An introduction to the analytic theory of Regularity Structures

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CHAPTER 1

Introduction

The aim of these lecture notes is to introduce the reader to some of the main tools in the analytic theory of regularity structures, in particular the notions of models, modelled distributions, reconstruction and multi-level Schauder estimates. We try to follow an original approach, giving a new and mostly self-contained presentation of these concepts, rather than referring to the existing literature.

Since the founding paper [**9**] by Martin Hairer there has been a lot of work in the field, but most of these articles are very hard to read for people outside a small group of experts. These notes are part of an ongoing project which aims to rethink these ideas and make them more intuitive and accessible. With [**4**] we started this project by rewriting one of the main results of the theory, the reconstruction theorem, in a more general setting; indeed our definitions and statements are purely in the domain of distribution theory and we do not need to define regularity structures. In [**4**] we introduced new notions, in particular that of *coherent germ*, and new results pertaining to them.

The second step of this project concerns another analytic cornerstone of the theory, namely the (multi-level) Schauder estimates, which we prove both in the more general setting of coherent germs and in the more restricted one of modelled distributions. These notes present several results in this setting, some of which are new; we are meanwhile writing a more detailed research paper [**1**] on the same topic, with a somewhat different approach.

In the process of rewriting the two main results of reconstruction and Schauder-estimates, we introduce in a simplified setting the fundamental notions of models and modelled distributions. Again, the aim is to give a pedagogical introduction with as little technical material as possible. The necessary structure and technical assumptions are given gradually, only when they are really needed.

Since we want to be essentially self-contained, we give (almost) complete proofs, which in some important cases contain new material. As a result, these notes are not exhaustive and some important topics in the analytic theory of regularity structures are not treated here. In particular the product of modelled distributions is the next step in our project and it will appear elsewhere.

CHAPTER 2

Reconstruction

In these lecture notes we want to present an introduction to (some of) the analytical aspects of regularity structures, with an emphasis on how to construct (some of) the most relevant objects.

2.1. Distributions

These lectures will concern the space $\mathscr{D}'(\mathbb{R}^d)$ of *distributions* or *generalised functions*. We consider the space $\mathscr{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support on $\mathbb{R}^d$. The Euclidean ball of radius r centered at x is denoted by $B(x,r) = \{z \in \mathbb{R}^d : |z-x| \leqslant r\}$.

A *distribution* on $\mathbb{R}^d$ is a linear functional $T : C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$ such that for every compact set $K \subset \mathbb{R}^d$ there is $r = r_K \in \mathbb{N}$

$$|T(\varphi)| \lesssim \|\varphi\|_{C^r} := \max_{|k|\leqslant r} \|\partial^k \varphi\|_\infty, \qquad \forall\, \varphi \in C_0^\infty(K) \tag{2.1.1}$$

where throughout these lecture notes $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leqslant C g$. If one can find a $r \in \mathbb{N}$ such that (2.1.1) holds for all compact set $K \subset \mathbb{R}^d$ then we say that T has *order* r.

Every locally integrable (in particular continuous) function $f : \mathbb{R}^d \to \mathbb{R}$ defines a distribution by integration:

$$f(\varphi) := \int_{\mathbb{R}^d} f(x)\, \varphi(x)\, \mathrm{d}x, \qquad \varphi \in \mathscr{D}(\mathbb{R}^d).$$

A famous example of distribution from quantum mechanics is the Dirac measure δ_x at $x \in \mathbb{R}^d$

$$\delta_x(\varphi) = \varphi(x), \qquad \varphi \in C^\infty(\mathbb{R}^d).$$

One can also differentiate any distribution $T \in \mathscr{D}'(\mathbb{R}^d)$ and obtain a new distribution: for $k \in \mathbb{N}^d$

$$\partial^k T(\varphi) := (-1)^{k_1+\cdots+k_d}\, T(\partial^k \varphi).$$

Distributions form a linear space. If $\varphi \in C^\infty(\mathbb{R}^d)$ and $T \in \mathscr{D}'(\mathbb{R}^d)$ then it is possible to define canonically the product $\varphi \cdot T = T \cdot \varphi$ as

$$\varphi \cdot T(\psi) = T \cdot \varphi(\psi) := T(\varphi\psi), \qquad \forall\, \psi \in C_c^\infty(\mathbb{R}^d).$$

However, if $T, T' \in \mathscr{D}'(\mathbb{R}^d)$, in general there is no canonical way of defining $T \cdot T'$.

One may use some form of regularisation of T, T' or both. Then, the result could heavily depend on the regularisation and thus be neither unique nor canonical. For example, there does not seem to exist a reasonable way to define the square $(\delta_x)^2$ of the Dirac function.

Regularity structures give a framework to define products of *certain* distributions, and to prove well-posedness of some PDEs where such distributions appear.

2.2. The main question of this chapter

For every $x \in \mathbb{R}^d$ we assign a distribution $F_x \in \mathscr{D}'(\mathbb{R}^d)$ and we call the family $(F_x)_{x\in\mathbb{R}^d}$ a germ if for all $\psi \in \mathscr{D}$, the map $x \mapsto F_x(\psi)$ is measurable. Measurability of the map $x \mapsto F_x(\psi)$ is a technical assumption, which is needed in the definition of suitable approximations to the reconstruction of $(F_z)_{z\in\mathbb{R}^d}$, see (2.3.2) below.

Problem: Can we find a distribution $f \in \mathscr{D}'(\mathbb{R}^d)$ which is locally "well approximated" by $(F_x)_{x\in\mathbb{R}^d}$? Before making this notion precise, we explore the familiar setting of Taylor expansions.

2.2.1. Taylor expansions.

For example, let us fix $f \in C^\infty(\mathbb{R}^d)$, and let us define for a fixed $\gamma > 0$

$$F_x(y) := \sum_{|k|<\gamma} \partial^k f(x) \, \frac{(y-x)^k}{k!}, \qquad x, y \in \mathbb{R}^d. \tag{2.2.1}$$

Note that for $j \in \mathbb{N}^d$, $w \in \mathbb{R}^d$, we use the notation

$$|j| := \sum_{k=1}^d j_k, \qquad w^j := \prod_{k=1}^d w_k^{j_k}, \qquad j! := \prod_{k=1}^d j_k!$$

with the convention $0^0 := 1$. Then the classical Taylor theorem says that there exists a function $R(x,y)$ such that

$$f(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma \tag{2.2.2}$$

uniformly for x, y on compact sets of $\mathbb{R}^d$. By (2.2.2) we say that the distribution defined by f is *locally well approximated* by the germ $(F_x)_{x\in\mathbb{R}^d}$ formed by its Taylor polynomials.

2.2.2. Scaling.

Let us introduce now the fundamental tool of *scaling*: for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$, $\lambda > 0$ and $y \in \mathbb{R}^d$ we set

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d} \, \varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d. \tag{2.2.3}$$

When $y = 0$ we write $\varphi^\lambda = \varphi_0^\lambda$,

Then the local approximation property (2.2.2) implies

Proposition 2.2.1. *Let* $f \in C^\infty(\mathbb{R}^d)$, $\gamma > 0$ *and* F_x *be defined by* (2.2.1). *Then*

$$\left|(f - F_y)(\varphi_y^\lambda)\right| \lesssim \lambda^\gamma, \tag{2.2.4}$$

uniformly for y *in compact sets of* $\mathbb{R}^d$, $\lambda \in]0,1]$ *and* $\varphi \in \mathscr{D}(B(0,1))$ *with* $\int|\varphi| \leqslant 1$.

PROOF. By (2.2.2) we have $f - F_y = R(y,\cdot)$ and $|R(y,w)| \lesssim |w-y|^\gamma$. Since φ_y^λ is supported by $B(y,\lambda)$ with $\int|\varphi_y^\lambda| = \int|\varphi|$,

$$\begin{aligned}\left|(f - F_y)(\varphi_y^\lambda)\right| &= \left|\int_{\mathbb{R}^d} R(y,w)\, \varphi_y^\lambda(w)\, \mathrm{d}w\right| \\ &\lesssim \sup_{w \in B(y,\lambda)} |w-y|^\gamma \int |\varphi_y^\lambda| \leqslant \lambda^\gamma\end{aligned}$$

uniformly for y in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$ and $\varphi \in \mathscr{D}(B(0,1))$ with $\int|\varphi| \leqslant 1$. □

In this context we have another simple formula, which does not seem so well known.

Proposition 2.2.2. *Let* $f \in C^\infty(\mathbb{R}^d)$, $\gamma > 0$ *and* F_x *be defined by* (2.2.1). *Then*

$$\left|(F_z - F_y)(\varphi_y^\lambda)\right| \lesssim (|y-z| + \lambda)^\gamma, \tag{2.2.5}$$

uniformly for y,z *in compact sets of* $\mathbb{R}^d$, $\lambda \in]0,1]$ *and* $\varphi \in \mathscr{D}(B(0,1))$ *with* $\int|\varphi| \leqslant 1$.

PROOF. Let us note that we can Taylor expand also the derivatives of f for $|k| < \gamma$

$$\partial^k f(y) = \sum_{|\ell| < \gamma - |k|} \partial^{k+\ell} f(z)\, \frac{(y-z)^\ell}{\ell!} + R^k(y,z), \qquad |R^k(y,z)| \lesssim |y-z|^{\gamma - |k|},$$

uniformly for x,y on compact sets of $\mathbb{R}^d$. Then we can write

$$\begin{aligned}F_y(w) &= \sum_{|k|<\gamma} \partial^k f(y)\, \frac{(w-y)^k}{k!} \\ &= \sum_{|k|<\gamma} \left(\sum_{|\ell|<\gamma-|k|} \partial^{k+\ell} f(z)\, \frac{(y-z)^\ell}{\ell!} + R^k(y,z) \right) \frac{(w-y)^k}{k!} \\ &= F_z(w) + \sum_{|k|<\gamma} R^k(y,z)\, \frac{(w-y)^k}{k!},\end{aligned}$$

having applied Newton's binomial. Therefore we obtain the expression

$$F_z(w) - F_y(w) = -\sum_{|k|<\gamma} R^k(y,z)\, \frac{(w-y)^k}{k!}. \tag{2.2.6}$$

In particular

$$\begin{aligned} |F_z(w) - F_y(w)| &\leqslant \sum_{|k|<\gamma} |R^k(y,z)|\, \frac{|w-y|^k}{k!} \\ &\lesssim \sum_{|k|<\gamma} |y-z|^{\gamma-|k|}\, |w-y|^k \lesssim (|y-z| + |w-y|)^\gamma \end{aligned}$$

since $a^t b^s \leqslant (a+b)^t (a+b)^s$ for $a,b,t,s \geqslant 0$. Now by (2.2.3), for all $\varphi \in \mathscr{D}(B(0,1))$ with $\int |\varphi| \leqslant 1$

$$\begin{aligned} \left| \int_{\mathbb{R}^d} (F_z(w) - F_y(w))\, \varphi_y^\lambda(w)\, \mathrm{d}w \right| &\lesssim \sup_{w \in B(y,\lambda)} (|y-z| + |w-y|)^\gamma \int |\varphi_y^\lambda| \\ &\leqslant (|y-z| + \lambda)^\gamma. \end{aligned}$$

We have obtained (2.2.5). □

2.3. Reconstruction

We define throughout the paper

$$\varepsilon_n := 2^{-n}, \qquad n \in \mathbb{N}.$$

We have seen in (2.2.4) that for the germ $(F_y)_{y \in \mathbb{R}^d}$ related to a Taylor expansion of order $\gamma > 0$

$$\left| (f - F_y)(\varphi_y^{\varepsilon_n}) \right| \lesssim \varepsilon_n^\gamma,$$

uniformly for y in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$ and $\varphi \in \mathscr{D}(B(0,1))$ with $\int |\varphi| \leqslant 1$. This property does not rely explicitly on the smoothness of f, and seems to be a promising way of expressing the fact that $(F_y)_{y \in \mathbb{R}^d}$ locally approximates well (at order $\gamma > 0$) the distribution f.

This motivates the following:

Definition 2.3.1. *Let $(F_y)_{y \in \mathbb{R}^d} \subseteq \mathscr{D}'(\mathbb{R}^d)$ a family of distributions. We say that $f \in \mathscr{D}'(\mathbb{R}^d)$ is a reconstruction of $(F_y)_{y \in \mathbb{R}^d}$ if there exists $\gamma > 0$ such that for all $\varphi \in \mathscr{D}$*

$$\left| (f - F_y)(\varphi_y^{\varepsilon_n}) \right| \lesssim \varepsilon_n^\gamma, \tag{2.3.1}$$

uniformly for y in compact sets of $\mathbb{R}^d$ and $n \in \mathbb{N}$.

We are going to see below sufficient conditions for a family $(F_y)_{y\in\mathbb{R}^d} \subseteq \mathscr{D}'(\mathbb{R}^d)$ of distributions to admit a reconstruction. A first important remark is that, with this definition, there is at most one reconstruction for a given $(F_y)_{y\in\mathbb{R}^d}$.

We are going to use a number of times the following formula: for all $T \in \mathscr{D}'$ and $\varphi, g \in \mathscr{D}$

$$T(\varphi * g) = \int_{\mathbb{R}^d} T(\varphi(\cdot - y))\, g(y)\, \mathrm{d}y\,.$$

With the notation $\varphi_y(x) := \varphi(x-y) = \varphi_y^1(x)$, recall (2.2.3), we obtain the basic formula

$$T(\varphi * g) = \int_{\mathbb{R}^d} T(\varphi_y)\, g(y)\, \mathrm{d}y\,, \tag{2.3.2}$$

Lemma 2.3.2 (Uniqueness). *Given any $(F_x)_{x\in\mathbb{R}^d} \subseteq \mathscr{D}'(\mathbb{R}^d)$ and $\gamma > 0$, there is at most one reconstruction of $(F_x)_{x\in\mathbb{R}^d}$ in the sense of Definition 2.3.1.*

PROOF. We fix a test function $\varphi \in \mathscr{D}$ with $\int \varphi = 1$, and two distributions $f, g \in \mathscr{D}'$ which satisfy, uniformly for y in compact sets,

$$\lim_{n\to\infty} |(f - F_y)(\varphi_y^{\varepsilon_n})| = \lim_{n\to\infty} |(g - F_y)(\varphi_y^{\varepsilon_n})| = 0\,. \tag{2.3.3}$$

We set $T := f - g$. Since $(\varphi^{\varepsilon_n})_{n\in\mathbb{N}}$ is a family of mollifiers, for any $\psi \in \mathscr{D}$ we have $T(\psi) = \lim_{n\to\infty} T(\psi * \varphi^{\varepsilon_n})$. If K is any compact set which contains the support of ψ we have by (2.3.2)

$$|T(\psi * \varphi^{\varepsilon_n})| = \left| \int_{\mathbb{R}^d} T(\varphi_y^{\varepsilon_n})\, \psi(y)\, \mathrm{d}y \right| \leqslant \|\psi\|_{L^1} \sup_{y\in K} |T(\varphi_y^{\varepsilon_n})|\,.$$

It remains to show that $\lim_{n\to\infty} T(\varphi_y^{\varepsilon_n}) = 0$ uniformly for $y \in K$, for which it is enough to observe that

$$|T(\varphi_y^{\varepsilon_n})| = |f(\varphi_y^{\varepsilon_n}) - g(\varphi_y^{\varepsilon_n})| \leqslant |(f - F_y)(\varphi_y^{\varepsilon_n})| + |(g - F_y)(\varphi_y^{\varepsilon_n})|$$

and these terms vanish as $n \to \infty$ uniformly for y in compact sets, by (2.3.3). □

2.4. Coherence

We have seen in (2.2.5) that for the germ related to a Taylor expansion we have for any $\gamma > 0$

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim (|y-z| + \varepsilon_n)^\gamma, \qquad \left|(f - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\gamma,$$

uniformly for y, z in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$ and $\varphi \in \mathscr{D}(B(0,1))$ with $\int |\varphi| \leqslant 1$.

However the first estimate implicitly relies on the information that the distribution $F_z - F_y$ is a locally bounded function: suppose indeed that this is not the case; then we expect that the quantity $(F_z - F_y)(\varphi_y^{\varepsilon_n})$ does not necessarily remain bounded as $n \to \infty$; this is the case for example if $F_z - F_y$ is a Dirac mass at y, where

$$(F_z - F_y)(\varphi_y^{\varepsilon_n}) = \frac{1}{\varepsilon_n^d}\,\varphi(0). \tag{2.4.1}$$

Therefore, if we want to consider more general families $(F_y)_{y\in\mathbb{R}^d}$ of genuine distributions, we expect (2.2.5) to be too strong a requirement.

Formula (2.4.1) suggests that a weaker version of (2.2.5), which could be convenient in this context, may be obtained by allowing a multiplicative factor ε_n^α with $\alpha \leqslant 0$ in (2.2.5):

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\alpha(|y-z|+\varepsilon_n)^\gamma. \tag{2.4.2}$$

However, it turns out that (2.4.2) may not be strong enough to obtain (2.3.1): the multiplicative factor ε_n^α, which explodes as $n \to \infty$ if $\alpha < 0$, makes a better control on the factor $(|y-z|+\varepsilon_n)$ necessary, as can be seen from the proof of Theorem 2.5.1 below. It turns out that a sufficient condition for the existence of a (unique) reconstruction is

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\alpha(|y-z|+\varepsilon_n)^{\gamma-\alpha},$$

uniformly for z, y in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$. We call this property *coherence*, see below.

Definition 2.4.1. *We say that a germ $(F_z)_{z\in\mathbb{R}^d} \subset \mathscr{D}'$ is (α,γ)-*coherent *for $\gamma \in \mathbb{R}$, and $\alpha \leqslant \gamma$, if there exists $\varphi \in \mathscr{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$, such that*

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\alpha(|y-z|+\varepsilon_n)^{\gamma-\alpha}, \tag{2.4.3}$$

uniformly for z, y in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$.

We denote by $\mathscr{G}^{\alpha,\gamma}$ the set of (α,γ)-coherent germs.

Remark 2.4.2. It is a non obvious (but true) fact, see [**4**, Proposition 13.1], that relation (2.4.3) actually holds uniformly over $\varphi \in \mathscr{D}(B(0,1))$ with bounded $\|\varphi\|_{C^{r_\alpha}}$, where $r_\alpha := \min\{k \in \mathbb{N} : k > -\alpha\}$. More precisely:

$$|(F_z - F_y)(\varphi_y^{\varepsilon_n})| \lesssim \|\varphi\|_{C^{r_\alpha}}\,\varepsilon_n^\alpha\,(|y-z|+\varepsilon_n)^{\gamma-\alpha}, \tag{2.4.4}$$

uniformly for x, y, z in compact sets, $n \in \mathbb{N}$ and $\varphi \in \mathscr{D}(B(0,2))$. This property is called *enhanced coherence*. In particular, $\mathscr{G}^{\alpha,\gamma}$ is a vector space.

2.5. Hairer's Reconstruction Theorem (without regularity structures)

We define the following family of test functions:

$$\mathscr{B}_r := \{\psi \in \mathscr{D}(B(0,1)) :\ \|\psi\|_{C^r} \leqslant 1\}. \tag{2.5.1}$$

THEOREM 2.5.1 (Reconstruction Theorem). *Suppose that* $(F_z)_{z\in\mathbb{R}^d} \subset \mathscr{D}'$ *is a* (α,γ)-coherent germ *in the sense of Definition 2.4.1 with* $\gamma > 0$*, namely there exist* $\gamma > 0$*,* $\alpha \leqslant \gamma$ *and a* $\varphi \in \mathscr{D}(\mathbb{R}^d)$ *with* $\int \varphi \neq 0$*, such that*

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\alpha}(|x-y| + \varepsilon_n)^{\gamma-\alpha},$$

uniformly for x, y *in compact sets of* $\mathbb{R}^d$*,* $n \in \mathbb{N}$*. Then there exists a unique distribution* $\mathscr{R}F \in \mathscr{D}'(\mathbb{R}^d)$ *such that*

$$|(\mathscr{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\gamma} \tag{2.5.2}$$

uniformly for x *in compact sets of* $\mathbb{R}^d$*,* $n \in \mathbb{N}$*,* $\psi \in \mathscr{B}_r$*, see* (2.5.1)*, for any fixed integer* $r > -\alpha$*.*

- This result was stated and proved by Martin Hairer in [**9**, Thm. 3.10] for a subclass of germs related to regularity structures. He used wavelets.
- Later Otto-Weber [**13**] proposed an approach based on a semigroup. This corresponds to a special choice of the test functions φ, ψ. See also [**12**].
- The above statement is a slight improvement of [**4**, Thm. 5.1]. It is more general and requires no knowledge of regularity structures. The improvement is due to [**15**] and concerns the fact that it is not necessary to impose a homogeneity condition on the germ (see below).
- This result can be seen as a generalisation of the Sewing Lemma in rough paths [**8, 7**]. See [**3**, section 5] for a discussion of the analogies between the Reconstruction Theorem and the Sewing Lemma.
- The construction is completely local: constants and even the exponent α may be allowed to depend on the compact set.
- We also cover the case $\gamma \leqslant 0$ (see below).

