ON DAVIE'S NON-EXISTENCE AND NON-UNIQUENESS EXAMPLES FOR EQUATIONS DRIVEN BY ROUGH PATHS

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ABSTRACT. We consider differential equations driven by rough paths, focusing on examples of non-existence and non-uniqueness of solutions, as provided by Davie [Dav08] under optimal regularity assumptions. We provide in this note complete proofs and explanations, together with some extensions and improvements, with the goal of making these examples better-known.

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1. Introduction

Rough Paths were introduced by Terry Lyons [Lyo98] as a mean to give a pathwise theory of *differential equations driven by irregular paths*, such as the sample paths of Brownian motion. This theory was then enriched by Massimiliano Gubinelli with the notion of Controlled Paths [Gub04] and the crucial Sewing Lemma, see also [FLP06]. We refer to [FH20] for a comprehensive introduction to the subject. It is worth stressing that the ideas at the basis of rough paths play a crucial role in the theory of Regularity Structures by Martin Hairer [Hai14], which allows to make sense of a large class of singular *stochastic partial differential equations*.

We focus here on the *finite-difference formulation* of differential equations driven by rough paths, proposed by Alexander M. Davie [Dav08], which leads to results of well-posedness (existence and uniqueness of solutions) with *sharp regularity assumptions*. See [CGZ24] for a recent pedagogical introduction to this approach.

In the same paper, Davie gave also examples of non-uniqueness and non-existence of solutions when the aforementioned assumptions fail, thus proving their optimality, see [Dav08, §5, examples 1-4]. These examples appear to be less well-known than they deserve, possibly because many details of the arguments involved are left to the reader. *The purpose of this note is to discuss these examples in depth, working out their construction in detail, and presenting novel generalisations.*

1.1. Main results. We recall the definition of an α-rough path for $\alpha \in \left]\frac{1}{3}, 1\right]$: For $n \geq 1$ we define the simplex

$$[0,T]^n_{\leq} := \{(t_1, \ldots, t_n) : 0 \leq t_1 \leq \ldots \leq t_n \leq T\}. \tag{1.1}$$

Some recurrent notation is recalled in Section 2.

Definition 1.1 (Rough path). *Let $\alpha \in \left]\frac{1}{3}, 1\right]$ and let $X : [0,T] \to \mathbb{R}^d$ be a path of class $\mathcal{C}^\alpha$. We call α-rough path over X a pair $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ such that:*

- *$\mathbb{X}^1 : [0,T]^2_{\leq} \to \mathbb{R}^d$ is simply $\mathbb{X}^1_{st} = X_t - X_s$, for all $0 \leq s \leq t \leq T$;*
- *$\mathbb{X}^2 : [0,T]^2_{\leq} \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfies, for all $0 \leq s \leq u \leq t \leq T$,*

$$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut};$$

- *the following analytic bounds hold, uniformly over $0 \leq s \leq t \leq T$:*

$$|\mathbb{X}^1_{st}| \lesssim |t-s|^\alpha, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}.$$

We recall that for $\alpha > \frac{1}{2}$ the second level $\mathbb{X}^2$ is uniquely determined and it is given explicitly by the Young integral

$$\mathbb{X}^2_{st} = \int_s^t (X_r - X_s) \otimes \mathrm{d}X_r.$$

On the other hand, for $\alpha \leq \frac{1}{2}$ the choice of $\mathbb{X}^2$ is non-unique (but any two choices differ by the increment of a 2α-Hölder function).

Given a path X of class $\mathcal{C}^\alpha$ with $\alpha \in]\frac{1}{2}, 1]$ and a function $\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ we study the following *controlled difference equation* for an unknown path $Z : [0,T] \to \mathbb{R}^k$:

$$Z_t - Z_s = \sigma(Z_s)(X_t - X_s) + o(t-s), \qquad 0 \le s \le t \le T. \tag{1.2}$$

The difference equation (1.2) is a natural generalised formulation of the *controlled differential equation*

$$\dot{Z}_t = \sigma(Z_t)\dot{X}_t, \qquad 0 \le t \le T. \tag{1.3}$$

Whenever we write $o(t-s)$, we always mean uniformly for $0 \le s \le t \le T$, i.e.

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \quad 0 \le s \le t \le T, \ t - s \le \delta \text{ implies } |o(t-s)| \le \varepsilon(t-s).$$

It can be easily proven that (1.2) is equivalent to (1.3) when X is in $\mathcal{C}^1$ and σ is continuous, however (1.2) is also meaningful when X is not differentiable.

We recall the results [Dav08] regarding local and global existence and uniqueness of solutions for the difference equation (1.2) (see also [CGZ24, Chapter 2]).

Theorem 1.2 (Well-posedness, Young case)**.** *Let $X : [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$ with $\alpha \in]\frac{1}{2}, 1]$ and let $\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$. Then we have:*

- ***local existence**: if σ is locally γ-Hölder with $\gamma > \frac{1}{\alpha} - 1$, then for every $z_0 \in \mathbb{R}^k$ there is a possibly shorter time horizon $T' = T'_{\alpha,X,\sigma}(z_0) \in]0,T]$ and a path $Z : [0,T'] \to \mathbb{R}^k$ starting from $Z_0 = z_0$ which solves* (1.2) *for $0 \le s \le t \le T'$;*
- ***global existence**: if σ is globally γ-Hölder with $\gamma > \frac{1}{\alpha} - 1$, then we can take $T'_{\alpha,X,\sigma}(z_0) = T$ for any $z_0 \in \mathbb{R}^k$;*
- ***uniqueness**: if σ is γ-Hölder with $\gamma > \frac{1}{\alpha}$ (i.e. σ is differentiable with $\nabla\sigma$ of class $\mathcal{C}^{\gamma-1}$), then for every $z_0 \in \mathbb{R}^k$ there is exactly one solution of* (1.2) *with $Z_0 = z_0$.*

When $\alpha < \frac{1}{2}$, in general (1.2) does not admit any solution. If $\alpha \in]\frac{1}{3}, \frac{1}{2}]$, we can enrich (1.2) and consider the *rough difference equation*

$$Z_t - Z_s = \sigma(Z_s)\mathbb{X}^1_{st} + \sigma_2(Z_s)\mathbb{X}^2_{st} + o(t-s), \qquad 0 \le s \le t \le T, \tag{1.4}$$

where $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ is an α-rough path over X and we define

$$\sigma_2(z) := \nabla\sigma(z)\sigma(z)\,. \tag{1.5}$$

When X is of class $\mathcal{C}^1$, we can consider the *canonical rough path*

$$\mathbb{X}^2_{st} = \int_s^t (X_u - X_s) \otimes \dot{X}_u \, du.$$

With this choice (1.4) is equivalent to (1.3), however (1.4) is meaningful also for X non differentiable. The construction of $\mathbb{X}^2$ is in general non canonical, as there are multiple choices of $\mathbb{X}^2$ for a given X. In section 4.3 we will see an example of non canonical rough path.

We now recall the results [Dav08] regarding local and global existence and uniqueness of solutions for the rough difference equation (1.4) (see also [CGZ24, Chapter 3]).

Theorem 1.3 (Well-posedness, rough case). *Let $X : [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$ with $\alpha \in]\frac{1}{3}, \frac{1}{2}]$, $\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ and let $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ be an α-rough path over X. Then we have:*

- ***local existence**: if σ is locally γ-Hölder with $\gamma > \frac{1}{\alpha} - 1$, then for every $z_0 \in \mathbb{R}^k$ there is a possibly shorter time horizon $T' = T'_{\alpha,X,\sigma}(z_0) \in]0,T]$ and a path $Z : [0,T'] \to \mathbb{R}^k$ starting from $Z_0 = z_0$ which solves (1.4) for $0 \le s \le t \le T'$;*
- ***global existence**: if σ is globally γ-Hölder with $\gamma > \frac{1}{\alpha} - 1$, then we can take $T'_{\alpha,X,\sigma}(z_0) = T$ for any $z_0 \in \mathbb{R}^k$;*
- ***uniqueness**: if σ is γ-Hölder with $\gamma > \frac{1}{\alpha}$, then for every $z_0 \in \mathbb{R}^k$ there is exactly one solution of (1.4) with $Z_0 = z_0$.*

We now discuss the possibly less known part of Davie's paper, see [Dav08, §5, examples 1-4], which shows that the assumptions of Theorems 1.2 and 1.3 are indeed sharp. We start with non-uniqueness.

Theorem 1.4 (Davie's non-uniqueness examples). *The following holds.*

- *(Young case) Let $\alpha \in]\frac{1}{2}, 1[$ and $\gamma < \frac{1}{\alpha}$. There exist a path $X : [0,T] \to \mathbb{R}^2$ of class $\mathcal{C}^\alpha$ and a non-linearity $\sigma : \mathbb{R}^2 \to \mathbb{R}^2 \otimes (\mathbb{R}^2)^*$ of class $\mathcal{C}^\gamma$ such that for any $T > 0$ the equation (1.2), with the initial condition $Z_0 = 0$, admits two different solutions on an arbitrary time interval.*
- *(rough case) Let $\alpha \in]\frac{1}{3}, \frac{1}{2}[$ and $\gamma < \frac{1}{\alpha}$. There exist an α-rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ and a non-linearity $\sigma : \mathbb{R}^2 \to \mathbb{R}^2 \otimes (\mathbb{R}^2)^*$ of class $\mathcal{C}^\gamma$ such that the equation (1.4), with the initial condition $Z_0 = 0$, admits two different solutions on an arbitrary time interval.*

Besides proving this theorem in full detail, we present in this paper two generalisations: we extend Davie's example to a rough path of *arbitrary low regularity* $\alpha \in]0,1[$ (excluding for simplicity the boundary cases $\alpha = \frac{1}{n}$ for some $n \in \mathbb{N}$), see Definition 4.4, and we show that the rough path can be taken *geometric*. Let us extend the definition (1.5) of σ_2 by setting recursively, for any $k \in \mathbb{N}$,

$$\sigma_1(z) := \sigma(z)\,, \qquad \sigma_k(z) = \nabla\sigma_{k-1}(z)\,\sigma(z)\,. \tag{1.6}$$

Theorem 1.5 (Improved non-uniqueness examples). *Let $\alpha \in]\frac{1}{n+1}, \frac{1}{n}[$ for some $n \in \mathbb{N}$ and let $\gamma < \frac{1}{\alpha}$. There exist an α-rough path $\mathbb{X} = (\mathbb{X}^1, \dots, \mathbb{X}^n)$ (see Definition 4.4) and a non-linearity $\sigma : \mathbb{R}^2 \to \mathbb{R}^2 \otimes (\mathbb{R}^2)^*$ of class $\mathcal{C}^\gamma$ such that the generalised difference equation*

$$Z_t - Z_s = \sum_{k=1}^{n} \sigma_k(Z_s)\mathbb{X}^k_{st} + o(t-s),$$

with the initial condition $Z_0 = 0$ admits two different solutions on an arbitrary time interval.

Moreover, for $n = 2$, that is $\alpha \in]\frac{1}{3}, \frac{1}{2}[$, the rough path $\mathbb{X}$ can be taken geometric, i.e. there exists a sequence X_n of paths of class $\mathcal{C}^1$ such that the associated canonical rough path $\mathbb{X}_n$ converges to $\mathbb{X}$ in the α-rough path topology.

Remark 1.6. *The fact that the rough path $\mathbb{X}$ in Theorem 1.5 can be taken geometric should work for any $n \geq 2$ and $\alpha \in]\frac{1}{n+1}, \frac{1}{n}[$ with a similar construction, however we limit ourselves for simplicity only to the "rough case" $n = 2$, i.e. $\alpha \in]\frac{1}{3}, \frac{1}{2}[$.*

We next turn to non-existence.

Theorem 1.7 (Davie's non-existence examples)**.** *The following holds.*

- *Let $\alpha \in]\frac{1}{2}, 1[$ and $\gamma = \frac{1}{\alpha} - 1$. There exist a path $X : [0, T] \to \mathbb{R}^3$ of class $\mathcal{C}^\alpha$ and a non-linearity $\sigma : \mathbb{R}^3 \to \mathbb{R}^3 \otimes (\mathbb{R}^3)^*$ such that the difference equation* (1.2) *does not admit any solution Z such that $Z_0 = 0$.*
- *Let $\alpha \in]\frac{1}{3}, \frac{1}{2}[$ and $\gamma = \frac{1}{\alpha} - 1$. There exist a path $X : [0, T] \to \mathbb{R}^3$ of class $\mathcal{C}^\alpha$ and a non-linearity $\sigma : \mathbb{R}^4 \to \mathbb{R}^4 \otimes (\mathbb{R}^4)^*$ such that, for any α-rough path $\mathbb{X}$ over X, the difference equation* (1.4) *does not admit any solution Z such that $Z_0 = 0$.*

We will prove Theorem 1.7 in detail and present a generalisation to rough paths of arbitrary low regularity $\alpha \in]0, 1[$.

Theorem 1.8 (Improved non-existence examples)**.** *Let $\alpha \in]\frac{1}{n+1}, \frac{1}{n}[$ for some $n \in \mathbb{N}$ and let $\gamma = \frac{1}{\alpha} - 1$. There exist a path $X : [0, T] \to \mathbb{R}^{n+2}$ of class $\mathcal{C}^\alpha$ and a function $\sigma : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2} \otimes (\mathbb{R}^{n+2})^*$ of class $\mathcal{C}^\gamma$ such that, for any α-rough path $\mathbb{X} = (\mathbb{X}^1, \ldots, \mathbb{X}^n)$ (see Definition 4.4) the generalised difference equation*

$$Z_t - Z_s = \sum_{k=1}^{n} \sigma_k(Z_s)\mathbb{X}_{st}^k + o(t - s),$$

does not admit any solution Z such that $Z_0 = 0$.

In conclusion, we provide in this note a detailed proof of Theorems 1.4-1.8. After introducing some notation in Section 2, in Sections 3-4 we give examples of difference equations that admit multiple solutions with σ in $\mathcal{C}^\gamma$ and $\gamma < \frac{1}{\alpha}$; then in Sections 5-6 we show that existence can fail for σ in $\mathcal{C}^\gamma$ with $\gamma = \frac{1}{\alpha} - 1$.

2. Preliminaries and notation

Given a time horizon $T > 0$ and two dimensions $k, d \in \mathbb{N}$, we use "path" as a synonym of "function defined on $[0, T]$" with values in $\mathbb{R}^d$. We denote by $|\cdot|$ the Euclidean norm. Linear maps from $\mathbb{R}^d$ to $\mathbb{R}^k$, identified by $k \times d$ real matrices, are denoted by $\mathbb{R}^k \otimes (\mathbb{R}^d)^*$ and equipped with the Hilbert-Schmidt norm $|\cdot|$.

Recall the definition (1.1) of the simplex $[0,T]^n_\leq$. We write $C_n = C([0,T]^n_\leq, \mathbb{R}^k)$ as a shorthand for the space of continuous functions from $[0,T]^n_\leq$ to $\mathbb{R}^k$:

$$C_n := C([0,T]^n_\leq, \mathbb{R}^k) = \left\{F : [0,T]^n_\leq \to \mathbb{R}^k : F \text{ is continuous}\right\}.$$

We will work with continuous functions of one (f_s), two (F_{st}) or three (G_{sut}) ordered variables in $[0,T]$, hence we focus on the spaces C_1, C_2, C_3. In particular

- On the spaces C_2 and C_3 we introduce a norm which controls the behaviour close to the diagonal: given $\eta \in]0,\infty[$, we define for $F \in C_2$ and $G \in C_3$

$$\|F\|_\eta := \sup_{0\leq s<t\leq T} \frac{|F_{st}|}{(t-s)^\eta}, \qquad \|G\|_\eta := \sup_{\substack{0\leq s\leq u\leq t\leq T\\ s<t}} \frac{|G_{sut}|}{(t-s)^\eta}, \tag{2.1}$$

  and we denote by C_2^η and C_3^η the corresponding function spaces:

$$C_2^\eta := \{F \in C_2 : \|F\|_\eta < \infty\}, \qquad C_3^\eta := \{G \in C_3 : \|G\|_\eta < \infty\}.$$

- On the space C_1 of continuous functions $f : [0,T] \to \mathbb{R}^k$ we consider the usual Hölder structure. We first introduce the *increment* δf by

$$(\delta f)_{st} := f_t - f_s, \quad 0 \leq s \leq t \leq T,$$

  and note that δf is in C_2 for any $f \in C_1$. Then, for $\alpha \in]0,1]$, we define the classical space $\mathcal{C}^\alpha = \mathcal{C}^\alpha([0,T], \mathbb{R}^k)$ of α-Hölder functions

$$\mathcal{C}^\alpha := \left\{ f : [0,T] \to \mathbb{R}^k : \|\delta f\|_\alpha = \sup_{0\leq s<t\leq T} \frac{|f_t - f_s|}{(t-s)^\alpha} < \infty \right\}$$

  (for $\alpha = 1$ it is the space of Lipschitz functions). Observe that $f \mapsto \|\delta f\|_\alpha$ is a semi-norm on $\mathcal{C}^\alpha$. The standard norm on $\mathcal{C}^\alpha$ is

$$\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha,$$

  where we define the standard sup norm

$$\|f\|_\infty := \sup_{t\in[0,T]} |f_t|.$$

Definition 2.1. *Let $\gamma > 0$. We say that a function $F : \mathbb{R}^k \to \mathbb{R}^N$ is (globally) γ-Hölder, or (globally) of class $\mathcal{C}^\gamma$, if*

- *for $\gamma \in]0,1]$, we have*

$$[F]_{\mathcal{C}^\gamma} := \sup_{x,y\in\mathbb{R}^k, x\neq y} \frac{|F(x) - F(y)|}{|x-y|^\gamma} < +\infty.$$

- *for $\gamma \in]n, n+1]$ and $n = \{1, 2, \dots\}$, F is n times continuously differentiable and*

$$[D^{(n)}F]_{\mathcal{C}^\gamma} := \sup_{x,y\in\mathbb{R}^k, x\neq y} \frac{|D^{(n)}F(x) - D^{(n)}F(y)|}{|x-y|^{\gamma-n}} < +\infty$$

  where $D^{(n)}$ is the n-fold differential of F.

