

An introduction to regularity structures

Lorenzo Zambotti (Sorbonne U, Paris)

8-12 August 2022
Campinas

These slides can be downloaded from my home page

In this course I want to discuss mainly the analytical aspects of the theory of regularity structures.

The plan for the four lectures is roughly:

- ▶ I. Reconstruction Theorem
- ▶ II. Models and modelled distributions
- ▶ III. Schauder estimates
- ▶ IV. Products and equations

Chapter 1: The Reconstruction Theorem

This talk is based on a paper (appeared in 2021 in the [EMS Surveys in Mathematics](#))

- ▶ *Hairer's Reconstruction Theorem without Regularity Structures*
by F. Caravenna and L.Z.

In this paper we have extracted a single result (the Reconstruction Theorem) from a larger theory (Regularity Structures).

We present the former in a simpler and more general version, without reference to the latter.

A later paper by [Pavel Zorin-Kranich](#), to appear in *Revista Matemática Iberoamericana*, has introduced introduced further simplifications and improvements to our results.

Prelude: The Sewing Lemma

You have seen in Sebastian's lectures last week that the main tool in rough analysis is the Theorem (Sewing Lemma)

Let $0 < \alpha \leq 1 < \beta$. There exists a unique map $\mathcal{I} : \mathcal{C}_2^{\alpha,\beta}([0, T] : \mathbb{R}^d) \rightarrow \mathcal{C}^\alpha([0, T] : \mathbb{R}^d)$ s.t.

$$(\mathcal{I}\Xi)_0 = 0, \quad |\mathcal{I}\Xi_t - \mathcal{I}\Xi_s - \Xi_{s,t}| \lesssim |t - s|^\beta, \quad s, t \in [0, T].$$

We recall that $\mathcal{C}_2^{\alpha,\beta}$ denotes the space of continuous $\Xi : \{(s, t) : 0 \leq s \leq t \leq T\} \rightarrow \mathbb{R}^d$ s.t.

$$\sup_{0 \leq s < t \leq T} \frac{|\Xi_{s,t}|}{|t - s|^\alpha} + \sup_{0 \leq s < u < t \leq T} \frac{|\Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}|}{|t - s|^\beta} < +\infty.$$

This theorem was proved around 2003 independently by **Gubinelli** and **Feyel-de la Pradelle**.

It is restricted to **functions depending on a one-dimensional parameter**.

It took ten years to find a version of this result in higher dimension... This is **Martin's Reconstruction Theorem**.

This talk will concern the space $\mathcal{D}'(\mathbb{R}^d)$ of **distributions** or **generalised functions**.

We consider the space $\mathcal{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support on \mathbb{R}^d .

A **distribution** on \mathbb{R}^d is a linear functional $T : C_c^\infty(\mathbb{R}^d) \rightarrow \mathbb{R}$ such that for every compact set $K \subset \mathbb{R}^d$ there is $r = r_K \in \mathbb{N}$

$$|T(\varphi)| \lesssim \|\varphi\|_{C^r} := \max_{|k| \leq r} \|\partial^k \varphi\|_\infty, \quad \forall \varphi \in C_0^\infty(K)$$

where throughout the lectures $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leq Cg$.

When r can be chosen uniformly over K we say that T has **order** r .

Every locally integrable (in particular continuous) function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ defines a distribution:

$$f(\varphi) := \int_{\mathbb{R}^d} f(x) \varphi(x) \, dx, \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

A famous example of distribution from quantum mechanics, which actually motivated the whole theory of distributions, is the **Dirac measure** δ_x at $x \in \mathbb{R}^d$

$$\delta_x(\varphi) = \varphi(x), \quad \varphi \in C^\infty(\mathbb{R}^d).$$

One can also differentiate any distribution $T \in \mathcal{D}'(\mathbb{R}^d)$: for $k \in \mathbb{N}^d$

$$\partial^k T(\varphi) := (-1)^{k_1 + \dots + k_d} T(\partial^k \varphi).$$

Products of distributions

Distributions form a linear space. If $\varphi \in C^\infty(\mathbb{R}^d)$ and $T \in \mathcal{D}'(\mathbb{R}^d)$ then it is possible to define **canonically** the product $\varphi \cdot T = T \cdot \varphi$ as

$$\varphi \cdot T(\psi) = T \cdot \varphi(\psi) := T(\varphi\psi), \quad \forall \psi \in C_c^\infty(\mathbb{R}^d).$$

However, if $T, T' \in \mathcal{D}'(\mathbb{R}^d)$, in general there is **no canonical way** of defining $T \cdot T'$.

