An introduction to regularity structures

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8-12 August 2022
Campinas

These slides can be downloaded from my home page

Plan of the course

In this course I want to discuss mainly the analytical aspects of the theory of regularity structures.

The plan for the four lectures is roughly:

- I. Reconstruction Theorem
- II. Models and modelled distributions
- III. Schauder estimates
- IV. Products and equations

Lecture notes and papers in collaboration with F. Caravenna and L. Broux.

Chapter 1: The Reconstruction Theorem

A theory, a theorem

This talk is based on a paper (appeared in 2021 in the EMS Surveys in Mathematics)

- *Hairer's Reconstruction Theorem without Regularity Structures*
  by F. Caravenna and L.Z.

In this paper we have extracted a single result (the Reconstruction Theorem) from a larger theory (Regularity Structures).

We present the former in a simpler and more general version, without reference to the latter.

A later paper by Pavel Zorin-Kranich, to appear in Revista Matematica Iberoamericana, has introduced introduced further simplifications and improvements to our results.

Prelude: The Sewing Lemma

You have seen in Sebastian's lectures last week that the main tool in rough analysis is the

Theorem (Sewing Lemma)

Let $0 < \alpha \leq 1 < \beta$. *There exists a unique map* $\mathcal{I} : \mathcal{C}_2^{\alpha,\beta}([0,T] : \mathbb{R}^d) \to \mathcal{C}^\alpha([0,T] : \mathbb{R}^d)$ *s.t.*

$$(\mathcal{I}\Xi)_0 = 0, \qquad |\mathcal{I}\Xi_t - \mathcal{I}\Xi_s - \Xi_{s,t}| \lesssim |t-s|^\beta, \qquad s,t \in [0,T].$$

We recall that $\mathcal{C}_2^{\alpha,\beta}$ denotes the space of continuous $\Xi : \{(s,t) : 0 \leq s \leq t \leq T\} \to \mathbb{R}^d$ s.t.

$$\sup_{0\leq s<t\leq T} \frac{|\Xi_{s,t}|}{|t-s|^\alpha} + \sup_{0\leq s<u<t\leq T} \frac{|\Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}|}{|t-s|^\beta} < +\infty.$$

This theorem was proved around 2003 indipendently by Gubinelli and Feyel-de la Pradelle.

It is restricted to functions depending on a one-dimensional parameter.

It took ten years to find a version of this result in higher dimension... This is Martin's Reconstruction Theorem.

Distributions

This talk will concern the space $\mathcal{D}'(\mathbb{R}^d)$ of distributions or generalised functions.

We consider the space $\mathcal{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support on $\mathbb{R}^d$.

A distribution on $\mathbb{R}^d$ is a linear functional $T : C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$ such that for every compact set $K \subset \mathbb{R}^d$ there is $r = r_K \in \mathbb{N}$

$$|T(\varphi)| \lesssim \|\varphi\|_{C^r} := \max_{|k|\le r} \|\partial^k \varphi\|_\infty, \qquad \forall\, \varphi \in C_0^\infty(K)$$

where throughout the lectures $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \le C\, g$.

When r can be chosen uniformly over K we say that T has order r.

Distributions

Every locally integrable (in particular continuous) function $f : \mathbb{R}^d \to \mathbb{R}$ defines a distribution:

$$f(\varphi) := \int_{\mathbb{R}^d} f(x)\, \varphi(x)\, \mathrm{d}x, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

A famous example of distribution from quantum mechanics, which actually motivated the whole theory of distributions, is the Dirac measure δ_x at $x \in \mathbb{R}^d$

$$\delta_x(\varphi) = \varphi(x), \qquad \varphi \in C^\infty(\mathbb{R}^d).$$

One can also differentiate any distribution $T \in \mathcal{D}'(\mathbb{R}^d)$: for $k \in \mathbb{N}^d$

$$\partial^k T(\varphi) := (-1)^{k_1+\cdots+k_d}\, T(\partial^k \varphi).$$

Products of distributions

Distributions form a linear space. If $\varphi \in C^\infty(\mathbb{R}^d)$ and $T \in \mathcal{D}'(\mathbb{R}^d)$ then it is possible to define canonically the product $\varphi \cdot T = T \cdot \varphi$ as

$$\varphi \cdot T(\psi) = T \cdot \varphi(\psi) := T(\varphi\psi), \qquad \forall\, \psi \in C_c^\infty(\mathbb{R}^d).$$

However, if $T, T' \in \mathcal{D}'(\mathbb{R}^d)$, in general there is no canonical way of defining $T \cdot T'$.