Given a function $\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ of class $\mathcal{C}^2$, that we represent by $\sigma^i_j(z)$ with $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, d\}$, we denote by $\nabla\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^k)^*$ its gradient, represented for $i, a \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, d\}$ by

$$(\nabla\sigma(z))^i_{ja} = \frac{\partial \sigma^i_j}{\partial z_a}(z).$$

Given a two-variable function $R = (R_{st})_{s\le t} \in C_2$, we define the three-variable function $\delta R = (\delta R_{sut})_{u\le s\le t} \in C_3$ by

$$\delta R_{sut} := R_{st} - R_{su} - R_{ut}\,.$$

The next *Sewing Bound* will be used in a sequel (recall the norm $\|\cdot\|_\eta$ from (2.1)).

Theorem 2.2 (Sewing Bound). *Given any $R \in C_2$ with $R_{st} = o(t-s)$, the following estimate holds for any $\eta \in]1, \infty[$:*

$$\|R\|_\eta \le K_\eta \|\delta R\|_\eta\,. \tag{2.2}$$

where $K_\eta := (1 - 2^{1-\eta})^{-1}$.

The proof follows as a corollary of the celebrated Sewing Lemma (but it can also be obtained in a more elementary way, see [CGZ24, Theorem 1.9]). Indeed, assume that $\|\delta R\|_\eta < \infty$ for some $\eta > 1$ (otherwise there is nothing to prove). Then the Sewing Lemma ensures the existence of a one-variable function $f \in C_1$ such that $\|R - \delta f\|_\eta \le K_\eta \|\delta R\|_\eta$, hence $R_{st} - (f_t - f_s) = O((t-s)^\eta) = o(t-s)$. Since $R_{st} = o(t-s)$ by assumption, it follows that $f_t - f_s = o(t-s)$, which implies $\delta f \equiv 0$ (i.e. f must be constant). Then $R - \delta f = R$ which yields (2.2).

3. Preparation for non-uniqueness

3.1. Whitney's Extension Theorem. In the following we will introduce functions defined on closed subsets of $\mathbb{R}^n$ and we will need to extend them to the whole space. In particular, we will want the extensions to be γ-Hölder functions, with γ possibly greater than 1 (recall Definition 2.1). To do so we will use the version of Whitney's Extension Theorem in Theorem 4 of section VI.2 of [Ste70], which we report in Theorem 3.2. Before stating this Theorem, we need to define the space $\mathcal{C}^\gamma(F)$ for $F \subset \mathbb{R}^n$ closed, that is the space of γ-Hölder functions on F.

Definition 3.1. *Given $k < \gamma < k+1$ with $k \in \mathbb{N}$, F a closed subset of $\mathbb{R}^n$ and a function $f : F \to \mathbb{R}$, we say that f is in $\mathcal{C}^\gamma(F)$ if there exists $M > 0$ and functions $\{f^{(j)} : F \to \mathbb{R}\}_{0\le |j| \le k}$ such that $f^{(0)} = f$ and if*

$$f^{(j)}(x) = \sum_{|j+l|\le k} f^{(j+l)}(y) \frac{(x-y)^l}{l!} + R_j(x,y)$$

then $f^{(j)}(x) \le M$ and $|R_j(x,y)| \le M|x-y|^{\gamma - |j|}$ for every $x, y \in F$ and $|j| \le k$.

Theorem 3.2. *Let F be a closed subset of $\mathbb{R}^n$ and $f : F \to \mathbb{R}$ a function in $\mathcal{C}^\gamma(F)$ with $k < \gamma < k+1$ for some $k \in \mathbb{N}$. Then there exists $h : \mathbb{R}^n \to \mathbb{R}$ such that*

(1) *$h(x) = f(x)$ for every $x \in F$*
(2) *h is in $\mathcal{C}^\gamma$*
(3) *$\frac{\partial^{(j)} h}{\partial x^{(j)}}(x) = f^{(j)}(x)$ for every $x \in F$.*

3.2. A key oscillatory integral. All our proofs concerning the non-uniqueness of solutions rely on an elementary (yet non-trivial) result involving a one-dimensional integral, which we now present. Let us fix $\beta, \gamma, \eta > 1$ such that

$$\gamma < \frac{\eta}{\beta} < \frac{\eta+1}{\beta} < \gamma + 1 . \tag{3.1}$$

Define $X_0^1 := 0, X_0^2 := 0$ and for $t > 0$:

$$X_t^1 := t^\beta \cos t^{-\eta}, \quad X_t^2 := t^\beta (2 + \sin t^{-\eta}), \qquad X_t := \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} . \tag{3.2}$$

The main result of this section is the following.

Theorem 3.3. *Let X be as in* (3.2) *for some $\beta, \gamma, \eta > 1$ satisfying* (3.1)*.*

(1) *The function $X : [0,1] \to \mathbb{R}^2$ is $\frac{\beta}{\eta+1}$-Hölder.*
(2) *The function $[0,1] \ni t \mapsto (X_t^2)^\gamma \in \mathbb{R}$ is $\frac{\beta\gamma}{\eta+1}$-Hölder.*
(3) *The function*

$$I : \mathbb{R}_+ \to \mathbb{R}, \qquad I_t := \int_0^t (X_u^2)^\gamma \dot{X}_u^1 \, \mathrm{d}u, \tag{3.3}$$

is well-defined.

(4) *There exist $C, c, T > 0$ such that*

$$ct^{\beta(\gamma+1)-\eta} \le I_t \le Ct^{\beta(\gamma+1)-\eta}, \qquad \forall t \in [0,T].$$

Moreover $I_t \ge X_t^2$ for every $t \in [0,T]$.

(5) *We have uniformly for $0 \le s \le t \le T$*

$$\delta I_{st} - (X_s^2)^\gamma \delta X_{st}^1 = o(t-s).$$

Before proving Theorem 3.3 let us make some important remarks:

- Observe that, for every $\varepsilon > 0$, X is $\mathcal{C}^\infty$ on $]\varepsilon, +\infty[$. Near the origin the fast oscillations of the sine and cosine give rise to the function's irregularity.
- In general, composing a Hölder function with a more regular one, does not improve its regularity. However, $[0,T] \ni t \mapsto (X_t^2)^\gamma \in \mathbb{R}$ is indeed more regular than X^2, since by the previous theorem the former is $\frac{\beta\gamma}{\eta+1}$-Hölder while the latter is $\frac{\beta}{\eta+1}$-Hölder and $\gamma > 1$. As we will see, this fact becomes very important in the proof of point 5 of Theorem 3.3.

- It might be surprising that $I_t \geq X_t^2$, as in general the integral of a positive continuous function over $[0,t]$ (even if raised to the γ), for t sufficiently small, is not greater than the function itself valued at point t, see 7.2 for more details. This does not apply to our case because X^1 is not a function of bounded variation and we cannot define I_t as the Lebesgue integral $\int_0^t (X_u^2)^\gamma \, dX_u^1$. However, we could define I_t as a Young integral, see 7.1 for further details.
- For our non uniqueness examples to work, it will be fundamental that $I_t \geq X_t^2$. If the sine and cosine in X^2 and X^1 did not resonate, or if $\gamma > \frac{\eta+1}{\beta}$, we could not prove that $I_t \geq X_t^2$ and the non-uniqueness phenomenon would not occur in our setting. We refer the interested reader to 7.3 to see what could happen without resonance.

Proof of Theorem 3.3. We prove the five points of the Theorem one by one.

Proof of 1. Let us study the regularity of X defined above. First we prove that X^1 is of class $\mathcal{C}^{\frac{\beta}{\eta+1}}$ on $[0,1]$. We want to show that there exists $C > 0$ such that

$$|X_{t+h}^1 - X_t^1| \leq C h^{\frac{\beta}{\eta+1}}$$

for every $t, h \geq 0$ such that $t + h \leq 1$. Observe that

$$\dot{X}_t^1 = \beta t^{\beta-1} \cos t^{-\eta} + \eta t^{\beta-\eta-1} \sin t^{-\eta}.$$

We consider two cases:

(1) If $h \geq t^{\eta+1}$, then

$$\begin{aligned}
|X_{t+h}^1 - X_t^1| &\leq |X_{t+h}^1| + |X_t^1| \leq (t+h)^\beta + t^\beta \\
&\leq 2(t+h)^\beta \leq 2(h^{\frac{1}{\eta+1}} + h)^\beta \\
&\leq 2(2h^{\frac{1}{\eta+1}})^\beta \leq 2\, 2^\beta h^{\frac{\beta}{\eta+1}}.
\end{aligned}$$

(2) If $h < t^{\eta+1}$, then $\frac{1}{t} < h^{-\frac{1}{\eta+1}}$ and

$$\begin{aligned}
|X_{t+h}^1 - X_t^1| &\leq \sup_{s\in[t,t+h]} |\dot{X}_s^1|\, h \leq 2\beta\eta t^{\beta-\eta-1} h \\
&= 2\beta\eta h \left(\frac{1}{t}\right)^{\eta+1-\beta} \leq 2\beta\eta h^{\frac{\beta}{\eta+1}}.
\end{aligned}$$

This proves that X^1 is in $\mathcal{C}^{\frac{\beta}{\eta+1}}$; it can be proven analogously that also X^2 is in $\mathcal{C}^{\frac{\beta}{\eta+1}}$.

Proof of 2. We now study the regularity of $t \mapsto (X_t^2)^\gamma$, in particular we are going to show that this function is of class $\mathcal{C}^{\frac{\beta\gamma}{\eta+1}}$. The proof is very similar to the one of point 1 of Theorem 3.3. We want to show that there exists $C > 0$ such that

$$|(X_{t+h}^2)^\gamma - (X_t^2)^\gamma| \leq C h^{\frac{\beta\gamma}{\eta+1}}$$

for every $t, h \geq 0$ such that $t + h \leq 1$. Observe that for $t > 0$

$$\frac{d}{dt}(X_t^2)^\gamma = \beta\gamma t^{\beta\gamma-1}(2+\sin t^{-\eta})^\gamma - \eta\gamma t^{\beta\gamma-\eta-1}(2+\sin t^{-\eta})^{\gamma-1}\cos t^{-\eta},$$

so, there exists $C > 0$ such that

$$|(X_t^2)^\gamma| \leq Ct^{\beta\gamma}, \qquad \left|\frac{d}{dt}(X_t^2)^\gamma\right| \leq Ct^{\beta\gamma-\eta-1}$$

for every $t \in [0,1]$. We consider two cases:

(1) If $h \geq t^{\eta+1}$, then

$$\begin{aligned}|(X_{t+h}^2)^\gamma - (X_t^2)^\gamma| &\leq |(X_{t+h}^2)^\gamma| + |(X_t^2)^\gamma| \\ &\leq C(t+h)^{\beta\gamma} + Ct^{\beta\gamma} \\ &\leq 2C(t+h)^{\beta\gamma} \leq 2C(h^{\frac{1}{\eta+1}} + h)^{\beta\gamma} \\ &\leq 2C(2h^{\frac{1}{\eta+1}})^{\beta\gamma} \leq 2^{\beta\gamma+1}Ch^{\frac{\beta\gamma}{\eta+1}}.\end{aligned}$$

(2) If $h < t^{\eta+1}$, then $\frac{1}{t} < h^{-\frac{1}{\eta+1}}$ and

$$\begin{aligned}|(X_{t+h}^2)^\gamma - (X_t^2)^\gamma| &\leq \sup_{s\in[t,t+h]}\left|\frac{d}{ds}(X_s^2)^\gamma\right| h \leq Ct^{\beta\gamma-\eta-1}h \\ &= Ch\left(\frac{1}{t}\right)^{\eta+1-\beta\gamma} \leq Ch^{\frac{\beta\gamma}{\eta+1}}.\end{aligned}$$

This proves that $(X^2)^\gamma$ is in $\mathcal{C}^{\frac{\beta\gamma}{\eta+1}}$.

Proof of 3. Recall that by (3.3)

$$I_t = \int_0^t (X_u^2)^\gamma \dot{X}_u^1 \,\mathrm{d}u.$$

A priori it is not obvious that $]0,1] \ni u \mapsto (X_u^2)^\gamma \dot{X}_u^1$ is integrable, nor that $I \neq 0$. We will prove in the following that $I \neq 0$, now we focus on the integrability. Observe that for $u \in]0,1]$

$$\begin{aligned}&(X_u^2)^\gamma \dot{X}_u^1 = u^{\beta\gamma}(2+\sin u^{-\eta})^\gamma(\beta u^{\beta-1}\cos u^{-\eta} + \eta u^{\beta-\eta-1}\sin u^{-\eta}) \\ &= \beta u^{\beta(\gamma+1)-1}(2+\sin u^{-\eta})^\gamma \cos u^{-\eta} + \eta u^{\beta(\gamma+1)-\eta-1}(2+\sin u^{-\eta})^\gamma \sin u^{-\eta}\end{aligned} \tag{3.4}$$

and both functions are integrable over $]0,1]$ since $\beta(\gamma+1)-1 \geq \beta(\gamma+1)-\eta-1 > -1$ by (3.1), because $\beta(\gamma+1) > \eta > 0$.

Proof of 4. We now prove that there exists $C > 0$ such that

$$I_t \leq Ct^{\beta(\gamma+1)-\eta} \tag{3.5}$$

for all $t \in [0,1]$. Recalling (3.4), we find

$$\begin{aligned}|I_t| &\leq 3^\gamma \int_0^t \left|\beta u^{\beta(\gamma+1)-1}\right| \mathrm{d}u + 3^\gamma \int_0^t \left|\eta u^{\beta(\gamma+1)-\eta-1}\right| \mathrm{d}u \\ &\leq \beta 3^\gamma t^{\beta(\gamma+1)} + \eta 3^\gamma t^{\beta(\gamma+1)-\eta} \\ &\leq \beta 3^\gamma t^{\beta(\gamma+1)-\eta} + \eta 3^\gamma t^{\beta(\gamma+1)-\eta}.\end{aligned}$$

Defining $C = 3^\gamma(\beta+\eta)$ we conclude the proof of (3.5).

In order to conclude the proof of point 4 of Theorem 3.3, we are going to prove that there exists $c > 0$ and $T \in]0,1[$ such that

$$I_t \geq c t^{\beta(\gamma+1)-\eta} \tag{3.6}$$

for all $t \in [0,T]$. The strategy is the following: by an integration by parts we obtain the integral of $\cos^2 s^{-\eta}$ multiplied for a power function with exponent $\beta(\gamma+1)-\eta-1$. All the other terms will be negligible respect to this integral for t small enough. The fact that $(X_t^2)^\gamma$ and $\dot{X}_t^1$ resonate is fundamental to obtain a significant lower bound on I_t. Observe that

$$\begin{aligned}I_t &= \lim_{\varepsilon\to 0} \int_\varepsilon^t (X_s^2)^\gamma \dot{X}_s^1 \,\mathrm{d}s \\ &= \lim_{\varepsilon\to 0} \left((X_t^2)^\gamma X_t^1 - (X_\varepsilon^2)^\gamma X_\varepsilon^1 - \int_\varepsilon^t \frac{d\,(X_s^2)^\gamma}{ds} X_s^1 \,\mathrm{d}s \right) \\ &= t^{\beta(\gamma+1)} \cos t^{-\eta} (2+\sin t^{-\eta})^\gamma - \int_0^t \beta\gamma s^{\beta(\gamma+1)-1} (2+\sin s^{-\eta})^\gamma \cos s^{-\eta} \,\mathrm{d}s + \\ &\qquad + \int_0^t \eta\gamma s^{\beta(\gamma+1)-\eta-1} (2+\sin s^{-\eta})^{\gamma-1} \cos^2 s^{-\eta} \,\mathrm{d}s \\ &\geq -2\, 3^\gamma t^{\beta(\gamma+1)} + \eta\gamma \int_0^t s^{\beta(\gamma+1)-\eta-1} \cos^2 s^{-\eta} \,\mathrm{d}s\end{aligned}$$

Intuitively $\cos^2 s^{-\eta}$ is a quickly oscillating function with mean $\frac{1}{2}$, hence the previous integral is morally equal to $\frac{\eta\gamma}{2}\int_0^t s^{\beta(\gamma+1)-\eta-1}\,\mathrm{d}s$. We now prove that this intuition is precise; note that

$$\begin{aligned}\eta\gamma \int_0^t s^{\beta(\gamma+1)-\eta-1} \cos^2 s^{-\eta} \,\mathrm{d}s &= \eta\gamma \int_0^t s^{\beta(\gamma+1)-\eta-1} \left(1 - \sin^2 s^{-\eta}\right) \mathrm{d}s \\ &= \eta\gamma \int_0^t s^{\beta(\gamma+1)-\eta-1} \left(1 - \frac{1}{2} + \frac{1}{2}\cos 2s^{-\eta}\right) \mathrm{d}s \\ &= \frac{\eta\gamma}{2} \int_0^t s^{\beta(\gamma+1)-\eta-1} \,\mathrm{d}s + \frac{\eta\gamma}{2} \int_0^t s^{\beta(\gamma+1)-\eta-1} \cos 2s^{-\eta} \,\mathrm{d}s.\end{aligned}$$

Now,

$$\int_0^t \eta s^{\beta(\gamma+1)-\eta-1}\cos 2s^{-\eta}\,\mathrm{d}s = -\frac{t^{\beta(\gamma+1)}}{2}\sin 2s^{-\eta} + \frac{\beta(\gamma+1)}{2}\int_0^t s^{\beta(\gamma+1)-1}\sin 2s^{-\eta}\,\mathrm{d}s$$
$$\geq -t^{\beta(\gamma+1)}.$$

This implies that

$$I_t \geq -(2\,3^\gamma + \frac{\gamma}{2})t^{\beta(\gamma+1)} + \frac{\eta\gamma}{2(\beta(\gamma+1)-\eta)}t^{\beta(\gamma+1)-\eta}$$

and if $t < (\frac{\eta\gamma}{4(\beta(\gamma+1)-\eta)(2\,3^\gamma+\frac{\gamma}{2})})^{\frac{1}{\eta}}$, this yields that

$$I_t \geq \frac{\eta\gamma}{4(\beta(\gamma+1)-\eta)}t^{\beta(\gamma+1)-\eta}.$$

This proves (3.6) choosing $T \leq \min\{\pi^{-\frac{1}{\eta}}, (\frac{\eta\gamma}{4(\beta(\gamma+1)-\eta)(2\,3^\gamma+\frac{\gamma}{2})})^{\frac{1}{\eta}}\}$ and $c = \frac{\eta\gamma}{4(\beta(\gamma+1)-\eta)}$.