One may use some form of **regularisation** of T, T' or both. Then, the result could **heavily depend** on the regularisation and thus be **neither unique nor canonical**.

For example, one can not define the **square** $(\delta_x)^2$ of the Dirac function.

The main question of reconstruction

For every $x \in \mathbb{R}^d$ we fix a distribution $F_x \in \mathcal{D}'(\mathbb{R}^d)$. If for all $\psi \in \mathcal{D}$ the map

$$\mathbb{R}^d \ni x \mapsto F_x(\psi)$$

is measurable, then we call $(F_x)_{x \in \mathbb{R}^d}$ a **germ**.

Problem:

Can we find a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ which is locally **well approximated** by $(F_x)_{x \in \mathbb{R}^d}$?

The main question of reconstruction

For every $x \in \mathbb{R}^d$ we fix a distribution $F_x \in \mathcal{D}'(\mathbb{R}^d)$. If for all $\psi \in \mathcal{D}$ the map

$$\mathbb{R}^d \ni x \mapsto F_x(\psi)$$

is measurable, then we call $(F_x)_{x \in \mathbb{R}^d}$ a **germ**.

Problem:

Can we find a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ which is locally **well approximated** by $(F_x)_{x \in \mathbb{R}^d}$?

Note that for $j \in \mathbb{N}^d$, $w \in \mathbb{R}^d$, we use the notation

$$|j| := \sum_{k=1}^d j_k, \quad w^j := \prod_{k=1}^d w_k^{j_k}, \quad j! := \prod_{k=1}^d j_k!$$

with the convention $0^0 := 1$.

Taylor expansions

For example, let us fix $f \in C^\infty(\mathbb{R}^d)$, and let us define for a fixed $\gamma > 0$

$$F_x(y) := \sum_{|k| < \gamma} \partial^k f(x) \frac{(y-x)^k}{k!}, \quad x, y \in \mathbb{R}^d.$$

Then the classical Taylor theorem says that there exists a function $R(x, y)$ such that

$$f(y) - F_x(y) = R(x, y), \quad |R(x, y)| \lesssim |x - y|^\gamma$$

uniformly for every x, y on compact sets of \mathbb{R}^d .

We say that the distribution f is **locally well approximated** by the germ $(F_x)_{x \in \mathbb{R}^d}$.

Scaling

Let us introduce now the following fundamental tool:

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\lambda > 0$ and $y \in \mathbb{R}^d$

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d} \varphi\left(\frac{w-y}{\lambda}\right), \quad w \in \mathbb{R}^d.$$

Then the local approximation property

$$f(y) - F_x(y) = R(x, y), \quad |R(x, y)| \lesssim |x - y|^\gamma$$

implies for any $\varphi \in \mathcal{D}$, uniformly for y in compact sets of \mathbb{R}^d , $\lambda \in]0, 1]$.

$$\begin{aligned} |(f - F_y)(\varphi_y^\lambda)| &= \left| \int_{\mathbb{R}^d} R(y, w) \varphi_y^\lambda(w) \, dw \right| \\ &\lesssim \frac{1}{\lambda^d} \int_{B_y(\lambda)} |w - y|^\gamma \, dw \lesssim \lambda^\gamma \end{aligned}$$

Another simple formula in this context is

$$\left| (F_z - F_y)(\varphi_y^\lambda) \right| \lesssim (|y - z| + \lambda)^\gamma,$$

for any $\varphi \in \mathcal{D}$, uniformly for y, z in compact sets of \mathbb{R}^d , $\lambda \in]0, 1]$.

We call this property **coherence**, see below.

This comes from a simple estimate of $F_z(w) - F_y(w)$.