One may use some form of regularisation of T, T' or both. Then, the result could heavily depend on the regularisation and thus be neither unique nor canonical.

For example, one can not define the square $(\delta_x)^2$ of the Dirac function.

The main question of reconstruction

For every $x \in \mathbb{R}^d$ we fix a distribution $F_x \in \mathcal{D}'(\mathbb{R}^d)$. If for all $\psi \in \mathcal{D}$ the map

$$\mathbb{R}^d \ni x \mapsto F_x(\psi)$$

is measurable, then we call $(F_x)_{x\in\mathbb{R}^d}$ a germ.

Problem:
Can we find a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ which is locally well approximated by $(F_x)_{x\in\mathbb{R}^d}$?

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Note that for $j \in \mathbb{N}^d$, $w \in \mathbb{R}^d$, we use the notation

$$|j| := \sum_{k=1}^{d} j_k, \qquad w^j := \prod_{k=1}^{d} w_k^{j_k}, \qquad j! := \prod_{k=1}^{d} j_k!$$

with the convention $0^0 := 1$.

Taylor expansions

For example, let us fix $f \in C^\infty(\mathbb{R}^d)$, and let us define for a fixed $\gamma > 0$

$$F_x(y) := \sum_{|k|<\gamma} \partial^k f(x) \, \frac{(y-x)^k}{k!}, \qquad x, y \in \mathbb{R}^d \, .$$

Then the classical Taylor theorem says that there exists a function $R(x,y)$ such that

$$f(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma$$

uniformly for every x, y on compact sets of $\mathbb{R}^d$.

We say that the distribution f is locally well approximated by the germ $(F_x)_{x \in \mathbb{R}^d}$.

Scaling

Let us introduce now the following fundamental tool:

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\lambda > 0$ and $y \in \mathbb{R}^d$

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d}\,\varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d.$$

Then the local approximation property

$$f(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma$$

implies for any $\varphi \in \mathcal{D}$, uniformly for y in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$.

$$\left|(f - F_y)(\varphi_y^\lambda)\right| = \left|\int_{\mathbb{R}^d} R(y,w)\,\varphi_y^\lambda(w)\,\mathrm{d}w\right|$$

$$\lesssim \frac{1}{\lambda^d}\int_{B_y(\lambda)} |w-y|^\gamma\,\mathrm{d}w \lesssim \lambda^\gamma$$

Taylor expansions

Another simple formula in this context is

$$\left|(F_z - F_y)(\varphi_y^\lambda)\right| \lesssim (|y-z| + \lambda)^\gamma,$$

for any $\varphi \in \mathcal{D}$, uniformly for y, z in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$.

We call this property coherence, see below.

This comes from a simple estimate of $F_z(w) - F_y(w)$.

Coherence of Taylor expansions

Let us note that we can Taylor expand also the derivatives of f: for $|k| < \gamma$

$$\partial^k f(y) = \sum_{|\ell|<\gamma-|k|} \partial^{k+\ell} f(z) \, \frac{(y-z)^\ell}{\ell!} + R^k(y,z), \qquad |R^k(y,z)| \lesssim |y-z|^{\gamma-|k|}.$$

Then we can write

$$\begin{aligned} F_y(w) &= \sum_{|k|<\gamma} \partial^k f(y) \, \frac{(w-y)^k}{k!} \\ &= \sum_{|k|<\gamma} \left(\sum_{|\ell|<\gamma-|k|} \partial^{k+\ell} f(z) \, \frac{(y-z)^\ell}{\ell!} + R^k(y,z) \right) \frac{(w-y)^k}{k!} \\ &= F_z(w) + \sum_{|k|<\gamma} R^k(y,z) \, \frac{(w-y)^k}{k!}. \end{aligned}$$

Coherence of Taylor expansions

Therefore

$$F_z(w) - F_y(w) = -\sum_{|k|<\gamma} R^k(y,z)\,\frac{(w-y)^k}{k!}.$$

In particular

$$\begin{aligned}|F_z(w) - F_y(w)| &\leq \sum_{|k|<\gamma} |R^k(y,z)|\,\frac{|w-y|^k}{k!} \\ &\lesssim \sum_{|k|<\gamma} |y-z|^{\gamma-|k|}\,|w-y|^k \\ &\lesssim (|y-z|+|w-y|)^\gamma\end{aligned}$$

since $a^t b^s \leq (a+b)^t(a+b)^s$ for $a, b, t, s \geq 0$.