Example 2.5.2. *Let* $A \subset \mathbb{R}$ *be a (locally) finite set such that* $\alpha := \inf A \in \mathbb{R}$*. Let* $F = (F_x)_{x\in\mathbb{R}^d}$ *be a germ such that, for some* $\gamma \geqslant \alpha$ *and* $\varphi \in \mathscr{D}(\mathbb{R}^d)$ *with* $\int \varphi \neq 0$*, we have*

$$|(F_z - F_y)(\varphi_y^{\varepsilon_n})| \lesssim \sum_{a\in A:\ a<\gamma} \varepsilon_n^a\, |z-y|^{\gamma-a}, \tag{2.5.3}$$

uniformly for z, y *in compact sets and for* $n \in \mathbb{N}$*.*

Then the germ F *is* (α,γ)*-coherent, since for* $\varepsilon \in (0,1]$

$$\varepsilon^a\, |z-y|^{\gamma-a} = \varepsilon^{\alpha}\, \varepsilon^{a-\alpha}\, |z-y|^{\gamma-a} \leqslant \varepsilon^{\alpha}\, (\varepsilon + |z-y|)^{\gamma-\alpha}.$$

For example we saw in (2.2.5) *that the Taylor expansions* (2.2.1) *satisfy* (2.5.3) *with* $A = \mathbb{N}$ *and* $\alpha = 0$*.*

Remark 2.5.3. If $(F_z)_{z\in\mathbb{R}^d} \subset \mathscr{D}'$ is a (α,γ)-coherent germ and $\alpha > 0$, then the map $z \mapsto F_z$ is constant, so that we implicitly assume from now on that $\alpha \leqslant 0$. In order to prove the claim, we apply the triangular inequality

$$|(F_y - F_x)(\varphi_z^{\varepsilon_n})| \leqslant |(F_y - F_z)(\varphi_z^{\varepsilon_n})| + |(F_z - F_x)(\varphi_z^{\varepsilon_n})| \to 0$$

as $n \to +\infty$ (uniformly for x, y, z in compact sets) by the coherence assumption. Then we obtain for all $\psi \in \mathscr{D}$ by (2.3.2)

$$\begin{aligned}(F_y - F_x)(\psi) &= \lim_{n\to+\infty} (F_y - F_x)(\psi * \varphi^{\varepsilon_n}) \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^d} (F_y - F_x)(\varphi_z^{\varepsilon_n})\, \psi(z)\, \mathrm{d}z = 0.\end{aligned}$$

2.6. Sketch of the proof

In this section we give a detailed sketch of the proof of Theorem 2.5.1. We use also in the following the notation

$$\bar{K}_r := \{x \in \mathbb{R}^d : \mathrm{dist}(x, K) \leqslant r\} \tag{2.6.1}$$

for $K \subset \mathbb{R}^d$ and $r > 0$.

We fix a (α,γ)-coherent germ $(F_z)_{z\in\mathbb{R}^d} \subset \mathscr{D}'$, i.e. we suppose that there exist $\gamma > 0, \alpha \leqslant 0$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$, such that

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^{\alpha} (|y - z| + \varepsilon_n)^{\gamma-\alpha}, \tag{2.6.2}$$

uniformly for z, y in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$. We fix an integer $r > -\alpha$ and we find in an elementary way a related $\hat{\varphi} \in \mathscr{D}(B(0,1))$ such that

$$\int_{\mathbb{R}^d} \hat{\varphi}(y)\, \mathrm{d}y = 1\,, \quad \int_{\mathbb{R}^d} y^k\, \hat{\varphi}(y)\, \mathrm{d}y = 0\,, \quad \forall\, k \in \mathbb{N}^d:\ 1 \leqslant |k| \leqslant r-1\,, \tag{2.6.3}$$

and (2.6.2) holds with φ replaced by $\hat{\varphi}$, see [**4**, Lemma 8.3]. Then we define

$$\rho := \hat{\varphi}^2 * \hat{\varphi} \qquad \text{and} \qquad \varepsilon_n := 2^{-n}, \quad n \in \mathbb{N}\,, \tag{2.6.4}$$

where we recall that $\psi^{\varepsilon_N} = \psi_0^{\varepsilon_N}$ is a scaling of ψ as in (2.2.3). Note that $\int \rho = \int \hat{\varphi}^2 \int \hat{\varphi} = 1$. This peculiar choice of ρ ensures that *the difference* $\rho^{\frac{1}{2}} - \rho$ *is a convolution*:

$$\rho^{\frac{1}{2}} - \rho = \hat{\varphi} * \check{\varphi}\,, \qquad \text{where we define} \qquad \check{\varphi} := \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2\,. \tag{2.6.5}$$

By (2.6.3),

$$\int_{\mathbb{R}^d} y^k\, \check{\varphi}(y)\, \mathrm{d}y = 0\,, \quad \forall\, k \in \mathbb{N}^d:\ 0 \leqslant |k| \leqslant r-1\,. \tag{2.6.6}$$

This will be used below to subtract suitable Taylor polynomials. Moreover it follows that

$$\rho^{\varepsilon_{n+1}} - \rho^{\varepsilon_n} = (\rho^{\frac{1}{2}} - \rho)^{\varepsilon_n} = \hat{\varphi}^{\varepsilon_n} * \check{\varphi}^{\varepsilon_n}\,. \tag{2.6.7}$$

With these definitions, we can define the function

$$f_n(z) := F_z(\rho_z^{\varepsilon_n})$$

that we may look at as a distribution, so that we write

$$f_n(\psi) := \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n})\,\psi(z)\,\mathrm{d}z\,, \qquad \psi \in \mathscr{D}\,. \tag{2.6.8}$$

The definition of f_n is inspired by (2.3.2): we show that f_n converges to a limiting distribution, which is the reconstruction $\mathscr{R}F$ we are looking for.

We study the function

$$f_{x,n}(z) := f_n(z) - F_x(\rho_z^{\varepsilon_n}) = (F_z - F_x)(\rho_z^{\varepsilon_n})\,, \qquad x, z \in \mathbb{R}^d. \tag{2.6.9}$$

We write $f_{x,n}$ as a telescoping sum:

$$\begin{aligned}
&f_{x,k+1}(z) - f_{x,k}(z) = (F_z - F_x)(\rho_z^{\varepsilon_{k+1}} - \rho_z^{\varepsilon_k}) \\
&= (F_z - F_x)(\hat{\varphi}^{\varepsilon_n} * \check{\varphi}_z^{\varepsilon_n}) = \int_{\mathbb{R}^d} (F_z - F_x)(\hat{\varphi}_y^{\varepsilon_k})\,\check{\varphi}^{\varepsilon_k}(y-z)\,\mathrm{d}y \\
&= \underbrace{\int_{\mathbb{R}^d} (F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})\,\check{\varphi}^{\varepsilon_k}(y-z)\,\mathrm{d}y}_{g'_{x,k}(z)} + \underbrace{\int_{\mathbb{R}^d} (F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})\,\check{\varphi}^{\varepsilon_k}(y-z)\,\mathrm{d}y}_{g''_k(z)}\,,
\end{aligned} \tag{2.6.10}$$

where again we use (2.3.2). We have first by (2.6.2), for all $z \in \mathbb{R}^d$,

$$|g''_k(z)| \leqslant \|\check{\varphi}^{\varepsilon_k}\|_{L^1} \sup_{|y-z|\leqslant\varepsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \lesssim \varepsilon_k^{\alpha}\,\varepsilon_k^{\gamma-\alpha} = \varepsilon_k^{\gamma}\,,$$

since $\|\check{\varphi}^{\varepsilon_k}\|_{L^1} = \|\check{\varphi}\|_{L^1}$. Then we obtain for all $\psi \in \mathscr{D}$

$$\left|\int_{\mathbb{R}^d} g''_k(z)\,\psi(z)\,\mathrm{d}z\right| \lesssim \varepsilon_k^{\gamma}\,\|\psi\|_{L^1}. \tag{2.6.11}$$

Now we want to estimate

$$\int_{\mathbb{R}^d} g'_{x,k}(z)\psi(z)\,\mathrm{d}z = \int_{\mathbb{R}^d} (F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})\,(\check{\varphi}^{\varepsilon_k} * \psi)\,(y)\,\mathrm{d}y\,. \tag{2.6.12}$$

If K is the support of ψ and $\bar{K}_1$ is the subset of $\mathbb{R}^d$ which has distance $\leqslant 1$ from K, we obtain that $\check{\varphi}^{\varepsilon} * \psi$ has support in $\bar{K}_1$ for $\varepsilon \leqslant \frac{1}{2}$. Then by the coherence condition

$$\left|\int_{\mathbb{R}^d} g'_{x,k}(z)\psi(z)\,\mathrm{d}z\right| \leqslant \sup_{y\in\bar{K}_1} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})|\,\|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1} \lesssim \varepsilon_k^{\alpha}\|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1}.$$

Note now that by (2.6.6)

$$(\check{\varphi}^{\varepsilon} * \psi)(y) = \int_{\mathbb{R}^d} \check{\varphi}^{\varepsilon}(y-z)\,\{\psi(z) - p_y(z)\}\,\mathrm{d}z\,,$$

where $p_y(\cdot) := \sum_{|k|\leqslant r-1} \frac{\partial^k \psi(y)}{k!}(\cdot - y)^k$ the Taylor polynomial of ψ of order $r-1$ based at y; since $|\psi(z) - p_y(z)| \lesssim \|\psi\|_{C^r} |z-y|^r$, we obtain

$$|(\check{\varphi}^\varepsilon * \psi)(y)| \leqslant \|\psi\|_{C^r} \int_{\mathbb{R}^d} |\check{\varphi}^\varepsilon(y-z)|\,|z-y|^r \,\mathrm{d}z \leqslant \|\psi\|_{C^r} \|\check{\varphi}\|_{L^1} \varepsilon^r, \quad y \in \mathbb{R}^d.$$

We obtain

$$\left| \int_{\mathbb{R}^d} g'_{x,k}(z)\,\psi(z)\,\mathrm{d}z \right| \lesssim \varepsilon_k^{\alpha+r} \|\psi\|_{C^r}. \tag{2.6.13}$$

In particular we obtain by (2.6.11)-(2.6.13), since $\gamma > 0$ and $\alpha + r > 0$ (recall that we fixed $r > -\alpha$)

$$f_{x,n}(\psi) = f_{x,0}(\psi) + \sum_{k=0}^{n-1} \left[g'_{x,k}(\psi) + g''_k(\psi) \right]$$

converges as $n \to +\infty$ to a distribution of order r. Note now that $F_x(\rho_\cdot^{\varepsilon_n})$ converges to F_x in $\mathscr{D}'$, since by (2.3.2)

$$\int_{\mathbb{R}^d} F_x(\rho_z^{\varepsilon_n})\,\psi(z)\,\mathrm{d}z = F_x(\rho^{\varepsilon_n} * \psi) \to F_x(\psi), \qquad \forall\, \psi \in \mathscr{D}.$$

We obtain by (2.6.9) that f_n converges to a distribution $\mathscr{R}F$ in $\mathscr{D}'$. Moreover, since for all $n \geqslant \ell$ we have

$$f_{x,n}(\psi) = f_{x,\ell}(\psi) + \sum_{k=\ell}^{n-1} \left[g'_{x,k}(\psi) + g''_k(\psi) \right], \tag{2.6.14}$$

letting $n \to +\infty$ we obtain that for all $x \in \mathbb{R}^d$, $\psi \in \mathscr{D}$ and $\ell \in \mathbb{N}$

$$\mathscr{R}F(\psi) = F_x(\psi) + f_{x,\ell}(\psi) + \sum_{k=\ell}^{\infty} \left[g'_{x,k}(\psi) + g''_k(\psi) \right]. \tag{2.6.15}$$

Formula (2.6.15) is due to [**15**].

We want now to prove the reconstruction bound (2.5.2). We recall the following result, proved in [**4**, Lemma 9.3]: let $k, N \in \mathbb{N}$ and $G : \mathbb{R}^d \to \mathbb{R}$ a measurable function; then for all $x \in \mathbb{R}^d$ and $\psi \in \mathscr{B}_r$, see (2.5.1),

$$\left| \int_{\mathbb{R}^d} G(y)\,(\check{\varphi}^{\varepsilon_k} * \psi_x^{\varepsilon_N})(y)\,\mathrm{d}y \right| \leqslant 4^d \|\check{\varphi}\|_{L^1} \min\{\varepsilon_k/\varepsilon_N, 1\}^r \sup_{B(x,\varepsilon_N+\varepsilon_k)} |G|. \tag{2.6.16}$$

By (2.6.12) and (2.6.16), for any $k, N \in \mathbb{N}$,

$$\left| \int_{\mathbb{R}^d} g'_{x,k}(z)\,\psi_x^{\varepsilon_N}(z)\,\mathrm{d}z \right| \leqslant 4^d \|\check{\varphi}\|_{L^1} \min\{\varepsilon_k/\varepsilon_N, 1\}^r \sup_{y \in B(x,\varepsilon_N+\varepsilon_k)} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})|.$$

For $y \in B(x, \varepsilon_N + \varepsilon_k)$, by (2.6.2) with φ replaced by $\hat{\varphi}$, we have

$$|(F_x - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \lesssim \varepsilon_k^{\alpha}(|x-y| + \varepsilon_k)^{\gamma-\alpha} \lesssim \varepsilon_k^{\alpha} \max\{\varepsilon_k, \varepsilon_N\}^{\gamma-\alpha}.$$

We have obtained

$$\left| \int_{\mathbb{R}^d} g'_{x,k}(z)\, \psi_x^{\varepsilon_N}(z)\, \mathrm{d}z \right| \lesssim \begin{cases} \varepsilon_N^{\gamma-\alpha-r}\, \varepsilon_k^{\alpha+r} & \text{if } k > N \\ \varepsilon_k^{\gamma} & \text{if } k \leqslant N \end{cases}. \tag{2.6.17}$$

We want now to estimate $J_x := F_x - \mathscr{R}F$, and in particular $J_x(\psi_x^{\varepsilon_N})$. We write

$$|J_x(\psi_x^{\varepsilon_N})| \leqslant |f_{x,N}(\psi_x^{\varepsilon_N})| + |(J_x - f_{x,N})(\psi_x^{\varepsilon_N})|.$$

First by (2.6.9) and (2.6.4)

$$f_{x,N}(\psi_x^{\varepsilon_N}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (F_z - F_x)(\hat{\varphi}_y^{\varepsilon_N})\, \hat{\varphi}^{2\varepsilon_N}(y-z)\, \psi_x^{\varepsilon_N}(z)\, \mathrm{d}y\, \mathrm{d}z,$$

so that, since ψ is supported in $B(0,1)$ and $\hat{\varphi}$ is supported in $B(0, \frac{1}{2})$,

$$|f_{x,N}(\psi_x^{\varepsilon_N})| \leqslant \|\hat{\varphi}^{2\varepsilon_N}\|_{L^1}\, \|\psi_x^{\varepsilon_N}\|_{L^1} \sup_{z \in B(x,\varepsilon_N), |y-z| \leqslant \varepsilon_N} |(F_z - F_x)(\hat{\varphi}_y^{\varepsilon_N})|.$$

Now we write $|(F_z - F_x)(\hat{\varphi}_y^{\varepsilon_N})| \leqslant |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_N})| + |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_N})|$ and

$$\sup_{z \in B(x,\varepsilon_N), |y-z| \leqslant \varepsilon_N} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_N})| \lesssim \varepsilon_N^{\alpha}\, \varepsilon_N^{\gamma-\alpha} \leqslant \varepsilon_N^{\gamma},$$

$$\sup_{z \in B(x,\varepsilon_N), |y-z| \leqslant \varepsilon_N} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_N})| \lesssim \varepsilon_N^{\alpha}\, (\varepsilon_N + 2\varepsilon_N)^{\gamma-\alpha} \lesssim \varepsilon_N^{\gamma},$$

so that we obtain

$$|f_{x,N}(\psi_x^{\varepsilon_N})| \lesssim \varepsilon_N^{\gamma}, \tag{2.6.18}$$

and this argument holds for any $\gamma \in \mathbb{R}$. Now by (2.6.15)

$$(J_x - f_{x,N})(\psi_x^{\varepsilon_N}) = -\sum_{k=N}^{\infty} \left[g'_{x,k}(\psi) + g''_k(\psi) \right],$$

and by (2.6.11) and (2.6.17),

$$\begin{aligned} |(J_x - f_{x,N})(\psi_x^{\varepsilon_N})| &\leqslant \sum_{k \geqslant N} \left[|g'_{x,k}(\psi_x^{\varepsilon_N})| + |g''_k(\psi_x^{\varepsilon_N})| \right] \\ &\lesssim \sum_{k \geqslant N} \left[\varepsilon_N^{\gamma-\alpha-r}\, \varepsilon_k^{\alpha+r} + \varepsilon_k^{\gamma} \right] \\ &\leqslant \frac{\varepsilon_N^{\gamma-\alpha-r}\, \varepsilon_N^{\alpha+r}}{1 - 2^{-(\alpha+r)}} + \frac{\varepsilon_N^{\gamma}}{1 - 2^{-\gamma}} \lesssim \varepsilon_N^{\gamma}, \end{aligned}$$

since $\gamma > 0$ and $\alpha + r > 0$. The proof is complete.

2.7. The Reconstruction Theorem for $\gamma \leqslant 0$.

In Theorem 2.5.1 we have proved the existence and the uniqueness of the reconstruction of a (α,γ)-coherent germ in the case of $\gamma > 0$. If $\gamma \leqslant 0$ then we have a weaker result.

THEOREM 2.7.1. *Suppose that for a given $F : \mathbb{R}^d \to \mathscr{D}'(\mathbb{R}^d)$ there exist $\gamma \leqslant 0$ and $\alpha \leqslant \gamma$, such that for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$*

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\alpha}(|x-y| + \varepsilon_n)^{\gamma-\alpha},$$

uniformly for x, y in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$, namely F is (α,γ)-coherent.

Then there exists a (non-unique) $\mathscr{R}F \in \mathscr{D}'(\mathbb{R}^d)$ such that

$$|(\mathscr{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \begin{cases} \varepsilon_n^{\gamma} & \text{if } \gamma < 0 \\ 1+n & \text{if } \gamma = 0 \end{cases}. \tag{2.7.1}$$

uniformly for x in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$, $\{\psi \in \mathscr{D}(B(0,1)) : \|\psi\|_{C^r} \leqslant 1\}$ with a fixed $r > -\alpha$.

PROOF. If one checks the proof of the case $\gamma > 0$, one sees that the convergence of the different terms depends either on $\gamma > 0$ or on $\alpha + r > 0$. More precisely, the estimate (2.6.11) on g_k'' is useful if $\gamma > 0$, while the estimate (2.6.13) on $g_{x,k}'$ is useful if $\alpha + r > 0$. If $\gamma \leqslant 0$, the estimate on g_k'' is simply not good enough.

On the other hand, for $\gamma \leqslant 0$ the reconstruction bound (2.7.1) is weaker, since ε_n^{γ} or n diverge as $n \to \infty$, and we do not state that there is a unique choice for $\mathscr{R}F$.

In fact, in order to prove the statement we can modify the approximating sequence f_n defined in (2.6.8), by eliminating the term g_k'' whose convergence is based on $\gamma > 0$. However, $g_{x,k}'$, given by (2.6.12) above, depends on $x \in \mathbb{R}^d$, while we want the approximating sequence $\bar{f}_n \in \mathscr{D}'$ to be independent of any base point.