Finally, the fact that $I_t \geq X_t^2$ follows from (3.6) by possibly choosing a smaller T since $\beta(\gamma+1)-\eta = \beta + (\beta\gamma - \eta) < \beta$ by the first inequality in (3.1) and since $X_t^2 \leq 3t^\beta$ by (3.2).

Proof of 5. Finally we show that

$$\delta I_{st} - (X_s^2)^\gamma \delta X_{st}^1 = o(t-s) \tag{3.7}$$

uniformly. Fix $\varepsilon \in]0,T]$ and observe that, for $s,t \in [\varepsilon, T]$,

$$\delta I_{st} - \dot{I}_s(t-s) = o(t-s).$$

Moreover,

$$\dot{I}_s(t-s) = (X_s^2)^\gamma \dot{X}_s^1(t-s)$$

and

$$\dot{X}_s^1(t-s) = \delta X_{st}^1 + o(t-s).$$

Putting everything together we find that

$$R_{st}^\varepsilon := \big(\delta I_{st} - (X_s^2)^\gamma \delta X_{st}^1\big)\,\mathbb{1}_{(\varepsilon\leq s\leq t\leq T)} = o(t-s)$$

uniformly for $\varepsilon \leq s \leq t \leq T$. Since $\frac{\beta(\gamma+1)}{\eta+1} > 1$ by the last inequality in (3.1), we can apply the Sewing Bound, see Theorem 2.2, obtaining

$$\|R^\varepsilon\|_{\frac{\beta(\gamma+1)}{\eta+1}} \leq K_{\frac{\beta(\gamma+1)}{\eta+1}}\|\delta(X^2)^\gamma\|_{\frac{\beta\gamma}{\eta+1}}\|\delta X^1\|_{\frac{\beta}{\eta+1}} < +\infty,$$

where $K_{\frac{\beta(\gamma+1)}{\eta+1}} = (1-2^{1-\frac{\beta(\gamma+1)}{\eta+1}})^{-1}$. Observing that the right hand side does not depend on ε and by taking the limit for $\varepsilon \to 0$ we find that

$$\|\delta I_{st} - (X_s^2)^\gamma \delta X_{st}^1\|_{\frac{\beta(\gamma+1)}{\eta+1}} \leq K_{\frac{\beta(\gamma+1)}{\eta+1}}\|\delta(X^2)^\gamma\|_{\frac{\beta\gamma}{\eta+1}}\|\delta X^1\|_{\frac{\beta}{\eta+1}} < +\infty$$

for every $0 \leq s < t \leq T$. Since $\frac{\beta(\gamma+1)}{\eta+1} > 1$ by (3.1), we have proved (3.7). $\square$

4. Non uniqueness of solutions

4.1. Introduction. The aim of this section is to present examples of non-uniqueness of solutions à la Davie for rough differential equations driven by a path X in $\mathcal{C}^\alpha$ with $\alpha \in]\frac{1}{n+1}, \frac{1}{n}[$ for $n \in \mathbb{N}$. In each case we will define an appropriate function σ of class $\mathcal{C}^\gamma$ with $\gamma < \frac{1}{\alpha}$.

We have three selected regimes for $\alpha \in]0,1[\setminus\{\frac{1}{n} : n \in \mathbb{N}\}$:

- for $\alpha \in]\frac{1}{2}, 1[$, that we call the *Young case*, we assume $1 < \gamma < \frac{1}{\alpha} < 2$
- for $\alpha \in]\frac{1}{3}, \frac{1}{2}[$, that we call the *Rough case*, we assume $2 < \gamma < \frac{1}{\alpha} < 3$
- for $\alpha < \frac{1}{3}$ and $\frac{1}{\alpha} \notin \mathbb{N}$, that we call the *general case*, we assume $\lfloor \frac{1}{\alpha} \rfloor < \gamma < \frac{1}{\alpha} < \lceil \frac{1}{\alpha} \rceil$.

In every regime we can find β and η large positive numbers such that

$$\gamma < \frac{\eta}{\beta} < \frac{\eta+1}{\beta} < \frac{1}{\alpha} \tag{4.1}$$

4.2. Non-uniqueness in the Young case. Suppose $\alpha \in]\frac{1}{2}, 1[$ and $\gamma \in]1, \frac{1}{\alpha}[$. We want to construct, for some $T > 0$, functions

- $X : [0,T] \to \mathbb{R}^2$ of class $\mathcal{C}^\alpha$,
- $\sigma : \mathbb{R}^2 \to L(\mathbb{R}^2; \mathbb{R}^2)$ of class $\mathcal{C}^\gamma$,
- $Z, \bar{Z} : [0,T] \to \mathbb{R}^2$

such that

$$Z_0 = \bar{Z}_0, \qquad \delta Z_{st} - \sigma(Z_s)\delta X_{st} = o(t-s), \qquad \delta \bar{Z}_{st} - \sigma(\bar{Z}_s)\delta X_{st} = o(t-s),$$

and Z is not identically equal to $\bar{Z}$.

Since $1 < \gamma < \frac{1}{\alpha} < 2$ we can find β and η large positive numbers that satisfy (4.1). Define X as in (3.2) and I as in (3.3). The main result of this section is the following.

Theorem 4.1. *There exists $f : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^\gamma$ such that*

$$f(x,y) = \begin{cases} y^\gamma & \text{if } |x| \geq y \geq 0 \text{ and } y \leq 3 \\ 0 & \text{if } x = 0. \end{cases} \tag{4.2}$$

Let us see why Theorem 4.1 implies the non-uniqueness phenomenon. We define

$$Z_t = \begin{pmatrix} 0 \\ X_t^2 \end{pmatrix} \text{ and } \bar{Z}_t = \begin{pmatrix} I_t \\ X_t^2 \end{pmatrix}$$

and $\sigma : \mathbb{R}^2 \to L(\mathbb{R}^2, \mathbb{R}^2)$ as

$$\sigma(x,y) = \begin{pmatrix} f(x,y) & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the equation

$$\delta Z_{st} = \sigma(Z_s)\delta X_{st} + o(t-s), \tag{4.3}$$

which is equivalent, if $Z_t = \begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix}$, to

$$\begin{cases} \delta Z^1_{st} & = \sigma^{1,1}(Z^1_s, Z^2_s)\delta X^1_{st} + o(t-s), \\ \delta Z^2_{st} & = \delta X^2_{st} + o(t-s). \end{cases}$$

Then $Z, \bar{Z}$ are two (different) solutions of (4.3) with the initial condition $Z_0 = \bar{Z}_0 = 0$. Recall that X is in $\mathcal{C}^{\frac{\beta}{\eta+1}}$; since $\alpha < \frac{\beta}{\eta+1}$, X is also an α-Hölder function. Moreover σ is in $\mathcal{C}^\gamma$, i.e. it is continuous and its gradient is $(\gamma-1)$-Hölder. Recall that $\gamma < \frac{1}{\alpha}$. We are in the situation in which the regularity of the driving path X and the regularity of σ are such that we cannot guarantee the uniqueness of the solution of (4.3). It is easy to show that Z is a solution; to prove that also $\bar{Z}$ is a solution we need to prove that there exists $T \in]0,1[$ and $C > 0$ such that

$$I_t \geq X^2_t \tag{4.4}$$

for all $t \in [0,T]$ (so that $\sigma^{1,1}(I_t, X^2_t) = (X^2_t)^\gamma$) and

$$\delta I_{st} - (X^2_s)^\gamma \delta X^1_{st} = o(t-s) \tag{4.5}$$

uniformly for $0 \leq s < t \leq T$. Observe that (4.4) implies that $I \neq 0$ and $Z \neq \bar{Z}$. Recall that both (4.5) and (4.4) have been proven in Theorem 3.3, hence $\bar{Z}$ is a solution of (4.3).

Proof of Theorem 4.1. Finally we show that $f : \mathbb{R}^2 \to \mathbb{R}$ as in Theorem 4.1 exists. We will use the version of Whitney's Extension Theorem in Theorem 4 of section VI.2 of [Ste70], which we have reported in Theorem 3.2 for convenience. To apply this result we have to show that f is in $\mathcal{C}^\gamma(F)$ where $F := \{(x,y) \in \mathbb{R}^2 : |x| \geq y \geq 0, y \leq 3\} \cup \{(x,y) \in \mathbb{R}^2 : x = 0\}$ and $\mathcal{C}^\gamma(F)$ is as in Definition 3.1. We define

$$f^{(j)}(x,y) = \begin{cases} y^\gamma \mathbb{1}_{|x| \geq y \geq 0} & \text{if } j = (0,0) \\ \gamma y^{\gamma-1} \mathbb{1}_{|x|\geq y \geq 0} & \text{if } j = (0,1) \\ 0 & \text{otherwise} \end{cases}$$

for every $(x,y) \in F$. Recall that we need to prove that there exists $M > 0$ such that that if

$$f^{(j)}(z) = \sum_{|j+l| \leq 1} f^{(j+l)}(z') \frac{(z-z')^l}{l!} + R_j(z,z')$$

then

$$f^{(j)}(z) \leq M \text{ and } |R_j(z,z')| \leq M|z-z'|^{\gamma-|j|} \tag{4.6}$$

for every $z, z' \in F$ and $|j| \leq 1$. Since the function $(x,y) \mapsto y^\gamma$ is in $\mathcal{C}^\gamma(\mathbb{R}^2)$, it is clear that we only need to check that (4.6) holds for $z \in F_1 := \{(x,y) \in F : |x| \geq$

$y \geq 0, y \leq 3\}$ and $z' \in F_2 := \{(x,y) \in F : x = 0\}$, or vice versa. In particular, for $j = (0,1)$,

$$\begin{aligned}
\sup_{z\in F_1, z'\in F_2} \frac{|f^{(0)}(z) - f^{(0)}(z') - f^{(j)}(z')(z-z')^j|}{|z-z'|^\gamma} &= \sup_{z=(x,y)\in F_1, z'=(0,y')\in F_2} \frac{|y^\gamma|}{|z-z'|^\gamma} \\
&= \sup_{z=(x,y)\in F_1, z'=(0,y)} \frac{|y^\gamma|}{|x|^\gamma} \\
&= \sup_{z=(x,x)\in F_1, z'=(0,y)} \frac{|x^\gamma|}{|x|^\gamma} \\
&= 1.
\end{aligned}$$

Analogously

$$\begin{aligned}
&\sup_{z\in F_2, z'\in F_1} \frac{|f^{(0)}(z) - f^{(0)}(z') - f^{(j)}(z')(z-z')^j|}{|z-z'|^\gamma} \\
&= \sup_{z=(0,y)\in F_2, z'=(x',y')\in F_1} \frac{|y'^\gamma + \gamma y'^{\gamma-1}(y-y')|}{|z-z'|^\gamma} \\
&\leq \sup_{z=(0,y)\in F_2, z'=(x',y')\in F_1} \frac{|y'|^\gamma + |\gamma y'^{\gamma-1}|\,|z-z'|}{|z-z'|^\gamma} \\
&= 1 + \sup_{z=(0,y')\in F_2, z'=(y',y')} \frac{|\gamma y'^{\gamma-1}|}{|y'|^{\gamma-1}} \\
&= 1 + \gamma.
\end{aligned}$$

Finally in a similar way we can prove that

$$\sup_{z\in F_1, z'\in F_2} \frac{|f^{(j)}(z) - f^{(j)}(z')(z-z')^j|}{|z-z'|^{\gamma-1}} \leq \gamma$$

and

$$\sup_{z\in F_2, z'\in F_1} \frac{|f^{(j)}(z) - f^{(j)}(z')(z-z')^j|}{|z-z'|^{\gamma-1}} \leq \gamma.$$

To conclude the proof observe that $f^{(0)}(x,y) \leq 3^\gamma$ and, for $j = (0,1)$, $f^{(j)}(x,y) \leq \gamma 3^{\gamma-1}$.

$\square$

4.3. Non-uniqueness in the Rough case. Suppose $2 < \gamma < \frac{1}{\alpha} < 3$. Let β and η be large positive numbers such that they satisfy (4.1). Let X be as in (3.2). Moreover let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2 \times (\mathbb{R}^2)^*$ and define $\sigma_2 : \mathbb{R}^2 \to \mathbb{R}^2 \times (\mathbb{R}^2)^* \times (\mathbb{R}^2)^*$ as

$$\sigma_2(z) = \nabla\sigma(z)\sigma(z).$$

Consider the equation

$$\delta Z_{st} = \sigma(Z_s)\mathbb{X}^1_{st} + \sigma_2(Z_s)\mathbb{X}^2_{st} + o(t-s) \tag{4.7}$$

where $\mathbb{X}^1 : [0,T]^2_\leq \to \mathbb{R}^2$ is

$$\mathbb{X}^1_{st} = X_t - X_s$$

and $\mathbb{X}^2 : [0,T]^2_\leq \to \mathbb{R}^2 \otimes \mathbb{R}^2$ satisfies

$$\delta\mathbb{X}^2_{sut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut} \tag{4.8}$$

$$|\mathbb{X}^2_{st}| \leq C|t-s|^{2\alpha}. \tag{4.9}$$

We will show that for a suitable choice of $\mathbb{X}^2$ the problem (4.7) with initial condition $Z_0 = 0$ admits two different solutions. The main result of this section is the following

Theorem 4.2. *Let X be as in* (3.2) *and define $\mathbb{X}^2$ as*

$$\mathbb{X}^2_{st} := -X_s \otimes \delta X_{st}. \tag{4.10}$$

(1) The function $\mathbb{X}^2 : [0,T]^2 \to \mathbb{R}^2 \otimes \mathbb{R}^2$ satisfies (4.8) *and* (4.9).
(2) There exists $f : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^\gamma$ such that

$$f(x,y) = \begin{cases} y^\gamma & \text{if } |x| \geq y \geq 0 \text{ and } y \leq 3 \\ 0 & \text{if } x = 0. \end{cases}$$

The definition (4.10) of $\mathbb{X}^2$ is rather unusual, and actually in general it would give a pair $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ satisfying (4.8) but possibly not (4.9). For $0 < s \leq t$ it would be possible to use the *canonical* rough path over X given by

$$\int_s^t (X_u - X_s) \otimes \dot{X}_u \, \mathrm{d}u$$

but this would satisfy the analytical property (4.9) only for $0 < \varepsilon \leq s \leq t$, and *not* for $0 \leq s \leq t$, see section 7.4 below for further details. The proof of (4.9) for (4.10) is given in Step 1 on page 18 below.

We note that the definition (4.10) corresponds to the choice $\int_0^t X_s \otimes \mathrm{d}X_s \equiv 0$ as a generalised integral, in the sense of [CGZ24, Definition 7.1].

Before giving its proof, let us see why Theorem 4.2 implies the non-uniqueness phenomenon. Let I_t be as in (3.3) and define

$$Z_t = \begin{pmatrix} 0 \\ X^2_t \end{pmatrix} \text{ and } \bar{Z}_t = \begin{pmatrix} (1-\gamma)I_t \\ X^2_t \end{pmatrix}$$

and $\sigma : \mathbb{R}^2 \to \mathbb{R}^2 \times (\mathbb{R}^2)^*$

$$\sigma(x,y) = \begin{pmatrix} f(x,y) & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.11}$$

Then $Z, \bar{Z}$ are two (different) solutions of (4.7) with the path $(\mathbb{X}^1, \mathbb{X}^2)$ and the initial condition $Z_0 = \bar{Z}_0 = 0$. Observe that Theorem 4.2 implies that $(\mathbb{X}^1, \mathbb{X}^2)$ is a rough path. Recall that

$$[\sigma_2(z)]^i_{jl} = \sum_{a=1}^{2} \frac{\partial \sigma^i_j(z)}{\partial z_a} \sigma^a_l(z)$$

and for $B \in \mathbb{R}^2 \otimes \mathbb{R}^2$

$$[\sigma_2(z)B]^i = \sum_{l,m=1}^{2} [\sigma_2(z)]^i_{lm} B^{ml}.$$

Then for $z = (x, y) \in \{(x, y) \in \mathbb{R}^2 : |x| \geq y \geq 0, y \leq 3\}$ we have

$$[\sigma_2(z)]^i_{jl} = \begin{cases} \gamma y^{\gamma-1} & \text{if } i = 1, j = 1, l = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The fact that we are in $\mathbb{R}^d$ with $d \geq 2$ here plays an important role: it is the reason why $[\sigma_2(z)]^1_{12} = \gamma y^{\gamma-1}$. If σ was simply the one dimensional function $y \mapsto y^\gamma$, then σ_2 would have been equal to $\gamma y^{2\gamma-1}$. Having y raised to the power $\gamma - 1$ is fundamental, as it will be clear from the following. The product $\sigma_2(z)\mathbb{X}^2_{st}$ is equal to

$$[\sigma_2(z)\mathbb{X}^2_{st}]^i = \begin{cases} \gamma y^{\gamma-1} (\mathbb{X}^2_{st})^{2,1} = -\gamma y^{\gamma-1} X^2_s \, \delta X^1_{st} & \text{if } i = 1 \\ 0 & \text{if } i = 2. \end{cases}$$

If z is such that $z = (0, y)$ for some $y \in \mathbb{R}$, then

$$[\sigma_2(z)]^i_{jl} = 0$$

for every $i, j, l \in \{1, 2\}$ because

$$\frac{\partial \sigma^1_1(z)}{\partial z_2} = 0 \text{ and } \sigma^1_1(z) = 0.$$

To prove that $\overline{Z}$ is a solution observe that by Theorem 3.3 there exists $T > 0$ such that $|(1 - \gamma)I_t| \geq X^2_t$ for every $t \in [0, T]$ which implies that $\sigma^{1,1}(\overline{Z}_s) = (X^2_s)^\gamma$ and (4.7) is equivalent to

$$\begin{cases} \delta Z^1_{st} = (X^2_s)^\gamma (\mathbb{X}^1_{st})^1 + \gamma (X^2_s)^{\gamma-1} (\mathbb{X}^2_{st})^{2,1} + o(t - s) \\ \delta Z^2_{st} = \delta X^2_{st} + o(t - s). \end{cases}$$

or, more explicitly

$$\begin{cases} \delta Z^1_{st} = (1 - \gamma)(X^2_s)^\gamma \delta X^1_{st} + o(t - s) \\ \delta Z^2_{st} = \delta X^2_{st} + o(t - s). \end{cases}$$

The second equation is obviously satisfied for $Z^2 = X^2$; the first equation admits $(1 - \gamma)I_t$ as a solution, as it follows from Theorem 3.3. On the other hand $Z_t =$

$\begin{pmatrix} 0 \\ X_t^2 \end{pmatrix}$ trivially satisfies (4.7). In fact, noting that $\sigma^{1,1}(Z_s) = 0$ for every $s \in [0,T]$ and recalling the computations above, equation (4.7) is equivalent to

$$\begin{cases} \delta Z_{st}^1 = o(t-s) \\ \delta Z_{st}^2 = \delta X_{st}^2 + o(t-s). \end{cases}$$

Proof of Theorem 4.2. Step 1. We prove that the function $\mathbb{X}^2$ defined as in (4.10) satisfies (4.9). We want to show that there exists $C > 0$ such that

$$|X_s^i|\,|X_{s+h}^j - X_s^j| \le C h^{\frac{2\beta}{\eta+1}}$$

for every $s, h \ge 0$ such that $s + h \le 1$ and $i, j \in \{1,2\}$. We state a more general lemma, which will be useful also below.