Coherence of Taylor expansions

Now recall that

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d}\,\varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d\,.$$

Then

$$\begin{aligned}\left|\int_{\mathbb{R}^d} (F_z(w) - F_y(w))\ \varphi_y^\lambda(w)\,\mathrm{d}w\right| &\lesssim \frac{1}{\lambda^d}\ \int_{B_y(\lambda)} (|y-z| + |w-y|)^\gamma\,\mathrm{d}w \\ &\lesssim (|y-z| + \lambda)^\gamma.\end{aligned}$$

We have obtained for the germ $(F_y)_{y\in\mathbb{R}^d}$ and for any $\varphi \in \mathcal{D}$, $y, z \in \mathbb{R}^d$

$$\left|(F_z - F_y)(\varphi_y^\lambda)\right| \lesssim (|y-z| + \lambda)^\gamma.$$

Coherence

Let us set from now on

$$\varepsilon_n := 2^{-n}, \qquad n \in \mathbb{N}.$$

In particular for the germ related to a Taylor expansion we have for $\lambda \in \{\varepsilon_n : n \in \mathbb{N}\}$

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim (|y-z| + \varepsilon_n)^\gamma, \qquad \left|(f - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\gamma,$$

for any $\varphi \in \mathcal{D}$, uniformly for y, z in compact sets of $\mathbb{R}^d$ and $n \in \mathbb{N}$.

We say that a germ $(F_z)_{z \in \mathbb{R}^d} \subset \mathcal{D}'$ is (α, γ)-coherent for $\alpha, \gamma \in \mathbb{R}$ with $\alpha \leq \gamma$, if there exists $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\alpha (|y-z| + \varepsilon_n)^{\gamma-\alpha}$$

uniformly for z, y in compact sets of $\mathbb{R}^d$ and $n \in \mathbb{N}$.

Hairer's Reconstruction Theorem (without regularity structures)

Theorem (Hairer 14, Caravenna-Z. 20)

Consider a (α,γ)*-coherent germ with* $\gamma > 0$*, namely we suppose that there exists* $\varphi \in \mathcal{D}(\mathbb{R}^d)$ *such that* $\int \varphi \neq 0$ *and*

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\alpha(|x-y| + \varepsilon_n)^{\gamma-\alpha},$$

uniformly for x, y *in compact sets of* $\mathbb{R}^d$ *and* $n \in \mathbb{N}$ *(coherence condition). Then there exists a unique* $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ *such that*

$$|(\mathcal{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$$

uniformly for x *in compact sets of* $\mathbb{R}^d$, $n \in \mathbb{N}$, $\{\psi \in \mathcal{D}(B(0,1)) : \|\psi\|_{C^r} \leq 1\}$ *with a fixed* $r > -\alpha$.

Comments

- This result was stated and proved by Martin in [Hai14] for a subclass of germs related to regularity structures. He used wavelets.
- Later Otto-Weber proposed an approach based on a semigroup. This corresponds to a special choice of the test functions φ, ψ.
- Our statement is more general and requires no knowledge of regularity structures.
- This result can be seen as a generalisation of the Sewing Lemma in rough paths (Gubinelli, Feyel-de La Pradelle).
- The construction is completely local: constants and even the exponent α can depend on the compact set.
- We also cover the case $\gamma \leq 0$ (see below).
- Pavel Zorin-Kranich recently showed how to simplify, shorten and (slightly) improve our proof.

Proof for $\gamma > 0$: Uniqueness

Suppose we have two distributions $f, g \in \mathcal{D}'$ which satisfy, uniformly for $x \in K$ for any compact $K \subset \mathbb{R}^d$,

$$\lim_{n \to +\infty} |(f - F_x)(\varphi_x^{\varepsilon_n})| = \lim_{n \to +\infty} |(g - F_x)(\varphi_x^{\varepsilon_n})| = 0 . \tag{1}$$

We may assume that $c := \int \varphi = 1$ (otherwise just replace φ by $c^{-1}\,\varphi$).