Recalling the definition of f_n and g_k'' from (2.6.8) and (2.6.10), we define (also recall (2.6.9))

$$\bar{f}_n := f_n - \sum_{k=0}^{n-1} g_k'',$$

$$\bar{f}_{x,n}(\psi) := \bar{f}_n(\psi) - F_x(\rho^{\varepsilon_n} * \psi) = f_{x,n}(\psi) - \sum_{k=0}^{n-1} g_k''(\psi).$$

Then, by (2.6.14), for all $n \geqslant \ell$,

$$\bar{f}_{x,n}(\psi) = f_{x,\ell}(\psi) + \sum_{k=\ell}^{n-1} g'_{k,x}(\psi) - \sum_{k=0}^{\ell-1} g''_k(\psi) = \bar{f}_{x,\ell}(\psi) + \sum_{k=\ell}^{n-1} g'_{k,x}(\psi)\,. \tag{2.7.2}$$

By the estimate (2.6.13) on $g'_{x,k}$, we obtain that $\bar{f}_{x,n}$, and therefore $\bar{f}_n$, converge in $\mathscr{D}'$ and we can write for all $\psi \in \mathscr{D}$, $x \in \mathbb{R}^d$ and $\ell \in \mathbb{N}$

$$\begin{aligned} \mathscr{R}F(\psi) := \lim_n \bar{f}_n(\psi) &= F_x(\psi) + \lim_n \bar{f}_{x,n}(\psi) \\ &= F_x(\psi) + \bar{f}_{x,\ell}(\psi) + \sum_{k=\ell}^{\infty} g'_{k,x}(\psi). \end{aligned} \tag{2.7.3}$$

For the reconstruction bound (2.7.1), we want to estimate $\tilde{f}_x := \mathscr{R}F - F_x$, and in particular $\tilde{f}_x(\psi_x^{\varepsilon_\ell})$. We write

$$|\tilde{f}_x(\psi_x^{\varepsilon_\ell})| \leqslant |\tilde{f}_{x,\ell}(\psi_x^{\varepsilon_\ell})| + |(\tilde{f}_x - \tilde{f}_{x,\ell})(\psi_x^{\varepsilon_\ell})|.$$

Now, by (2.6.17) and (2.7.3), if $\gamma \leqslant 0$

$$\begin{aligned} |(\tilde{f}_x - \tilde{f}_{x,\ell})(\psi_x^{\varepsilon_\ell})| &\leqslant \sum_{k\geqslant\ell} |g'_{x,k}(\psi_x^{\varepsilon_\ell})| \\ &\lesssim \sum_{k\geqslant\ell} \varepsilon_\ell^{\gamma-\alpha-r}\,\varepsilon_k^{\alpha+r} \lesssim \varepsilon_\ell^{\gamma-\alpha-r}\,\varepsilon_\ell^{\alpha+r} = \varepsilon_\ell^{\gamma}\,, \end{aligned}$$

since $\alpha + r > 0$. By (2.6.18) and by (2.6.17), if $\gamma < 0$

$$\begin{aligned} |\tilde{f}_{x,\ell}(\psi_x^{\varepsilon_\ell})| &\leqslant |f_{x,\ell}(\psi_x^{\varepsilon_\ell})| + \sum_{k=0}^{\ell-1} \left|g''_k(\psi_x^{\varepsilon_\ell})\right| \\ &\lesssim \varepsilon_\ell^{\gamma} + \sum_{k=0}^{\ell-1} 2^{|\gamma|k} \lesssim 2^{|\gamma|\ell} = \varepsilon_\ell^{\gamma}. \end{aligned}$$

In the case $\gamma = 0$ we have rather

$$|\tilde{f}_{x,\ell}(\psi_x^{\varepsilon_\ell})| \leqslant |f_{x,\ell}(\psi_x^{\varepsilon_\ell})| + \sum_{k=0}^{\ell-1} \left|g''_k(\psi_x^{\varepsilon_\ell})\right| \lesssim 1 + \ell\,.$$

The proof is complete. □

2.8. Homogeneity

Definition 2.8.1. *Let F be a germ. We say that F satisfies a homogeneity bound with exponent $\bar{\alpha} \in \mathbb{R}$ if there exists $r > -\bar{\alpha}$ such that*

$$|F_x(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\bar{\alpha}}\,,$$

uniformly for x in compact sets, $n \in \mathbb{N}$ and $\psi \in \mathscr{B}_r$, see (2.5.1).

We recall the following result, which is proved in [**4**, Lemma 4.12].

Lemma 2.8.2 (Homogeneity). *Let $F = (F_x)_{x \in \mathbb{R}^d}$ be a (α, γ)-coherent germ. For any compact set $K \subseteq \mathbb{R}^d$, there is a real number $\bar{\alpha}_K < \gamma$ such that*

$$|F_x(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\bar{\alpha}_K} \quad \textit{uniformly for } x \in K \textit{ and } n \in \mathbb{N}, \tag{2.8.1}$$

with φ as in Definition 2.4.1.

Therefore coherence of a germ implies a local form of homogeneity of the same germ. However in Definition 2.8.1 we require the coefficient $\bar{\alpha}$ to be uniform over the compact set K.

If a germ satisfies a homogeneity bound with exponent $\bar{\alpha} \in \mathbb{R}$, then it satisfies a homogeneity bound with exponent $\bar{\alpha}'$ for all $\bar{\alpha}' \leqslant \bar{\alpha}$. Therefore the set of $\bar{\alpha} \in \mathbb{R}$ such that a fixed germ satisfies a homogeneity bound with exponent $\bar{\alpha}$ takes the form $]-\infty, b]$ or $]-\infty, b[$.

Definition 2.8.3. *We denote by $\mathscr{G}^{\bar{\alpha};\alpha,\gamma}$ the set of (α, γ)-coherent germs which satisfy a homogeneity bound with exponent $\bar{\alpha}$.*

Remark 2.8.4. Let F be a (α, γ)-coherent germ with respect to a test function $\varphi \in \mathscr{D}$ such that $\int \varphi \neq 0$. If there is $\bar{\alpha} \in \mathbb{R}$ such that for all compact set $K \subset \mathbb{R}^d$

$$|F_x(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\bar{\alpha}} \quad \text{uniformly for } x \in K \text{ and } n \in \mathbb{N},$$

then F satisfies a homogeneity bound with exponent $\bar{\alpha} \in \mathbb{R}$ and with $r = r_{\alpha \wedge \bar{\alpha}} = \min\{n \in \mathbb{N} : n > -(\alpha \wedge \bar{\alpha})\}$ as in Definition 2.8.1. This property is called *enhanced homogeneity*, see [**4**, Theorem 12.4], and is the analog of the enhanced coherence of Remark 2.4.2.

2.9. Negative Hölder (Besov) spaces

Given $\alpha \in]-\infty, 0[$, we define $\mathscr{C}^\alpha = \mathscr{C}^\alpha(\mathbb{R}^d)$ as the space of distributions $T \in \mathscr{D}'$ such that

$$\frac{|T(\psi_x^{\varepsilon_n})|}{\|\psi\|_{C^{r_\alpha}}} \lesssim \varepsilon_n^\alpha, \tag{2.9.1}$$

uniformly for x in compact sets, $\psi \in \mathscr{B}_{r_\alpha} \setminus \{0\}$ and $n \in \mathbb{N}$, where we define r_α as the smallest integer $r \in \mathbb{N}$ such that $r > -\alpha$. For any distribution $T \in \mathscr{D}'$ and $\alpha < 0$, we define $\|T\|_{\mathscr{C}^\alpha(K)}$ as the best constant in (2.9.1):

$$\|T\|_{\mathscr{C}^\alpha(K)} := \sup_{z \in K, n \in \mathbb{N}, \psi \in \mathscr{B}_{r_\alpha}} \frac{|T(\psi_x^{\varepsilon_n})|}{\varepsilon_n^\alpha \|\psi\|_{C^{r_\alpha}}}. \tag{2.9.2}$$

Then $T \in \mathscr{C}^\alpha$ if and only if $\|T\|_{\mathscr{C}^\alpha(K)} < \infty$, for all compact sets $K \subseteq \mathbb{R}^d$.

We want now to show that a coherent germ which satisfies a homogeneity bound with exponent $\bar{\alpha} < 0$ has a reconstruction (unique or not) which belongs to the Besov space $\mathscr{C}^{\bar{\alpha}}$, and the map $F \mapsto \mathscr{R}F$ is linear continuous.

We introduce the semi-norms

$$|||F|||^{\mathrm{coh}}_{K,\varphi,\alpha,\gamma} := \sup_{y,z\in K,\ n\in\mathbb{N}} \frac{|(F_z - F_y)(\varphi_y^{\varepsilon_n})|}{\varepsilon_n^{\alpha}\,(|z-y|+\varepsilon_n)^{\gamma-\alpha}}\,, \tag{2.9.3}$$

$$|||F|||^{\mathrm{hom}}_{K,\varphi,\bar\alpha} := \sup_{x\in K,\ n\in\mathbb{N}} \frac{|F_x(\varphi_x^{\varepsilon_n})|}{\varepsilon_n^{\bar\alpha}}\,, \tag{2.9.4}$$

where φ is as in Definition 2.4.1. We can now state the following result.

THEOREM 2.9.1 (Reconstruction Theorem and Hölder spaces). *Let $\alpha \leqslant \gamma$ and $\gamma \neq 0$. Let $(F_x)_{x\in\mathbb{R}^d}$ be a (α,γ)-coherent germ with local homogeneity bound $\bar\alpha \leqslant \gamma$, namely $F \in \mathscr{G}^{\bar\alpha;\alpha,\gamma}$. If $\bar\alpha > 0$, then $\mathscr{R}F = 0$. If $\bar\alpha < 0$, then $\mathscr{R}F$ belongs to $\mathscr{C}^{\bar\alpha}$ and for every compact set $K \subseteq \mathbb{R}^d$*

$$\|\mathscr{R}F\|_{\mathscr{C}^{\bar\alpha}(K)} \leqslant \mathfrak{C}_{\alpha,\gamma,\bar\alpha,d,\varphi}\left(|||F|||^{\mathrm{coh}}_{\bar K_4,\varphi,\alpha,\gamma} + |||F|||^{\mathrm{hom}}_{\bar K_2,\varphi,\bar\alpha}\right), \tag{2.9.5}$$

where φ is the test function in the coherence condition (2.4.3)*, $\mathfrak{C}_{\alpha,\gamma,\bar\alpha,d,\varphi} < \infty$ is a constant which depends neither on F nor on K and we use the notation* (2.6.1).

PROOF. We fix a compact set $K \subset \mathbb{R}^d$ and $y \in K$. By the reconstruction bounds (2.5.2) for $\gamma > 0$ and (2.7.1) for $\gamma < 0$, $\mathscr{R}F$ satisfies

$$\left|(\mathscr{R}F - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^{\gamma}\,.$$

It follows by the homogeneity bound (2.8.1) and the triangle inequality that

$$\left|\mathscr{R}F(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^{\bar\alpha} + \varepsilon_n^{\gamma}\,.$$

When $\bar\alpha > 0$ then also $\gamma > 0$ and the r.h.s. vanishes as $n \to \infty$, which yields $\mathscr{R}F \equiv 0$, because $\mathscr{R}F(\psi) = \lim_n \mathscr{R}F(\psi * \varphi^{\varepsilon_n}) = \lim_n \int \mathscr{R}F(\varphi_y^{\varepsilon_n})\,\psi(y)\,\mathrm{d}y$.

Henceforth we fix $\bar\alpha < 0$. Let φ be the test function in the coherence condition (2.4.3). Let $f = \mathscr{R}F$ be a reconstruction of F. Fix a compact set K: we want to show that

$$\sup_{x\in\bar K_2,\,N\in\mathbb{N},\,\psi\in\mathscr{B}_{r_{\alpha\wedge\bar\alpha}}} \frac{|f(\psi_x^{\varepsilon_N})|}{\varepsilon_N^{\bar\alpha}} \leqslant \mathfrak{C}'\left(|||F|||^{\mathrm{coh}}_{\bar K_4,\varphi,\alpha,\gamma} + |||F|||^{\mathrm{hom}}_{\bar K_2,\varphi,\bar\alpha}\right) \tag{2.9.6}$$

for some $\mathfrak{C}' = \mathfrak{C}'_{\alpha,\gamma,\bar\alpha,d,\varphi} < \infty$, where $r_{\alpha\wedge\bar\alpha} = \min\{n \in \mathbb{N} : n > -(\alpha\wedge\bar\alpha)\}$. Note that in (2.9.6) we have a supremum over $\psi \in \mathscr{B}_{r_{\alpha\wedge\bar\alpha}}$, while in (2.9.2) we had a supremum over $\psi \in \mathscr{B}_{r_\alpha}$, so that it seems that (2.9.6) does not imply that $f \in \mathscr{C}^{\bar\alpha}(K)$. However, the definition (2.9.2) gives the same space if r_α is replaced by any $r > -\alpha$, see e.g. [**4**, Theorem 12.4].

Now we have, uniformly for $x \in \bar K_2$, $\psi \in \mathscr{B}_{r_{\alpha\wedge\bar\alpha}}$ and $N \in \mathbb{N}$,

$$|(f - F_x)(\psi_x^{\varepsilon_N})| \leqslant \mathfrak{c}\,|||F|||^{\mathrm{coh}}_{\bar K_4,\varphi,\alpha,\gamma}\,\varepsilon_N^{\gamma}$$

for a suitable $\mathfrak{c} = \mathfrak{c}_{\alpha,\gamma,\bar\alpha,d,\varphi}$, where the constant $|||F|||^{\mathrm{coh}}_{\bar K_4,\varphi,\alpha,\gamma}$ arises by tracking carefully the constants in the estimates in proof of the Reconstruction

Theorem, see Section 2.6. Since $\bar\alpha \leqslant \gamma \neq 0$, we bound $\varepsilon_N^\gamma \leqslant \varepsilon_N^{\bar\alpha}$, for all $n \in \mathbb{N}$. Recalling (2.9.4), by the triangle inequality we obtain

$$\sup_{x\in\bar K_2,\, N\in\mathbb{N}} \frac{|f(\psi_x^{\varepsilon_N})|}{\varepsilon_N^{\bar\alpha}} \leqslant \sup_{x\in\bar K_2,\, N\in\mathbb{N}} \frac{|(f-F_x)(\psi_x^{\varepsilon_N})| + |F_x(\psi_x^{\varepsilon_N})|}{\varepsilon_N^{\bar\alpha}}$$
$$\leqslant (1+c_{\bar\alpha})\,\mathfrak{c}'\,|||F|||^{\mathrm{coh}}_{\bar K_4,\varphi,\alpha,\gamma} + c_{\bar\alpha,\varphi}\,|||F|||^{\mathrm{hom}}_{\bar K_2,\varphi,\bar\alpha}\,,$$

by the enhanced homogeneity of Remark 2.8.4. This completes the proof of (2.9.6). □

2.10. Singular product

Let $f\in\mathscr{C}^\alpha$ with $\alpha>0$ and $F_y(w)=\sum_{|k|<\alpha}\partial^k f(y)\,\frac{(w-y)^k}{k!}$. Let also $g\in\mathscr{C}^\beta$ with $\beta\leqslant 0$. We define the germ $P=(P_x := g\cdot F_x)_{x\in\mathbb{R}^d}$ as

$$P_x(\varphi) = (g\cdot F_x)(\varphi) := g(\varphi F_x), \qquad \varphi\in\mathscr{D}\,.$$

Note that this makes sense and defines a distribution in $\mathscr{D}'$ since $\varphi F_x\in\mathscr{D}$ for all $\varphi\in\mathscr{D}$.

THEOREM 2.10.1. *If $f\in\mathscr{C}^\alpha$ and $g\in\mathscr{C}^\beta$, with $\alpha>0$ and $\beta\leqslant 0$, then the germ $P=(P_x)_{x\in\mathbb{R}^d}$ is $(\beta,\alpha+\beta)$-coherent and satisfies a homogeneity bound with exponent β,*

$$\left|(P_z-P_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\beta\,(|y-z|+\varepsilon_n)^\alpha, \qquad \left|P_y(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\beta\,,$$

uniformly over z,y in compact sets, $n\in\mathbb{N}$ and $\varphi\in\mathscr{B}_r$, with $r>-\beta$.

PROOF. Since $g\in\mathscr{C}^\beta$ we have for all $\varepsilon\in(0,1]$, $\psi\in\mathscr{D}(B(0,1))$ and $y\in K$

$$\left|g(\psi_y^\varepsilon)\right| \leqslant \|g\|_{\mathscr{C}^\beta(K)}\,\|\psi\|_{C^r}\,\varepsilon^\beta\,. \tag{2.10.1}$$

Fix now any $\varphi\in\mathscr{D}(B(0,1))$ with $\int\varphi\neq 0$ and $\|\varphi\|_{C^r}\leqslant 1$. By (2.2.6), for any $y,z\in K$ (and γ replaced by α)

$$(P_z-P_y)(\varphi_y^\varepsilon) = -\sum_{0\leqslant|k|<\alpha} g\left((\cdot-y)^k\,\varphi_y^\varepsilon\right)\frac{R^k(y,z)}{k!}$$

where $|R^k(y,z)|\lesssim \|f\|_{\mathscr{C}^\alpha(K)}\,|z-y|^{\alpha-|k|}$. We have for fixed $y\in\mathbb{R}^d$, $k\in\mathbb{N}^d$ and $\varepsilon>0$

$$(w-y)^k\,\varphi_y^\varepsilon(w) = \varepsilon^{|k|}\,\psi_y^\varepsilon(w)\,, \quad \text{where} \quad \psi(w) := w^k\,\varphi(w)\,.$$

Then $\psi\in\mathscr{D}(B(0,1))$ and $\|\psi\|_{C^r}\lesssim\|\varphi\|_{C^r}\leqslant 1$, hence it follows by (2.10.1) that

$$|g\left((\cdot-y)^k\,\varphi_y^\varepsilon\right)| = \varepsilon^{|k|}\,g\left(\psi_y^\varepsilon\right) \lesssim \|g\|_{\mathscr{C}^\beta(K)}\,\varepsilon^{\beta+|k|}\,. \tag{2.10.2}$$

We thus obtain, uniformly for $z, y \in K$ and $\varepsilon \in (0,1]$,

$$\begin{aligned} |(P_z - P_y)(\varphi_y^\varepsilon)| &\lesssim \|f\|_{\mathscr{C}^\alpha(K)} \|g\|_{\mathscr{C}^\beta(K)} \sum_{0 \leqslant |k| < \alpha} \varepsilon^{\beta+|k|} |z-y|^{\alpha-|k|} \\ &\lesssim \|f\|_{\mathscr{C}^\alpha(K)} \|g\|_{\mathscr{C}^\beta(K)} \varepsilon^\beta (|z-y|+\varepsilon)^\alpha, \end{aligned}$$

which completes the proof of coherence. We next prove homogeneity. By (2.10.2), we obtain

$$\begin{aligned} |P_x(\varphi_x^\varepsilon)| &\leqslant \sum_{0 \leqslant |k| < \gamma} \left| g\left((\cdot - x)^k \varphi_x^\varepsilon \right) \right| \left| \frac{\partial^k f(x)}{k!} \right| \\ &\lesssim \|g\|_{\mathscr{C}^\beta(K)} \sum_{0 \leqslant |k| < \gamma} \varepsilon^{\beta+|k|} \left| \frac{\partial^k f(x)}{k!} \right| \\ &\lesssim \|f\|_{\mathscr{C}^\alpha(K)} \|g\|_{\mathscr{C}^\beta(K)} \sum_{0 \leqslant |k| < \gamma} \varepsilon^{\beta+|k|} \\ &\lesssim \|f\|_{\mathscr{C}^\alpha(K)} \|g\|_{\mathscr{C}^\beta(K)} \varepsilon^\beta, \end{aligned}$$

uniformly for x in compact sets and $\varepsilon \in (0,1]$. This completes the proof. □

If $\alpha + \beta > 0$ the (unique) distribution $\mathscr{R}P$ can be used to construct a *canonical product* of f and g. Moreover $\mathscr{R}P \in \mathscr{C}^\beta$.