Lemma 4.3. *Let $n \ge 2$, $j_1, \dots, j_n \in \{1,2\}$ and $\eta, \beta > 0$ such that $\eta + 1 > n\beta$. Define X^1, X^2 as in (3.2), then*

$$\left(\prod_{k=1}^{n-1} |X_s^{j_k}|\right) |X_{s+h}^{j_n} - X_s^{j_n}| \lesssim h^{\frac{n\beta}{\eta+1}}$$

uniformly over $s, h \ge 0$ such that $s + h \le 1$.

Proof. Recall that

$$X_s^1 = s^\beta \cos s^{-\eta}, \quad X_s^2 = s^\beta (2 + \sin s^{-\eta})$$

and

$$\dot{X}_s^1 = \beta s^{\beta-1} \cos s^{-\eta} + \eta s^{\beta-\eta-1} \sin s^{-\eta}, \quad \dot{X}_s^2 = \beta s^{\beta-1}(2 + \sin s^{-\eta}) - \eta s^{\beta-\eta-1} \cos s^{-\eta}.$$

Hence, there exists $C > 0$ such that

$$|X_s^j| \le C s^\beta, \quad |\dot{X}_s^j| \le C s^{\beta-\eta-1}$$

for every $s \in [0,1]$, $j \in \{1,2\}$. We consider two cases:

(1) Suppose that $h \ge s^{\eta+1}$, then

$$\begin{aligned} \left(\prod_{k=1}^{n-1} |X_s^{j_k}|\right) |X_{s+h}^{j_n} - X_s^{j_n}| &\lesssim s^{(n-1)\beta}(|X_{s+h}^{j_n}| + |X_s^{j_n}|) \\ &\lesssim s^{(n-1)\beta}((s+h)^\beta + s^\beta) \\ &\lesssim (s+h)^{n\beta} \lesssim (h^{\frac{1}{\eta+1}} + h)^{n\beta} \\ &\lesssim (2h^{\frac{1}{\eta+1}})^{n\beta} \lesssim h^{\frac{n\beta}{\eta+1}}. \end{aligned}$$

(2) Suppose that $h < s^{\eta+1}$, then $\frac{1}{s} < h^{-\frac{1}{\eta+1}}$ and

$$\left(\prod_{k=1}^{n-1} \left|X_s^{j_k}\right|\right) \left|X_{s+h}^{j_n} - X_s^{j_n}\right| \lesssim s^{(n-1)\beta} \sup_{t\in[s,s+h]} |\dot{X}_t^{j_n}|\, h \lesssim s^{(n-1)\beta} s^{\beta-\eta-1} h$$
$$= h\left(\frac{1}{s}\right)^{\eta+1-n\beta} \lesssim h^{\frac{n\beta}{\eta+1}}.$$

This proves the Lemma. □

Step 2. Finally we show that $f : \mathbb{R}^2 \to \mathbb{R}$ as in Theorem 4.2 exists. We will use the version of Whitney's Extension Theorem in Theorem 4 of section VI.2 of [Ste70], which we have reported in Theorem 3.2 for convenience. To apply this result we have to show that f is in $\mathcal{C}^\gamma(F)$ where $F = \{(x, y) \in \mathbb{R}^2 : |x| \geq$ and $y \leq 3y\} \cup \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and $\mathcal{C}^\gamma(F)$ is as in Definition 3.1. The proof is very similar to the one of Theorem 4.1. □

4.4. Non-uniqueness in the general case. Suppose $N < \gamma < \frac{1}{\alpha} < N + 1$ for some $N \in \mathbb{N}$, $N \geq 3$. Let β and η be large positive numbers such that they satisfy (4.1). Let X be as in (3.2). Moreover let $\sigma : \mathbb{R}^2 \to L(\mathbb{R}^2, \mathbb{R}^2)$ and define $\sigma_1 := \sigma$ and for $n \geq 2$ $\sigma_n : \mathbb{R}^2 \to \mathbb{R}^2 \times ((\mathbb{R}^2)^*)^n$ as

$$\sigma_n(z) = \nabla\sigma_{n-1}(z)\sigma(z).$$

We can define a generalised α-rough path over X as follows.

Definition 4.4. *Let $\alpha \in]\frac{1}{N+1}, \frac{1}{N}]$ and $X : [0, T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$. We say that $\mathbb{X} = (\mathbb{X}^1, \ldots, \mathbb{X}^N)$ is a generalised α-rough path over X if $\mathbb{X}^1 : [0, T]^2_\leq \to \mathbb{R}^2$ is*

$$\mathbb{X}^1_{st} = X_t - X_s$$

and for every $n \geq 2$, $\mathbb{X}^n : [0, T]^2_\leq \to (\mathbb{R}^2)^{\otimes n}$ satisfies Chen's relation, that is

$$\delta\mathbb{X}^n_{sut} = \sum_{i=1}^{n-1} \mathbb{X}^i_{su} \otimes \mathbb{X}^{n-i}_{ut} \tag{4.12}$$

and the analytical relation

$$|\mathbb{X}^n_{st}| \leq C|t - s|^{n\alpha}.$$

Consider the generalised difference equation

$$\delta Z_{st} = \sum_{i=1}^{N} \sigma_i(Z_s)\mathbb{X}^i_{st} + o(t - s) \tag{4.13}$$

where $\mathbb{X} = (\mathbb{X}^1, \ldots, \mathbb{X}^N)$ is a generalised α-rough path over X. We will show that for a suitable choice of $\mathbb{X}^2, \ldots, \mathbb{X}^N$ the problem (4.13) with initial condition $Z_0 = 0$ admits two different solutions. We will need the following analogue of Theorem 4.2:

Theorem 4.5. *Let X be as in (3.2) and for every $n \in \{1,\dots,N\}$ let $\mathbb{X}^n : [0,T]^2 \to (\mathbb{R}^2)^{\otimes n}$ be*

$$\mathbb{X}^n_{st} := (-1)^{n-1} X_s^{\otimes n-1} \otimes \delta X_{st}. \tag{4.14}$$

1. *Let $n \geq 2$, then the function $\mathbb{X}^n : [0,T]^2 \to (\mathbb{R}^2)^{\otimes n}$ satisfies (4.12) for every $0 \leq s \leq t \leq 1$.*
2. *For every $n \in \{1,\dots,N\}$ there exists $C > 0$ such that*
$$|\mathbb{X}^n_{st}| \leq C|t-s|^{\frac{n\beta}{\eta+1}}$$
for every $0 \leq s \leq t \leq 1$.
3. *There exists $f : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^\gamma$ such that*
$$f(x,y) = \begin{cases} y^\gamma & \text{if } |x| \geq y \geq 0 \text{ and } y \leq 3 \\ 0 & \text{if } x = 0. \end{cases}$$
4. *Define $\sigma : \mathbb{R}^2 \to \mathbb{R}^2 \times (\mathbb{R}^2)^*$ as*
$$\sigma(x,y) = \begin{pmatrix} f(x,y) & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.15}$$
Then for every $n \in \{1,\dots,N\}$, if $z = (x,y) \in \{(x,y) \in \mathbb{R}^2 : |x| \geq y \geq 0, y \leq 3\}$,
$$[\sigma_n(z)]^i_{j_1\cdots j_n} = \begin{cases} C_{\gamma_n} y^{\gamma+1-n} & \text{if } i = j_1 = 1 \text{ and } j_2 = j_3 = \ldots = j_n = 2 \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$
where $C_{\gamma_n} := \prod_{i=0}^{n-2}(\gamma - i)$.

Let us see why Theorem 4.5 implies the non-uniqueness phenomenon. Let I_t be as in (3.3) and define

$$Z_t = \begin{pmatrix} 0 \\ X_t^2 \end{pmatrix} \text{ and } \bar{Z}_t = \begin{pmatrix} C_\gamma I_t \\ X_t^2 \end{pmatrix}$$

where

$$C_\gamma := 1 + \sum_{i=2}^{N} (-1)^{i-1} C_{\gamma_i}.$$

Let $\sigma : \mathbb{R}^2 \to L(\mathbb{R}^2, \mathbb{R}^2)$ be defined as in (4.15), then $Z, \bar{Z}$ are two (different) solutions of (4.13) with the initial condition $Z_0 = \bar{Z}_0 = 0$ and the path $\mathbb{X} := (\mathbb{X}^1, \mathbb{X}^2, \dots, \mathbb{X}^N)$, where, for every $i = 1,\dots,N$, $\mathbb{X}^i$ is defined as in (4.14). Observe that Theorem 4.5 implies that $\mathbb{X}$ is a rough path. Recall that for $B \in (\mathbb{R}^2)^{\otimes n}$

$$[\sigma_n(z)B]^i = \sum_{j_1,\dots,j_n=1}^{2} [\sigma_n(z)]^i_{j_1\cdots j_n} B^{j_n\cdots j_1}.$$

Then for $z = (x, y) \in \{(x, y) \in \mathbb{R}^2 : |x| \geq y \geq 0, y \leq 3\}$ the product $\sigma_n(z)\mathbb{X}^n_{st}$ is equal to

$$[\sigma_n(z)\mathbb{X}^n_{st}]^i = \begin{cases} (-1)^{n-1} C_{\gamma_n} y^{\gamma+1-n} (X^2_s)^{n-1} \delta X^1_{st} & \text{if } i = 1 \\ 0 & \text{if } i = 2. \end{cases}$$

If z is such that $z = (0, y)$ for some $y \in \mathbb{R}$, then

$$[\sigma_n(z)]^i_{j_1 \cdots j_n} = 0$$

for every $i, j_1, \dots, j_n \in \{1, 2\}$. To prove that $\overline{Z}$ is a solution observe that by Theorem 3.3 there exists $T > 0$ such that $|C_\gamma I_t| \geq X^2_t$ for every $t \in [0, T]$ which implies that $\sigma^{1,1}(\overline{Z}_s) = (X^2_s)^\gamma$ and (4.13) is equivalent to

$$\begin{cases} \delta Z^1_{st} = (X^2_s)^\gamma (\mathbb{X}^1_{st})^1 + \sum_{i=2}^n C_{\gamma_i} (X^2_s)^{\gamma+1-i} (\mathbb{X}^i_{st})^{2,\dots,2,1} + o(t-s) \\ \delta Z^2_{st} = \delta X^2_{st} + o(t-s). \end{cases}$$

or, more explicitly

$$\begin{cases} \delta Z^1_{st} = C_\gamma (X^2_s)^\gamma \delta X^1_{st} + o(t-s) \\ \delta Z^2_{st} = \delta X^2_{st} + o(t-s). \end{cases}$$

The second equation is obviously satisfied for $Z^2 = X^2$; the first equation admits $C_\gamma \delta I_{st}$ as a solution, as it follows from Theorem 3.3. On the other hand $Z_t = \begin{pmatrix} 0 \\ X^2_t \end{pmatrix}$ trivially satisfies (4.13). In fact, noticing that $\sigma^{1,1}(Z_s) = 0$ for every $s \in [0, T]$ and recalling the computations above, equation (4.13) is equivalent to

$$\begin{cases} \delta Z^1_{st} = o(t-s) \\ \delta Z^2_{st} = \delta X^2_{st} + o(t-s). \end{cases}$$

Let us now prove Theorem 4.5.

Proof of Theorem 4.5. Step 1. We prove (4.12) by induction. Let $n = 2$, then

$$\mathbb{X}^n_{st} = -X_s \otimes \delta X_{st}$$

and (4.12) becomes

$$\delta \mathbb{X}^2_{sut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}.$$

Simply observe that

$$\begin{aligned} \delta \mathbb{X}^2_{sut} &= -X_s \otimes \delta X_{st} + X_s \otimes \delta X_{su} + X_u \otimes \delta X_{ut} \\ &= -X_s \otimes \delta X_{ut} + X_u \otimes \delta X_{ut} \\ &= \delta X_{su} \otimes \delta X_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}. \end{aligned}$$

Assume that (4.12) holds for n and let us prove it for $n+1$.

$$\begin{aligned}
\delta\mathbb{X}^{n+1}_{sut} &= \mathbb{X}^{n+1}_{st} - \mathbb{X}^{n+1}_{su} - \mathbb{X}^{n+1}_{ut} \\
&= (-1)^n(X_s^{\otimes n}\otimes \delta X_{st} - X_s^{\otimes n}\otimes \delta X_{su} - X_s\otimes X_u^{\otimes n-1}\otimes \delta X_{ut} + \\
&\qquad\qquad + X_s\otimes X_u^{\otimes n-1}\otimes \delta X_{ut} - X_u^{\otimes n}\otimes \delta X_{ut}) \\
&= -(X_s\otimes \delta\mathbb{X}^n_{sut}) + (-1)^n(X_s\otimes X_u^{\otimes n-1}\otimes \delta X_{ut} - X_u^{\otimes n}\delta X_{ut}) \\
&= -X_s\otimes \sum_{i=1}^{n-1}\mathbb{X}^i_{su}\otimes \mathbb{X}^{n-i}_{ut} + (-1)^n(\delta X_{us}\otimes X_u^{\otimes n-1}\otimes \delta X_{ut}) \\
&= \sum_{i=2}^{n}\mathbb{X}^i_{su}\otimes \mathbb{X}^{n+1-i}_{ut} + \mathbb{X}^1_{su}\otimes \mathbb{X}^n_{ut} \\
&= \sum_{i=1}^{n}\mathbb{X}^i_{su}\otimes \mathbb{X}^{n+1-i}_{ut}.
\end{aligned}$$

This proves that $\mathbb{X}^n$ satisfies (4.12) for every $n \geq 2$.

Step 2. This follows from Lemma 4.3.

Step 3. Finally we show that $f : \mathbb{R}^2 \to \mathbb{R}$ as in Theorem 4.5 exists. We will use the version of Whitney's Extension Theorem in Theorem 4 of section VI.2 of [Ste70], which we have reported in Theorem 3.2 for convenience. To apply this result we have to show that f is in $\mathcal{C}^\gamma(F)$ where $F = \{(x,y) \in \mathbb{R}^2 : |x| \geq y \geq 0 \text{ or } x = 0\}$ and $\mathcal{C}^\gamma(F)$ is as in Definition 3.1. The proof is very similar to the one of Theorem 4.1.

Step 4. Recall that

$$[\sigma_n(z)]^i_{j_1\cdots j_n} = \sum_{a=1}^{2}\frac{\partial[\sigma_{n-1}(z)]^i_{j_1\cdots j_{n-1}}}{\partial z_a}\sigma^a_{j_n}(z).$$

Let $z = (x,y) \in \{(x,y) \in \mathbb{R}^2 : |x| \geq y \geq 0, y \leq 3\}$. We already proved in the previous section that (4.16) holds for $\sigma_2(z)$. Assume that $\sigma_n(z)$ satisfies (4.16) and let us prove it for $\sigma_{n+1}(z)$. Observe that $[\sigma_n(z)]^i_{j_1\cdots j_n}$ is not null if and only if $i, j_1 = 1$ and $j_2, \dots, j_n = 2$, in which case it is a function only of y. Moreover $\sigma^2_1 = 0$ and $\sigma^2_2 = 1$, hence

$$\begin{aligned}
[\sigma_{n+1}(z)]^i_{j_1\cdots j_{n+1}} &= \begin{cases} \frac{\partial[\sigma_n(z)]^i_{j_1\cdots j_n}}{\partial z_2} & \text{if } i, j_1 = 1, j_2\dots, j_{n+1} = 2 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} (\gamma + 2 - (n+1))C_{\gamma_n}y^{\gamma+1-(n+1)} & \text{if } i, j_1 = 1, j_2\dots, j_{n+1} = 2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and (4.16) is proved. □

4.5. Geometric Rough Path. In Section 4.3 we presented an example of rough difference equation that admitted two different solutions starting at the origin. In particular we defined a function σ in $\mathcal{C}^\gamma$ and an α-rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ over $X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$ where

$$X^1_t = t^\beta \cos t^{-\eta},\ X^2_t = t^\beta(3 + \sin t^{-\eta}),\ \mathbb{X}^2_{st} = -X_s \otimes (X_t - X_s)$$

and

$$2 < \gamma < \frac{\eta}{\beta} < \frac{\eta + 1}{\beta} < \frac{1}{\alpha} < 3.$$

The definition (4.10), namely $\mathbb{X}^2_{st} := -X_s \otimes \delta X_{st}$, is easily seen not to produce a weakly geometric rough path, namely it does *not* satisfy $\mathbb{X}^2 + (\mathbb{X}^2)^T = \mathbb{X}^1 \otimes \mathbb{X}^1$. It is however possible to define $\mathbb{X}^2$ so that $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ is weakly geometric. Observe that to show the non uniqueness phenomenon for the Rough case, the only component of $\mathbb{X}^2$ that played a role is $(\mathbb{X}^2)^{2,1}$. Hence using the shuffle relation we can define

$$\mathbb{X}^2 := \begin{pmatrix} \frac{1}{2}(X^1_t - X^1_s)^2 & X^2_t(X^1_t - X^1_s) \\ -X^2_s(X^1_t - X^1_s) & \frac{1}{2}(X^2_t - X^2_s)^2 \end{pmatrix}. \tag{4.17}$$

Observe that $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ is weakly geometric. Classical results guarantee that given a weakly geometric d dimensional α-rough path $\mathbb{X}$ over X there exists a succession of canonical rough paths over smooth paths converging to $\mathbb{X}$ in $\mathcal{R}_{\alpha\prime,d}$ (the set of d dimensional $\alpha\prime$-rough paths) for every $\frac{1}{3} < \alpha' < \alpha$. In our case we defined a $\frac{\beta}{\eta+1}$-weakly geometric rough path over X, see Theorem 3.3 and Lemma 4.3, so there exists a succession of canonical rough paths over smooth paths converging to $\mathbb{X}$ in $\mathcal{R}_{\alpha,2}$ (the set of 2 dimensional α-rough paths).