We set $T := f - g$, we fix a test function $\psi \in \mathcal{D}$. We recall the definition of the convolution

$$\psi * \varphi(w) = \int_{\mathbb{R}^d} \psi(y)\,\varphi(w-y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \psi(w-y)\,\varphi(y)\,\mathrm{d}y ,$$

for $w \in \mathbb{R}^d$. This implies

$$T(\psi * \varphi) = \int_{\mathbb{R}^d} \psi(y)\,T(\varphi(\cdot - y))\,\mathrm{d}y = \int_{\mathbb{R}^d} T(\psi(\cdot - y))\,\varphi(y)\,\mathrm{d}y . \tag{2}$$

Proof for $\gamma > 0$: Uniqueness

It follows that

$$T(\psi) = \lim_{n \to +\infty} T(\psi * \varphi_0^{\varepsilon_n}).$$

Moreover

$$T(\psi * \varphi_0^{\varepsilon_n}) = \int_{\mathbb{R}^d} T(\varphi_0^{\varepsilon_n}(\cdot - y))\, \psi(y)\, \mathrm{d}y = \int_{\mathbb{R}^d} T(\varphi_y^{\varepsilon_n})\, \psi(y)\, \mathrm{d}y\,,$$

$$|T(\psi * \varphi_0^{\varepsilon_n})| = \left| \int_{\mathbb{R}^d} T(\varphi_y^{\varepsilon_n})\, \psi(y)\, \mathrm{d}y \right| \leq \|\psi\|_{L^1} \sup_{y \in \mathrm{supp}(\psi)} \left| T(\varphi_y^{\varepsilon_n}) \right| .$$

It remains to show that $\lim_{n \to +\infty} \sup_{y \in \mathrm{supp}(\psi)} \left| T(\varphi_y^{\varepsilon_n}) \right| = 0$. Now

$$|T(\varphi_y^{\varepsilon_n})| = |f(\varphi_y^{\varepsilon_n}) - g(\varphi_y^{\varepsilon_n})| \leq |(f - F_y)(\varphi_y^{\varepsilon_n})| + |(g - F_y)(\varphi_y^{\varepsilon_n})|$$

which vanishes as $n \to +\infty$ uniformly for $y \in \mathrm{supp}(\psi)$, by the reconstruction bound (1).

Proof for $\gamma > 0$: Existence

We fix a test function $\varphi \in \mathcal{D}$ with $\int \varphi \neq 0$ which makes the germ F coherent.

We can find in an elementary way a related $\hat{\varphi} \in \mathcal{D}(B(0,1))$ such that

$$\int_{\mathbb{R}^d} y^k \, \hat{\varphi}(y) \, \mathrm{d}y = 0 \,, \qquad \forall\, k \in \mathbb{N}_0^d : \; 1 \leq |k| \leq r - 1 \,,$$

for a given $r > -\alpha$. Then we define

$$\rho := \hat{\varphi}^2 * \hat{\varphi} \qquad \text{and} \qquad \check{\varphi} := \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2,$$

where by $\hat{\varphi}^{\frac{1}{2}}, \hat{\varphi}^2$ we mean $\hat{\varphi}^{\lambda}(z) = \lambda^{-d}\hat{\varphi}(\lambda^{-1}z)$ for $\lambda = \frac{1}{2}, 2$, respectively.

This peculiar choice of ρ ensures that the difference $\rho^{\frac{1}{2}} - \rho$ is a convolution:

$$\rho^{\frac{1}{2}} - \rho = \hat{\varphi} * \check{\varphi} \,.$$

It follows that

$$\rho^{\varepsilon_{n+1}} - \rho^{\varepsilon_n} = (\rho^{\frac{1}{2}} - \rho)^{\varepsilon_n} = \hat{\varphi}^{\varepsilon_n} * \check{\varphi}^{\varepsilon_n} \,.$$

Proof for $\gamma > 0$: Existence

Finally we define

$$f_n(z) := F_z(\rho_z^{\varepsilon_n}), \qquad f_n(\psi) := \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n})\,\psi(z)\,\mathrm{d}z\,, \qquad z \in \mathbb{R}^d,\ \psi \in \mathcal{D}\,.$$