Coherence of Taylor expansions

Let us note that we can Taylor expand also the derivatives of f : for $|k| < \gamma$

$$\partial^k f(y) = \sum_{|\ell| < \gamma - |k|} \partial^{k+\ell} f(z) \frac{(y-z)^\ell}{\ell!} + R^k(y, z), \quad |R^k(y, z)| \lesssim |y-z|^{\gamma-|k|}.$$

Then we can write

$$\begin{aligned} F_y(w) &= \sum_{|k| < \gamma} \partial^k f(y) \frac{(w-y)^k}{k!} \\ &= \sum_{|k| < \gamma} \left(\sum_{|\ell| < \gamma - |k|} \partial^{k+\ell} f(z) \frac{(y-z)^\ell}{\ell!} + R^k(y, z) \right) \frac{(w-y)^k}{k!} \\ &= F_z(w) + \sum_{|k| < \gamma} R^k(y, z) \frac{(w-y)^k}{k!}. \end{aligned}$$

Coherence of Taylor expansions

Therefore

$$F_z(w) - F_y(w) = - \sum_{|k| < \gamma} R^k(y, z) \frac{(w - y)^k}{k!}.$$

In particular

$$\begin{aligned} |F_z(w) - F_y(w)| &\leq \sum_{|k| < \gamma} |R^k(y, z)| \frac{|w - y|^k}{k!} \\ &\lesssim \sum_{|k| < \gamma} |y - z|^{\gamma - |k|} |w - y|^k \\ &\lesssim (|y - z| + |w - y|)^\gamma \end{aligned}$$

since $a^t b^s \leq (a + b)^t (a + b)^s$ for $a, b, t, s \geq 0$.

Coherence of Taylor expansions

Now recall that

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d} \varphi\left(\frac{w-y}{\lambda}\right), \quad w \in \mathbb{R}^d.$$

Then

$$\begin{aligned} \left| \int_{\mathbb{R}^d} (F_z(w) - F_y(w)) \varphi_y^\lambda(w) \, dw \right| &\lesssim \frac{1}{\lambda^d} \int_{B_y(\lambda)} (|y-z| + |w-y|)^\gamma \, dw \\ &\lesssim (|y-z| + \lambda)^\gamma. \end{aligned}$$

We have obtained for the germ $(F_y)_{y \in \mathbb{R}^d}$ and for any $\varphi \in \mathcal{D}$, $y, z \in \mathbb{R}^d$

$$\left| (F_z - F_y)(\varphi_y^\lambda) \right| \lesssim (|y-z| + \lambda)^\gamma.$$

Let us set from now on

$$\varepsilon_n := 2^{-n}, \quad n \in \mathbb{N}.$$

In particular for the germ related to a Taylor expansion we have for $\lambda \in \{\varepsilon_n : n \in \mathbb{N}\}$

$$|(F_z - F_y)(\varphi_y^{\varepsilon_n})| \lesssim (|y - z| + \varepsilon_n)^\gamma, \quad |(f - F_y)(\varphi_y^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma,$$

for any $\varphi \in \mathcal{D}$, uniformly for y, z in compact sets of \mathbb{R}^d and $n \in \mathbb{N}$.

We say that a germ $(F_z)_{z \in \mathbb{R}^d} \subset \mathcal{D}'$ is **(α, γ) -coherent** for $\alpha, \gamma \in \mathbb{R}$ with $\alpha \leq \gamma$, if there exists $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and

$$|(F_z - F_y)(\varphi_y^{\varepsilon_n})| \lesssim \varepsilon_n^\alpha (|y - z| + \varepsilon_n)^{\gamma - \alpha}$$

uniformly for z, y in compact sets of \mathbb{R}^d and $n \in \mathbb{N}$.

Hairer's Reconstruction Theorem (without regularity structures)

Theorem (Hairer 14, Caravenna-Z. 20)

Consider a (α, γ) -coherent germ with $\gamma > 0$, namely we suppose that there exists $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\alpha (|x - y| + \varepsilon_n)^{\gamma - \alpha},$$

uniformly for x, y in compact sets of \mathbb{R}^d and $n \in \mathbb{N}$ (*coherence condition*). Then there exists a *unique* $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$|(\mathcal{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$$

uniformly for x in compact sets of \mathbb{R}^d , $n \in \mathbb{N}$, $\{\psi \in \mathcal{D}(B(0, 1)) : \|\psi\|_{C^r} \leq 1\}$ with a fixed $r > -\alpha$.