We want to build a sequence of canonical rough paths $(\mathbb{X}_n) = (\mathbb{X}^1_n, \mathbb{X}^2_n)$ over smooth paths $X_n = \begin{pmatrix} X^1_n \\ X^2_n \end{pmatrix}$ such that

$$\lim_{n\to\infty} \|\mathbb{X}^1 - \mathbb{X}^1_n\|_\alpha + \|\mathbb{X}^2 - \mathbb{X}^2_n\|_{2\alpha} = 0. \tag{4.18}$$

(This result is not present in [Dav08], it is an original contribution of our paper.) We would like to find functions X^1_n, X^2_n of class $\mathcal{C}^1$ such that

$$X^1_{n,t} \to X^1_t, \quad X^2_{n,t} \to X^2_t, \quad \lim_{n\to\infty} \int_0^t X^2_{n,u}\, \dot{X}^1_{n,u}\, \mathrm{d}u = 0. \tag{4.19}$$

In fact (4.19) would imply that

$$\lim_{n\to\infty} \int_s^t (X^2_{n,u} - X^2_{n,s})\, \dot{X}^1_{n,u}\, \mathrm{d}u = -\lim_{n\to\infty} X^2_{n,s} \delta X^1_{n,s,t} = -X^2_s \delta X^1_{st} \tag{4.20}$$

and

$$\lim_{n\to\infty}\int_s^t (X^i_{n,u}-X^i_{n,s})\,\dot X^i_{n,u} = \lim_{n\to\infty}\frac{1}{2}(X^i_{n,t}-X^i_{n,s})^2 = \frac{1}{2}(X^i_t - X^i_s)^2 \tag{4.21}$$

for $i=1,2$, which means that $\mathbb{X}_n$ converges to $\mathbb{X}$ at least pointwise. The first idea is to shift X^1, X^2 by n^{-p} for some $p>0$, in order to obtain two sequences of $\mathcal{C}^1$ functions that converge (at least pointwise) respectively to X^1 and X^2. This would be enough to satisfy (4.21), but (4.20) would not hold as, for $t>0$,

$$\lim_{n\to+\infty}\int_0^t X^2_{u+n^{-p}}\,\dot X^1_{u+n^{-p}}\,\mathrm{d}u \geq \lim_{n\to+\infty}\left(n^{\rho(\eta-2\beta)} - 5(n^{-\rho}+t)^{2\beta-\eta}\right) = +\infty.$$

To overcome this problem, the idea is to add to $X^1_{t+n^{-p}}$ and to $X^2_{t+n^{-p}}$ respectively some functions C_n, S_n such that

$$C_{n,t}\to 0,\quad S_{n,t}\to 0,\quad \int_s^t S_{n,u}\,\dot C_{n,u}\,\mathrm{d}u = -\int_s^t X^2_{u+n^{-p}}\,\dot X^1_{u+n^{-p}}\,\mathrm{d}u + o(1).$$

We will see that we can actually take $o(1)=O\left(\frac{1}{n}\right)$. Finding such functions might seem complicated, but there are two classical candidates. In general, if we want

$$\int_s^t S_{n,u}\,\dot C_{n,u}\,\mathrm{d}u = -\int_s^t G_u\,\mathrm{d}u + o(1)$$

for some $G:\mathbb{R}\to\mathbb{R}$, we can take

$$S_{n,t} = \frac{1}{\sqrt{n}}\sin\left(2n\int_0^t G_u\,\mathrm{d}u\right),\quad C_{n,t} = \frac{1}{\sqrt{n}}\cos\left(2n\int_0^t G_u\,\mathrm{d}u\right).$$

In fact, with this choice,

$$S_{n,u}\,\dot C_{n,u} = -2G_u\sin^2\left(2n\int_0^t G_u\,\mathrm{d}u\right) = -2G_u\left(\frac{1}{2}-\frac{1}{2}\cos(4n\int_0^t G_u\,\mathrm{d}u)\right),$$

and

$$\int_s^t G_u\cos\left(4n\int_0^t G_u\,\mathrm{d}u\right) = \left.\frac{\sin\left(4n\int_0^t G_u\,\mathrm{d}u\right)}{4n}\right|_s^t = O\left(\frac{1}{n}\right) = o(1).$$

By adding these terms (4.21) would continue to hold and we might be able to prove (4.20). In fact, for (4.20) to hold, it only remains to control the mixed terms in the product $(X^1_{t+n^{-p}}+C_{n,t})(X^2_{t+n^{-p}}+S_{n,t})$, that is

$$\int_s^t (X^2_{u+n^{-p}} - X^2_{u+n^{-p}})\,\dot C_{n,u}\,\mathrm{d}u,\quad \int_s^t (S_{n,u}-S_{n,s})\,\dot X^1_{u+n^{-p}}\,\mathrm{d}u,$$

and in particular we would like both integrals to converge to 0. This is where the choice of the power p becomes important. Fix $0<\rho<\frac{1}{2}$ such that

$$\alpha < \frac{1}{2(1+\rho)}.$$

Recall that

$$X_t^1 = t^\beta \cos t^{-\eta}, \quad X_t^2 = t^\beta(2 + \sin t^{-\eta}) \tag{4.22}$$

and define, for $t \in [0, T]$,

$$\begin{aligned}
I_n(t) &:= 2\int_0^t X^2_{r+w_n}\dot{X}^1_{r+w_n}\,\mathrm{d}r, \\
C_{n,t} &:= \frac{1}{\sqrt{n}}\cos(nI_n(t)), &(4.23)\\
S_{n,t} &:= \frac{1}{\sqrt{n}}\sin(nI_n(t)), &(4.24)\\
X^1_{n,t} &:= X^1_{t+w_n} + C_{n,t}, \qquad X^2_{n,t} := X^2_{t+w_n} + S_{n,t}, \\
X_{n,t} &:= \begin{pmatrix} X^1_{n,t} \\ X^2_{n,t} \end{pmatrix}, &(4.25)
\end{aligned}$$

where $w_n := n^{-\frac{\rho}{\eta+1}}$. To prove (4.18) we need the following

Theorem 4.6. *Let X, X_n be as in* (4.22) *and* (4.25)*. Let $\mathbb{X}_n = (\mathbb{X}^1_n, \mathbb{X}^2_n)$ be the canonical rough path over X_n and $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ the rough path over X in which $\mathbb{X}^2$ is as in* (4.17)*. Then for $i = 1, 2$*

(1) $\mathbb{X}^i_{n,s,t} \to \mathbb{X}^i_{st}$ uniformly.

(2) There exists $C > 0$ and $\varepsilon > 0$ such that

$$|\mathbb{X}^i_{n,s,t}| \le C|t-s|^{i\alpha+\varepsilon}$$

for every $0 \le s < t \le 1$.

Let us see how we can prove (4.18) using Theorem 4.6.

$$\begin{aligned}
\|\mathbb{X}^1_n - \mathbb{X}^1\|_\alpha &= \sup_{0\le s<t\le T}\frac{|\mathbb{X}^1_{n,s,t} - \mathbb{X}^1_{st}|}{|t-s|^\alpha} \\
&\le \sup_{0\le s<t\le T}|\mathbb{X}^1_{n,s,t} - \mathbb{X}^1_{st}|^{\frac{\varepsilon}{\alpha+\varepsilon}}\sup_{0\le s<t\le T}\frac{|\mathbb{X}^1_{n,s,t} - \mathbb{X}^1_{n,s,t}|^{\frac{\alpha}{\alpha+\varepsilon}}}{|t-s|^\alpha} \\
&\le \|\mathbb{X}^1_n - \mathbb{X}^1\|_\infty^{\frac{\varepsilon}{\alpha+\varepsilon}}\sup_{0\le s<t\le T}\frac{C|t-s|^\alpha}{|t-s|^\alpha} \\
&\le C\|\mathbb{X}^1_n - \mathbb{X}^1\|_\infty^{\frac{\varepsilon}{\alpha+\varepsilon}}
\end{aligned}$$

which goes to 0 for $n \to \infty$. Analogously we prove convergence for $\mathbb{X}^2_n$.

Proof of Theorem 4.6. In the following C is a positive constant whose value might chance every line, but does not depend on n, s, t. We will often use the fact that

X^1, X^2 are $\frac{\beta}{\eta+1}$-Hölder and the following inequalities

$$\begin{aligned}
|X^2_s\,\delta X^1_{st}| &\le C|t-s|^{\frac{2\beta}{\eta+1}},\\
|\dot X^1_{t+w_n}| &\le Cn^\rho,\\
|\dot X^2_{t+w_n}| &\le Cn^\rho,\\
|X^1_{n,t}| + |X^2_{n,t}| &\le C,\\
|\dot I_n(t)| &\le Cn^\rho. \qquad (4.26)
\end{aligned}$$

The only non trivial inequality is the first one which has been proven in Theorem 4.2.

Step 1. Observe that for $i = 1, 2$,

$$(\mathbb{X}^2_n)^{i,i}_{st} = \int_s^t (X^i_{n,u} - X^i_{n,s})\,\dot X^i_{n,u}\,\mathrm{d}u = \frac{(X^i_{n,t} - X^i_{n,s})^2}{2},$$

and

$$\begin{aligned}
|(\mathbb{X}^2_{n,s,t})^{1,1} - (\mathbb{X}^2_{st})^{1,1}| &\le |X^1_{n,t} - X^1_{n,s} - X^1_t + X^1_s|\,|X^1_{n,t} - X^1_{n,s} + X^1_t - X^1_s|\\
&\le C|X^1_{n,t} - X^1_t| + C|X^1_s - X^1_{n,s}|\\
&\le C\left(w_n^\alpha + |C_{n,t}| + |C_{n,s}|\right)\\
&\le C\left(\frac{1}{n^{\frac{\alpha\rho}{\eta+1}}} + \frac{1}{\sqrt{n}}\right).
\end{aligned}$$

This proves that $(\mathbb{X}^2_n)^{1,1}$ converges uniformly to $(\mathbb{X}^2)^{1,1}$; analogously $(\mathbb{X}^2)^{2,2}_n$ converges uniformly to $(\mathbb{X}^2)^{2,2}$. Moreover, for every $0 \le s < t \le 1$,

$$\begin{aligned}
(\mathbb{X}^2_{n,s,t})^{2,1} &= \int_s^t (X^2_{n,u} - X^2_{n,s})\,\dot{X^1_{n,u}}\,\mathrm{d}u\\
&= \int_s^t (X^2_{u+w_n} - X^2_{s+w_n} + S_{n,u} - S_{n,s})(\dot X^1_{u+w_n} + \dot C_{n,u})\,\mathrm{d}u\\
&= \int_s^t X^2_{u+w_n}\,\dot X^1_{u+w_n}\,\mathrm{d}u + \int_s^t S_{n,u}\,\dot C_{n,u}\,\mathrm{d}u\\
&\quad + \int_s^t (X^2_{u+w_n} - X^2_{s+w_n})\,\dot C_{n,u}\,\mathrm{d}u + \int_s^t (S_{n,u} - S_{n,s})\,\dot X^1_{u+w_n}\,\mathrm{d}u\\
&\quad - X^2_{s+w_n}\delta X^1_{(s+w_n)(t+w_n)} - S_{n,s}\,\delta C_{n,s,t}
\end{aligned}$$

and defining $J_{n,s,t} := \int_s^t \delta X^2_{(s+w_n)(u+w_n)}\, \dot{C}_{n,u}\,\mathrm{d}u + \int_s^t \delta S_{n_{ut}}\, \dot{X}^1_{u+w_n}\,\mathrm{d}u - S_{n,s}\delta C_{n,s,t}$, we obtain

$$
\begin{aligned}
&(\mathbb{X}^2_{n,s,t})^{2,1} = \\
&= \int_s^t X^2_{u+w_n}\, \dot{X}^1_{u+w_n}\,\mathrm{d}u - \int_s^t \dot{I}_n(u)\sin^2(nI_n(u))\,\mathrm{d}u + J_{n,s,t} - X^2_{s+w_n}\delta X^1_{(s+w_n)(t+w_n)} \\
&= \int_s^t \frac{1}{2}\dot{I}_n(u)\,\mathrm{d}u - \int_s^t \dot{I}_n(u)\left(\frac{1}{2} - \frac{1}{2}\cos(2nI_n(u))\right) + J_{n,s,t} - X^2_{s+w_n}\delta X^1_{(s+w_n)(t+w_n)} \\
&= \frac{1}{2}\int_s^t \cos(2nI_n(u))\,\dot{I}_n(u)\,\mathrm{d}u + J_{n,s,t} - X^2_{s+w_n}\delta X^1_{(s+w_n)(t+w_n)} \\
&= \frac{1}{4n}\delta(\sin(2nI_n(\cdot)))_{st} + J_{n,s,t} - X^2_{s+w_n}\delta X^1_{(s+w_n)(t+w_n)} \\
&= \frac{1}{2}\delta(S_nC_n)_{st} + J_{n,s,t} - X^2_{s+w_n}\delta X^1_{(s+w_n)(t+w_n)} \qquad (4.27)\\
&= -X^2_{s+w_n}\delta X^1_{(s+w_n)(t+w_n)} + O(n^{-(\frac{1}{2}-\rho)}).
\end{aligned}
$$

In fact

$$
\begin{aligned}
|\int_s^t \delta X^2_{(s+w_n)(u+w_n)}\, \dot{C}_{n,u}\,\mathrm{d}u| &\le |\Big(\delta X^2_{(s+w_n)(u+w_n)} C_{n,u}\Big|_s^t| + |\int_s^t \dot{X}^2_{u+w_n} C_{n,u}\,\mathrm{d}u| \\
&\le Cn^{-\frac{1}{2}} + Cn^{-(\frac{1}{2}-\rho)}, \\
|\int_s^t \delta S_{n,s,u}\, \dot{X}^1_{u+w_n}\,\mathrm{d}u| &\le Cn^{-(\frac{1}{2}-\rho)}
\end{aligned}
$$

and

$$|S_{n,s}\,\delta C_{n,s,t}| \le n^{-1}.$$

Finally observe that

$$
\begin{aligned}
|(\mathbb{X}^2_{n,s,t})^{2,1} - (\mathbb{X}^2_{st})^{2,1}| &\le |X^2_{s+w_n}\delta X^1_{(s+w_n)(t+w_n)} - X^2_s\delta X^1_{st}| + Cn^{-(\frac{1}{2}-\rho)} \\
&\le C|X^2_{s+w_n} - X^2_s| + C|\delta X^1_{st} - \delta X^1_{(s+w_n)(t+w_n)}| + Cn^{-\frac{1}{2}+\rho} \\
&\le Cn^{-\frac{\alpha\rho}{\eta+1}} + C|X^1_t - X^1_{t+w_n}| + C|X^1_{s+w_n} - X^1_s| + Cn^{-\frac{1}{2}+\rho} \\
&\le Cn^{-\frac{\alpha\rho}{\eta+1}} + Cn^{-\frac{1}{2}+\rho} \\
&\le Cn^{-\frac{\alpha\rho}{\eta+1}}.
\end{aligned}
$$

This shows that $(\mathbb{X}^2_n)^{2,1}$ converges uniformly to $(\mathbb{X}^2)^{2,1}$. Observe that since $\mathbb{X}^2$ is weakly geometric, by the shuffle relation, we also have that $(\mathbb{X}^2_n)^{1,2}$ converges uniformly to $(\mathbb{X}^2)^{1,2}$. The uniform convergence of $\mathbb{X}^1_n$ to $\mathbb{X}^1$ is simple to prove; we only show it for the first component:

$$\begin{aligned}|\delta X^1_{n,s,t} - \delta X^1_{st}| &\le |X^1_{t+w_n} - X^1_t| + |X^1_s - X^1_{s+w_n}| + |\delta C_{n,s,t}| \\ &\le Cn^{-\frac{\alpha\rho}{\eta+1}} + 2n^{-\frac{1}{2}}.\end{aligned}$$

Step 2. We have shown the uniform convergence

$$\mathbb{X}^2_{n,s,t} \to \mathbb{X}^2_{st}$$

To prove convergence in $\mathcal{C}^{2\alpha}$ it is sufficient to show that there exists $C > 0$ and $\varepsilon > 0$ such that

$$|\mathbb{X}^2_{n,s,t}| \le C|t-s|^{2\alpha+\varepsilon} \tag{4.28}$$

for every $0 \le s < t \le 1$. We will use the following

Lemma 4.7. *Let C_n, S_n be as in* (4.23) *and* (4.24), *then*

(1) C_n is $\frac{1}{2(1+\rho)}$-Hölder.
(2) S_nC_n and $\frac{1}{\sqrt{n}}C_n$ are $\frac{1}{1+\rho}$-Hölder.

Proof. We will use (4.26) and the fact that

$$\min\{x,y\} \le x^\xi y^{1-\xi} \tag{4.29}$$

for every $\xi \in [0,1]$.

(1) Recall that $C_{n,t} = \frac{1}{\sqrt{n}}\cos(nI_n(t))$ and observe that

$$\|C_n\|_\infty \le n^{-\frac{1}{2}}, \quad \|C'_n\|_\infty \le Cn^{\frac{1}{2}+\rho}.$$

So,

$$\begin{aligned}|C_{n,t} - C_{n,s}| &\le \min\{\|C'_n\|_\infty|t-s|, 2\|C_n\|_\infty\} \\ &\le \min\{Cn^{\frac{1}{2}+\rho}|t-s|, 2n^{-\frac{1}{2}}\}\end{aligned}$$

and applying (4.29) with $\xi = \frac{1}{2(1+\rho)}$

$$|C_{n,t} - C_{n,s}| \le C|t-s|^{\frac{1}{2(1+\rho)}}.$$

(2) Observe that

$$\|S_nC_n\|_\infty \le n^{-1}, \quad \|(S_nC_n)'\|_\infty \le Cn^\rho.$$

So,

$$\begin{aligned}|S_{n,t}C_{n,t} - S_{n,s}C_{n,s}| &\le \min\{\|(S_nC_n)'\|_\infty|t-s|, 2\|S_nC_n\|_\infty\} \\ &\le \min\{Cn^\rho|t-s|, 2n^{-1}\}\end{aligned}$$

and applying (4.29) with $\xi = \frac{1}{1+\rho}$

$$\le C|t-s|^{\frac{1}{1+\rho}}.$$

Analogously we can prove that $\frac{1}{\sqrt{n}}C_n$ is $\frac{1}{1+\rho}$-H"older.