Then we want to prove that $f_n(\psi) \to f(\psi)$ and $|(f - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\gamma}$ for all $\psi \in \mathcal{D}$, namely that

$$\mathcal{R}F = \lim_{n\to+\infty} f_n \qquad \text{in } \mathcal{D}'.$$

We study the function

$$f_{x,n}(z) := f_n(z) - F_x(\rho_z^{\varepsilon_n}) = (F_z - F_x)(\rho_z^{\varepsilon_n}), \qquad x, z \in \mathbb{R}^d. \tag{3}$$

Proof for $\gamma > 0$: Existence

We write $f_{x,n}$ as a telescoping sum:

$$
\begin{aligned}
&f_{x,k+1}(z) - f_{x,k}(z) = (F_z - F_x)(\rho_z^{\varepsilon_{k+1}} - \rho_z^{\varepsilon_k}) \\
&= (F_z - F_x)(\hat{\varphi}^{\varepsilon_n} * \check{\varphi}_z^{\varepsilon_n}) = \int_{\mathbb{R}^d} (F_z - F_x)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y-z)\, \mathrm{d}y \\
&= \underbrace{\int_{\mathbb{R}^d} (F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y-z)\, \mathrm{d}y}_{g'_{x,k}(z)} + \underbrace{\int_{\mathbb{R}^d} (F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y-z)\, \mathrm{d}y}_{g''_k(z)}, \qquad (4)
\end{aligned}
$$

where again we use (2). By coherence we have

$$
|g''_k(z)| \le \|\check{\varphi}^{\varepsilon_k}\|_{L^1} \sup_{|y-z|\le\varepsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \lesssim \varepsilon_k^{\alpha}\, \varepsilon_k^{\gamma-\alpha} = \varepsilon_k^{\gamma},
$$

$$
\left| \int_{\mathbb{R}^d} g'_{x,k}(z)\, \psi(z)\, \mathrm{d}z \right| \le \sup_{y\in\bar{K}_1} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})|\, \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1} \lesssim \varepsilon_k^{\alpha} \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1}.
$$

Proof for $\gamma > 0$: Existence

By the properties of $\check\varphi$ we can write

$$(\check\varphi^\varepsilon * \psi)(y) = \int_{\mathbb{R}^d} \check\varphi^\varepsilon(y-z)\,\{\psi(z) - p_y(z)\}\,\mathrm{d}z\,,$$

where $p_y(z) := \sum_{|k|\leq r-1} \frac{\partial^k\psi(y)}{k!}\,(z-y)^k$ is the Taylor polynomial of ψ of order $r-1$ based at y; since $|\psi(z) - p_y(z)| \lesssim \|\psi\|_{C^r}\,|z-y|^r$, we obtain

$$\|\check\varphi^{\varepsilon_k} * \psi\|_{L^1} \lesssim \int_{\mathbb{R}^d} |\check\varphi^{\varepsilon_k}(y-z)|\,|z-y|^r\,\mathrm{d}z \lesssim \varepsilon_k^r\,.$$

We obtain

$$\left|\int_{\mathbb{R}^d} g'_{x,k}(z)\,\psi(z)\,\mathrm{d}z\right| \lesssim \varepsilon_k^{\alpha+r}\,, \qquad \left|\int_{\mathbb{R}^d} g''_k(z)\,\psi(z)\,\mathrm{d}z\right| \lesssim \varepsilon_k^{\gamma}\,.$$

Now we have by assumptions $\gamma > 0$ and $\alpha + r > 0$.

Proof for $\gamma > 0$: Existence

In particular, as $n \to +\infty$,

$$f_{x,n}(\psi) = f_{x,0}(\psi) + \sum_{k=0}^{n-1} \left[g'_{x,k}(\psi) + g''_k(\psi)\right]$$

converges to a distribution of order r. Now that $F_x(\rho_\cdot^{\varepsilon_n})$ converges to F_x in $\mathcal{D}'$. We obtain $f_n = f_{x,n} + F_x(\rho_\cdot^{\varepsilon_n})$ converges to a distribution $\mathcal{R}F$ in $\mathcal{D}'$. We also obtain for all ℓ

$$\mathcal{R}F(\psi) = F_x(\psi) + f_{x,\ell}(\psi) + \sum_{k=\ell}^{\infty} \left[g'_{x,k}(\psi) + g''_k(\psi)\right] ,$$

and the latter formula yields similarly the reconstruction bound $|(f - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$.