- ▶ This result was stated and proved by [Martin](#) in [[Hai14](#)] for a **subclass of germs** related to **regularity structures**. He used **wavelets**.
- ▶ Later [Otto-Weber](#) proposed an approach based on a semigroup. This corresponds to a **special choice** of the test functions φ, ψ .
- ▶ Our statement is more general and requires no knowledge of regularity structures.
- ▶ This result can be seen as a generalisation of the Sewing Lemma in rough paths ([Gubinelli, Feyel-de La Pradelle](#)).
- ▶ The construction is completely local: constants and even the exponent α can depend on the compact set.
- ▶ We also cover the case $\gamma \leq 0$ (see below).
- ▶ [Pavel Zorin-Kranich](#) recently showed how to simplify, shorten and (slightly) improve our proof.

Proof for $\gamma > 0$: Uniqueness

Suppose we have two distributions $f, g \in \mathcal{D}'$ which satisfy, uniformly for $x \in K$ for any compact $K \subset \mathbb{R}^d$,

$$\lim_{n \rightarrow +\infty} |(f - F_x)(\varphi_x^{\varepsilon_n})| = \lim_{n \rightarrow +\infty} |(g - F_x)(\varphi_x^{\varepsilon_n})| = 0. \quad (1)$$

We may assume that $c := \int \varphi = 1$ (otherwise just replace φ by $c^{-1} \varphi$).

We set $T := f - g$, we fix a test function $\psi \in \mathcal{D}$. We recall the definition of the **convolution**

$$\psi * \varphi(w) = \int_{\mathbb{R}^d} \psi(y) \varphi(w - y) dy = \int_{\mathbb{R}^d} \psi(w - y) \varphi(y) dy,$$

for $w \in \mathbb{R}^d$. This implies

$$T(\psi * \varphi) = \int_{\mathbb{R}^d} \psi(y) T(\varphi(\cdot - y)) dy = \int_{\mathbb{R}^d} T(\psi(\cdot - y)) \varphi(y) dy. \quad (2)$$

Proof for $\gamma > 0$: Uniqueness

It follows that

$$T(\psi) = \lim_{n \rightarrow +\infty} T(\psi * \varphi_0^{\varepsilon_n}).$$

Moreover

$$T(\psi * \varphi_0^{\varepsilon_n}) = \int_{\mathbb{R}^d} T(\varphi_0^{\varepsilon_n}(\cdot - y)) \psi(y) \, dy = \int_{\mathbb{R}^d} T(\varphi_y^{\varepsilon_n}) \psi(y) \, dy,$$

$$|T(\psi * \varphi_0^{\varepsilon_n})| = \left| \int_{\mathbb{R}^d} T(\varphi_y^{\varepsilon_n}) \psi(y) \, dy \right| \leq \|\psi\|_{L^1} \sup_{y \in \text{supp}(\psi)} |T(\varphi_y^{\varepsilon_n})|.$$

It remains to show that $\lim_{n \rightarrow +\infty} \sup_{y \in \text{supp}(\psi)} |T(\varphi_y^{\varepsilon_n})| = 0$. Now

$$|T(\varphi_y^{\varepsilon_n})| = |f(\varphi_y^{\varepsilon_n}) - g(\varphi_y^{\varepsilon_n})| \leq |(f - F_y)(\varphi_y^{\varepsilon_n})| + |(g - F_y)(\varphi_y^{\varepsilon_n})|$$

which vanishes as $n \rightarrow +\infty$ uniformly for $y \in \text{supp}(\psi)$, by the reconstruction bound (1).

Proof for $\gamma > 0$: Existence

We fix a test function $\varphi \in \mathcal{D}$ with $\int \varphi \neq 0$ which makes the germ F coherent.

We can find in an elementary way a related $\hat{\varphi} \in \mathcal{D}(B(0, 1))$ such that

$$\int_{\mathbb{R}^d} y^k \hat{\varphi}(y) \, dy = 0, \quad \forall k \in \mathbb{N}_0^d : 1 \leq |k| \leq r - 1,$$

for a given $r > -\alpha$. Then we define

$$\rho := \hat{\varphi}^2 * \hat{\varphi} \quad \text{and} \quad \check{\varphi} := \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2,$$

where by $\hat{\varphi}^{\frac{1}{2}}, \hat{\varphi}^2$ we mean $\hat{\varphi}^\lambda(z) = \lambda^{-d} \hat{\varphi}(\lambda^{-1}z)$ for $\lambda = \frac{1}{2}, 2$, respectively.