If $\alpha + \beta \leqslant 0$, the (non-unique) distribution $\mathscr{R}P$ can be used to construct a *non-canonical product* of f and g. Moreover $\mathscr{R}P \in \mathscr{C}^\beta$.

2.11. A special case

Let us assume that $F_x \in C(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$ and moreover that the map $\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto F_x(y)$ is continuous. We recall that in Section 2.6 we proved that for all $\psi \in \mathscr{D}$

$$\mathscr{R}(F)(\psi) = \lim_{n \to +\infty} \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n})\, \psi(z)\, \mathrm{d}z.$$

Now if $(x,y) \mapsto F_x(y)$ is continuous, we obtain by dominated convergence that

$$\mathscr{R}(F)(\psi) = \int_{\mathbb{R}^d} F_z(z)\, \psi(z)\, \mathrm{d}z,$$

namely $\mathscr{R}(F)$ is also a continuous function and coincides with $z \mapsto F_z(z)$.

For an example one can consider the germ F defined by the Taylor expansion of a smooth function f, see Section 2.2.1. In this case it is clear that $\mathscr{R}(F) = f$ is a function and $f(x) = F_x(x)$, $x \in \mathbb{R}^d$.

2.12. Recent developments

We mention that the approach to the Reconstruction Theorem of [**4**] has been recently developed in further directions:

- on smooth manifolds [**14**]
- in the direction of Besov Reconstruction [**2**], [**15**]
- as a *stochastic reconstruction theorem* [**10**], akin to the stochastic sewing lemma [**11**].
- in a microlocal setting [**5, 6**]

CHAPTER 3

Models and modelled distributions

In the previous chapter we have introduced the notion of coherent germs and the operation of reconstruction. In this chapter we define a special class of germs which arise in regularity structures.

3.1. Pre-models and modelled distributions

We are going to study germs which can be written as suitable linear combinations of a fixed finite family of germs. First we introduce the notion of *pre-models*:

Definition 3.1.1. *A* pre-model *is a pair* (Π, Γ) *where*

(1) $\Pi = (\Pi^i)_{i\in I}$ *is a family of germs* $\Pi^i = (\Pi^i_x)_{x\in\mathbb{R}^d}$ *labelled by a finite index set I,*

(2) $\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto (\Gamma^{ij}_{xy})_{i,j\in I}$ *is a matrix-valued function such that*

$$\Pi^j_y = \sum_{i\in I} \Pi^i_x \Gamma^{ij}_{xy}, \qquad j \in I,\ x,y \in \mathbb{R}^d, \tag{3.1.1}$$

and we suppose that

(3) there exist $(\alpha_i)_{i\in I} \subset \mathbb{R}$ *and a* $\varphi \in \mathscr{D}(\mathbb{R}^d)$ *with* $\int \varphi \neq 0$ *such that*

$$|\Pi^i_x(\varphi^{\varepsilon_n}_x)| \lesssim \varepsilon_n^{\alpha_i},$$

uniformly over x in compact sets of $\mathbb{R}^d$*,* $n \in \mathbb{N}$*.*

We denote $\bar\alpha := \min(\alpha_i, i \in I)$*.*

Example 3.1.2. *For a fixed* $\gamma > 0$*, the family of classical monomials*

$$\Pi^j_y(w) = \frac{(w-y)^j}{j!}, \qquad j \in \mathbb{N}^d, \quad y,w \in \mathbb{R}^d, \quad j \in I := \{i \in \mathbb{N}^d : |i| \leqslant \gamma\},$$

with $\alpha_i = |i|$*, any* $\varphi \in \mathscr{D}$ *and*

$$\Gamma^{ij}_{xy} = \mathbb{1}_{(i\leqslant j)} \frac{(x-y)^{j-i}}{(j-i)!}, \qquad i \in \mathbb{N}^d,$$

forms a pre-model.

Now we can define the notion of *modelled distribution*.

Definition 3.1.3. *Let (Π,Γ) be a pre-model, and let $\gamma > \max(\alpha_i, i \in I)$. If $f : \mathbb{R}^d \to \mathbb{R}^I$ satisfies for all $i \in I$*

$$\left| f_x^i \right| \lesssim 1, \qquad \left| f_x^i - \sum_{j \in I} \Gamma_{xy}^{ij} f_y^j \right| \lesssim |x-y|^{\gamma-\alpha_i},$$

uniformly for x, y in compact subsets of $\mathbb{R}^d$, then we call f a distribution modelled *by (Π,Γ), or simply a* modelled distribution*, and we write $f \in \mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$.*

Given a pre-model (Π,Γ) and a modelled distribution $f \in \mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$, we define the germ

$$\langle \Pi, f \rangle_x := \sum_{i \in I} \Pi_x^i f_x^i, \qquad x \in \mathbb{R}^d. \tag{3.1.2}$$

We want to show that $\langle \Pi, f \rangle$ is $(\bar{\alpha},\gamma)$-coherent, where $\bar{\alpha} := \min(\alpha_i, i \in I)$. Using the reexpansion property (3.1.1) we have

$$\langle \Pi, f \rangle_z - \langle \Pi, f \rangle_y = \sum_{j \in I} \Pi_z^i f_z^i - \sum_{i \in I} \Pi_y^i f_y^i = -\sum_{i \in I} \Pi_y^i \left(f_y^i - \sum_{j \in I} \Gamma_{yz}^{ij} f_z^j \right).$$

Therefore

$$(\langle \Pi, f \rangle_z - \langle \Pi, f \rangle_y)(\varphi_y^{\varepsilon}) = -\sum_{i \in I} \Pi_y^i(\varphi_y^{\varepsilon}) \left(f_y^i - \sum_{j \in I} \Gamma_{yz}^{ij} f_z^j \right),$$

namely

$$\left| (\langle \Pi, f \rangle_z - \langle \Pi, f \rangle_y)(\varphi_y^{\varepsilon}) \right| \lesssim \sum_{i \in I} \varepsilon^{\alpha_i} |z-y|^{\gamma-\alpha_i} \lesssim \varepsilon^{\bar{\alpha}} (\varepsilon + |z-y|)^{\gamma-\bar{\alpha}},$$

uniformly for y, z in compact sets. Moreover

$$\left| \langle \Pi, f \rangle_y (\varphi_y^{\varepsilon}) \right| \leqslant \sum_{i \in I} f_y^i \, |\Pi_y^i(\varphi_y^{\varepsilon})| \lesssim \sum_{i \in I} \varepsilon^{\alpha_i} \lesssim \varepsilon^{\bar{\alpha}},$$

uniformly over y in compact subsets of $\mathbb{R}^d$. In other words we have proved that

Proposition 3.1.4. *If (Π,Γ) is a pre-model and $f \in \mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$, then $\langle \Pi, f \rangle$ is a $(\bar{\alpha},\gamma)$-coherent germs with uniform homogeneity bound with exponent $\bar{\alpha}$. In other words, $\langle \Pi, f \rangle$ belongs to $\mathscr{G}^{\bar{\alpha};\bar{\alpha},\gamma}$.*

3.2. A special case

We have seen in Section 2.11 that under certain sufficient conditions on the coherent germ $(F_x)_{x\in\mathbb{R}^d}$, the reconstruction $\mathscr{R}F$ is a function and has an explicit form. An important example of this setting, where moreover $\mathscr{R}F$ is a (locally) Hölder-continuous function, is the following:

Example 3.2.1. Suppose we have a pre-model (Π,Γ) and a modelled distribution $f\in\mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$ as in Definition 3.1.3. We suppose that each germ Π^i_x is (locally) Hölder-continuous, for some exponent $\beta_i\in]0,1[$, uniformly for x in compact sets: more explicitly, we assume that

$$|\Pi_x(y)-\Pi_x(y')|\lesssim|y-y'|^{\beta_i}$$

uniformly for x,y,y' in compact sets. Then we can write unambiguously $y\mapsto\Pi^i_x(y)$ and

$$y\mapsto F_x(y):=\sum_{i\in I}f^i_x\Pi^i_x(y).$$

Now by the reexpansion property (3.1.1)

$$F_{x'}(y)-F_x(y)=-\sum_{i\in I}\Pi^i_x(y)\left(f^i_x-\sum_{j\in I}\Gamma^{ij}_{xx'}f^j_{x'}\right).$$

Then

$$\begin{aligned}|F_x(y)-F_{x'}(y')|&\leqslant|F_x(y)-F_{x'}(y)|+|F_{x'}(y)-F_{x'}(y')|\\&\lesssim\sum_{i\in I}\left(|\Pi^i_x(y)|\left|x-x'\right|^{\gamma-\alpha_i}+|f^i_x|\,|y-y'|^{\beta_i}\right)\end{aligned}$$

which shows that $(x,y)\mapsto F_x(y)$ is continuous. Therefore, in this case the reconstruction of F is equal to $x\mapsto F_x(x)$. Moreover setting $y=x$ and $y'=x'$ we obtain

$$|F_x(x)-F_{x'}(x')|\lesssim\sum_{i\in I}\left(|\Pi^i_x(x)|\left|x-x'\right|^{\gamma-\alpha_i}+|f^i_x|\,|x-x'|^{\beta_i}\right),$$

namely the reconstruction of $F=\langle\Pi,f\rangle$ is even locally Hölder-continuous.

3.3. Models

We now define the notion of a *model*.

Definition 3.3.1. *A* model *is a pre-model* (Π,Γ) *as in Definition 3.1.1, such that moreover*

(1) $\Gamma^{ii}_{xy}=1$ *for all* $i\in I$*,*
(2) $\Gamma^{ij}_{xy}=0$ *if* $\alpha_i\geqslant\alpha_j$ *and* $i\neq j$*,*
(3) $|\Gamma^{ij}_{xy}|\lesssim|x-y|^{\alpha_j-\alpha_i}$ *if* $\alpha_i<\alpha_j$*.*

If (Π,Γ) is a model, then spaces $\mathscr{D}^\gamma_{(\Pi,\Gamma)}$ of modelled distributions satisfy the following properties.

Lemma 3.3.2. *Let (Π,Γ) be a model as in Definition 3.3.1. Fix an exponent $\gamma > \max(\alpha_i\colon i\in I)$ and set $\bar\alpha := \min(\alpha_i\colon i\in I)$. Then*

(1) The space $\mathscr{D}^\gamma_{(\Pi,\Gamma)}$ is not reduced to the null vector.

(2) For any $\gamma' > \bar\alpha$, the restricted family $(\Pi',\Gamma') := (\Pi^i,\Gamma^{ij})_{i,j\in I'}$ labelled by $I' := \{i\in I\colon \alpha_i < \gamma'\}$ is a model. If $\gamma > \gamma'$, the projection

$$f = (f^i)_{i\in I} \mapsto f' = (f^i)_{i\in I'}$$

maps $\mathscr{D}^\gamma_{(\Pi,\Gamma)}$ to $\mathscr{D}^{\gamma'}_{(\Pi',\Gamma')}$.

PROOF. For the first assertion, we consider an element Π^i_x of minimal homogeneity $\bar\alpha = \min_I\alpha$. In this case by the properties (1)-(2) in Definition 3.3.1 we see that $\Gamma^{ij}_{xy} = \delta_{ij}$ for all $j\in I$, where δ is the Kronecker symbol, and the function $f^j_x = \delta_{ij}$ is a modelled distribution.

Let us prove now the second assertion. Assume that $\gamma' \leqslant \max(\alpha_i\colon i\in I)$, hence $I' \subsetneq I$, otherwise there is nothing to prove. By property (2) in Definition 3.3.1, relation (3.1.1) holds for the restricted family (Π',Γ'), because for $j\in I'$ we can restrict the sum in (3.1.1) to $i\in I'$ (otherwise $\Gamma^{ij}=0$). The other properties of a model are easily checked, hence (Π',Γ') is a model. Given a modelled distribution $f = (f^i)_{i\in I}\in \mathscr{D}^\gamma_{(\Pi,\Gamma)}$, we need to check that $f' = (f^i)_{i\in I'}\in\mathscr{D}^{\gamma'}_{(\Pi',\Gamma')}$. We write for $i\in I$

$$\begin{aligned}\Big|f^i_x - \sum_{i\in I'}\Gamma^{ij}_{xy} f^j_y\Big| &\leqslant \Big|f^i_x - \sum_{i\in I}\Gamma^{ij}_{xy} f^j_y\Big| + \sum_{i\in I\setminus I'}\big|\Gamma^{ij}_{xy} f^j_y\big| \\ &\lesssim |x-y|^{\gamma-\alpha_i} + \sum_{\gamma'\leqslant\alpha_j<\gamma}|x-y|^{\alpha_j-\alpha_i} \\ &\lesssim |x-y|^{\gamma-\alpha_i} + |x-y|^{\gamma'-\alpha_i} \lesssim |x-y|^{\gamma'-\alpha_i},\end{aligned}$$

uniformly for x,y in compact sets, by the property (3) in Definition 3.3.1. □

We also have another instructive remark. Suppose that (Π,Γ) is a model. Then for every $j\in I$, the germ $\Pi^j = (\Pi^j_x)_{x\in\mathbb{R}^d}$ is $(\bar\alpha,\alpha_j)$-coherent. Indeed

$$\Pi^j_y - \Pi^j_x = \sum_{i\in I}\Pi^i_x\Gamma^{ij}_{xy} - \Pi^j_x = \sum_{i\neq j}\Pi^i_x\Gamma^{ij}_{xy},$$

so that

$$\begin{aligned}|(\Pi_y^j - \Pi_x^j)(\varphi_x^{\varepsilon_n})| &\leqslant \sum_{\alpha_i<\alpha_j} |\Pi_x^i(\varphi_x^{\varepsilon_n})||x-y|^{\alpha_j-\alpha_i} \\ &\lesssim \sum_{\alpha_i<\alpha_j} \varepsilon_n^{\alpha_i}|x-y|^{\alpha_j-\alpha_i} \\ &\lesssim \varepsilon_n^{\bar\alpha}(|x-y|+\varepsilon_n)^{\alpha_j-\bar\alpha}.\end{aligned}$$

Moreover, by property (3) in Definition 3.1.1, this germ satisfies a homogeneity bound with exponent α_j. The same property can in fact be viewed as a reconstruction bound for this germ, with $\mathscr{R}(\Pi^j) = 0$. If $\alpha_j > 0$ then the reconstruction is unique.

Note that we can write, as in notation (3.1.2), $\Pi^j = \langle \Pi, f\rangle$ with $f_x^i := \delta_{ij}$, with δ the Kronecker symbol. However in this setting f does not belong to $\mathscr{D}^{\alpha_j}_{(\Pi,\Gamma)}$; indeed, if it did, by Definition 3.1.3 we should have $\alpha_j > \max(\alpha_i, i \in I)$, which is clearly false.

3.4. Hölder functions as modelled distributions

We have see in Example 3.1.2 that the classical polynomial family

$$\Pi_y^i(w) = \frac{(w-y)^i}{i!}, \quad i \in \mathbb{N}^d,\ \alpha_i = |i| < \gamma,$$

$$\Gamma_{xy}^{ij} = \mathbb{1}_{(i\leqslant j)} \frac{(x-y)^{j-i}}{(j-i)!}, \quad i,j \in \mathbb{N}^d,$$

forms a pre-model and actually a model. It is an interesting exercise to check that modelled distributions with respect to this model are actually classical Hölder functions.

This model belongs to the class that we have considered in Section 3.2, namely the function $(x,y) \mapsto \Pi_x^i(y)$ is continuous for all i and each Π_x^i is locally β-Hölder continuous, uniformly for x in compact sets, for any $\beta \in]0,1[$. Therefore by the discussion in Section 3.2 we know that any modelled distribution $f \in \mathscr{D}^\gamma_{(\Pi,\Gamma)}$ gives rise to a $(0,\gamma)$-coherent germ $\langle \Pi, f\rangle$ and that the reconstruction of $\langle \Pi, f\rangle$ is a locally Hölder-continuous function.

Let us consider for simplicity the case $\gamma \notin \mathbb{N}$. Now, a modelled distribution $f \in \mathscr{D}^\gamma_{(\Pi,\Gamma)}$ satisfies

$$\left| f_x^i - \sum_{j\geqslant i, |j|<\gamma} \frac{(x-y)^{j-i}}{(j-i)!} f_y^j \right| \lesssim |x-y|^{\gamma-|i|}, \qquad \forall\ |i| < \gamma.$$

This is in fact a Taylor expansion of f^i at order $\lfloor \gamma - |i| \rfloor$ with a remainder of order $\gamma - |i|$, and this implies that f^i is of class $C^{\gamma-|i|}$ and

$$f^j = \partial_{j-i} f^i, \qquad \forall\ j \geqslant i.$$

In particular, for $i = 0$ we see that f^0 is of class C^γ and satisfies (2.2.2); in particular by Proposition 2.2.1 we have that f^0 is a reconstruction of $\langle \Pi, f \rangle$, and since $\gamma > 0$ it is the unique reconstruction. In other words we have seen that

$$f^0 = \mathscr{R}\langle \Pi, f \rangle \in C^\gamma, \qquad f^i = \partial_i f^0, \quad \forall |i| < \gamma.$$

The fact that f^0 is the reconstruction of $\langle \Pi, f \rangle$ also follows by Section 3.2, because we must have $\mathscr{R}\langle \Pi, f \rangle = \{x \mapsto \langle \Pi, f \rangle_x(x)\} = \{x \mapsto f_x^0\}$.

3.5. Semi-norms

Back to the general case, for a fixed pre-model (Π, Γ) we can interpret, by analogy with the case of Hölder functions of the previous section, the space $\mathscr{D}^\gamma_{(\Pi,\Gamma)}$ of all distributions modelled by (Π, Γ) as the collection of *generalised derivatives* of $u := \mathscr{R}\langle \Pi, f \rangle$ with respect to the model (Π, Γ).

We can define a system of seminorms for $f \in \mathscr{D}^\gamma_{(\Pi,\Gamma)}$

$$[f]_{\mathscr{D}^\gamma_{(\Pi,\Gamma)}, K} = \sup_{i \in I} \sup_{x,y \in K, x \neq y} \frac{\left| f_x^i - \sum_{j \in I} \Gamma_{xy}^{ij} f_y^j \right|}{|x - y|^{\gamma - \alpha_i}},$$

where K is a compact subset of $\mathbb{R}^d$.

This is rather original with respect to the standard situation in ODEs or PDEs, where one sets an equation in a fixed Banach space. Here the Banach (Fréchet) space depends on an external parameter, the model (Π, Γ). For SDEs and SPDEs, the model (Π, Γ) is actually *random*.

CHAPTER 4

Schauder estimates for coherent germs

In this chapter we discuss one of the most important operations on coherent germs: the convolution with a regularising integration kernel.

4.1. Integration kernels

Definition 4.1.1 (Regularising kernel)**.** *Fix a dimension $d \in \mathbb{N}$ and an exponent $\beta \in (0,d)$. A measurable function $\mathsf{K} : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ is called a* β-regularizing kernel up to degree $m \in \mathbb{N}$ *if the following conditions hold:*

- *the function $x \mapsto \mathsf{K}(x)$ is of class C^m on $\mathbb{R}^d \backslash \{0\}$;*
- *the following upper bound holds:*

$$\forall\, k \in \mathbb{N}^d \ \ \textit{with} \ \ |k| \leqslant m : \qquad |\partial^k \mathsf{K}(x)| \lesssim \frac{1}{|x|^{d-\beta+|k|}} \mathbb{1}_{\{|x|\leqslant 1\}} \quad \textit{uniformly for } x \textit{ in compact sets}\,. \tag{4.1.1}$$

In particular, note that for $k = 0$ equation (4.1.1) reduces to

$$|\mathsf{K}(x)| \lesssim \frac{1}{|x|^{d-\beta}} \mathbb{1}_{\{|x|\leqslant 1\}}\,. \tag{4.1.2}$$

This shows that a β-regularizing kernel is locally integrable on $\mathbb{R}^d$.