□

Let us go back to our problem. Observe that

$$\begin{aligned}(\mathbb{X}^2_{n,s,t})^{1,1} &= \frac{(X^1_{n,t} - X^1_{n,s})^2}{2} \\ &\leq |X^1_{t+w_n} - X^1_{s+w_n}|^2 + |C_{n,t} - C_{n,s}|^2 \\ &\leq |t-s|^{\frac{2\beta}{\eta+1}} + |t-s|^{\frac{1}{1+\rho}}\end{aligned}$$

where $w_n := n^{-\frac{\rho}{\eta+1}}$ and, since $\frac{\beta}{\eta+1} > \alpha$ and $\frac{1}{2(1+\rho)} > \alpha$, we have proved that (4.28) holds for $(\mathbb{X}^2_n)^{1,1}$. We can proceed analogously for $(\mathbb{X}^2_n)^{2,2}$. We now focus on $(\mathbb{X}^2)^{2,1}$. Recall that, from (4.27),

$$(\mathbb{X}^2_{n,s,t})^{2,1} = \frac{1}{2}\delta(S_n C_n)_{st} + J_{n,s,t} - X^2_{s+w_n}\delta X^1_{(s+w_n)(t+w_n)}$$

where $J_{n,s,t} := \int_s^t \delta X^2_{(s+w_n)(u+w_n)}\, \dot{C}_{n,u}\,\mathrm{d}u + \int_s^t \delta S_{n_{ut}}\, \dot{X}^1_{u+w_n}\,\mathrm{d}u - S_{n,s}\delta C_{n,s,t}$. Now, by Lemma 4.7

$$|\delta(S_n C_n)_{st}| \leq C|t-s|^{\frac{1}{1+\rho}}.$$

Moreover

$$\begin{aligned}|\int_s^t \delta X^2_{(s+w_n)(u+w_n)}\, \dot{C}_{n,u}\,\mathrm{d}u| &\leq |\delta X^2_{(s+w_n)(t+w_n)} C_{n,t}| + |\int_s^t \dot{X}^2_{u+w_n} C_{n,t}\,\mathrm{d}u| \\ &\leq |\frac{1}{\sqrt{n}}\int_s^t \dot{X}^2_{u+w_n}\,\mathrm{d}u| + \frac{1}{\sqrt{n}}\int_s^t |\dot{X}^2_{u+w_n}|\,\mathrm{d}u \\ &\leq C n^{-(\frac{1}{2}-\rho)}|t-s| \leq C|t-s|\end{aligned}$$

and analogously

$$|\int_s^t \delta S_{n,s,t}\, \dot{X}^1_{u+w_n}\,\mathrm{d}u| \leq C\int_s^t n^{-(\frac{1}{2}-\rho)}\,\mathrm{d}u \leq C|t-s|.$$

From Lemma 4.7,

$$\begin{aligned}|S_{n,s}\,\delta C_{n,s,t}| &\leq \frac{1}{\sqrt{n}}\delta C_{n,s,t} \\ &\leq C|t-s|^{\frac{1}{1+\rho}}.\end{aligned}$$

Finally,

$$|X^2_{s+w_n}\delta X^1_{(s+w_n)(t+w_n)}| \leq |t-s|^{\frac{2\beta}{\eta+1}}.$$

Since $\frac{\beta}{\eta+1} > \alpha$ and $\frac{1}{2(1+\rho)} > \alpha$, we have proved that (4.28) holds for $(\mathbb{X}^2_n)^{2,1}$. Using the shuffle relation a similar estimate can be proven for $(\mathbb{X}^2_n)^{1,2}$. This concludes the proof of (4.28). We are left to prove a Hölder like estimate for $\mathbb{X}^1_n$. This is simple, in fact

$$
\begin{aligned}
|(\mathbb{X}^1_{n,s,t})^1| = |\delta X^1_{n,s,t}| &\leq |\delta X^1_{(s+w_n)(t+w_n)}| + |\delta C_{n,s,t}| \\
&\leq C|t-s|^{\frac{\beta}{\eta+1}} + C|t-s|^{\frac{1}{2(1+\rho)}} \\
&\leq C|t-s|^{\alpha}.
\end{aligned}
$$

A similar estimate holds for $(\mathbb{X}^1_n)^2$. □

5. Preparation for non-existence

We present here some elementary, but not trivial, results that we will use in both the Young and Rough case and hold for any choice of $\alpha \in]0,1[$. For $t \geq 0$ define

$$
X^1_t := \sum_{k \in \mathbb{T}_t} 2^{-\alpha k} \sin(2^k t), \quad G_t := \sum_{k=1}^{+\infty} 2^{-(1-\alpha)k} \cos 2^k t, \tag{5.1}
$$

where conceptually, $\mathbb{T}_t = \{k \leq \frac{1}{t}\}$. However, defining $\mathbb{T}_t$ simply as $\{k \leq \frac{1}{t}\}$ would result in X^1_t being discontinuous. This issue can be resolved with a slight modification to the definition of $\mathbb{T}_t$.

Specifically, for $k \in \mathbb{N}$, we define:

$$
n_k := \inf \left\{ n \in \mathbb{N} : n \geq \frac{2^{k-1}}{k\pi} \right\}
$$

and

$$
t_k := \pi n_k 2^{1-k}.
$$

Observe that $\frac{1}{k} + \pi 2^{1-k} \geq t_k \geq \frac{1}{k}$ and define:

$$
\mathbb{T}_t = \{k \in \mathbb{N} : t_k \geq t\}.
$$

Notice that X^1_t is continuous. Indeed,

$$
X^1_{t_k} - \lim_{t \downarrow t_k} X^1_t = 2^{-\alpha k} \sin(2^k t_k)
$$

and we need $2^k t_k \in \mathbb{N}\pi$ for X^1 to be continuous. For $0 < s \leq t$ define

$$
I_{st} = \int_s^t G_u \dot{X}^1_u \, \mathrm{d}u. \tag{5.2}
$$

The main result of this section is the following

Theorem 5.1. *Let X^1, G and I be as in* (5.1) *and* (5.2) *for some $\alpha \in]0,1[$.*

(1) The functions X^1_t and G_t are respectively α-Hölder and $(1-\alpha)$-Hölder.
(2) The function X^1_t is locally Lipschitz on $]0,\pi]$.

(3) Fix $T > 0$, then

$$I_{sT} \geq -\frac{1}{2}\log s + O(1)$$

for $s \to 0$.

Proof. We prove the three points of the Theorem one by one.

Step 1. We begin by proving that G is in $\mathcal{C}^{1-\alpha}$. We will then use a similar approach to prove that X^1 is in $\mathcal{C}^\alpha$. We set $G_n(t) := \sum_{k=1}^n 2^{-(1-\alpha)k}\cos 2^k t$. Note that G_n is in $\mathcal{C}^1$ and

$$\|G - G_n\|_\infty \leq \sum_{k=n+1}^\infty 2^{-(1-\alpha)k} \leq \frac{2^{-(1-\alpha)(n+1)}}{1-2^{-(1-\alpha)}},$$

$$\|G_n'\|_\infty \leq \sum_{k=0}^n 2^{\alpha k} \leq \frac{2^{\alpha(n+1)}}{2^\alpha - 1}.$$

Then, for $s, t \in [0,1]$

$$\begin{aligned}|G(t) - G(s)| &\leq |G(t) - G_n(t)| + |G(s) - G_n(s)| + |G_n(t) - G_n(s)| \\ &\leq \frac{2}{1-2^{-(1-\alpha)}}\, 2^{-(1-\alpha)(n+1)} + \frac{2^{\alpha(n+1)}}{2^\alpha - 1}|t-s|\end{aligned}$$

and choosing n so that $2^{-(n+1)} \leq |t-s| \leq 2^{-n}$, we obtain

$$\leq \left(\frac{2}{1-2^{-(1-\alpha)}} + \frac{2^\alpha}{2^\alpha - 1}\right)|t-s|^{1-\alpha}.$$

To prove that X^1 is in $\mathcal{C}^\alpha$ we set

$$g_k : \mathbb{R}_+ \to \mathbb{R}, \qquad g_k(t) := \mathbb{1}_{(t \leq t_k)}\, 2^{-\alpha k}\, \sin(2^k t), \qquad t \geq 0,$$

and $f_n := \sum_{k=1}^n g_k$. Notice that g_k is continuous, but has a corner point in $t = t_k$, hence f_n is Lipschitz but not $\mathcal{C}^1$. Denoting $f := X^1$

$$\|f - f_n\|_\infty \leq \sum_{k=n+1}^\infty 2^{-\alpha k} \leq \frac{2^{-\alpha(n+1)}}{1-2^{-\alpha}},$$

$$\sup_{s,t\in[0,1]} \frac{|f_n(t) - f_n(s)|}{|t-s|} \leq \sum_{k=0}^n \sup_{t\in[0,t_k]} |g_k'(t)| \leq \sum_{k=0}^n 2^{(1-\alpha)k} \leq \frac{2^{(1-\alpha)(n+1)}}{2^{1-\alpha}-1}.$$

Then for $s, t \in [0,1]$

$$\begin{aligned}|f(t) - f(s)| &\leq |f(t) - f_n(t)| + |f(s) - f_n(s)| + |f_n(t) - f_n(s)| \\ &\leq \frac{2}{1-2^{-\alpha}} 2^{-\alpha(n+1)} + \frac{2^{(1-\alpha)(n+1)}}{2^{1-\alpha}-1}|t-s|.\end{aligned}$$

If n is chosen so that $2^{-(n+1)} \leq |t-s| \leq 2^{-n}$, then we obtain

$$|f(t) - f(s)| \leq C|t-s|^\alpha,$$

where $C = \left(\frac{2}{1-2^{-\alpha}} + \frac{2^{(1-\alpha)}}{2^{1-\alpha}-1}\right)$.

Step 2. To prove that X^1 is locally Lipschitz on $]0,\pi]$ observe that $\{t_k\}_{k\in\mathbb{N}}$ is in $]0,+\infty[$ and tends to 0 as $k \to +\infty$, more precisely

$$t_k = \pi 2^{1-k} \left\lceil \frac{2^{k-1}}{k\pi} \right\rceil \sim \frac{1}{k}.$$

Hence, for $\varepsilon > 0$ the set

$$\mathbb{T}_\varepsilon = \{k \in \mathbb{N} : t_k \geq \varepsilon\}$$

is finite and, on $[\varepsilon,\pi]$, $X^1 = \sum_{k\in\mathbb{T}_\varepsilon} g_k$ is a finite sum of Lipschitz functions. This implies that X^1 is locally Lipschitz on $]0,\pi]$.

Step 3. Let $0 < s < T$, then

$$\begin{aligned}
I_{sT} &= \int_s^T G_u \dot{X}^1_u \,\mathrm{d}u \\
&= \int_s^T \sum_{k\in\mathbb{T}_u} \sum_{l=1}^{+\infty} 2^{(1-\alpha)(k-l)} \cos(2^k u)\cos(2^l u)\,\mathrm{d}u \\
&= \frac{1}{2} \sum_{k\in\mathbb{T}_s} \sum_{l=1}^{+\infty} \int_s^{t_k\wedge T} 2^{(1-\alpha)(k-l)} (\cos((2^k+2^l)u) + \cos((2^k-2^l)u))\,\mathrm{d}u \\
&= \sum_{k\in\mathbb{T}_s} \sum_{l=1}^{+\infty} \frac{2^{(1-\alpha)(k-l)}}{2} \frac{\sin((2^k+2^l)(t_k\wedge T)) - \sin((2^k+2^l)s)}{2^k+2^l} + \\
&\qquad + \sum_{k\in\mathbb{T}_s} \sum_{l\neq k} \frac{2^{(1-\alpha)(k-l)}}{2} \frac{\sin((2^k-2^l)(t_k\wedge T)) - \sin((2^k-2^l)s)}{2^k-2^l} + \\
&\qquad + \sum_{k\in\mathbb{T}_s} (t_k \wedge T - s) \\
&\geq -\frac{1}{2}\log(s) + O(1)
\end{aligned}$$

as $s \to 0$. The last inequality follows from the fact that $t_k \geq \frac{1}{k}$, for s small $\{k \in \mathbb{T}_s\} \supset \{k \leq \lfloor \frac{1}{s} \rfloor\}$ and

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + O(1).$$

Moreover,

$$\begin{aligned}\sum_{k\in\mathbb{T}_s}\sum_{k=1}^{+\infty}\frac{2^{(1-\alpha)(k-l)}}{2^k+2^l} &\le \sum_{l=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{2^{(1-\alpha)(k-l)}}{2^k}\\ &= \frac{1}{1-2^{-\alpha}}\sum_{l=0}^{+\infty}2^{(\alpha-1)l}\\ &= \frac{1}{(1-2^{-\alpha})(1-2^{\alpha-1})},\end{aligned}$$

and

$$\begin{aligned}\sum_{k\in\mathbb{T}_s}\sum_{l\neq k}\left|\frac{2^{(1-\alpha)(k-l)}}{2^k-2^l}\right| &\le \sum_{k=1}^{+\infty}\sum_{l=k+1}^{+\infty}\frac{2^{(1-\alpha)(k-l)}}{2^k}+\sum_{k=1}^{+\infty}\sum_{l=0}^{k-1}\frac{2^{(1-\alpha)(k-l)}}{2^k-2^l}\\ &\le \frac{1}{1-2^{\alpha-1}}\sum_{k=0}^{+\infty}2^{-\alpha k}+\sum_{k=1}^{+\infty}2^{-\alpha k}\sum_{l=0}^{k-1}\frac{2^{(\alpha-1)l}}{1-2^{l-k}}\\ &\le \frac{1}{(1-2^{-\alpha})(1-2^{\alpha-1})}+\sum_{k=1}^{+\infty}2^{-\alpha k}\,2\sum_{l=0}^{+\infty}2^{(\alpha-1)l}\\ &\le \frac{3}{(1-2^{-\alpha})(1-2^{\alpha-1})}.\end{aligned}$$

Lastly, observe that $\pi 2^{1-k}+\frac{1}{k}\ge t_k$, so $\mathbb{T}_s\subset\{k:\pi 2^{1-k}+\frac{1}{k}\ge t\}\subset\{k:\frac{5}{k}\ge s\}$ and

$$\sum_{k\in\mathbb{T}_s}s\le\sum_{k=1}^{\lfloor\frac{5}{s}\rfloor}s\le 5.$$

$\square$

6. Non existence of solutions

The aim of this section is to present examples of non-existence of solutions for equations driven by a path $X\in\mathcal{C}^\alpha$ with $\alpha\in]\frac{1}{2},1[$ or $\alpha\in]\frac{1}{3},\frac{1}{2}[$. In each case we will define an appropriate function σ of class $\mathcal{C}^\gamma$ with $\gamma=\frac{1}{\alpha}-1$.

6.1. Young case. Suppose $\alpha\in]\frac{1}{2},1[$ and $\gamma=\frac{1}{\alpha}-1$. We want to construct functions

- $X:[0,T]\to\mathbb{R}^3$ of class $\mathcal{C}^\alpha$
- $\sigma:\mathbb{R}^3\to L(\mathbb{R}^3,\mathbb{R}^3)$ of class $\mathcal{C}^\gamma$

such that for every $0 < T < \pi$ the system

$$\begin{cases} \delta Z_{st} = \sigma(Z_s)\delta X_{st} + o(t-s) \\ Z_0 = 0 \end{cases} \tag{6.1}$$

does not admit any solution. The main result of this section is the following

Theorem 6.1. *Let G be as in* (5.1)*. Fix $\alpha \in]\frac{1}{2}, 1[$.*

(1) There exists $X^2, X^3 : [0,1] \to \mathbb{R}$ in $\mathcal{C}^\alpha$ and $C > 0$ such that

$$|(X_t^2, X_t^3) - (X_s^2, X_s^3)| \geq C|t-s|^\alpha$$

for every $0 \geq s \geq t \geq T$.

(2) There exists $f : \mathbb{R}^2 \to \mathbb{R}$ in $\mathcal{C}^\gamma$ such that

$$G_t = f(X_t^2, X_t^3).$$

Let us see with Theorem 6.1 implies that (6.1) does not admit any solution. We define

$$X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix}$$

where X^1 is as in (5.1) and X^2, X^3 as in Theorem (6.1). Moreover define $\sigma : \mathbb{R}^3 \to L(\mathbb{R}^3, \mathbb{R}^3)$ as

$$\sigma(x,y,z) = \begin{pmatrix} f(y,z) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then σ is of class $\mathcal{C}^\gamma$ and X of class $\mathcal{C}^\alpha$. Suppose that we have a solution

$$Z_t = \begin{pmatrix} Z_t^1 \\ Z_t^2 \\ Z_t^3 \end{pmatrix}$$

on $[0,T]$ for some $T \in]0,\pi[$, then

$$Z_t^2 = X_t^2, \quad Z_t^3 = X_t^3$$

and, for $0 < s < T$,

$$Z_T^1 - Z_s^1 = \int_s^T G_u \dot{X}_u^1 \, \mathrm{d}u$$

and recalling (5.2) and Theorem 5.1

$$\geq -\frac{1}{2}\log(s) + O(1).$$

as $s \to 0$. This means that $Z_s \not\to 0$ as $s \to 0$ and the initial condition can not be satisfied. Let us now prove Theorem 6.1.

Step 1. We want to prove that there exists $X^2, X^3 : [0,1] \to \mathbb{R}$ in $\mathcal{C}^\alpha$ and $C > 0$ such that

$$|(X_t^2, X_t^3) - (X_s^2, X_s^3)| \geq C|t-s|^\alpha \tag{6.2}$$

for every $0 \geq s \geq t \geq T$. The existence of such functions follows from the following

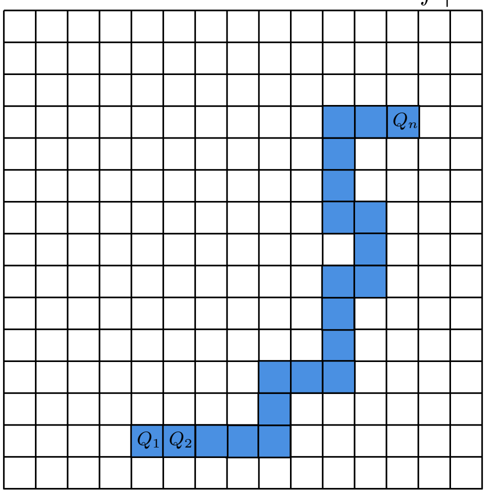
Lemma 6.2. *Let $\alpha \in]\frac{1}{2}, 1[$, then there exists $c_1, c_2 > 0$ and a function $u : [0,1] \to \mathbb{R}^2$ such that*

$$c_1|t-s|^\alpha \leq |u(t) - u(s)| \leq c_2|t-s|^\alpha$$

for all $s, t \in [0,1]$.

