent $u = \mathscr{R}_\Pi f$ with $f \in \mathscr{D}^\gamma(\Pi, \Gamma)$, and the fixed point becomes

$$f = \hat{I}(f), \qquad \hat{I} : \mathscr{D}^\gamma \to \mathscr{D}^\gamma.$$

The renormalisation procedure (subtraction of infinities) intervenes only at the level of (Π, Γ), which is an explicit object, while the non-linearity $\hat{I}$ is essentially standard.

6.4. Modelled distributions

More precisely, $\hat{I}$ takes the form

$$\hat{I}(f) = \mathscr{K}(P(f) + \Xi)$$

for the equation $u = G * (P(u) + \xi)$, where $\mathscr{K}$ is the integration operator we defined yesterday.

We still have to define the product of modelled distributions which is needed for $P(f)$.

In any case, let us stress again that $P(f)$ is a standard polynomial in f, for example f^3, rather than $: f^3 :$

6.5. Models

A model is a finite family $(\Pi_x^1, \dots, \Pi_x^N)$ of germs, such that for all $x \in \mathbb{R}^d$ Π_x^i has homogeneity $\alpha_i \in \mathbb{R}$, i.e.

$$|\Pi_x^i(\varphi_x^{\varepsilon_N})| \lesssim \lambda^{\alpha_i},$$

and there exists a matrix-valued function $(x, y) \mapsto (\Gamma_{xy}^{ij})_{i,j \in I}$ such that

$$\Pi_y^j = \sum_{i \in I} \Pi_x^i \Gamma_{xy}^{ij}, \qquad j \in I,\ x, y \in \mathbb{R}^d.$$

6.6. Modelled distributions

For a fixed model (Π, Γ), we call any function $f : \mathbb{R}^d \to \mathbb{R}^I$ such that

$$\left| f_x^i - \sum_{j \in I} \Gamma_{xy}^{ij} f_y^j \right| \lesssim |x - y|^{\gamma - \alpha_i}, \qquad \forall\, i \in I$$

with $\gamma > \max_i \alpha_i$ a modelled distribution in $\mathscr{D}^\gamma = \mathscr{D}^\gamma(\Pi, \Gamma)$. We have seen that the germ

$$F_x := \sum_{i \in I} \Pi_x^i f_x^i$$

turns out to be (α, γ)-coherent with $\alpha = \min_i \alpha_i$.

Then $u := \mathscr{R}F$ is locally well approximated by F_x (say $\gamma > 0$).

We introduce a new notation:

$$\mathscr{R}_\Pi f := \mathscr{R} \sum_i \Pi^i f^i .$$

6.7. Fixed points

In the space $\mathscr{D}^\gamma(\Pi,\Gamma)$ we want to set an equation in the form of a fixed point

$$f = \hat{I}_{(\Pi,\Gamma)}(f), \qquad \hat{I}_{(\Pi,\Gamma)} : \mathscr{D}^\gamma(\Pi,\Gamma) \to \mathscr{D}^\gamma(\Pi,\Gamma)$$

with $\hat{I}_{(\Pi,\Gamma)}(f) = \mathscr{K}(P(f)+\Xi)$ a non-linear map.

Then one defines $u := \mathscr{R}_\Pi f$ (uniquely defined for $\gamma > 0$).

The fixed point for f replaces the fixed point for u

$$u = \widetilde{I}(u),$$

which is ill-defined.

Therefore it is very important to define several operations on modelled distributions. Two main ingredients are needed:

- an integration of modelled distributions with respect to a kernel
- a product of modelled distributions.

6.8. Multilevel Schauder estimates

Let K be a β-regularising integration kernel with $\beta > 0$.