This peculiar choice of ρ ensures that **the difference $\rho^{\frac{1}{2}} - \rho$ is a convolution**:

$$\rho^{\frac{1}{2}} - \rho = \hat{\varphi} * \check{\varphi}.$$

It follows that

$$\rho^{\varepsilon_{n+1}} - \rho^{\varepsilon_n} = (\rho^{\frac{1}{2}} - \rho)^{\varepsilon_n} = \hat{\varphi}^{\varepsilon_n} * \check{\varphi}^{\varepsilon_n}.$$

Proof for $\gamma > 0$: Existence

Finally we define

$$f_n(z) := F_z(\rho_z^{\varepsilon_n}), \quad f_n(\psi) := \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n}) \psi(z) \, dz, \quad z \in \mathbb{R}^d, \psi \in \mathcal{D}.$$

Then we want to prove that $f_n(\psi) \rightarrow f(\psi)$ and $|(f - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$ for all $\psi \in \mathcal{D}$, namely that

$$\mathcal{R}F = \lim_{n \rightarrow +\infty} f_n \quad \text{in } \mathcal{D}'.$$

We study the function

$$f_{x,n}(z) := f_n(z) - F_x(\rho_z^{\varepsilon_n}) = (F_z - F_x)(\rho_z^{\varepsilon_n}), \quad x, z \in \mathbb{R}^d. \quad (3)$$

Proof for $\gamma > 0$: Existence

We write $f_{x,n}$ as a telescoping sum:

$$\begin{aligned} f_{x,k+1}(z) - f_{x,k}(z) &= (F_z - F_x)(\rho_z^{\varepsilon_{k+1}} - \rho_z^{\varepsilon_k}) \\ &= (F_z - F_x)(\hat{\varphi}^{\varepsilon_n} * \check{\varphi}_z^{\varepsilon_n}) = \int_{\mathbb{R}^d} (F_z - F_x)(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y - z) \, dy \\ &= \underbrace{\int_{\mathbb{R}^d} (F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y - z) \, dy}_{g'_{x,k}(z)} + \underbrace{\int_{\mathbb{R}^d} (F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y - z) \, dy}_{g''_k(z)}, \end{aligned} \quad (4)$$

where again we use (2). By coherence we have

$$\begin{aligned} |g''_k(z)| &\leq \|\check{\varphi}^{\varepsilon_k}\|_{L^1} \sup_{|y-z| \leq \varepsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \lesssim \varepsilon_k^\alpha \varepsilon_k^{\gamma-\alpha} = \varepsilon_k^\gamma, \\ \left| \int_{\mathbb{R}^d} g'_{x,k}(z) \psi(z) \, dz \right| &\leq \sup_{y \in \bar{K}_1} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})| \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1} \lesssim \varepsilon_k^\alpha \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1}. \end{aligned}$$

Proof for $\gamma > 0$: Existence

By the properties of $\check{\varphi}$ we can write

$$(\check{\varphi}^\varepsilon * \psi)(y) = \int_{\mathbb{R}^d} \check{\varphi}^\varepsilon(y-z) \{\psi(z) - p_y(z)\} dz,$$

where $p_y(\cdot) := \sum_{|k| \leq r-1} \frac{\partial^k \psi(y)}{k!} (\cdot - y)^k$ the Taylor polynomial of ψ of order $r-1$ based at y ; since $|\psi(z) - p_y(z)| \lesssim \|\psi\|_{C^r} |z-y|^r$, we obtain

$$\|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1} \leq \varepsilon_k^r.$$

We obtain

$$\left| \int_{\mathbb{R}^d} g'_{x,k}(z) \psi(z) dz \right| \lesssim \varepsilon_k^{\alpha+r}, \quad \left| \int_{\mathbb{R}^d} g''_k(z) \psi(z) dz \right| \lesssim \varepsilon_k^\gamma.$$

Now we have by assumptions $\gamma > 0$ and $\alpha + r > 0$.

Proof for $\gamma > 0$: Existence

In particular, as $n \rightarrow +\infty$,

$$f_{x,n}(\psi) = f_{x,0}(\psi) + \sum_{k=0}^{n-1} [g'_{x,k}(\psi) + g''_k(\psi)]$$

converges to a distribution of order r . Now that $F_x(\rho^{\varepsilon_n})$ converges to F_x in \mathcal{D}' . We obtain $f_n = f_{x,n} + F_x(\rho^{\varepsilon_n})$ converges to a distribution $\mathcal{R}F$ in \mathcal{D}' . We also obtain for all ℓ

$$\mathcal{R}F(\psi) = F_x(\psi) + f_{x,\ell}(\psi) + \sum_{k=\ell}^{\infty} [g'_{x,k}(\psi) + g''_k(\psi)] ,$$

and the latter formula yields similarly the reconstruction bound $|(f - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$.