4.1.1. Singular convolution. We want to consider the convolution $\mathsf{K} * f \in \mathscr{D}'$ between a kernel K and a distribution $f \in \mathscr{D}'$. This is *formally* defined by

$$(\mathsf{K} * f)(x) := f(\mathsf{K}(x - \cdot)) = \int_{\mathbb{R}^d} \mathsf{K}(x-y)\, f(\mathrm{d}y)\,, \tag{4.1.3}$$

but we stress that in general $\mathsf{K} * f$ is ill-defined. Under suitable conditions, $\mathsf{K} * f$ can be defined as a distribution by duality: for a test function $\psi \in \mathscr{D}$ we set

$$(\mathsf{K} * f)(\psi) := f(\mathsf{K}^* \psi) \quad \text{where} \quad (\mathsf{K}^* \psi)(y) := \int_{\mathbb{R}^d} \psi(x)\, \mathsf{K}(x-y)\, \mathrm{d}x\,, \tag{4.1.4}$$

provided $f(\mathsf{K}^* \psi)$ makes sense, of course. We are going to study the convolution $\mathsf{K}^* \psi$ between the kernel K and a test function ψ, to ensure that $f(\mathsf{K}^* \psi)$ is well-defined.

We start with an elementary observation: if $\mathsf{K}(\cdot)$ is β-regularizing up to some degree m, then $(\mathsf{K}^*\psi)(\cdot)$ *is a well-defined compactly supported measurable function*, because $\mathsf{K}(x-y)$ is jointly measurable, locally integrable and compactly supported in the difference $|x-y|$. The delicate point is that $\mathsf{K}^*\psi$ *needs not be smooth*, hence we cannot hope to define $f(\mathsf{K}^*\psi)$ for arbitrary $(f,\psi)\in\mathscr{D}'\times\mathscr{D}$.

4.1.2. Partition of unity. Let us introduce the usual dyadic sequence

$$\varepsilon_n := 2^{-n}, \qquad n\in\mathbb{Z}.$$

We call *dyadic partition of unity* a family of functions $(\rho_n)_{n\in\mathbb{Z}}$ such that:

- $\rho_n(z)$ is supported in the annulus $\{\frac{1}{2}\varepsilon_n \leqslant |z| \leqslant 2\varepsilon_n\}$ and
$$\forall z\in\mathbb{R}^d\backslash\{0\}: \qquad \sum_{n\in\mathbb{Z}}\rho_n(z) = 1\,;$$
- for any given $k\in\mathbb{N}^d$, one has
$$\|\partial^k\rho_n\|_\infty \lesssim \varepsilon_n^{-|k|} \quad \text{uniformly in } n\in\mathbb{N}\,.$$

It is easy to build a dyadic partition of unity. Given any smooth function $\chi:\mathbb{R}^d\to[0,1]$ such that

$$\chi(z)\begin{cases} = 1 & \text{if } |z|\leqslant 1\\ \in[0,1] & \text{if } 1\leqslant|z|\leqslant 2\\ = 0 & \text{if } |z|\geqslant 2\end{cases},$$

we obtain a dyadic partition of unity $(\rho_n)_{n\in\mathbb{Z}}$ by setting

$$\rho_n(z) := \chi(\varepsilon_n^{-1}z) - \chi(\varepsilon_{n+1}^{-1}z)\,.$$

Such a partition of unity is *scale invariant*, since $\rho_n(z) = \rho_0(\varepsilon_n^{-1}z)$. We set

$$\begin{aligned} &\mathsf{K}_n:\mathbb{R}^d\to\mathbb{R}, \qquad \mathsf{K}_n(x) := \rho_n(x)\,\mathsf{K}(x)\,,\\ &\text{so that} \qquad \mathsf{K}(x) = \sum_{n=0}^{\infty}\mathsf{K}_n(x) \quad \forall\, x\in\mathbb{R}^d\backslash\{0\}. \end{aligned} \tag{4.1.5}$$

We stress that $\mathsf{K}_n(x)$ is supported in the annulus $\{\frac{1}{2}\varepsilon_n\leqslant|x|\leqslant 2\varepsilon_n\}$. Moreover

$$\begin{aligned} &\forall k\in\mathbb{N}^d \text{ with } |k|\leqslant m:\\ &\qquad |\partial^k\mathsf{K}_n(x)| \lesssim \frac{1}{|x|^{d-\beta-|k|}}\,\mathbb{1}_{\{\frac{1}{2}\varepsilon_n\leqslant|x|\leqslant2\varepsilon_n\}}\\ &\qquad\qquad \lesssim \varepsilon_n^{\beta-d-|k|}\,\mathbb{1}_{\{\frac{1}{2}\varepsilon_n\leqslant|x|\leqslant2\varepsilon_n\}}\\ &\qquad \text{uniformly for } n\in\mathbb{N}\,. \end{aligned} \tag{4.1.6}$$

Finally we have for all $y \in \mathbb{R}^d$ and $|\ell| < |k|$

$$\int_{\mathbb{R}^d} x^\ell\, \partial^k \mathsf{K}_n(x-y)\, \mathrm{d}x = (-1)^{|k|} \int_{\mathbb{R}^d} (\partial^k x^\ell)\, \mathsf{K}_n(x-y)\, \mathrm{d}x = 0\,, \tag{4.1.7}$$

because $\partial^k x^\ell = 0$ for $|\ell| < |k|$.

4.2. Convolution with distributions

We show now that $\mathsf{K}^* \psi$ in (4.1.4) is well-defined and differentiable.

Proposition 4.2.1. *Given a kernel* K *which is* β*-regularizing up to degree* $m \in \mathbb{N}$ *and a test function* $\psi \in \mathscr{D}$*, the convolution* $\mathsf{K}^* \psi$ *defined in* (4.1.4) *belongs to* C^m.

More precisely, recalling K_n *defined in* (4.1.5)*, we have the following bound:*

$$\begin{aligned} \forall r \in \{0,1,\ldots,m\}: \qquad & \|\mathsf{K}_n^* \psi\|_{C^r} \lesssim \|\psi\|_{C^r}\, \varepsilon_n^\beta \\ & \textit{uniformly for } n \in \mathbb{N} \textit{ and } \psi \in \mathscr{D}(B(0,1))\,, \end{aligned} \tag{4.2.1}$$

hence the series $\mathsf{K}^* \psi = \sum_{n=0}^\infty \mathsf{K}_n^* \psi$ *converges in* C^m *(recall that* $\beta > 0$*).*

PROOF. We recall that $\mathsf{K}(x-y) = \sum_{n=0}^\infty \mathsf{K}_n(x-y)$ for all $x, y \in \mathbb{R}^d$ with $x \neq y$, by (4.1.5). Then by dominated convergence, thanks to (4.1.2), for any $y \in \mathbb{R}^d$ we can write

$$(\mathsf{K}^* \psi)(y) = \sum_{n=0}^\infty (\mathsf{K}_n^* \psi)(y) \quad \text{where} \quad (\mathsf{K}_n^* \psi)(y) := \int_{\mathbb{R}^d} \psi(x)\, \mathsf{K}_n(x-y)\, \mathrm{d}x\,.$$

To prove (4.2.1), it is sufficient to show that

$$\begin{aligned} \forall k \in \mathbb{N}^d \text{ with } |k| \leqslant m: \qquad & \|\partial^k(\mathsf{K}_n^* \psi)\|_\infty \lesssim \|\psi\|_{C^{|k|}}\, \varepsilon_n^\beta \\ & \text{uniformly for } n \in \mathbb{N} \text{ and } \psi \in \mathscr{D}(B(0,1))\,. \end{aligned} \tag{4.2.2}$$

By Definition 4.1.1, for any $n \in \mathbb{N}$ the function $y \mapsto \mathsf{K}_n(x-y)$ is of class C^m *on the whole* $\mathbb{R}^d$ (including $y = x$, because $\mathsf{K}_n(x-y)$ vanishes for $|y-x| \leqslant \frac{1}{2}\varepsilon_n$, see (4.1.5)). Exchanging derivatives and integral by dominated convergence, thanks to (4.1.1), we see that $\mathsf{K}_n^* \psi \in C^m$ and

$$\forall k \in \mathbb{N}^d \ \text{ with } |k| \leqslant m: \quad \partial^k(\mathsf{K}_n^* \psi)(y) = (-1)^{|k|} \int_{\mathbb{R}^d} \psi(x)\, \partial^k \mathsf{K}_n(x-y)\, \mathrm{d}x\,. \tag{4.2.3}$$

We now estimate $\partial^k(\mathsf{K}_n^* \psi)(y)$ for fixed $n \in \mathbb{N}$ and $y \in \mathbb{R}^d$, $k \in \mathbb{N}^d$ with $|k| \leqslant m$. Denote by $Q^{[y,k]}(\cdot)$ the Taylor polynomial of ψ of degree $|k|-1$ based at y, that is

$$Q^{[y,k]}(x) := \sum_{|\ell| \leqslant |k|-1} \frac{\partial^\ell \psi(y)}{\ell!}\, (x-y)^\ell\,,$$

where we agree that for $k=0$ we set $Q^{[y,0]}(x) \equiv 0$. Then we can bound

$$|\psi(x) - Q^{[y,k]}(x)| \lesssim \|\psi\|_{C^{|k|}} \, |y-x|^{|k|} \,. \tag{4.2.4}$$

Starting from (4.2.3), we decompose

$$\begin{aligned}\partial^k(\mathsf{K}_n^* \psi)(y) =& (-1)^{|k|} \underbrace{\int_{\mathbb{R}^d} (\psi - Q^{[y,k]})(x) \, \partial^k \mathsf{K}_n(x-y) \, \mathrm{d}x}_{A_{n,k}(y)} \\ & + (-1)^{|k|} \underbrace{\int_{\mathbb{R}^d} Q^{[y,k]}(x) \, \partial^k \mathsf{K}_n(x-y) \, \mathrm{d}x}_{B_{n,k}(y)} \,.\end{aligned}$$

By (4.1.7) we have that $B_{n,k}(y) = 0$. By (4.2.4) and (4.1.6), for $|k| \leqslant m$, the first term is bounded by

$$|A_{n,k}(y)| \lesssim \|\psi\|_{C^{|k|}} \int_{|y-x| \leqslant \varepsilon_n} |y-x|^{|k|} \, |y-x|^{\beta - |k| - d} \, \mathrm{d}x \lesssim \|\psi\|_{C^{|k|}} \, \varepsilon_n^{\beta} \,,$$

uniformly for y in compact sets and $n \in \mathbb{N}$. This completes the proof of (4.2.2). ☐

We obtain the following useful

Proposition 4.2.2. *Given a kernel* K *which is* β*-regularizing up to degree* $m \in \mathbb{N}$ *and a distribution* $T \in \mathscr{D}'$ *of order* $r \leqslant m$*, the distribution*

$$\mathscr{D} \ni \psi \mapsto \mathsf{K} * T(\psi) := T(\mathsf{K}^* \psi),$$

where $\mathsf{K}^* \psi \in C^m$ *is as in Proposition 4.2.1, is well-defined in* $\mathscr{D}'$ *and has order* r*.*

4.3. Schauder estimate for coherent germs

4.3.1. Coherent germs. Fix two real numbers α, γ such that

$$\alpha \leqslant \gamma, \quad \gamma \neq 0.$$

Let $F = (F_x)_{x \in \mathbb{R}^d}$ be a (α, γ)-coherent germ, i.e. we have

$$\begin{gathered} |(F_z - F_y)(\varphi_y^{\varepsilon_n})| \lesssim \varepsilon_n^{\alpha} \, (|y-z| + \varepsilon_n)^{\gamma - \alpha} \\ \text{uniformly for } y, z \text{ in compact sets and } n \in \mathbb{N}, \end{gathered} \tag{4.3.1}$$

for some test function $\varphi \in \mathscr{D}$ with $\int \varphi \neq 0$. We define r_α as the smallest integer larger than $-\alpha$, namely

$$r_\alpha := \min\{k \in \mathbb{N} : \; k > -\alpha\} \,. \tag{4.3.2}$$

By the Reconstruction Theorem 2.5.1-2.7.1 there is a distribution $\mathscr{R}F \in \mathscr{D}'$ such that

$$\begin{gathered}|(\mathscr{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \|\psi\|_{C^{r_\alpha}}\, \varepsilon_n^\gamma \\ \text{uniformly for } x \text{ in compact sets, } n \in \mathbb{N} \text{ and } \psi \in \mathscr{D}(B(0,1))\,.\end{gathered} \tag{4.3.3}$$

If $\gamma > 0$ then $\mathscr{R}F$ is unique.

4.3.2. Singular convolution. Fix a kernel K which is β-regularizing up to degree m for some $\beta \in (0,d)$, see Definition 4.1.1. We now want to "lift the convolution with K on the space of coherent germs", i.e. to find a coherent germ $H = (H_x)_{x\in\mathbb{R}^d}$ with the property that

$$\mathscr{R}H = \mathsf{K} * \mathscr{R}F\,. \tag{4.3.4}$$

A simple solution of (4.3.4) is the constant germ $H_x \equiv \mathsf{K} * \mathscr{R}F$, which is trivially coherent, but this does not allow to construct a fixed-point theory for PDEs. The naive guess $H_x = \mathsf{K} * F_x$ needs not give a coherent germ, therefore we need to enrich it. To this purpose, we look for H_x of the following special form:

$$\forall x \in \mathbb{R}^d: \qquad H_x = \mathsf{K} * F_x + R_x \qquad \text{where } R_x(\cdot) \text{ is a } \textit{polynomial}\,. \tag{4.3.5}$$

Remarkably, this is possible with the following explicit solution:

$$H_x := \mathsf{K} * F_x + \underbrace{\sum_{|\ell|<\gamma+\beta} (\mathscr{R}F - F_x)\left(\partial^\ell \mathsf{K}(x-\cdot)\right) \mathbb{X}_x^\ell}_{R_x(\cdot)}, \tag{4.3.6}$$

where we denote for $x \in \mathbb{R}^d$, $\ell \in \mathbb{N}^d$ the classical monomials

$$\mathbb{X}_x^\ell : \mathbb{R}^d \to \mathbb{R}, \qquad \mathbb{X}_x^\ell(w) := \frac{(w-x)^\ell}{\ell!} \tag{4.3.7}$$

and where we agree that

$$R_x(\cdot) \equiv 0 \qquad \text{if} \qquad \gamma + \beta \leqslant 0.$$

Note that $R_x(\cdot)$ is a family of polynomials labelled by x, whose coefficients depend on F_x, on $\mathscr{R}F$ and on the derivatives $\partial^k \mathsf{K}$ for $|k| < \gamma + \beta$. Then we also assume that $\gamma + \beta \notin \mathbb{N}$ and we suppose that the integer m which appears in Definition 4.1.1 satisfies

$$m > \gamma + \beta + r_\alpha. \tag{4.3.8}$$

THEOREM 4.3.1 (Schauder estimate for coherent germs). *Fix a dimension $d \in \mathbb{N}$ and real numbers $\alpha, \gamma, \beta \in \mathbb{R}$ such that*

$$\alpha \leqslant \gamma, \qquad \gamma \neq 0, \qquad \beta > 0\,,$$

where we further assume for simplicity that

$$\{\alpha+\beta,\ \gamma+\beta\} \cap \mathbb{N} = \varnothing\,.$$

Consider the following ingredients:

- $F = (F_x)_{x\in\mathbb{R}^d} \in \mathscr{G}^{\alpha,\gamma}$ *is a* (α,γ)*-coherent germ;*
- K *is a* β*-regularizing kernel (see Definition 4.1.1) up to degree* m *given in* (4.3.8).

Then

(1) *the germ* $H = (H_x)_{x\in\mathbb{R}^d}$ *in* (4.3.6) *is locally well-defined, i.e.* $H_x(\varphi)$ *is well-defined for all* $\varphi \in \mathscr{D}(B(x,1))$.
(2) H *is* $((\alpha+\beta)\wedge 0, \gamma+\beta)$*-coherent, namely* $H \in \mathscr{G}^{(\alpha+\beta)\wedge 0,\gamma+\beta}$.
(3) H *satisfies* $\mathscr{R}H = \mathsf{K} * \mathscr{R}F$.

In other words, setting $\mathscr{K}F := H$*, we have a linear operator satisfying*

$$\mathscr{K} : \mathscr{G}^{\alpha,\gamma} \to \mathscr{G}^{(\alpha+\beta)\wedge 0,\gamma+\beta}, \qquad \mathscr{R}\circ\mathscr{K} = \mathsf{K}*\mathscr{R}.$$

Let us define the new germ

$$J_x := F_x - \mathscr{R}F\,,$$

which allows to rewrite (4.3.6) as

$$H_x = \mathsf{K} * \mathscr{R}F + L_x, \qquad \text{where} \qquad L_x := \mathsf{K} * J_x + R_x\,. \tag{4.3.9}$$

From (4.3.6), observe that

$$L_x = \mathsf{K} * J_x - \sum_{|\ell|<\gamma+\beta} J_x(\partial^\ell \mathsf{K}(x-\cdot))\,\mathbb{X}_x^\ell. \tag{4.3.10}$$

We are going to prove that L_x is $((\alpha+\beta)\wedge 0, \gamma+\beta)$-coherent, more precisely

$$\begin{gathered} |(L_z - L_y)(\psi_y^{\varepsilon_n})| \lesssim \|\psi\|_{C^{r_\alpha}}\, \varepsilon_n^{(\alpha+\beta)\wedge 0}\,(|y-z|+\varepsilon_n)^{\gamma+\beta-(\alpha+\beta)\wedge 0}, \\ \text{uniformly for } y,z \text{ in compact sets, } n\in\mathbb{N} \text{ and } \psi\in\mathscr{D}(B(0,1))\,. \end{gathered} \tag{4.3.11}$$

More explicitly:

$$|(L_z - L_y)(\psi_y^{\varepsilon_n})| \lesssim \|\psi\|_{C^{r_\alpha}} \times \begin{cases} \varepsilon_n^{\alpha+\beta}\,(|y-z|+\varepsilon_n)^{\gamma-\alpha} & \text{if } \alpha+\beta<0\,, \\ (|y-z|+\varepsilon_n)^{\gamma+\beta} & \text{if } \alpha+\beta>0\,. \end{cases}$$

Then we are going to prove that L has homogeneity bound with exponent $\gamma+\beta$, that is,

$$|L_x(\psi_x^{\varepsilon_n})| \lesssim \|\psi\|_{C^{r_\alpha}}\, \varepsilon_n^{\gamma+\beta} \qquad \begin{array}{l} \text{uniformly for } x \text{ in compact sets,} \\ n\in\mathbb{N} \text{ and } \psi\in\mathscr{D}(B(0,1)). \end{array}$$

Recalling (4.3.9), this implies that $\mathscr{R}H = \mathsf{K} * \mathscr{R}F$; indeed we recall that $h = \mathscr{R}H$ means precisely $|(h-H_x)(\psi_x^{\varepsilon_n})| \lesssim \|\psi\|_{C^{r_\alpha}}\, \varepsilon_n^{\gamma+\beta}$, as the coherence exponent of H_x is $\gamma+\beta$.

One of the tools in the proof of Theorem 4.3.1 is the following simple result.