THEOREM 6.8.1 (Hairer 14, Broux-Caravenna-Z. 21+). *There exist a suitable model* $(\bar{\Pi},\bar{\Gamma})$ *and a linear continuous map*

$$\mathscr{K} : \mathscr{D}^\gamma(\Pi,\Gamma) \to \mathscr{D}^{\gamma+\beta}(\bar{\Pi},\bar{\Gamma})$$

sastisfying the commutation relation

$$\mathscr{R}_{\bar{\Pi}} \mathscr{K} = K * \mathscr{R}_\Pi .$$

In the setting of Ilya's lectures, in fact $(\bar{\Pi},\bar{\Gamma}) = (\Pi,\Gamma)$. We restrict to this situation from now on.

6.9. Products

Let $\mathscr{D}^\gamma_\alpha(\Pi,\Gamma)$ be the set of modelled distributions in $\mathscr{D}^\gamma(\Pi,\Gamma)$ which have homogeneities bounded below by α.

For $f_1 \in \mathscr{D}^{\gamma_1}_{\alpha_1}(\Pi,\Gamma)$ and $f_2 \in \mathscr{D}^{\gamma_2}_{\alpha_2}(\Pi,\Gamma)$, we define

$$\gamma = (\gamma_1+\alpha_2) \wedge (\gamma_2+\alpha_1), \qquad \alpha = \alpha_1+\alpha_2,$$

$$(f_1 \star f_2)^i_x = \sum_{j,k} \mathbb{1}_{(\alpha_j+\alpha_k=\alpha_i<\gamma)} (f_1)^j_x\,(f_2)^k_x.$$

THEOREM 6.9.1 (Hairer 14). *With this definition,* $f_1 \star f_2 \in \mathscr{D}^{\gamma}_{\alpha}(\Pi,\Gamma)$ *and the map*

$$\mathscr{D}^{\gamma_1}_{\alpha_1}(\Pi,\Gamma) \times \mathscr{D}^{\gamma_2}_{\alpha_2}(\Pi,\Gamma) \ni (f_1, f_2) \mapsto f_1 \star f_2 \in \mathscr{D}^{\gamma}_{\alpha}(\Pi,\Gamma)$$

is bilinear and continuous.

6.10. Renormalised Products

As I mentioned before, there is no renormalisation in this product.

However if we reconstruct the product, renormalisation appears. For example

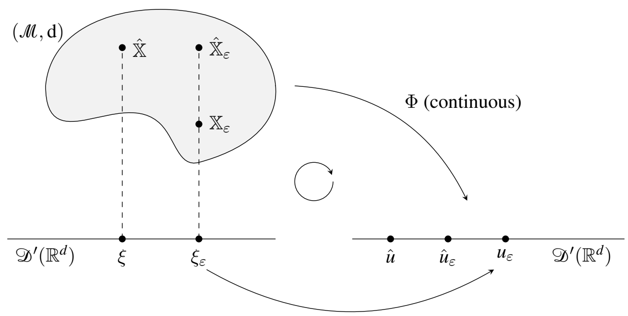
$$\mathscr{R}_{\Pi}\left(f^3\right) \quad = \quad :(\mathscr{R}_{\Pi} f)^3:$$

This explains why one can give sense to

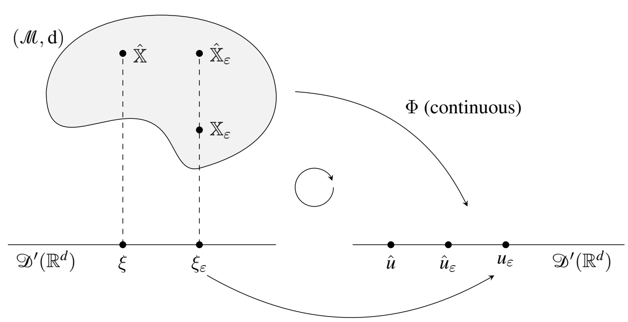
$$f = \hat{I}_{(\Pi,\Gamma)}(f), \qquad \hat{I} : \mathscr{D}^{\gamma}(\Pi,\Gamma) \to \mathscr{D}^{\gamma}(\Pi,\Gamma)$$

but not to

$$u = \widetilde{I}(u), \qquad u = \mathscr{R}_{\Pi} f.$$



6.11. Explanation



$\xi_\varepsilon = \rho_\varepsilon * \xi$, $\quad \mathbb{X}_\varepsilon :=$ canonical model associated to ξ_ε, $\quad \hat{\mathbb{X}}_\varepsilon =$ renormalised $\mathbb{X}_\varepsilon$,