The Reconstruction Theorem for $\gamma \leq 0$.

Theorem (Hairer 14, Caravenna-Z. 20)

Let $F : \mathbb{R}^d \to \mathcal{D}'(\mathbb{R}^d)$ *be a* (α, γ)*-coherent germ, with* $\alpha \leq \gamma \leq 0$, *namely there exists a* $\varphi \in \mathcal{D}(\mathbb{R}^d)$ *with* $\int \varphi \neq 0$ *s.t.*

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\alpha (|x-y| + \varepsilon_n)^{\gamma-\alpha}, \qquad n \in \mathbb{N},\ x, y \in \mathbb{R}^d,$$

(coherence condition). Then there exists a non-unique $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ *such that*

$$|(\mathcal{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \begin{cases} \varepsilon_n^\gamma & \text{if } \gamma < 0 \\ (1 + |\log \varepsilon_n|) & \text{if } \gamma = 0 \end{cases}.$$

uniformly for x *in compact sets of* $\mathbb{R}^d$, $n \in \mathbb{N}$, $\{\psi \in \mathcal{D}(B(0,1)) : \|\psi\|_{C^r} \leq 1\}$ *with a fixed* $r > -\alpha$.

Proof for $\gamma \le 0$

In the proof with $\gamma > 0$, we wrote, see (4) and (3),

$$f_{x,n} := f_n - F_x(\rho^{\varepsilon_n}_{\cdot}) = g'_{x,n} + g''_n, \qquad |g'_{x,n}| \lesssim \varepsilon_n^{\alpha+r}, \qquad |g''_n| \le \varepsilon_n^{\gamma}.$$

Now we can choose r such that $\alpha + r > 0$, but $\gamma \le 0$ is fixed.

The solution is to define a different approximation sequence, eliminating the term g''_n whose convergence depends on $\gamma > 0$, and the proof follows with the same estimates. Namely

$$\bar{f}_n := f_n - \sum_{k=0}^{n-1} g''_k, \qquad \bar{f}_{x,n}(\psi) := \bar{f}_n(\psi) - F_x(\rho^{\varepsilon_n} * \psi) = f_{x,0}(\psi) + \sum_{k=0}^{n-1} g'_{x,k}(\psi).$$

Then with the same arguments $\bar{f}_n(\psi) \to \bar{f}(\psi)$ and $|(\bar{f} - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\gamma}$.

Homogeneity

The coherence assumption only concerns $F_z - F_y$, never F_y alone.

Under coherence alone, the reconstruction $\mathcal{R}F$ exists in $\mathcal{D}'$ but we have little more information.

Another crucial notion for germs is homogeneity (with exponent $\bar{\alpha}$)

$$|F_x(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\bar{\alpha}}$$

uniformly for x in compact sets, $n \in \mathbb{N}$ and $\psi \in \mathcal{D}(B(0,1))$ with $\|\psi\|_{C^r} \leq 1$, for some fixed $r > -\bar{\alpha}$.

Negative Hölder (Besov) spaces

Given $\bar{\alpha} \in]-\infty, 0[$, we define $\mathcal{C}^{\bar{\alpha}} = \mathcal{C}^{\bar{\alpha}}(\mathbb{R}^d)$ as the space of distributions $T \in \mathcal{D}'$ such that for all $\psi \in \mathcal{D} \setminus \{0\}$

$$\frac{|T(\psi_x^\varepsilon)|}{\|\psi\|_{C^{r_{\bar{\alpha}}}}} \lesssim \varepsilon^{\bar{\alpha}}$$

uniformly for x in compact sets and $\varepsilon \in (0, 1]$,

where we define $r_{\bar{\alpha}}$ as the smallest integer $r \in \mathbb{N}$ such that $r > -\bar{\alpha}$.

Theorem

The reconstruction $\mathcal{R}F$ of a (α, γ)-coherent germ F with homogeneity exponent $\bar{\alpha}$ is in $\mathcal{C}^{\bar{\alpha}}$ (and the map $F \mapsto \mathcal{R}F \in \mathcal{C}^{\bar{\alpha}}$ is linear continuous).