Observe that a function with the same property but taking values in $\mathbb{R}$ does not exists, see section 7.6 for further details. Let X^2, X^3 be respectively the first and second component of such u. Before proving the Lemma we give the following definition.

Definition 6.3. *Given a lattice of squares of side l, we define a chain of squares of side l as a sequence $Q_1, \dots, Q_n$ of squares in the lattice, such that for every $i, j \in \{1, \dots, n\}$, Q_i and Q_{i+1} have one side in common, Q_i and Q_j are disjoint in $|i-j| > 2$ and have at most a corner in common if $|i-j| = 2$.*

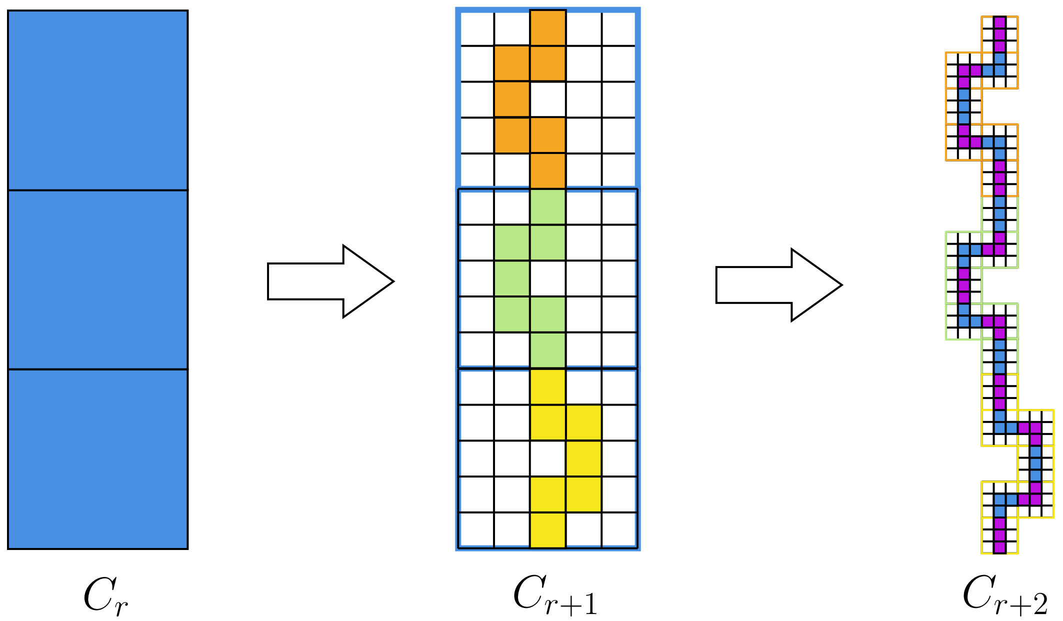


Proof. Since $\frac{1}{2} < \alpha < 1$ we can find two bounded sequences $(k_r)_{r\in\mathbb{N}}$, $(m_r)_{r\in\mathbb{N}}$ such that:

(1) $k_r \geq 2$
(2) m_r is odd
(3) $n_r \leq m_r \leq k_r^2$ where $n_r := 2k_r + 1$
(4) there exists $c, C > 0$ such that, defining $\varepsilon_0 := 1$, $\varepsilon_r := (n_1 n_2 \cdots n_r)^{-1}$ and $\delta_0 := 1$, $\delta_r := (m_1 m_2 \cdots m_r)^{-1}$,
$$c \leq \frac{\varepsilon_r}{\delta_r^\alpha} \leq C$$
for every $r \in \mathbb{N}$.

We now construct a sequence $(C_r)_{r\in\mathbb{N}}$ of chains of squares of side ε_r. Let C_0 be a square of side 1. Given $C_r = (Q_1, \dots, Q_{l_r})$ a chain of squares of side ε_r we build C_{r+1} with the following construction.

- Divide each square Q_i of C_r into a $n_{r+1} \times n_{r+1}$ grid of squares of side ε_{r+1}.
- If $1 < i < l_r$, Q_i has one side in common with Q_{i-1} and one with Q_{i+1}. Join the middle points of these two sides with a chain of squares of side ε_{r+1} consisting of m_{r+1} squares and containing no other edge squares. If $i = 1$ or $i = l_r$ join respectively the middle point of the bottom edge and the middle point of the upper edge with the middle points of the edges of Q_2 and Q_{l_r-1} touching Q_i.
- Let C_{r+1} be the chain of squares of side ε_{r+1} obtained by joining all the chains of squares of each Q_i.



The sequence $(C_r)_{r\in\mathbb{N}}$ converges to a curve which we parametrise by $t \mapsto u(t)$, with $t \in [0,1]$, so that $u(t)$ spends time δ_r in each square of C_r. Observe that $(\delta_r)_{r\in\mathbb{N}}$ is a sequence starting at 1, strictly decreasing and converging to 0, hence given $s,t \in [0,1]$ there exists $r \in \mathbb{N}$ such that

$$\delta_r \leq |t-s| \leq \delta_{r-1}.$$

This means that $u(t)$ and $u(s)$ belong either to the same square or to adjoining squares of C_{r-1}, so

$$|u(t) - u(s)| \leq 3\varepsilon_{r-1}.$$

Moreover, since $(k_r)_{r\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that $n_r \leq M$ for every $r \in \mathbb{N}$ and

$$\begin{aligned}\frac{|u(t)-u(s)|}{|t-s|^\alpha} &\leq \frac{3\varepsilon_{r-1}}{\delta_r^\alpha}\frac{n_r}{n_r}\\ &\leq \frac{3\varepsilon_r}{\delta_r^\alpha} n_r\\ &\leq 3CM\end{aligned}$$

for every $r \in \mathbb{N}$. On the other hand $u(t)$ and $u(s)$ can not be in the same square of C_r and can not be in adjoining squares of C_{r+1}, so

$$|u(t)-u(s)| \geq \varepsilon_{r+1}.$$

To conclude the proof observe that

$$\begin{aligned}\frac{|u(t)-u(s)|}{|t-s|^\alpha} &\geq \frac{\varepsilon_{r+1}}{\delta_{r-1}^\alpha}\frac{n_r n_{r+1}}{n_r n_{r+1}}\\ &\geq \frac{\varepsilon_{r-1}}{\delta_{r-1}^\alpha}\frac{1}{n_r n_{r+1}}\\ &\geq \frac{c}{M^2}.\end{aligned}$$

□

Step 2. We now prove that there exists $f \in \mathcal{C}^\gamma$ such that $G_t = f(X_t^2, X_t^3)$. Define the set

$$A = \{(X_t^2, X_t^3) \in \mathbb{R}^2 : t \in [0,1]\}.$$

We will first define a γ-Hölder function from $A \to \mathbb{R}$ and then extend it to $\mathbb{R}^2$. To this purpose we could apply Whitney's Extension Theorem, but since $\gamma < 1$ we can use the following more elementary result which gives an explicit definition of the extension.

Lemma 6.4. *Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}$ a γ-Hölder function with $\gamma \in]0,1[$. Then there exists $h : \mathbb{R}^n \to \mathbb{R}$ such that*

(1) $h(x) = f(x)$ for every $x \in A$
(2) h is γ-Hölder with $[h]_{\mathcal{C}^\gamma} = [f]_{\mathcal{C}^\gamma}$.

We postpone the proof of Lemma 6.4 and focus on the construction of $f : A \to \mathbb{R}$. Observe that the function $t \mapsto (X_t^2, X_t^3)$ is injective, in fact if $(X_t^2, X_t^3) = (X_s^2, X_s^3)$ for some $s \neq t$, then (6.2) could not hold. We can define

$$f(x,y) := G((X^2, X^3)^{-1}(x,y))$$

for every $(x,y) \in A$, so that

$$f(X_t^2, X_t^3) = G(t)$$

for every $t \in [0,1]$. Let $(x,y),(x',y') \in A$, then there exists $s,t \in [0,1]$ such that $(X^2_t, X^3_t) = (x,y), (X^2_s, X^3_s) = (x',y')$, so, recalling that G is $(1-\alpha)$-Hölder,

$$\begin{aligned} |f(x,y) - f(x',y')| &= |G(t) - G(s)| \\ &\leq C|t-s|^{1-\alpha} \\ &\leq C|(X^2_t, X^3_t) - (X^2_s, X^3_s)|^{\frac{1}{\alpha}-1} \\ &= C|(x,y) - (x',y')|^{\gamma} \end{aligned}$$

for some $C > 0$ that does not depend on (x,y) or (x',y'). We have proved that $f : A \to \mathbb{R}$ is γ-Hölder; to conclude the proof we now prove Lemma 6.4.

Proof of Lemma 6.4. Set $L = [f]_{\mathcal{C}^\gamma}$ and define

$$h(x) = \inf\{f(y) + L|x-y|^\gamma : y \in A\}$$

for every $x \in \mathbb{R}^n$. We start by proving that h is an extension of f. Let $y_0, y \in A, x \in \mathbb{R}^n$ and observe that

$$\begin{aligned} f(y) - f(y_0) + L|x-y|^\gamma &\geq -L|y-y_0|^\gamma + L|x-y|^\gamma \\ &\geq -L|x-y_0|^\gamma. \end{aligned}$$

So,

$$h(x) \geq f(y_0) - L|x-y_0|^\gamma$$

and if $x \in A$, choosing $y_0 = x$, we find

$$h(x) \geq f(x).$$

On the other hand it follows immediately from the definition of h that

$$h(x) \leq f(x)$$

for every $x \in A$. This proves that h is an extension of f. To prove that h is γ-Hölder consider $x,y \in \mathbb{R}^n$ and $\varepsilon > 0$, then there exists $y_0 \in A$ such that

$$h(x) \geq f(y_0) + L|x-y_0|^\gamma - \varepsilon.$$

Moreover

$$h(y) \leq f(y_0) + L|y-y_0|^\gamma.$$

Putting the inequalities together we find

$$\begin{aligned} h(y) - h(x) &\leq L|y-y_0|^\gamma - L|x-y_0|^\gamma + \varepsilon \\ &\leq L|y-x|^\gamma + \varepsilon. \end{aligned}$$

Exchanging the roles of x and y, and letting $\varepsilon \to 0$, we find

$$|h(x) - h(y)| \leq L|x-y|^\gamma.$$

□

6.2. Rough and general case. Suppose $\alpha \in]\frac{1}{n+1}, \frac{1}{n}[$, for some $n \geq 2$, and $\gamma = \frac{1}{\alpha} - 1$. We want to construct functions

- $X : [0, T] \to \mathbb{R}^{n+2}$ of class $\mathcal{C}^\alpha$
- $\sigma : \mathbb{R}^{n+2} \to L(\mathbb{R}^{n+2}, \mathbb{R}^{n+2})$ of class $\mathcal{C}^\gamma$

such that for every $0 < T < \pi$ the system

$$\begin{cases} \delta Z_{st} = \sum_{k=1}^n \sigma_k(Z_s)\mathbb{X}^k_{st} + o(t-s) \\ Z_0 = 0 \end{cases} \tag{6.3}$$

does not admit any solution, where

$$\sigma_1(z) := \sigma(z), \quad \sigma_k(z) = \nabla\sigma_{k-1}(z)\sigma(z)$$

and $\mathbb{X} = (\mathbb{X}^1, \ldots, \mathbb{X}^n)$ is as in Definition 4.4. The main result of this section is the following

Theorem 6.5. *Let G be as in (5.1). Fix $\alpha > 0$ and set $\gamma = \frac{1}{\alpha} - 1$.*

(1) If $\alpha > \frac{1}{3}$, there exists $X^2, X^3, X^4 : [0,1] \to \mathbb{R}$ in $\mathcal{C}^\alpha$ and $C > 0$ such that

$$|(X^2_t, X^3_t, X^4_t) - (X^2_s, X^3_s, X^4_t)| \geq C|t-s|^\alpha$$

for every $0 \geq s \geq t \geq T$.

(2) More generally, if $\alpha \in]\frac{1}{n+1}, \frac{1}{n}[$ for some $n \geq 3$, there exist $X^2, \ldots, X^{n+2} : [0,1] \to \mathbb{R}$ in $\mathcal{C}^\alpha$ and $C > 0$ such that

$$|(X^2_t, \ldots, X^{n+2}_t) - (X^2_s, \ldots, X^{n+2}_t)| \geq C|t-s|^\alpha$$

for every $0 \geq s \geq t \geq T$.

(3) If $\alpha > \frac{1}{3}$, there exists $f : \mathbb{R}^3 \to \mathbb{R}$ in $\mathcal{C}^\gamma$ such that

$$G_t = f(X^2_t, X^3_t, X^4_t),$$

where X^2, X^3, X^4 are defined as in point (1). Moreover the gradient of f vanishes along (X^2, X^3, X^4), i.e. $\nabla f(X^2_t, X^3_t, X^4_t) = 0$ for all $0 \leq t \leq T$.

(4) If $\alpha \in]\frac{1}{n+1}, \frac{1}{n}[$ for some $n \geq 3$, there exists $f : \mathbb{R}^{n+2} \to \mathbb{R}$ in $\mathcal{C}^\gamma$ such that

$$G_t = f(X^2_t, \ldots, X^{n+2}_t),$$

where $X^2, \ldots, X^{n+2}$ are defined as in point (2). Moreover for all $0 \leq t \leq T$ and $j \in \{1, \ldots, n-1\}$ we have $D^{(j)} f(X^2_t, \ldots, X^{n+2}_t) = 0$.

Let us see with Theorem 6.5 implies that (6.3) does not admit any solution. We discuss the rough case in detail, that is $\alpha \in]\frac{1}{2}, \frac{1}{3}[$ and then show how to generalise the result to any $\alpha > 0$. Define

$$X_t = \begin{pmatrix} X^1_t \\ X^2_t \\ X^3_t \\ X^4_t \end{pmatrix}$$

where X^1 is as in (5.1) and X^2, X^3, X^4 as in Theorem 6.5. Moreover define $\sigma : \mathbb{R}^4 \to L(\mathbb{R}^4, \mathbb{R}^4)$ as

$$\sigma(x,y,z,w) = \begin{pmatrix} f(y,z,w) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then σ is of class $\mathcal{C}^\gamma$ and X of class $\mathcal{C}^\alpha$. Let $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ be any α-rough path over X. Then (6.3) is equivalent to

$$\begin{cases} \delta Z^1_{st} = f(Z^2_s, Z^3_s, Z^4_s)\delta X^1_{st} + \nabla f(Z^2_s, Z^3_s, Z^4_s) \cdot \Big((\mathbb{X}^2)^{21}, (\mathbb{X}^2)^{31}, (\mathbb{X}^2)^{41} \Big) + o(t-s) \\ \delta Z^2_{st} = \delta X^2_{st} + o(t-s) \\ \delta Z^3_{st} = \delta X^3_{st} + o(t-s) \\ \delta Z^4_{st} = \delta X^4_{st} + o(t-s). \end{cases} \tag{6.4}$$

Suppose that we have a solution

$$Z_t = \begin{pmatrix} Z^1_t \\ Z^2_t \\ Z^3_t \\ Z^4_t \end{pmatrix}$$

on $[0,T]$ for some $T \in]0,\pi[$, then

$$Z^2_t = X^2_t, \quad Z^3_t = X^3_t, \quad Z^4_t = X^4_t$$

and, by Theorem 6.5, system (6.4) becomes

$$\begin{cases} \delta Z^1_{st} = f(Z^2_s, Z^3_s, Z^4_s)\delta X^1_{st} + o(t-s) \\ \delta Z^2_{st} = \delta X^2_{st} + o(t-s) \\ \delta Z^3_{st} = \delta X^3_{st} + o(t-s) \\ \delta Z^4_{st} = \delta X^4_{st} + o(t-s). \end{cases}$$

Finally, for $0 < s < T$,

$$Z^1_T - Z^1_s = \int_s^T G_u \dot{X}^1_u \, \mathrm{d}u$$

and recalling (5.2) and Theorem 5.1

$$\geq -\frac{1}{2}\log(s) + O(1).$$

as $s \to 0$. This means that $Z_s \not\to 0$ as $s \to 0$ and the initial condition can not be satisfied.

Let us now discuss how to generalise this example to $\alpha \in]\frac{1}{n+1}, \frac{1}{n}[$ for $n \geq 3$. Define

$$X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \\ \vdots \\ X_t^{n+2} \end{pmatrix}$$

where X^1 is as in (5.1) and $X^2, \ldots, X^{n+2}$ as in Theorem 6.5. Moreover define $\sigma : \mathbb{R}^{n+2} \to L(\mathbb{R}^{n+2}, \mathbb{R}^{n+2})$ as

$$[\sigma(x_1, \ldots, x_{n+2})]^i_j = \begin{cases} f(x_2, \ldots, x_{n+2}) & \text{if } i = 1, j = 1 \\ 1 & \text{if } i \geq 2, j = i \\ 0 & \text{otherwise} \end{cases},$$

where f is as in Theorem 6.5. Then σ is of class $\mathcal{C}^\gamma$ and X of class $\mathcal{C}^\alpha$, with $\gamma = \frac{1}{\alpha} - 1$. Let $\mathbb{X} = (\mathbb{X}^1, \ldots, \mathbb{X}^n)$ be any α-rough path over X, see Definition 4.4. Then (6.3) is equivalent to

$$\begin{cases} \delta Z^1_{st} = \sum_{k=1}^n [\sigma_k(Z^1_s, \ldots, Z^{n+2}_s)\mathbb{X}^k_{st}]^1 + o(t-s) \\ \delta Z^2_{st} = \delta X^2_{st} + o(t-s) \\ \vdots \\ \delta Z^{n+2}_{st} = \delta X^{n+2}_{st} + o(t-s). \end{cases} \tag{6.5}$$

Suppose that we have a solution

$$Z_t = \begin{pmatrix} Z_t^1 \\ \vdots \\ Z_t^{n+2} \end{pmatrix}$$

on $[0, T]$ for some $T \in]0, \pi[$, then

$$Z^2_t = X^2_t, \ldots, \quad Z^{n+2}_t = X^{n+2}_t$$

and, by Theorem 6.5, system (6.5) becomes

$$\begin{cases} \delta Z^1_{st} = f(Z^2_s, \ldots, Z^{n+2}_s)\delta X^1_{st} + o(t-s) \\ \delta Z^2_{st} = \delta X^2_{st} + o(t-s) \\ \vdots \\ \delta Z^{n+2}_{st} = \delta X^{n+2}_{st} + o(t-s). \end{cases}$$

Finally, for $0 < s < T$,

$$Z^1_T - Z^1_s = \int_s^T G_u \dot{X}^1_u \, \mathrm{d}u$$

and recalling (5.2) and Theorem 5.1

$$\geq -\frac{1}{2}\log(s) + O(1).$$

as $s \to 0$. This means that $Z_s \not\to 0$ as $s \to 0$ and the initial condition can not be satisfied.