$(\mathscr{M}, \mathrm{d}) =$ Space of models, $\quad \Phi : \mathscr{M} \to \mathscr{D}'(\mathbb{R}^d)$ continuous Solution Map,

$$\Phi(\mathbb{X}) = u \qquad \text{where} \qquad u = \hat{I}_{\mathbb{X}}(u).$$

6.12. The space of models and its topology

Models are $\mathbb{X} = (\Pi, \Gamma) \in \mathscr{M}$, with

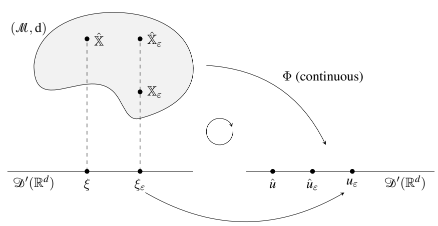
$$\Pi = (\Pi_x^i)_{x \in \mathbb{R}^d, i \in I} \subset \mathscr{D}'(\mathbb{R}^d), \qquad |\Pi_x^i(\varphi_x^{\varepsilon_N})| \lesssim \lambda^{\alpha_i},$$

$$\Gamma = (\Gamma_{xy}^{ij})_{x,y \in \mathbb{R}^d, i,\, j \in I} \subset \mathbb{R}, \qquad |\Gamma_{xy}^{ij} - \delta_{ij}| \lesssim \mathbb{1}_{(\alpha_i > \alpha_j)}\, |x-y|^{\alpha_i - \alpha_j}.$$

The $(\alpha_i)_{i \in I}$ are fixed.

The metric d on $\mathscr{M}$ is obtained by taking differences between (Π, Γ) and $(\bar{\Pi}, \bar{\Gamma})$ and choosing the best constants in the above inequalities.

6.13. An example: Φ_d^4



$$u_\varepsilon = \Phi(\mathbb{X}_\varepsilon), \qquad u_\varepsilon = G * \left(-u_\varepsilon^3 + \xi_\varepsilon\right),$$
$$\hat{u}_\varepsilon = \Phi(\hat{\mathbb{X}}_\varepsilon), \qquad \hat{u}_\varepsilon = G * \left(-\hat{u}_\varepsilon^3 + C_\varepsilon \hat{u}_\varepsilon + \xi_\varepsilon\right).$$

The renormalisation group acts on $\mathbb{X}_\varepsilon \mapsto \hat{\mathbb{X}}_\varepsilon$ and on the coefficients of the equation satisfied by u_ε and $\hat{u}_\varepsilon$, respectively.

On the other hand

$$f_\varepsilon = \mathscr{K}_{\mathbb{X}_\varepsilon}\left(-f_\varepsilon^3 + \Xi\right), \qquad \hat{f}_\varepsilon = \mathscr{K}_{\hat{\mathbb{X}}_\varepsilon}\left(-\hat{f}_\varepsilon^3 + \Xi\right).$$

The diverging constants C_ε appear only in $\hat{\mathbb{X}}_\varepsilon$.

6.14. The crucial result

We recall that $\Phi : \mathscr{M} \to \mathscr{D}'(\mathbb{R}^d)$ is given by

$$(\mathscr{M}, \mathrm{d}) = \text{Space of models}, \qquad \Phi(\mathbb{X}) = u \qquad \text{where} \qquad u = \hat{I}_{\mathbb{X}}(u).$$

The crucial results are

- $\mathscr{M} \ni \mathbb{X} \mapsto \Phi(\mathbb{X}) \in \mathscr{D}'(\mathbb{R}^d)$ is continuous
- $\hat{\mathbb{X}}_\varepsilon$ converges in probability to $\hat{\mathbb{X}}$ in $\mathscr{M}$.

This shows that $\hat{u}_\varepsilon \to \hat{u}$ converges in probability.

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