Sewing versus reconstruction

In dimension $d = 1$, the Sewing Lemma and the Reconstruction are almost equivalent.

For a continuous $\Xi : \{(s,t) : 0 \le s \le t \le T\} \to \mathbb{R}$ which vanishes on the diagonal we can define the germ $F_t(\cdot) := \partial_s \Xi_{\cdot,t}$.

Let $z > y > x$ and $\varphi := \mathbb{1}_{(-1,0)}$, so that $\varphi_y^{y-x} = \frac{1}{y-x}\mathbb{1}_{(x,y)}$. Then

$$\begin{aligned}(F_z - F_y)(\varphi_y^{y-x}) &= \frac{1}{y-x}\int_x^y \left(\partial_s \Xi_{s,z} - \partial_s \Xi_{s,y}\right) \mathrm{d}s \\ &= -\frac{1}{y-x}\left(\Xi_{x,z} - \Xi_{x,y} - \Xi_{y,z}\right).\end{aligned}$$

Then

$$\left|(F_z - F_y)(\varphi_y^{y-x})\right| \lesssim |y-x|^{-1}(|z-y| + |y-x|)^{\beta-1+1} \iff |\Xi_{x,z} - \Xi_{x,y} - \Xi_{y,z}| \lesssim |z-x|^{\beta}$$

namely $(-1, \beta-1)$-coherence of F is equivalent to $\delta\Xi \in \mathcal{C}_3^{\beta}$.

Sewing versus reconstruction

In particular, we can interpret the conditions

$$\underbrace{\sup_{0\leq s<t\leq T} \frac{|\Xi_{s,t}|}{|t-s|^{\alpha}} < +\infty}_{\text{homogeneity}} \qquad \underbrace{\sup_{0\leq s<u<t\leq T} \frac{|\Xi_{s,t}-\Xi_{s,u}-\Xi_{u,t}|}{|t-s|^{\beta}} < +\infty}_{\text{coherence}}.$$

As for reconstruction, also Sewing is possible under mere coherence

- coherence implies existence of $\mathcal{I}\Xi$
- homogeneity implies that $\mathcal{I}\Xi \in \mathcal{C}^{\alpha}$.

Moreover for $\beta \leq 1$ we still have a version of the Sewing Lemma, as for Reconstruction with $\gamma = \beta - 1 \leq 0$ (see Broux/Z.).

Singular product

Let $f \in \mathcal{C}^\alpha$ with $\alpha > 0$ and $F_y(w) = \sum_{|k|<\alpha} \partial^k f(y)\, \frac{(w-y)^k}{k!}$.

Let also $g \in \mathcal{C}^\beta$ with $\beta \le 0$. We define the germ $P = (P_x := g \cdot F_x)_{x\in\mathbb{R}^d}$, that is

$$P_x(\varphi) = (g \cdot F_x)(\varphi) := g(\varphi\, F_x), \qquad \varphi \in \mathcal{D}\,.$$

Theorem

If $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, with $\alpha > 0$ and $\beta \le 0$, then the germ $P = (P_x)_{x\in\mathbb{R}^d}$ is $(\beta, \alpha+\beta)$-coherent, namely

$$\left|(P_z - P_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\beta (|y-z| + \varepsilon_n)^\alpha.$$

If $\alpha + \beta > 0$, the unique distribution $\mathcal{R}P$ can be used to construct a canonical product of f and g. Moreover $\mathcal{R}P \in \mathcal{C}^\beta$.

If $\alpha + \beta \le 0$, the (non-unique) distribution $\mathcal{R}P$ can be used to construct a non-canonical product of f and g. Moreover $\mathcal{R}P \in \mathcal{C}^\beta$.

Recent developments

- *Reconstruction Theorem for Germs of Distributions on Smooth Manifolds*
  by Paolo Rinaldi and Federico Sclavi
- *On a Microlocal Version of Young's Product Theorem*
  by Claudio Dappiaggi, Paolo Rinaldi and Federico Sclavi
- *Besov Reconstruction*
  by Lucas Broux and David Lee
- *Reconstruction theorem in quasinormed spaces*
  by Pavel Zorin-Kranich
- *A stochastic reconstruction theorem*
  by Hannes Kern
- *The Sewing lemma for* $0 < \gamma \leq 1$
  by Lucas Broux and L.Z.