Lemma 4.3.2. *Fix $\gamma \in \mathbb{R}$, $\beta > 0$ such that $\gamma + \beta > 0$ and a point $x \in \mathbb{R}^d$. Let $T \in \mathscr{D}'$ have order r_α and homogeneity bound γ at a given point x, i.e. for some $\mathsf{C}_x < \infty$*

$$|T(\varphi_x^\varepsilon)| \leqslant \mathsf{C}_x \|\varphi\|_{C^{r_\alpha}} \varepsilon^\gamma \quad \text{uniformly for } \varepsilon \in (0,1] \text{ and } \varphi \in \mathscr{D}(B(0,1)). \tag{4.3.12}$$

Let K be a β-regularizing kernel up to degree $m > \gamma + \beta + r_\alpha$. Then for all $\ell \in \mathbb{N}^d$ with $|\ell| < \gamma + \beta$,

$$|T(\partial^\ell \mathsf{K}_n(x - \cdot))| \lesssim \mathsf{C}_x \varepsilon_n^{\gamma+\beta-|\ell|}. \tag{4.3.13}$$

In particular, writing $\partial^\ell \mathsf{K} = \sum_{n=0}^\infty \partial^\ell \mathsf{K}_n$ as in (4.1.5), we see that

$$T(\partial^\ell \mathsf{K}(x-\cdot)) := \sum_n T(\partial^\ell \mathsf{K}_n(x-\cdot))$$

is well-defined, and we have the tail estimate

$$\forall N \in \mathbb{N}: \qquad \sum_{n=N}^\infty \left|T\left(\partial^\ell \mathsf{K}_n(x-\cdot)\right)\right| \lesssim \mathsf{C}_x \varepsilon_N^{\gamma+\beta-|\ell|}. \tag{4.3.14}$$

Before proving Lemma 4.3.2 we need the following simple

Lemma 4.3.3. *Let K be a β-regularizing kernel up to degree $m > \gamma + \beta + r_\alpha$. We introduce the function*

$$\varphi^{[k,n]}(w) := (2\varepsilon_n)^d \, \partial^k \mathsf{K}_n(-2\varepsilon_n w), \tag{4.3.15}$$

so that

$$\partial^k \mathsf{K}_n(x-\cdot) = \left(\varphi^{[k,n]}\right)_x^{2\varepsilon_n}. \tag{4.3.16}$$

Then

$$\operatorname{supp}\left(\varphi^{[k,n]}\right) \subset B(0,1), \qquad \forall\, |k| < \gamma+\beta, \tag{4.3.17}$$

$$\left\|\varphi^{[k,n]}\right\|_{C^{r_\alpha}} \lesssim \varepsilon_n^{\beta-|k|}, \qquad \forall\, |k| < \gamma+\beta, \tag{4.3.18}$$

PROOF. Observe that (4.3.16) is straightforward from the definition of $\varphi^{[k,n]}$. One has $\operatorname{supp}\left(\partial^k \mathsf{K}_n(\cdot)\right) \subset B(0, 2\varepsilon_n)$ and thus one has as announced $\operatorname{supp}\left(\varphi^{[k,n]}\right) \subset B(0,1)$. Now, if $1 \leqslant |l| \leqslant r_\alpha$ then $\partial^l \varphi^{[k,n]} = (-1)^{|l|}(2\varepsilon_n)^{d+|l|} \partial^{k+l} \mathsf{K}_n(-2\varepsilon_n w)$. Thus from (4.1.6), one obtains (4.3.18). □

PROOF OF LEMMA 4.3.2. By (4.3.16) and by the homogeneity bound at x (4.3.12), using the properties (4.3.17) and (4.3.18) of $\varphi^{[\ell,n]}$ we can bound

$$|T(\partial^\ell \mathsf{K}_n(x-\cdot))| \leqslant \mathsf{C}_x \|\varphi^{[\ell,n]}\|_{C^{r_\alpha}}\, \varepsilon_n^\gamma \lesssim \mathsf{C}_x\, \varepsilon_n^{\gamma+\beta-|\ell|}.$$

Thus $T(\partial^\ell \mathsf{K}(x-\cdot)) := \sum_{n=0}^\infty T(\partial^\ell \mathsf{K}_n(x-\cdot))$ is well-defined in $\mathscr{D}'$ and moreover we obtain (4.3.14). □

4.4. Proof of Schauder estimates for coherent germs

In this section we prove Theorem 4.3.1.

Lemma 4.4.1. *L_x in* (4.3.10) *is a well-defined distribution.*

PROOF. We want first to show that the distribution $J_x = F_x - \mathscr{R}F$ has order r_α. By the reconstruction theorem, J_x is homogeneous with exponent γ; moreover $(J_x)_x$ is also coherent because $J_y - J_x = F_y - F_x$, i.e. $J \in \mathscr{G}^{\gamma;\alpha,\gamma}$. Moreover $(J_x)_x$ satisfies the enhanced coherence of Remark 2.4.2. By the triangle inequality, J thus satisfies the estimate

$$|J_y(\psi_x^\lambda)| \leqslant c_{K,\alpha,\gamma} \|\psi\|_{C^{r_\alpha}} \lambda^\alpha (|y-x|+\lambda)^{\gamma-\alpha},$$

uniformly over $x, y \in K$, $\lambda \in (0,1)$, whence the order r_α after plugging $x = 0$, $\lambda = 1$.

Then by Proposition 4.2.2 the distribution $\mathsf{K} * J_x$ is well defined and has order r_α. If we apply Lemma 4.3.2 to the distribution $T = J_x$ then we know that $T(\partial^\ell \mathsf{K}(x-\cdot)) \in \mathbb{R}$ is well-defined for all $\ell \in \mathbb{N}^d$ such that $|\ell| < \gamma + \beta$. Then L_x is a well-defined distribution. □

Remark 4.4.2. We will write $(L_z - L_y)(\psi_y^\lambda)$ for $\lambda \in]0,1]$ as a sum of various terms and show that

$$\text{each term is } \lesssim \lambda^a (|y-z|+\lambda)^{\gamma+\eta-a} \quad \text{for a suitable } a \geqslant (\alpha+\eta) \wedge 0 .$$

This implies (4.3.11) because $a \mapsto \lambda^a(|y-z|+\lambda)^{\gamma+\eta-a}$ is decreasing (note that we can write $\lambda^a(|y-z|+\lambda)^{\gamma+\eta-a} = A^a B$ with $A = \frac{\lambda}{\lambda+|y-z|} \leqslant 1$).

We take a compact set $K \subseteq \mathbb{R}^d$ and fix $y, z \in K$ as well as $N \in \mathbb{N}$. We set

$$M_{y,z,N} := \min\{n \in \mathbb{N} : \ \varepsilon_n \leqslant |y-z| + \varepsilon_N\},$$

and note that $0 \leqslant M_{y,z,N} \leqslant N < \infty$. Then we decompose

$$\mathsf{K} = \underbrace{\sum_{n=0}^{M_{y,z,N}-1} \mathsf{K}_n}_{\mathsf{K}_{[0,M)}} + \underbrace{\sum_{n=M_{y,z,N}}^{N-1} \mathsf{K}_n}_{\mathsf{K}_{[M,N)}} + \underbrace{\sum_{n=N}^{\infty} \mathsf{K}_n}_{\mathsf{K}_{[N,\infty)}},$$

where we stress that in this decomposition the sum is split at the points $M_{y,z,N}$ and N, for the fixed values of y, z, N, irrespective of the argument of $\mathsf{K}(\cdot)$.

We also define for $A \in \{\mathbb{N}, [M,\infty)\}$, $y \in \mathbb{R}^d$, $\psi \in \mathscr{D}$

$$J_y\left(P_y^A(\psi)\right) := \sum_{n\in A}\sum_{|\ell|<\gamma+\beta} J_y\left(\partial^\ell \mathsf{K}_n(y-\cdot)\right)\mathbb{X}_y^\ell(\psi), \tag{4.4.1}$$

where the sum is well defined by Lemma 4.3.2 and we recall the notation

$$\mathbb{X}_y^\ell(\psi) := \int_{\mathbb{R}^d} \mathbb{X}_y^\ell(w)\,\psi(w)\,\mathrm{d}w\,.$$

Moreover for *finite* $A \subset \mathbb{N}$ and $z, y \in \mathbb{R}^d$

$$J_z\left(P_y^A(\psi)\right) := \sum_{n\in A}\sum_{|\ell|<\gamma+\beta} J_z\left(\partial^\ell \mathsf{K}_n(y-\cdot)\right)\mathbb{X}_y^\ell(\psi)\,,$$

where the sum is well defined since K_n is smooth for each n. In particular, recalling (4.3.10), we can write

$$L_x(\psi) = J_x(\mathsf{K}^*\psi) - J_x\left(P_x^{\mathbb{N}}(\psi)\right). \tag{4.4.2}$$

Then, with the decomposition

$$\mathbb{N} = [0,M)\cup[M,N)\cup[N,\infty)$$

we bound for $\psi \in \mathscr{D}(B(0,1))$

$$\begin{aligned}
\left|(L_z-L_y)(\psi_y^{\varepsilon_N})\right| &\leqslant \left|(J_z-J_y)\left(\mathsf{K}^*\psi_y^{\varepsilon_N}\right) - J_z\left(P_z^{\mathbb{N}}(\psi_y^{\varepsilon_N})\right) + J_y\left(P_y^{\mathbb{N}}(\psi_y^{\varepsilon_N})\right)\right| \\
&\leqslant \underbrace{\left|(J_z-J_y)\left(\mathsf{K}^*_{[N,\infty)}\psi_y^{\varepsilon_N}\right)\right|}_{A} + \underbrace{\left|(J_z-J_y)\left(\mathsf{K}^*_{[M,N)}\psi_y^{\varepsilon_N}\right)\right|}_{B} \\
&\quad+ \underbrace{\left|J_z\left(P_z^{[M,\infty)}(\psi_y^{\varepsilon_N})\right)\right| + \left|J_y\left(P_y^{[M,\infty)}(\psi_y^{\varepsilon_N})\right)\right|}_{C} \\
&\quad+ \underbrace{\left|(J_z-J_y)\left(\mathsf{K}_{[0,M)}(\psi_y^{\varepsilon_N}) - P_y^{[0,M)}(\psi_y^{\varepsilon_N})\right)\right|}_{D} \\
&\quad+ \underbrace{\left|J_z\left(P_y^{[0,M)}(\psi_y^{\varepsilon_N}) - P_z^{[0,M)}(\psi_y^{\varepsilon_N})\right)\right|}_{E}.
\end{aligned}$$

We are going to need the following technical result, which can be proved as Lemma 4.3.2.

Lemma 4.4.3. *Let $\zeta^{[n,N,y]} : \mathbb{R}^d \to \mathbb{R}$ for $n \geqslant N$ and $y \in \mathbb{R}^d$ be defined by*

$$\zeta^{[n,N,y]}(w) := (3\varepsilon_N)^d\left(\mathsf{K}_n^*\psi_y^{\varepsilon_N}\right)(y+(3\varepsilon_N)w)\,. \tag{4.4.3}$$

Then $\zeta^{[n,N,y]}$ is supported in $B(0,1)$, and

$$\left\|\zeta^{[n,N,y]}\right\|_{C^{r\alpha}} \lesssim \|\psi\|_{C^{r\alpha}}\, \varepsilon_n^{\beta}, \qquad n \geqslant N,\ \psi \in \mathscr{D}(B(0,1)), \tag{4.4.4}$$

uniformly over y in compacts.

Let $\varphi^{[n,z]} : \mathbb{R}^d \to \mathbb{R}$

$$\varphi^{[n,z]}(w) := (3\varepsilon_n)^d\, \mathsf{K}_n(z - 3\varepsilon_n w)\,. \tag{4.4.5}$$

Then $\varphi^{[n,z]}$ is supported in $B(0,1)$ for all $|z| \leqslant \varepsilon_n$ and

$$\left\|\varphi^{[n,z]}\right\|_{C^{r\alpha}} \lesssim \varepsilon_n^{\beta}, \qquad \text{uniformly over } |z| \leqslant \varepsilon_n. \tag{4.4.6}$$

If $\gamma+\beta > 0$, let $\xi^{[k,n,z,t]} : \mathbb{R}^d \to \mathbb{R}$ for $k,n \in \mathbb{N}$, $z \in \mathbb{R}^d$, $t \in [0,1]$, with $|k| < \gamma+\beta$

$$\xi^{[k,n,z,t]}(w) := (3\varepsilon_n)^d\, \frac{d^{\lceil\gamma+\beta\rceil-|k|}}{dt^{\lceil\gamma+\beta\rceil-|k|}}\partial^k \mathsf{K}_n((1-t)z - 3\varepsilon_n w)\,. \tag{4.4.7}$$

Then $\xi^{[k,n,z,t]}$ is supported in $B(0,1)$ and

$$\left\|\xi^{[k,n,z,t]}\right\|_{C^{r\alpha}} \lesssim |z|^{\lceil\gamma+\beta\rceil-|k|}\varepsilon_n^{\beta-\lceil\gamma+\beta\rceil} \tag{4.4.8}$$

uniformly over z in compacts, $|k| < \gamma+\beta$, $t \in [0,1]$, $n \in \mathbb{N}$.

Estimate of A. We analyse

$$(J_z - J_y)\left(\mathsf{K}^*_{[N,\infty)}\psi_y^{\varepsilon_N}\right) = \sum_{n=N}^{\infty}(J_z - J_y)\left(\mathsf{K}^*_n\psi_y^{\varepsilon_N}\right)\,. \tag{4.4.9}$$

Note that we can write by (4.4.3)

$$\mathsf{K}^*_n\psi_y^{\varepsilon_N} = \left(\zeta^{[n,N,y]}\right)_y^{3\varepsilon_N}\,.$$

Then, by coherence (2.4.4) and by (4.4.4), we can bound

$$\begin{aligned}\left|(J_z - J_y)\left(\mathsf{K}^*_n\psi_y^{\varepsilon_N}\right)\right| &= \left|(J_z - J_y)\left(\left(\zeta^{[n,N,y]}\right)_y^{3\varepsilon_N}\right)\right| \\ &\lesssim \left\|\zeta^{[n,N,y]}\right\|_{C^{r\alpha}}(3\varepsilon_N)^{\alpha}(|y-z| + 3\varepsilon_N)^{\gamma-\alpha} \\ &\lesssim \|\psi\|_{C^{r\alpha}}\,\varepsilon_n^{\beta}\varepsilon_N^{\alpha}(|y-z| + \varepsilon_N)^{\gamma-\alpha}.\end{aligned}$$

Plugging this bound into (4.4.9) we finally obtain since $\beta > 0$ and $n \geqslant N$

$$\left|\mathsf{K}_{[N,\infty)} * (J_z - J_y)\left(\psi_y^{\varepsilon_N}\right)\right| \lesssim \|\psi\|_{C^{r\alpha}}\,\varepsilon_N^{\alpha+\beta}(|y-z| + \varepsilon_N)^{\gamma-\alpha},$$

which coincides with (4.3.11) for $\alpha+\beta \leqslant 0$, while for $\alpha+\beta > 0$ it is even better than (4.3.11), by Remark 4.4.2.

Estimate of B. Then we analyse

$$\begin{aligned}(J_z - J_y)\left(\mathsf{K}^*_{[M,N)}\psi_y^{\varepsilon_N}\right) &= \sum_{n=M_{y,z,N}}^{N-1}(J_z - J_y)\left(\mathsf{K}^*_n\psi_y^{\varepsilon_N}\right)\\ &= \sum_{n=M_{y,z,N}}^{N-1}\int_{\mathbb{R}^d}\psi_y^{\varepsilon_N}(x)\,(J_z - J_y)(\mathsf{K}_n(x-\cdot))\,\mathrm{d}x.\end{aligned} \tag{4.4.10}$$

Note now that one can write $\mathsf{K}_n(x-\cdot) = \left(\varphi^{[n,x-y]}\right)_y^{3\varepsilon_n}$ where $\varphi^{[n,z]}$ is defined in (4.4.5). Then, by coherence (2.4.4), and using the property (4.4.6) of $\varphi^{[n,x-y]}$ we can bound

$$\begin{aligned}|(J_z - J_y)(\mathsf{K}_n(x-\cdot))| &= \left|(J_z - J_y)\left(\left(\varphi^{[n,x-y]}\right)_y^{3\varepsilon_n}\right)\right|\\ &\lesssim \left\|\varphi^{[n,x-y]}\right\|_{C^{r_\alpha}}(3\varepsilon_n)^{\alpha}\,(|y-z|+3\varepsilon_n)^{\gamma-\alpha}\\ &\lesssim \varepsilon_n^{\beta}(3\varepsilon_n)^{\alpha}\,(|y-z|+3\varepsilon_n)^{\gamma-\alpha}\\ &\leqslant \varepsilon_n^{\beta}\,(3\varepsilon_n)^{\alpha}\,(4|y-z|+3\varepsilon_N)^{\gamma-\alpha},\end{aligned}$$

where in the last inequality we used the fact that $\varepsilon_n \leqslant |y-z| + \varepsilon_N$ for $n \geqslant M_{y,z,N}$. We plug this bound into (4.4.10). Note that

$$\sum_{n=M_{y,z,N}}^{N}\varepsilon_n^{\alpha+\beta} \lesssim \begin{cases}\displaystyle\sum_{n=0}^{N}\varepsilon_n^{\alpha+\beta} \lesssim \varepsilon_N^{\alpha+\beta} & \text{if } \alpha+\beta<0,\\ \displaystyle\sum_{n=M_{y,z,N}}^{\infty}\varepsilon_n^{\alpha+\beta} \lesssim (|y-z|+\varepsilon_N)^{\alpha+\beta} & \text{if } \alpha+\beta>0.\end{cases}$$

Moreover $\int_{\mathbb{R}^d}|\psi_y^{\varepsilon_N}(w)|\,\mathrm{d}w = \int_{\mathbb{R}^d}|\psi(w)|\,\mathrm{d}w \lesssim \|\psi\|_\infty \leqslant \|\psi\|_{C^{r_\alpha}}$ for any $\psi \in \mathscr{D}(B(0,1))$, hence

$$\frac{|(J_z - J_y)(\mathsf{K}^*_{[M,N)}\psi_y^{\varepsilon_N})|}{\|\psi\|_{C^{r_\alpha}}} \lesssim \begin{cases}\varepsilon_N^{\alpha+\beta}\,(|y-z|+\varepsilon_N)^{\gamma-\alpha} & \text{if } \alpha+\beta<0\\ (|y-z|+\varepsilon_N)^{\gamma+\beta} & \text{if } \alpha+\beta>0\end{cases},$$

which coincides with (4.3.11).

Estimate of C. If $\gamma+\beta \leqslant 0$ then $C = 0$. Let us consider the case $\gamma+\beta>0$. By (4.3.3) and Lemma 4.3.2, see in particular (4.3.14), we have

$$\sum_{n=M_{y,z,N}}^{\infty}\left|J_y\left(\partial^{\ell}\mathsf{K}_n(y-\cdot)\right)\right| \lesssim \varepsilon_{M_{y,z,N}}^{\gamma+\beta-|\ell|},$$

while

$$\left|\mathbb{X}_y^\ell\left(\psi_y^{\varepsilon_N}\right)\right| = \int_{\mathbb{R}^d}\left|\mathbb{X}_y^\ell(w)\,\psi_y^{\varepsilon_N}(w)\right|\mathrm{d}w \lesssim \varepsilon_N^{|\ell|}.$$

Then, recalling (4.4.1) and bounding $\varepsilon_{M_{y,z,N}} \leqslant |y-z| + \varepsilon_N$,

$$\left|J_y\left(P_y^{[M,\infty)}(\psi_y^{\varepsilon_N})\right)\right| \lesssim \sum_{|\ell|<\gamma+\beta} \varepsilon_{M_{y,z,N}}^{\gamma+\beta-|\ell|}\varepsilon_N^{|\ell|} \lesssim (|y-z|+\varepsilon_N)^{\gamma+\beta}\,. \tag{4.4.11}$$

Similarly

$$\sum_{n=M_{y,z,N}}^{\infty}\left|J_z\left(\partial^\ell \mathsf{K}_n(z-\cdot)\right)\right| \lesssim \varepsilon_{M_{y,z,N}}^{\gamma+\beta-|\ell|},$$

$$\left|\mathbb{X}_z^\ell\left(\psi_y^{\varepsilon_N}\right)\right| = \int_{\mathbb{R}^d}\left|\mathbb{X}_z^\ell(w)\,\psi_y^{\varepsilon_N}(w)\right|\mathrm{d}w \lesssim (|y-z|+\varepsilon_N)^{|\ell|},$$

so that

$$\left|J_z\left(P_z^{[M,\infty)}(\psi_y^{\varepsilon_N})\right)\right| \lesssim (|y-z|+\varepsilon_N)^{\gamma+\beta}\,. \tag{4.4.12}$$

Note that both (4.4.11) and (4.4.12) are better than (4.3.11), by Remark 4.4.2.