Let us now prove Theorem 6.5. We will prove only the third point of the Theorem, as the proofs of the first and the second are similar to the one of Lemma 6.2. The proof of the last point is similar to the one of the third, so we omit it.

Proof of Theorem 6.5 point 3. We prove that there exists $f \in \mathcal{C}^\gamma$ such that $G_t = f(X_t^2, X_t^3, X_t^4)$. Define the set

$$A = \{(X_t^2, X_t^3, X_t^4) \in \mathbb{R}^3 : t \in [0,1]\}.$$

We will first define a γ-Hölder function from $A \to \mathbb{R}$ and then extend it to $\mathbb{R}^2$ using Whitney's Extension Theorem. Observe that the function $t \mapsto (X_t^2, X_t^3, X_t^4)$ is injective, in fact if $(X_t^2, X_t^3, X_t^4) = (X_s^2, X_s^3, X_s^4)$ for some $s \neq t$, then the first point of the Theorem could not hold. We can define

$$f(x,y,z) := G((X^2, X^3, X^4)^{-1}(x,y,z))$$

for every $(x,y,z) \in A$, so that

$$f(X_t^2, X_t^3, X_t^4) = G(t)$$

for every $t \in [0,1]$. Let $(x,y,z),(x',y',z') \in A$, then there exists $s,t \in [0,1]$ such that $(X_t^2, X_t^3, X_t^4) = (x,y,z), (X_s^2, X_s^3, X_s^4) = (x',y',z')$, so, recalling that G is $(1-\alpha)$-Hölder,

$$\begin{aligned} |f(x,y,z) - f(x',y',z')| &= |G(t) - G(s)| \\ &\leq C|t-s|^{1-\alpha} \\ &\leq C|(X_t^2, X_t^3, X_t^4) - (X_s^2, X_s^3, X_s^4)|^{\frac{1}{\alpha}-1} \\ &= C|(x,y,z) - (x',y',z')|^\gamma \end{aligned}$$

for some $C > 0$ that does not depend on (x,y,z) or (x',y',z'). Observe that $\gamma > 1$ and, recalling Definition 3.1, we just proved that f is in $\mathcal{C}^\gamma(A)$ choosing $f^{(j)} = 0$ for every $|j| = 1$. We can now extend f to $\mathbb{R}^3$ using Whitney's Extension Theorem; in particular we apply Theorem 4 of section VI.2 of [Ste70], which we have reported in Theorem 3.2 for convenience. Moreover we obtain that $\nabla f(x,y,z) = 0$ for every $(x,y,z) \in A$, or equivalently $\nabla f(X_t^2, X_t^3, X_t^4) = 0$ for every $t \in [0,1]$. $\square$

7. Additional remarks

7.1. Young Integral. In section 3.2 we introduced a one-dimensional integral and studied its properties, see Theorem 3.3. We could define I as

$$I_t = \int_0^t (X_u^2)^\gamma dX_u^1$$

where the previous integral is a Young integral, which is well defined since X^1 and $(X^2)^\gamma$ are respectively in $\mathcal{C}^{\frac{\beta}{\eta+1}}$ and in $\mathcal{C}^{\frac{\beta\gamma}{\eta+1}}$ and $\frac{\beta(\gamma+1)}{\eta+1} > 1$. Once more we need to show that point 3 and 4 of Theorem 3.3 hold with this definition of I. The proof of point 3 is the same as before recalling that integration by parts holds for Young integrals. To prove point 4 it is enough to observe that, by definition of Young integral, I is the only function $I : [0,T] \to \mathbb{R}$ which satisfies

$$I_0 = 0, \qquad I_t - I_s = (X_s^2)^\gamma (X_t^1 - X_s^1) + o(t-s)$$

and point 4 holds trivially. Observe that $\int_0^t (X_u^2)^\gamma \,\mathrm{d}X_u^1$ cannot be interpreted as a Lebesgue integral because X^1 is not a function of bounded variation.

7.2. Integral inequality. Let $\gamma > 1$ and let $f : [0,+\infty) \to \mathbb{R}$ be a positive function such that $f(0) = 0$; fix $T > 0$ such that $f(x) \leq 1$ for every $x \in [0,T]$ and define $M = \max_{x\in[0,T]} f(x)$. Let g be a continuous function of bounded variation on $[0,T]$ and assume that $\max_{x\in[0,T]} g(x) - \min_{x\in[0,T]} g(x) \leq 1$. Then, we can write

$$g(x) = g_1(x) - g_2(x), \qquad \forall x \in [0,T],$$

where g_1 and g_2 are two monotone non decreasing functions. It is well defined the Lebesgue-Stieltjes integral of f with respect to g defined as

$$\int_a^b f(x)dg(x) = \int_a^b f(x)dg_1(x) - \int_a^b f(x)dg_2(x),$$

where dg_1 and dg_2 are the Lebesgue-Stieltjes measures associated respectively to g_1 and g_2. Since f is continuous, there exists $x_0 \in [0,T]$ such that $f(x_0) = M$. Fix $0 < c < M$, then there exists $0 < a < x_0$ such that $f(x) < c$ for every $x \in [0,a]$. So,

$$\begin{aligned}
\int_0^a (f(x))^\gamma dg(x) &= \int_0^a (f(x))^\gamma dg_1(x) - \int_0^a (f(x))^\gamma dg_2(x) \\
&< c^\gamma (g_1(a) - g_1(0)) - \int_0^a (f(x))^\gamma dg_2(x) \\
&< M^\gamma (g_1(a) - g_1(0)) \\
&< M(g_1(a) - g_1(0)).
\end{aligned}$$

So,

$$\begin{aligned}\int_0^{x_0}(f(x))^\gamma dg(x) &= \int_0^a (f(x))^\gamma dg(x) + \int_a^{x_0}(f(x))^\gamma dg(x)\\ &< M(g_1(a)-g_1(0)) + M(g_1(x_0)-g_1(a)) - \int_a^{x_0}(f(x))^\gamma dg_2(x)\\ &\le M(g_1(x_0)-g_1(0))\\ &\le M = f(x_0).\end{aligned}$$

This proves that it is not true that $\int_0^t (f(x))^\gamma dg(x) \ge f(t)$ for every $t \in [0,T]$.

7.3. Non resonant example. In section 4.2 we presented an example of a controlled difference equation which admitted two different solutions. In particular we considered a path $X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$ where

$$X_t^1 = t^\beta \cos t^{-\eta}, \qquad X_t^2 = t^\beta(2+\sin t^{-\eta}),$$

for β, η that satisfy (4.1). We are interested in what happens changing the definition of X. For example, we could define

$$X_t^1 = t^\xi$$

for some $\xi \ge \alpha$ (so that X remains α-Hölder). Then

$$I_t = \int_0^t \xi u^{\beta\gamma+\xi-1}(2+\sin u^{-\eta})\,\mathrm{d}u$$

and it is not true anymore that there exists $T \in]0,1[$ such that

$$I_t \ge X_t^2$$

for every $t \in [0,T]$. The best we can prove is that there exists $T \in]0,1[$ such that

$$I_t \ge (X_t^2)^\rho$$

for some $\rho > 1$. This suggests to modify the definition of f in (4.2) to

$$f(x,y) = \begin{cases} y^\gamma & \text{if } |x| \ge y^\rho \ge 0 \\ 0 & \text{if } x = 0. \end{cases}$$

However the resulting σ is not in $\mathcal{C}^\gamma$ anymore. In fact, for $(x,y) \in A := \{(x,y) \in \mathbb{R}^2 : |x| \ge y^\rho \ge 0 \text{ or } x = 0\}$,

$$\partial_y f(x,y) = \gamma y^{\gamma-1}\mathbb{1}_{|x|\ge y^\rho \ge 0}.$$

Let $x = 0, y > 0, x' = y^\rho$ and $y' = y$, then

$$\frac{|\partial_y f(x,y) - \partial_y f(x',y')|}{|(x,y)-(x',y')|^{\gamma-1}} = \frac{\gamma y^{\gamma-1}}{y^{\rho(\gamma-1)}}$$

which tends to infinity as $y \to 0$. This underlines once more the importance that the sine and cosine in X^2 and X^1 resonate. In fact it is only thanks to the resonance that $I_t \geq X_t^2$.

7.4. Canonical Rough Path. In section 4.3 we presented an example of rough difference equation driven by a path X, which admitted two different solutions. In particular we defined $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ as a non-canonical rough path. We made such choice because the canonical rough path for which

$$\mathbb{X}_{st}^2 = \int_s^t (X_u - X_s) \otimes \dot{X}_u \, \mathrm{d}u$$

does not satisfy the analytical condition (4.9). In fact

$$\begin{aligned}
(\mathbb{X}_{st}^2)^{2,1} &= \int_s^t (X_u^2 - X_s^2)\, \dot{X}_u^1 \, \mathrm{d}u \\
&= \int_s^t X_u^2 \dot{X}_u^1 \, \mathrm{d}u - X_s^2 \delta X_{st}^1 \\
&\geq \int_s^t X_u^2 \dot{X}_u^1 \, \mathrm{d}u - C(t-s)^{\frac{2\beta}{\eta+1}} \\
&= \int_s^t \eta u^{2\beta-\eta-1}(2+\sin u^{-\eta}) \sin u^{-\eta} \, \mathrm{d}u + \\
&\qquad + \int_s^t \beta u^{2\beta-1}(2+\sin u^{-\eta}) \cos u^{-\eta} \, \mathrm{d}u - C(t-s)^{\frac{2\beta}{\eta+1}} \\
&\geq \int_s^t \eta u^{2\beta-\eta-1} \sin^2 u^{-\eta} \, \mathrm{d}u + 2\int_s^t \eta u^{2\beta-\eta-1} \sin u^{-\eta} \, \mathrm{d}u + \\
&\qquad - 3(t^{2\beta} - s^{2\beta}) \, \mathrm{d}u - C(t-s)^{\frac{2\beta}{\eta+1}} \\
&\geq \frac{1}{2}\int_s^t \eta u^{2\beta-\eta-1} \, \mathrm{d}u - \frac{1}{2}\int_s^t \eta u^{2\beta-\eta-1} \cos 2u^{-\eta} \, \mathrm{d}u + \big(u^{2\beta} \cos u^{-\eta}\big|_s^t + + \\
&\qquad + \int_s^t 2\beta u^{2\beta-1} \cos u^{-\eta} \, \mathrm{d}u - 3(t^{2\beta} - s^{2\beta}) \, \mathrm{d}u - C(t-s)^{\frac{2\beta}{\eta+1}} \\
&\geq \frac{1}{2}\int_s^t \eta u^{2\beta-\eta-1} \, \mathrm{d}u + (\frac{1}{4} u^{2\beta} \sin 2u^{-\eta}\big|_s^t - \frac{\beta}{2}\int_s^t u^{2\beta-1} \sin 2u^{-\eta} \, \mathrm{d}u \\
&\qquad - 5(t^{2\beta} + s^{2\beta}) \, \mathrm{d}u - C(t-s)^{\frac{2\beta}{\eta+1}} \\
&\geq \frac{1}{2}\int_s^t \eta u^{2\beta-\eta-1} \, \mathrm{d}u - (6+\beta)(t^{2\beta} + s^{2\beta}) \, \mathrm{d}u - C(t-s)^{\frac{2\beta}{\eta+1}}
\end{aligned}$$

and the first integral diverges as $s \to 0$ since $2\beta - \eta - 1 < -1$. Analogously we can prove that $(\mathbb{X}^2)^{1,2}$ is not $O(t-s)^{2\alpha}$. On the other hand $(\mathbb{X}_{st}^2)^{1,1}$ and $(\mathbb{X}_{st}^2)^{2,2}$

are both $O(t-s)^{2\alpha}$, in fact for $i = 1, 2$

$$\int_s^t (X_u^i - X_s^i)\, \dot{X}_u^i \, \mathrm{d}u = \frac{1}{2}(X_t^i - X_s^i)^2$$

and X is α-Hölder.

7.5. Geometric Rough Path - Solutions. In section 4.5 we built a sequence of canonical rough paths $(\mathbb{X}_n) = (\mathbb{X}_n^1, \mathbb{X}_n^2)$ over smooth paths $X_n = \begin{pmatrix} X_n^1 \\ X_n^2 \end{pmatrix}$ such that $\mathbb{X}_n$ converges to a rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ over $X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$ for which the problem

$$\begin{cases} \delta Z_{st} = \sigma(Z_s)\mathbb{X}_{st}^1 + \sigma_2(Z_s)\mathbb{X}_{st}^2 + o(t-s) \\ Z_0 = 0, \end{cases} \tag{7.1}$$

with σ as in (4.11) does not admit a unique solution in $[0, T]$. In particular, we proved in section 4.3 that (7.1) admits

$$Z_t = \begin{pmatrix} 0 \\ X_t^2 \end{pmatrix} \text{ and } \bar{Z}_t = \begin{pmatrix} (1-\gamma)I_t \\ X_t^2 \end{pmatrix}$$

as solutions, where I_t is defined as in (3.3). For every $n \in \mathbb{N}$ consider the problem

$$\begin{cases} \delta Z_{st} = \sigma(Z_s)\mathbb{X}_{n,s,t}^1 + \sigma_2(Z_s)\mathbb{X}_{n,s,t}^2 + o(t-s) \\ Z_0 = 0 \end{cases} \tag{7.2}$$

and define a sequence of functions $(Z_n)_{n\in\mathbb{N}}$ in which each element is the solution to (7.2) for the corresponding n. Observe that $(Z_n)_{n\in\mathbb{N}}$ is well defined because σ is in $\mathcal{C}^\gamma$ with $\gamma \in]2, 3[$ and, for every n, X_n is a smooth path, hence for every n (7.2) admits a unique solution. We want to understand to which solution of (7.1) this sequence converges.

Recall that If z is such that $z = (0, y)$ for some $y \in \mathbb{R}$, then

$$[\sigma_2(z)]_{jl}^i = 0$$

for every $i, j, l \in \{1, 2\}$ because

$$\frac{\partial \sigma_1^1(z)}{\partial z_2} = 0 \text{ and } \sigma_1^1(z) = 0.$$

Now, $Z_{n,t} = \begin{pmatrix} 0 \\ X_{n,t}^2 \end{pmatrix}$ trivially satisfies (7.2). In fact, noticing that $\sigma^{1,1}(Z_{n,s}) = 0$ for every $s \in [0, T]$ and recalling the computations above, problem (7.2) is equivalent to

$$\begin{cases} \delta Z^1_{st} = o(t-s) \\ \delta Z^2_{st} = \delta X^2_{n,s,t} + o(t-s) \\ Z^1_0 = 0 \\ Z^2_0 = 0. \end{cases}$$

This proves that $(Z_n)_{n\in\mathbb{N}}$ converges to $Z_t = \begin{pmatrix} 0 \\ X^2_t \end{pmatrix}$. Observe that to prove this result we only used properties of σ, hence for any sequence of rough paths $(\mathbb{X}_n)$ over paths X_n in $\mathcal{C}^\alpha$ with $\gamma > \frac{1}{\alpha}$ (so that the solution of (7.2) is unique) converging to $\mathbb{X}$, the solutions to the corresponding problems will converge to $Z_t = \begin{pmatrix} 0 \\ X^2_t \end{pmatrix}$. We stress that this does not depend on the fact that $\mathbb{X}$ is a geometric rough path.

Let us finally explain why the the approximating rough difference equations fail to admit a solution similar to $\overline{Z}$. Recall that we were able to prove that $\overline{Z}$ was a solution of (7.1) by demonstrating that $I_t \geq X^2_t$ for t small enough. This inequality primary relied on two key factors: firstly, X^1 oscillates rapidly near the origin and in particular it is not a function of bounded variation; secondly, the sine and cosine in X^1 and X^2 resonate causing the integral to behave like a power function with exponent $\beta(\gamma+1)-\eta$ for sufficiently small t. Since X^2_t is bounded by $3t^\beta$ this ensured that $I_t \geq X^2_t$. If we define

$$I_{n,t} := \int_0^t (X^2_{n,u})^\gamma \dot{X}^1_{n,u}\, du,$$

then there cannot exist $T > 0$ such that $I_{n,t} \geq X^2_{n,t}$ for every $t \in [0,T]$. This holds because X^1_n is of bounded variation, see section 7.2 for a more detailed discussion.

7.6. Reverse Hölder functions. Let $\alpha \in [0,1]$. There cannot exist a function $f : [0,1] \to \mathbb{R}$ in $\mathcal{C}^\alpha$ and a constant $C > 0$ such that

$$|f(t) - f(s)| \geq C|t-s|^\alpha.$$

Suppose such a function exists. Then f must be injective. Indeed, if $f(t) = f(s)$ for some $t \neq s$, we would have $|f(t) - f(s)| = 0$, which cannot be greater than $C|t-s|^\alpha$ for any $C > 0$. Since f is in $\mathcal{C}^\alpha$, it is continuous. Being both continuous and injective, f must be strictly monotone.

Consider the partition $\mathbb{P}_n$ of $[0,1]$ defined as

$$\mathbb{P}_n = \left\{ \frac{k}{n} : k = 0, \dots, n \right\}.$$

Then,

$$\sum_{k=0}^{n-1} \left| f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right| \geq \sum_{k=0}^{n-1} \frac{C}{n^\alpha} = Cn^{1-\alpha}.$$

This implies that f is not a function of bounded variation. However, this is a contradiction because a strictly monotone function is always of bounded variation.

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