The Reconstruction Theorem for $\gamma \leq 0$.

Theorem (Hairer 14, Caravenna-Z. 20)

Let $F : \mathbb{R}^d \rightarrow \mathcal{D}'(\mathbb{R}^d)$ be a (α, γ) -coherent germ, with $\alpha \leq \gamma \leq 0$, namely there exists a $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$ s.t.

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\alpha (|x - y| + \varepsilon_n)^{\gamma - \alpha}, \quad n \in \mathbb{N}, x, y \in \mathbb{R}^d,$$

(coherence condition). Then there exists a *non-unique* $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$|(\mathcal{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \begin{cases} \varepsilon_n^\gamma & \text{if } \gamma < 0 \\ (1 + |\log \varepsilon_n|) & \text{if } \gamma = 0 \end{cases}.$$

uniformly for x in compact sets of \mathbb{R}^d , $n \in \mathbb{N}$, $\{\psi \in \mathcal{D}(B(0, 1)) : \|\psi\|_{C^r} \leq 1\}$ with a fixed $r > -\alpha$.

Proof for $\gamma \leq 0$

In the proof with $\gamma > 0$, we wrote, see (4) and (3),

$$f_{x,n} := f_n - F_x(\rho^{\varepsilon_n}) = g'_{x,n} + g''_n, \quad |g'_{x,n}| \lesssim \varepsilon_n^{\alpha+r}, \quad |g''_n| \leq \varepsilon_n^\gamma.$$

Now we can choose r such that $\alpha + r > 0$, but $\gamma \leq 0$ is fixed.

The solution is to define a **different** approximation sequence, eliminating the term g''_n whose convergence depends on $\gamma > 0$, and the proof follows with the same estimates. Namely

$$\bar{f}_n := f_n - \sum_{k=0}^{n-1} g''_k, \quad \bar{f}_{x,n}(\psi) := \bar{f}_n(\psi) - F_x(\rho^{\varepsilon_n} * \psi) = f_{x,0}(\psi) + \sum_{k=0}^{n-1} g'_{x,k}(\psi).$$

Then with the same arguments $\bar{f}_n(\psi) \rightarrow \bar{f}(\psi)$ and $|(\bar{f} - F_x)(\psi^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$.

Homogeneity

The coherence assumption only concerns $F_z - F_y$, never F_y alone.

Under coherence alone, the reconstruction $\mathcal{R}F$ exists in \mathcal{D}' but we have little more information.

Another crucial notion for germs is **homogeneity** (with exponent $\bar{\alpha}$)

$$|F_x(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\bar{\alpha}}$$

uniformly for x in compact sets, $n \in \mathbb{N}$ and $\psi \in \mathcal{D}(B(0, 1))$ with $\|\psi\|_{C^r} \leq 1$, for some fixed $r > -\bar{\alpha}$.

Negative Hölder (Besov) spaces

Given $\bar{\alpha} \in]-\infty, 0[$, we define $\mathcal{C}^{\bar{\alpha}} = \mathcal{C}^{\bar{\alpha}}(\mathbb{R}^d)$ as the space of distributions $T \in \mathcal{D}'$ such that for all $\psi \in \mathcal{D} \setminus \{0\}$

$$\frac{|T(\psi_x^\varepsilon)|}{\|\psi\|_{\mathcal{C}^{r_{\bar{\alpha}}}}} \lesssim \varepsilon^{\bar{\alpha}}$$

uniformly for x in compact sets and $\varepsilon \in (0, 1]$,

where we define $r_{\bar{\alpha}}$ as the smallest integer $r \in \mathbb{N}$ such that $r > -\bar{\alpha}$.

Theorem

The reconstruction $\mathcal{R}F$ of a (α, γ) -coherent germ F with homogeneity exponent $\bar{\alpha}$ is in $\mathcal{C}^{\bar{\alpha}}$ (and the map $F \mapsto \mathcal{R}F \in \mathcal{C}^{\bar{\alpha}}$ is linear continuous).