Estimate of D. We now focus now on

$$(J_z - J_y)\left(\mathsf{K}_{[0,M)}^*\psi_y^{\varepsilon_N} - P_y^{[0,M)}(\psi_y^{\varepsilon_N})\right). \tag{4.4.13}$$

We first assume that $\gamma+\beta>0$. Observe that one can write

$$\mathsf{K}_n(w-\cdot) - \sum_{|\ell|<\gamma+\beta}\partial^\ell \mathsf{K}_n(y-\cdot)\,\frac{(w-y)^\ell}{\ell!} = \int_0^1 \frac{(1-t)^m}{m!}\left(\xi^{[0,n,w-y,t]}\right)_y^{3\varepsilon_n}(\cdot)\,\mathrm{d}t, \tag{4.4.14}$$

where $\xi^{[k,n,z,t]}$ is defined as in (4.4.7). Therefore:

$$\begin{aligned}&(J_z - J_y)\left(\mathsf{K}_{[0,M)}^*\psi_y^{\varepsilon_N} - P_y^{[0,M)}(\psi_y^{\varepsilon_N})\right) = \\ &= \int_{\mathbb{R}^d}\psi_y^{\varepsilon_N}(w)\sum_{n=0}^{M_{y,z,N}-1}\int_0^1\frac{(1-t)^m}{m!}(J_z-J_y)\left(\left(\xi^{[0,n,w-y,t]}\right)_y^{3\varepsilon_n}\right)\mathrm{d}t\,\mathrm{d}w\,.\end{aligned}$$

Applying the coherence bound (2.4.4), we can estimate

$$\begin{aligned}\left|(J_z-J_y)\left(\left(\xi^{[0,n,w-y,t]}\right)_y^{3\varepsilon_n}\right)\right| &\lesssim \left\|\xi^{[0,n,w-y,t]}\right\|_{C^{r_\alpha}}(3\varepsilon_n)^\alpha(|z-y|+\varepsilon_n)^{\gamma-\alpha}\\ &\lesssim \left\|\xi^{[0,n,w-y,t]}\right\|_{C^{r_\alpha}}\varepsilon_n^\gamma,\end{aligned}$$

because for $n \leqslant M_{y,z,N}$ we have $(|z-y|+\varepsilon_n)^{\gamma-\alpha} \leqslant (2\varepsilon_n)^{\gamma-\alpha}$. If $|w-y|>\varepsilon_N$ then $\psi_y^{\varepsilon_N}(w)=0$, so that we can assume $|w-y|\leqslant\varepsilon_N\leqslant\varepsilon_n$. From the property (4.4.8) of $\xi^{[0,n,w-y,t]}$ one obtains

$$\left\|\xi^{[0,n,w-y,t]}\right\|_{C^{r_\alpha}} \lesssim |y-w|^{\lceil\gamma+\beta\rceil}\varepsilon_n^{\beta-\lceil\gamma+\beta\rceil} \leqslant \varepsilon_N^{\lceil\gamma+\beta\rceil}\varepsilon_n^{\beta-\lceil\gamma+\beta\rceil},$$

uniformly for $n \leqslant N$ and $t \in [0,1]$. Collecting all those estimates,

$$\begin{aligned}
|(J_z - J_y)\left(\mathsf{K}^*_{[0,M)}\psi^{\varepsilon_N}_y - P^{[0,M)}_y(\psi^{\varepsilon_N}_y)\right)| &\lesssim \varepsilon_N^{m+1} \sum_{n=0}^{M_{y,z,N}-1} \varepsilon_n^{\gamma+\beta-m-1} \\
&\lesssim \varepsilon_N^{m+1}\,(|z-y|+\varepsilon_N)^{\gamma+\beta-m-1} \leqslant (|z-y|+\varepsilon_N)^{\gamma+\beta}\,,
\end{aligned}$$

which, recalling (4.4.13), is better than (4.3.11) by Remark 4.4.2.

We next assume that $\gamma+\beta<0$. In this case we have $P^{[0,M)}_y \equiv 0$ in (4.4.13). Then, recall from (4.4.5) that one can write

$$\mathsf{K}_n\,(w-\cdot) = \left(\varphi^{[n,w-y]}\right)^{3\varepsilon_n}_y(\cdot).$$

Thus, from the coherence bound (2.4.4), and the property (4.4.6) of $\varphi^{[n,w-y]}$ one can estimate (recall that $\varepsilon_N \leqslant \varepsilon_n$ and $\beta>0$)

$$\begin{aligned}
\left|(J_z - J_y)\left(\mathsf{K}^*_n\psi^{\varepsilon_N}_y\right)\right| &\lesssim \sup_{|w-y|\leqslant\varepsilon_N}\left|(J_z-J_y)\left(\left(\varphi^{[n,w-y]}\right)^{3\varepsilon_n}_y\right)\right| \\
&\lesssim \sup_{|w-y|\leqslant\varepsilon_N}\left\|\varphi^{[n,w,y]}\right\|_{C^{r_\alpha}} (3\varepsilon_n)^{\alpha}\,(|z-y|+\varepsilon_n)^{\gamma-\alpha} \\
&\lesssim \varepsilon_n^{\beta}\,(3\varepsilon_n)^{\alpha}\,(|z-y|+\varepsilon_n)^{\gamma-\alpha}.
\end{aligned}$$

For $n \leqslant M_{y,z,N}$ we have $(|z-y|+\varepsilon_n)^{\gamma-\alpha} \leqslant (2\varepsilon_n)^{\gamma-\alpha}$, hence

$$\left|(J_z-J_y)\left(\mathsf{K}^*_{[0,M)}\psi^{\varepsilon_N}_y\right)\right| \lesssim \sum_{n=0}^{M_{y,z,N}-1} \varepsilon_n^{\gamma+\beta} \lesssim (|z-y|+\varepsilon_N)^{\gamma+\beta}$$

which, recalling (4.4.13), is better than (4.3.11) by Remark 4.4.2.

Estimate of E. We have

$$P^{\{n\}}_z(\psi^{\varepsilon_N}_y) - P^{\{n\}}_y(\psi^{\varepsilon_N}_y) = -\sum_{|k|<\gamma+\beta} R^k(y,z,\cdot)\,\mathbb{X}^{\ell}_y\left(\psi^{\varepsilon_N}_y\right)\,,$$

see [**4**, formula (4.7)], where

$$\begin{aligned}
R^k(y,z,\zeta) &:= \partial^k\,\mathsf{K}_n(y-\zeta) - \sum_{|\ell|<\gamma+\beta-|k|} \partial^{k+\ell}\,\mathsf{K}_n(z-\zeta)\,\frac{(y-z)^{\ell}}{\ell!} \\
&= \int_0^1 \frac{(1-t)^{m-|k|}}{(m-|k|)!}\left(\xi^{[k,n,y-z,t]}\right)^{3\varepsilon_n}_z(\zeta)\,\mathrm{d}t,
\end{aligned}$$

where $\xi^{[k,n,z,t]}$ is the function defined in (4.4.7). Then

$$J_z\left(P_y^{[0,M)}\left(\psi_y^{\varepsilon_N}\right)-P_z^{[0,M)}\left(\psi_y^{\varepsilon_N}\right)\right)=$$
$$=-\sum_{|k|<\gamma+\beta}\sum_{n=0}^{M_{y,z,N}-1}\int_0^1\frac{(1-t)^{m-|k|}}{(m-|k|)!}J_z\left(\left(\xi^{[k,n,y-z,t]}\right)_z^{3\varepsilon_n}\right)\,\mathrm{d}t\,\mathbb{X}_y^k(\psi_y^{\varepsilon_N}).$$

Applying the coherence bound (2.4.4), and the property (4.4.8) of $\xi^{[k,n,y-z,t]}$, since for $n\leqslant M_{y,z,N}$ we have $|y-z|\leqslant\varepsilon_n$, we can estimate

$$\begin{aligned}\left|J_z\left(\left(\xi^{[k,n,y-z,t]}\right)_z^{3\varepsilon_n}\right)\right|&\lesssim\|\xi^{[k,n,y-z,t]}\|_{C^{r_\alpha}}(3\varepsilon_n)^\alpha\left(|z-y|+\varepsilon_n\right)^{\gamma-\alpha}\\&\lesssim|y-z|^{\lceil\gamma+\beta\rceil-|k|}\varepsilon_n^{\beta-\lceil\gamma+\beta\rceil}(3\varepsilon_n)^\alpha\left(|z-y|+\varepsilon_n\right)^{\gamma-\alpha}.\end{aligned}$$

Recalling that $\left|\mathbb{X}_y^\ell\left(\psi_y^{\varepsilon_N}\right)\right|\lesssim\varepsilon_N^{|k|}$ and that $(|z-y|+\varepsilon_n)^{\gamma-\alpha}\leqslant(2\varepsilon_n)^{\gamma-\alpha}$, we bound

$$\begin{aligned}\left|J_z\left(P_y^{[0,M)}\left(\psi_y^{\varepsilon_N}\right)-P_z^{[0,M)}\left(\psi_y^{\varepsilon_N}\right)\right)\right|&\lesssim\varepsilon_N^{|k|}\,|y-z|^{\lceil\gamma+\beta\rceil-|k|}\sum_{n=0}^{M_{y,z,N}-1}\varepsilon_n^{\gamma+\beta-\lceil\gamma+\beta\rceil}\\&\lesssim\left(|y-z|+\varepsilon_N\right)^{\lceil\gamma+\beta\rceil}\sum_{n=0}^{M_{y,z,N}-1}\varepsilon_n^{\gamma+\beta-\lceil\gamma+\beta\rceil}\\&\lesssim\left(|y-z|+\varepsilon_N\right)^{\gamma+\beta}.\end{aligned}$$

which, recalling (4.4.13), is better than (4.3.11) by Remark 4.4.2.

L has homogeneity $\gamma+\beta$. Finally we prove that

$$|L_x(\psi_x^{\varepsilon_N})|\lesssim\varepsilon_N^{\gamma+\beta}$$

uniformly for $x\in K$ and $n\in\mathbb{N}$. This is a consequence of the following

Lemma 4.4.4. *Fix $\gamma\in\mathbb{R}$, $\beta>0$ and a point $x\in\mathbb{R}^d$. Let $T\in\mathscr{D}'$ have order r_α and homogeneity bound γ at the point x, i.e. for some $r\in\mathbb{N}$ and $\mathsf{C}_x<\infty$*

$$\begin{aligned}&|T(\varphi_x^\varepsilon)|\leqslant\mathsf{C}_x\,\|\varphi\|_{C^{r_\alpha}}\,\varepsilon^\gamma\\&\qquad\text{uniformly for }\varepsilon\in(0,1]\text{ and }\varphi\in\mathscr{D}(B(0,1)).\end{aligned}\tag{4.4.15}$$

Let K be a β-regularizing kernel up to degree $m>\gamma+\beta+r_\alpha$. Then

$$\left|T\left(\mathsf{K}^*\psi_x^{\varepsilon_N}-P_x^{\mathbb{N}}(\psi_x^{\varepsilon_N})\right)\right|\lesssim C_x\varepsilon_N^{\gamma+\beta},$$

recall (4.4.1) *and* (4.4.2).

PROOF. We consider the decomposition

$$T\left(\mathsf{K}^*\psi_x^{\varepsilon_N} - P_x^{\mathbb{N}}(\psi_x^{\varepsilon_N})\right) = \underbrace{T\left(\mathsf{K}^*_{[N,+\infty)}\psi_x^{\varepsilon_N}\right)}_{F} - \underbrace{T\left(P_x^{[N,+\infty)}(\psi_x^{\varepsilon_N})\right)}_{G} + \underbrace{T\left(\mathsf{K}^*_{[0,N)}\psi_x^{\varepsilon_N} - P_x^{[0,N)}(\psi_x^{\varepsilon_N})\right)}_{H}.$$

We shall estimate F, G, H separately. We analyse first

$$F = \sum_{n=N}^{\infty} T(\mathsf{K}^*_n\psi_x^{\varepsilon_N}). \tag{4.4.16}$$

Recall from (4.4.3) that one can write $\mathsf{K}^*_n\psi_x^{\varepsilon_N} = (\zeta^{[n,N,x]})_x^{3\varepsilon_N}$. Then, by the homogeneity bound (4.3.3) for J, and using the property (4.4.4) of $\zeta^{[n,N,x]}$, we can bound for $n \geqslant N$:

$$\begin{aligned}|T(\mathsf{K}^*_n\psi_x^{\varepsilon_N})| &= \left|T\left(\left(\zeta^{[n,N,x]}\right)_x^{3\varepsilon_N}\right)\right| \\ &\lesssim \left\|\zeta^{[n,N,x]}\right\|_{C^{r\alpha}} (3\varepsilon_N)^\gamma \\ &\lesssim \|\psi\|_{C^{r\alpha}}\, \varepsilon_n^\beta (3\varepsilon_N)^\gamma.\end{aligned}$$

Plugging this bound into (4.4.16) we finally obtain

$$|F| \lesssim \|\psi\|_{C^{r\alpha}}\, \varepsilon_N^{\gamma+\beta},$$

as required. The quantity G is treated in the same way as (4.4.11), so that:

$$|G| \lesssim \varepsilon_N^{\gamma+\beta}.$$

We are ready to control the contribution of H. As in the estimate of D above, we distinguish two cases. First assume that $\gamma+\beta > 0$, then we use (4.4.14) again. Therefore:

$$H = \sum_{n=0}^{N-1}\int_{\mathbb{R}^d}\int_0^1 \frac{(1-t)^m}{m!} T\left(\left(\xi^{[0,n,w-x,t]}\right)_x^{3\varepsilon_n}\right) \mathrm{d}t\, \psi_x^{\varepsilon_N}(w)\,\mathrm{d}w.$$

By the homogeneity bound (4.3.3) for J, and using the property (4.4.8) of $\xi^{[0,n,w-x,t]}$ (note that here $|x-w| \leqslant \varepsilon_N \leqslant \varepsilon_n$), we can bound

$$\left|T\left(\left(\xi^{[0,n,w-x,t]}\right)_x^{3\varepsilon_n}\right)\right| \lesssim \left\|\xi^{[0,n,w-x,t]}\right\|_{C^{r\alpha}} (3\varepsilon_n)^\gamma \lesssim \varepsilon_N^{m+1}\,\varepsilon_n^{\gamma+\beta-m-1}.$$

And thus after summing the geometric series one obtains since $\gamma+\beta < m+1$

$$|H| \lesssim \varepsilon_N^{\gamma+\beta}.$$

Finally, we bound H in the case when $\gamma+\beta<0$. In this case, $P_x^{[0,N)} \equiv 0$. Then, recall from (4.4.5) that one can write $\mathsf{K}_n(w-\cdot) = \left(\varphi^{[n,w-x]}\right)_x^{3\varepsilon_n}$, so that

$$H = \int_{\mathbb{R}^d} T\left(\left(\varphi^{[n,w-x]}\right)_x^{3\varepsilon_n}\right) \psi_x^{\varepsilon_N}(w)\,\mathrm{d}w.$$

Thus, from the homogeneity bound (4.3.3) for J, and the property (4.4.6) of $\varphi^{[n,w-x]}$ one can estimate (note that here $|w-x| \leqslant \varepsilon_N \leqslant \varepsilon_n$)

$$\left|T\left(\left(\varphi^{[n,w-x]}\right)_x^{3\varepsilon_n}\right)\right| \lesssim \left\|\varphi^{[n,w-x]}\right\|_{C^{r\alpha}} (3\varepsilon_n)^\gamma \lesssim \varepsilon_n^{\beta+\gamma}.$$

And thus after summing the geometric series one obtains as announced $|H| \lesssim \varepsilon_N^{\gamma+\beta}$. The proof of Lemma 4.4.4 is complete. □

Conclusion. We have shown that L is $((\alpha+\beta)\wedge 0, \gamma+\beta)$-coherent and that it has homogeneity bound with exponent $\gamma+\beta$. Then its $(\gamma+\beta)$-reconstruction is 0, and therefore the $(\gamma+\beta)$-reconstruction of H is $\mathsf{K} * \mathscr{R}F$.

CHAPTER 5

Multi-level Schauder estimates for modelled distributions

In this chapter we discuss one of the most important operations on modelled distributions: the convolution with a regularising integration kernel.

We fix a pre-model (Π,Γ) as in Definition 3.1.1 and we consider $f \in \mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$ as in Definition 3.1.3. We have seen in Theorem 4.3.1 how we can build a linear operator

$$\mathscr{K} : \mathscr{G}^{\alpha,\gamma} \to \mathscr{G}^{(\alpha+\beta)\wedge 0,\gamma+\beta}, \qquad \mathscr{R}\circ\mathscr{K} = \mathsf{K} * \mathscr{R}.$$

Now we want to address an analogous question for $F = \langle \Pi, f\rangle$. In other words, we want to show that it is possible to construct

(1) *another* pre-model $(\hat{\Pi},\hat{\Gamma})$, such that

(2) for every $f \in \mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$ there is a modelled distribution $\hat{f} \in \mathscr{D}^{\gamma+\beta}_{(\hat{\Pi},\hat{\Gamma})}$

such that

$$\mathscr{K}\langle \Pi, f\rangle = \langle \hat{\Pi}, \hat{f}\rangle.$$

5.1. The pre-model

We need an additional property for a pre-model (see Definition 3.1.1).

Definition 5.1.1. *A pre-model is* good *if there exists $r \in \mathbb{N}$ such that*

$$|\Pi^i_x(\varphi^{\varepsilon_n}_x)| \lesssim \varepsilon_n^{\alpha_i},$$

uniformly over x in compact subsets of $\mathbb{R}^d$, $n \in \mathbb{N}$ and $\varphi \in \mathscr{B}_r$.

Remark 5.1.2. A model (Definition 3.1.3) is *a fortiori* a good pre-model. Indeed, any germ Π^i in a model is coherent, as we discussed in Section 3.3, and for a coherent and homogeneous germ on can replace the single $\varphi \in \mathscr{D}$ by a generic $\psi \in \mathscr{B}_r$ for any $r > -(\alpha \wedge \bar{\alpha})$, see Remark 2.4.2.

We fix throughout this chapter an integration kernel K, which is supposed to be β-regularising up to order m where $m \in \mathbb{N}$ satisfies

$$m > \gamma + \beta + r, \tag{5.1.1}$$

where r is as in Definition 5.1.1.

We work from now on with a good pre-model (Π,Γ), and we want to construct a pre-model $(\hat{\Pi},\hat{\Gamma})$ with the property discussed at the beginning of this chapter. We suppose, as in Definition 3.1.3, that $\gamma > \max(\alpha_i, i \in I)$.

We start discussing the family $(\hat{\Pi}_x^i)_{i\in\hat{I},x\in\mathbb{R}^d}$. A reasonable guess would be to set $\hat{I} = I$ and $\hat{\Pi}_x^i = \mathsf{K} * \Pi_x^i$, recall (4.1.3). However we expect $\hat{\Pi}_x^i(\psi_x^{\varepsilon_n})$ to be small as $n \to +\infty$, at least if the homogeneity $\alpha_i + \beta$ which is expected for $\hat{\Pi}^i$ is positive.

However $\mathsf{K} * \Pi_x^i(\psi_x^{\varepsilon_n})$ has no reason to become small for large n. To this aim we can subtract a Taylor polynomial which can yield the desired behaviour. We are going to set for $i \in I$

$$\hat{\Pi}_x^i = \mathsf{K} * \Pi_x^i - \sum_{|k|<\alpha_i+\beta} \Pi_x^i\left(\partial^k \mathsf{K}(x-\cdot)\right)\mathbb{X}_x^k, \tag{5.1.2}$$

where we recall that $\mathbb{X}_x^k(w) := \frac{(w-x)^k}{k!}$. If Π_x^i is a polynomial, this definition yields $\hat{\Pi}_x^i \equiv 0$.

Proposition 5.1.3. *The distribution $\hat{\Pi}_x^i$ in* (5.1.2) *for $i \in I$ is well defined, has order r and satisfies for all compact set $K \subset \mathbb{R}^d$*

$$\sup_{x\in K}\sup_{\ell\in\mathbb{N}}\sup_{\psi\in\mathscr{B}_r} \frac{|\hat{\Pi}_x^i(\psi_x^{\varepsilon_\ell})|}{\varepsilon_\ell^{\alpha_i+\beta}} < +\infty. \tag{5.1.3}$$

PROOF. Since (Π,Γ) is a good pre-model, then Π_x^i is a distribution with order r. Then by Proposition 4.2.2 the distribution $\mathsf{K} * \Pi_x^i$ is well defined and has order r. By applying Lemma 4.3.2 to $T := \Pi_x^i$ and $\gamma = \alpha_i$, we obtain that $\Pi_x^i\left(\partial^k \mathsf{K}(x-\cdot)\right)$ is well defined for all $|k| < \alpha_i + \beta$.