Sewing versus reconstruction

In dimension $d = 1$, the Sewing Lemma and the Reconstruction are **almost** equivalent.

For a continuous $\Xi : \{(s, t) : 0 \leq s \leq t \leq T\} \rightarrow \mathbb{R}$ which vanishes on the diagonal we can define the germ $F_t(\cdot) := \partial_s \Xi_{\cdot, t}$.

Let $z > y > x$ and $\varphi := \mathbb{1}_{(-1,0)}$, so that $\varphi_y^{y-x} = \frac{1}{y-x} \mathbb{1}_{(x,y)}$. Then

$$\begin{aligned}(F_z - F_y)(\varphi_y^{y-x}) &= \frac{1}{y-x} \int_x^y (\partial_s \Xi_{s,z} - \partial_s \Xi_{s,y}) \, ds \\ &= -\frac{1}{y-x} (\Xi_{x,z} - \Xi_{x,y} - \Xi_{y,z}).\end{aligned}$$

Then

$$|(F_z - F_y)(\varphi_y^{y-x})| \lesssim |y-x|^{-1} (|z-y| + |y-x|)^{\beta-1+1} \iff |\Xi_{x,z} - \Xi_{x,y} - \Xi_{y,z}| \lesssim |z-x|^\beta$$

namely $(-1, \beta - 1)$ -coherence of F is equivalent to $\delta\Xi \in \mathcal{C}_3^\beta$.

Sewing versus reconstruction

In particular, we can interpret the conditions

$$\underbrace{\sup_{0 \leq s < t \leq T} \frac{|\Xi_{s,t}|}{|t-s|^\alpha} < +\infty}_{\text{homogeneity}} \quad \underbrace{\sup_{0 \leq s < u < t \leq T} \frac{|\Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}|}{|t-s|^\beta} < +\infty}_{\text{coherence}}.$$

As for reconstruction, also Sewing is possible under mere coherence

- ▶ coherence implies existence of $\mathcal{I}\Xi$
- ▶ homogeneity implies that $\mathcal{I}\Xi \in \mathcal{C}^\alpha$.

Moreover for $\beta \leq 1$ we still have a version of the Sewing Lemma, as for Reconstruction for $\gamma = \beta - 1 \leq 0$ (see Broux/Z.).

Singular product

Let $f \in \mathcal{C}^\alpha$ with $\alpha > 0$ and $F_y(w) = \sum_{|k| < \alpha} \partial^k f(y) \frac{(w-y)^k}{k!}$.

Let also $g \in \mathcal{C}^\beta$ with $\beta \leq 0$. We define the germ $P = (P_x := g \cdot F_x)_{x \in \mathbb{R}^d}$, that is

$$P_x(\varphi) = (g \cdot F_x)(\varphi) := g(\varphi F_x), \quad \varphi \in \mathcal{D}.$$

Theorem

If $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, with $\alpha > 0$ and $\beta \leq 0$, then the germ $P = (P_x)_{x \in \mathbb{R}^d}$ is $(\beta, \alpha + \beta)$ -coherent, namely

$$|(P_z - P_y)(\varphi_y^{\varepsilon_n})| \lesssim \varepsilon_n^\beta (|y - z| + \varepsilon_n)^\alpha.$$

If $\alpha + \beta > 0$, the unique distribution \mathcal{RP} can be used to construct a **canonical product** of f and g . Moreover $\mathcal{RP} \in \mathcal{C}^\beta$.

If $\alpha + \beta \leq 0$, the (non-unique) distribution \mathcal{RP} can be used to construct a **non-canonical product** of f and g . Moreover $\mathcal{RP} \in \mathcal{C}^\beta$.

Recent developments

- ▶ *Reconstruction Theorem for Germs of Distributions on Smooth Manifolds*
by Paolo Rinaldi and Federico Sclavi
- ▶ *On a Microlocal Version of Young's Product Theorem*
by Claudio Dappiaggi, Paolo Rinaldi and Federico Sclavi
- ▶ *Besov Reconstruction*
by Lucas Broux and David Lee
- ▶ *Reconstruction theorem in quasinormed spaces*
by Pavel Zorin-Kranich
- ▶ *A stochastic reconstruction theorem*
by Hannes Kern
- ▶ *The Sewing lemma for $0 < \gamma \leq 1$*
by Lucas Broux and L.Z.