Finally, (5.1.3) follows from Lemma 4.4.4. □

We can therefore associate to $\hat{\Pi}^i$ the homogeneity $\alpha_i + \beta$. Then we construct a new basis by setting

$$\hat{I} := I \sqcup I_{\mathrm{Poly}}, \quad I_{\mathrm{Poly}} := \{k \in \mathbb{N}^d : |k| < \gamma + \beta\},$$
$$\hat{\Pi}_x^k := \mathbb{X}_x^k, \qquad k \in I_{\mathrm{Poly}}.$$

recall (4.3.7); of course the homogeneity of $\hat{\Pi}_x^k$ is $|k|$.

Once this choice is made, it remains to construct $\hat{\Gamma}$ and $\hat{f}$. It turns out that there are very natural choices for these objects. Let us set for notational convenience

$$A_x^{i,\ell} := \mathbb{1}_{(|\ell|<\alpha_i+\beta)}\,\Pi_x^i\left(\partial^\ell \mathsf{K}(x-\cdot)\right), \qquad x \in \mathbb{R}^d,\ i \in I,\ \ell \in \mathbb{N}^d,$$

so that (5.1.2) becomes

$$\hat{\Pi}_x^i = \mathsf{K} * \Pi_x^i - \sum_{k\in I_{\mathrm{Poly}}} A_x^{i,k}\,\mathbb{X}_x^k,$$

and we have already seen in the proof of Proposition 5.1.3 that $A_x^{i,\ell}$ is well defined.

We define now the coefficients $(\hat\Gamma^{ij}_{xy})_{i,j\in\hat I}$. These are straightforward when

(1) $i,j\in I$
(2) $i,j\in I_{\rm Poly}$
(3) $i\in I$ and $j\in I_{\rm Poly}$,

see (5.1.4) below for the precise values. The less simple case is that of $i\in I_{\rm Poly}$ and $j\in I$, to which we turn now. By the definition of $(\hat\Pi^i_x)_{i\in\hat I}$ we find that for $j\in I$

$$\hat\Pi^j_y-\sum_{i\in I}\hat\Pi^i_x\,\Gamma^{ij}_{xy}=\sum_{k\in I_{\rm Poly}}\left(-A^{j,k}_y\,\mathbb{X}^k_y+\sum_{i\in I}\Gamma^{ij}_{xy}A^{i,k}_x\,\mathbb{X}^k_x\right).$$

Since $\mathbb{X}^k_y=\sum_{\ell\leqslant k}\mathbb{X}^{k-\ell}_y(x)\,\mathbb{X}^\ell_x$, the right-hand side of the latter expression is equal to (after renaming some indices)

$$\sum_{i\in I_{\rm Poly}}\mathbb{X}^i_x\left(\sum_{k\in I}\Gamma^{kj}_{xy}A^{k,i}_x-\sum_{\ell\in\mathbb{N}^d}\mathbb{X}^\ell_y(x)\,A^{j,i+\ell}_y\right),$$

namely a linear combination of elements in $I_{\rm Poly}$. Therefore we set for $j\in I$ and $i\in I_{\rm Poly}$

$$\hat\Gamma^{ij}_{xy}:=\sum_{k\in I}\Gamma^{kj}_{xy}A^{k,i}_x-\sum_{\ell\in\mathbb{N}^d}\mathbb{X}^\ell_y(x)\,A^{j,i+\ell}_y.$$

To resume we have

$$\hat\Gamma^{ij}_{xy}=\begin{cases}\Gamma^{ij}_{xy}, & \text{if } i,j\in I,\\[2ex] \mathbb{1}_{(i\leqslant j)}\,\mathbb{X}^{j-i}_y(x), & \text{if } i,j\in I_{\rm Poly},\\[2ex] \displaystyle\sum_{k\in I}\Gamma^{kj}_{xy}A^{k,i}_x-\sum_{\ell\in\mathbb{N}^d}\mathbb{X}^\ell_y(x)\,A^{j,i+\ell}_y, & \text{if } i\in I_{\rm Poly},\ j\in I,\\[2ex] 0 & \text{if } i\in I,\ j\in I_{\rm Poly}.\end{cases}\tag{5.1.4}$$

Then we have the desired property for $\hat\Gamma$

$$\sum_{i\in\hat I}\hat\Pi^i_x\,\hat\Gamma^{ij}_{xy}=\hat\Pi^j_y,\qquad j\in\hat I$$

and we have proved the following

THEOREM 5.1.4. *If* (Π,Γ) *is a good pre-model, then* $(\hat\Pi,\hat\Gamma)$ *is also a good pre-model.*

5.2. The modelled distribution

For a modelled distribution $f:\mathbb{R}^d \to \mathbb{R}^I$ we define now a new function $\hat{f}:\mathbb{R}^d \to \mathbb{R}^{\hat{I}}$

$$\hat{f}^i_x := \begin{cases} f^i_x & \text{if} \quad i \in I, \\ \left(\mathscr{R}F - \sum_{\alpha_a \leqslant |i|-\beta} f^a_x\, \Pi^a_x\right)(\partial^i \mathsf{K}(x-\cdot)) & \text{if} \quad i \in I_{\mathrm{Poly}}, \end{cases} \tag{5.2.1}$$

where $F = \langle \Pi, f\rangle$ and we recall that $I_{\mathrm{Poly}} = \{k \in \mathbb{N}^d : |k| < \gamma + \beta\}$. If Π^i_x is a polynomial, then $\hat{\Pi}^i_x = 0$, hence the value of $\hat{f}^i_x$ is immaterial.

Remark 5.2.1. Note that we have

$$\mathscr{K}\langle f, \Pi\rangle = \langle \hat{f}, \hat{\Pi}\rangle,$$

where $\mathscr{K}$ is the operator of Theorem 4.3.1. Indeed, observe that from the definitions and the notation (3.1.2)

$$\begin{aligned}
\langle \hat{f}, \hat{\Pi}\rangle &= \sum_{i\in I} f^i_x \left(\left(\mathsf{K} * \Pi^i_x\right) - \sum_{|k|<\alpha_i+\beta} \Pi^i_x \left(\partial^k \mathsf{K}(x-\cdot)\right) \mathbb{X}^k_x \right) \\
&\quad + \sum_{|i|<\gamma+\beta} \left(\mathscr{R}F - \sum_{\alpha_a \leqslant |i|-\beta} f^a_x\, \Pi^a_x\right)(\partial^i \mathsf{K}(x-\cdot))\, \mathbb{X}^i_x \\
&= \mathsf{K} * \left(\sum_{i\in I} f^i_x \Pi^i_x\right) + \sum_{|i|<\gamma+\beta} \left(\mathscr{R}F - \sum_{a\in I} f^a_x\, \Pi^a_x\right) \left(\partial^i \mathsf{K}(x-\cdot)\right) \mathbb{X}^i_x \\
&= \mathscr{K}\langle f, \Pi\rangle_x.
\end{aligned}$$

In particular, if $f \in \mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$ then we have already proved in Theorem 4.3.1 that $\mathscr{R}\langle \hat{f}, \hat{\Pi}\rangle = \mathsf{K} * \mathscr{R}F$.

We have seen in Theorem 5.1.4 that $(\hat{\Pi}, \hat{\Gamma})$ is a pre-model. It remains to show that $\hat{f}$ defined in (5.2.1) is in $\mathscr{D}^{\gamma+\beta}_{(\hat{\Pi},\hat{\Gamma})}$. For that however, we need the following additional assumption:

$$\forall\, i \in I: \qquad \alpha_i + \beta \notin \mathbb{N}.$$

More generally, it is enough to impose the following requirement:

$$\begin{aligned}
&\forall\, i \in I, \quad \text{if} \quad \alpha_i + \beta \in \mathbb{N} \quad \text{then} \\
&\quad \Pi^i_x \left(\partial^k_x \mathsf{K}(x-\cdot)\right) = 0 \quad \forall\, k \in \mathbb{N}^d \text{ with } |k| = \alpha_i + \beta,\ x \in \mathbb{R}^d.
\end{aligned} \tag{5.2.2}$$

One can check that this condition always holds if Π_x^i is a monomial of degree $\leqslant \alpha_i$, hence it effectively applies only to non-polynomial germs Π_x^i.

THEOREM 5.2.2. *Let* (Π,Γ) *be a pre-model satisfying* (5.2.2), $\gamma \in \mathbb{R}$ *and* $f \in \mathscr{D}^{\gamma}_{(\Pi,\Gamma)}$. *Then* $\hat{f}$ *defined in* (5.2.1) *is in* $\mathscr{D}^{\gamma+\beta}_{(\hat\Pi,\hat\Gamma)}$.

PROOF. We want $\hat{f}$ to be a modelled distribution of order $\gamma+\beta$ with respect to $\hat\Gamma$: the condition is obvious for $i \in I$, since it is equivalent to the condition on f with respect to Γ. We have to check the correct bound for $i \in I_{\mathrm{Poly}} = \{k \in \mathbb{N}^d : |k| < \gamma+\beta\}$. Fix $x, y \in \mathbb{R}^d$ and introduce the quantity

$$N_{x,y} := \min\{n \in \mathbb{N} : \ \varepsilon_n \leqslant |y-x|\}.$$

We recall the notation $J_x = F_x - \mathscr{R}F$, and we write the decomposition:

$$\begin{aligned}
\hat{f}_x^i - \sum_{j\in\hat I} \hat\Gamma_{xy}^{ij} \hat{f}_y^j = \\
= -\underbrace{\sum_{n=0}^{N_{x,y}-1} J_y \left(\partial^i \mathsf{K}_n(x-\cdot) - \sum_{|k|<\gamma+\beta-|i|} \partial^{i+k}\mathsf{K}_n(y-\cdot)\,\mathbb{X}_y^k(x)\right)}_{A} \\
- \underbrace{\sum_{n=0}^{N_{x,y}-1} \sum_{\alpha_a \leqslant |i|-\beta} \Pi_x^a(\partial^i \mathsf{K}_n(x-\cdot)) \left(f_x^a - \sum_{j\in I} \Gamma_{xy}^{aj} f_y^j\right)}_{B} \\
- \underbrace{\sum_{n=N_{x,y}}^{+\infty} J_x\left(\partial^i \mathsf{K}_n(x-\cdot)\right)}_{C} + \underbrace{\sum_{n=N_{x,y}}^{+\infty} \sum_{|k|<\gamma+\beta-|i|} J_y(\partial^{i+k}\mathsf{K}_n(y-\cdot))\,\mathbb{X}_y^k(x)}_{D} \\
+ \underbrace{\sum_{n=N_{x,y}}^{+\infty} \sum_{\alpha_a > |i|-\beta} \Pi_x^a(\partial^i \mathsf{K}_n(x-\cdot)) \left(f_x^a - \sum_{j\in I} \Gamma_{xy}^{aj} f_y^j\right)}_{E}.
\end{aligned}$$

Now, with the multiscale techniques of the proof of Theorem 4.3.1, we shall prove that each of those terms is bounded by $|x-y|^{\gamma+\beta-|i|}$.

Estimate of A. In view of (4.4.14), we rewrite:

$$A = \sum_{n=0}^{N_{x,y}-1} \int_0^1 \frac{(1-t)^{\lfloor\gamma+\beta\rfloor-|i|}}{(\lfloor\gamma+\beta\rfloor-|i|)!} J_y\left(\left(\xi^{[i,n,x-y,t]}\right)_y^{3\varepsilon_n}\right) \mathrm{d}t,$$

where $\xi^{[i,n,z,t]}$ is the function defined in (4.4.7). Note that because $n \leqslant N_{x,y}$ we are in the regime $|y-x| \leqslant \varepsilon_n$ and thus from (4.4.8) and the reconstruction bound on F, see (4.3.3), one obtains:

$$\left| J_y\left(\left(\xi^{[i,n,x-y,t]}\right)_y^{3\varepsilon_n}\right)\right| \lesssim \left\|\xi^{[i,n,x-y,t]}\right\|_{C^r}(3\varepsilon_n)^\gamma$$
$$\lesssim |y-x|^{\lceil\gamma+\beta\rceil-|i|}\varepsilon_n^{\beta-\lceil\gamma+\beta\rceil}(3\varepsilon_n)^\gamma.$$

Thus, summing a geometric series and since $\gamma+\beta<\lceil\gamma+\beta\rceil$,

$$|A|\lesssim|y-x|^{\gamma+\beta-|i|}.$$

Estimate of B. Because of the assumption (5.2.2) that $\Pi_x^a\left(\partial^i\mathsf{K}(x-\cdot)\right)=0$ when $|i|=\alpha_a+\beta\in\mathbb{N}$, only the terms with $\alpha_a<|i|-\beta$ contribute to the sum defining B. In view of (4.3.16), we rewrite

$$\Pi_x^a(\partial^i\mathsf{K}_n(x-\cdot))=\Pi_x^a\left(\left(\varphi^{[i,n]}\right)_x^{3\varepsilon_n}\right),$$

where $\varphi^{[i,n]}$ is defined in (4.3.15). Thus from the property (4.3.18) of $\varphi^{[i,n]}$ and the fact that Π^a has homogeneity bound α_a, we obtain:

$$\left|\Pi_x^a(\partial^i\mathsf{K}_n(x-\cdot))\right\|\lesssim\left\|\varphi^{[i,n]}\right\|_{C^r}(3\varepsilon_n)^{\alpha_a}$$
$$\lesssim\varepsilon_n^{\beta-|i|}(3\varepsilon_n)^{\alpha_a}\lesssim\varepsilon_n^{\beta+\alpha_a-|i|}.$$

Now since f is a modelled distribution with respect to Γ one can bound B by:

$$|B|\lesssim\sum_{n=0}^{N_{x,y}-1}\sum_{\alpha_a<|i|-\beta}\varepsilon_n^{\beta+\alpha_a-|i|}|x-y|^{\gamma-\alpha_a}. \tag{5.2.3}$$

Summing the geometric sums yields as announced

$$|B|\lesssim|y-x|^{\gamma+\beta-|i|}.$$

Estimate of C. As just above, we rewrite

$$C=\sum_{n=N_{x,y}}^{+\infty}J_x\left(\left(\varphi^{[i,n]}\right)_x^{3\varepsilon_n}\right),$$

where $\varphi^{[i,n]}$ satisfies (4.3.17), (4.3.18), and thus from the reconstruction bound on F, see (4.3.3), one obtains:

$$\left|J_x\left(\left(\varphi^{[i,n]}\right)_x^{3\varepsilon_n}\right)\right|\lesssim\left\|\varphi^{[i,n]}\right\|_{C^r}(3\varepsilon_n)^\gamma$$
$$\lesssim\varepsilon_n^{\beta-|i|}(3\varepsilon_n)^\gamma\lesssim\varepsilon_n^{\gamma+\beta-|i|}.$$

Hence, summing a geometric series and since $\gamma+\beta>|i|$, we obtain

$$|C|\lesssim|y-x|^{\gamma+\beta-|i|}.$$

Estimate of D. Here we use the estimate proved just above:

$$\left|J_y(\partial^{i+k}\mathsf{K}_n(y-\cdot))\right| \lesssim \varepsilon_n^{\gamma+\beta-|i|-|k|}.$$

Thus by summing a geometric series, one obtains:

$$|D| \lesssim \sum_{n=N_{x,y}}^{+\infty} \sum_{|k|<\gamma+\beta-|i|} \varepsilon_n^{\gamma+\beta-|i|-|k|}|y-x|^k \lesssim |y-x|^{\gamma+\beta-|i|}.$$

Estimate of E. Finally, for the term E, the estimates are the same as for the term B, but are summed over different indices. Indeed, similarly to (5.2.3), we get:

$$|E| \lesssim \sum_{n=N_{x,y}}^{+\infty} \sum_{\alpha_a>|i|-\beta} \varepsilon_n^{\beta+\alpha_a-|i|}|x-y|^{\gamma-\alpha_a},$$

and summing the geometric series yields as announced:

$$|E| \lesssim |y-x|^{\gamma+\beta-|i|}.$$

This concludes the proof. □

5.3. Recursive properties

In this section we consider a good pre-model (Π,Γ) and the good pre-model $(\hat{\Pi},\hat{\Gamma})$ of Theorem 5.1.4. We want to show that certain properties are inherited by $(\hat{\Pi},\hat{\Gamma})$ from (Π,Γ).

Recall that we have not imposed a group property on the reexpansion operators Γ. The following proposition however establishes that if Γ enjoys such a property, then so does $\hat{\Gamma}$.

Proposition 5.3.1. *The following assertions are equivalent:*

(1) For all $x,y,z \in \mathbb{R}^d$, $\Gamma_{x,y}\Gamma_{y,z} = \Gamma_{x,z}$.

(2) For all $x,y,z \in \mathbb{R}^d$, $\hat{\Gamma}_{xy}\hat{\Gamma}_{yz} = \hat{\Gamma}_{xz}$.

(Here the product is understood as the matrix product.)

PROOF. The implication (2) ⇒ (1) is straightforward. Now assume (1) and let us establish (2). We have to prove that for all $i,j \in \hat{I}$,

$$\sum_{k\in\hat{I}} \hat{\Gamma}^{ik}_{xy}\hat{\Gamma}^{kj}_{yz} = \hat{\Gamma}^{ij}_{xy}. \tag{5.3.1}$$

We distinguish the different possible cases for $i,j \in \hat{I}$. If $i,j \in I$, (5.3.1) is straighforward from the definition of $\hat{\Gamma}$ and (1). If $i,j \in I_{\mathrm{Poly}}$, then (5.3.1) is also straighforward from Newton's binomial formula. In the case when $i \in I,\ j \in I_{\mathrm{Poly}}$, the left-hand side and the right-hand side of (5.3.1) vanish.

It remains to tackle the case when $i \in I_{\text{Poly}},\ j \in I$. In this case, we can calculate explicitly

$$\begin{aligned}\sum_{k\in\hat{I}} \hat{\Gamma}^{ik}_{xy}\hat{\Gamma}^{kj}_{yz} &= \sum_{k\in I} \hat{\Gamma}^{ik}_{xy}\hat{\Gamma}^{kj}_{yz} + \sum_{k\in I_{\text{Poly}}} \hat{\Gamma}^{ik}_{xy}\hat{\Gamma}^{kj}_{yz} \\ &= \sum_{k\in I}\left(\sum_{a\in I} A^{a,i}_x \Gamma^{ak}_{xy} - \sum_{\ell\in I_{\text{Poly}}} \mathbb{X}^{\ell}_y(x) A^{k,i+\ell}_y\right)\Gamma^{kj}_{yz} \\ &\quad + \sum_{k\in I_{\text{Poly}}} \mathbb{X}^{k-i}_y(x)\left(\sum_{a\in I} A^{a,k}_y \Gamma^{aj}_{yz} - \sum_{\ell\in I_{\text{Poly}}} \mathbb{X}^{\ell}_z(y) A^{j,k+\ell}_z\right).\end{aligned}$$

Using the fact that $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ in the first term:

$$\begin{aligned}\sum_{k\in\hat{I}} \hat{\Gamma}^{ik}_{xy}\hat{\Gamma}^{kj}_{yz} &= \sum_{a\in I} A^{a,i}_x \Gamma^{aj}_{xz} - \sum_{k\in I}\sum_{\ell\in I_{\text{Poly}}} \mathbb{X}^{\ell}_y(x) A^{k,i+\ell}_y \Gamma^{kj}_{yz} \\ &\quad + \sum_{k\in I_{\text{Poly}}}\sum_{a\in I} \mathbb{X}^{k-i}_y(x) A^{a,k}_y \Gamma^{aj}_{yz} - \sum_{k\in I_{\text{Poly}}} \mathbb{X}^{k-i}_y(x) \sum_{\ell\in I_{\text{Poly}}} A^{j,k+\ell}_z \mathbb{X}^{\ell}_z(y).\end{aligned}$$

Observe that the second and third term cancel out, and from Newton's binomial formula in the last term, we obtain

$$\sum_{k\in\hat{I}} \hat{\Gamma}^{ik}_{xy}\hat{\Gamma}^{kj}_{yz} = \sum_{a\in I} A^{a,i}_x \Gamma^{aj}_{xz} - \sum_{a\in I_{\text{Poly}}} \mathbb{X}^{a}_z(y) A^{j,i+a}_z = \hat{\Gamma}^{ij}_{xz}.$$

The proof is complete □

Analogously, one can prove the following:

Proposition 5.3.2. *If* (Π, Γ) *is a model in the sense of Definition 3.3.1 then* $(\hat{\Pi}, \hat{\Gamma})$ *is also a model.*

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