Ten lectures on rough paths
(work in progress)

F. Caravenna   M. Gubinelli   L. Zambotti

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Part I

Rough Equations

Chapter 1

The Sewing Bound

The problem of interest in this book is the study of differential equations driven by *irregular functions* (more specifically: continuous but not differentiable). This will be achieved through the powerful and elegant theory of *rough paths*. A key motivation comes from stochastic differential equations driven by Brownian motion, but the goal is to develop a general theory which does not rely on probability.

This first chapter is dedicated to an elementary but fundamental tool, the *Sewing Bound*, that will be applied extensively throughout the book. It is a general Hölder-type bound for functions of two real variables that can be understood by itself, see Theorem 1.9 below. To provide motivation, we present it as a natural a priori estimate for solutions of differential equations.

Notation. We fix a time horizon $T > 0$ and two dimensions $k, d \in \mathbb{N}$. We use "path" as a synonymous of "function defined on $[0, T]$" with values in $\mathbb{R}^d$. We denote by $|\cdot|$ the Euclidean norm. The space of linear maps from $\mathbb{R}^d$ to $\mathbb{R}^k$, identified by $k \times d$ real matrices, is denoted by $\mathbb{R}^k \otimes (\mathbb{R}^d)^* \simeq \mathbb{R}^{k \times d}$ and is equipped with the Hilbert-Schmidt norm $|\cdot|$ (i.e. the Euclidean norm on $\mathbb{R}^{k \times d}$). For $A \in \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ we have $|A\,v| \leqslant |A|\,|v|$.

1.1. Controlled differential equation

Consider the following *controlled ordinary differential equation (ODE)*: given a continuously differentiable path $X: [0, T] \to \mathbb{R}^d$ and a continuous function $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$, we look for a differentiable path $Z: [0, T] \to \mathbb{R}^k$ such that

$$\dot{Z}_t = \sigma(Z_t)\,\dot{X}_t\,, \qquad t \in [0, T]. \tag{1.1}$$

By the fundamental theorem of calculus, this is equivalent to

$$Z_t = Z_0 + \int_0^t \sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s\,, \qquad t \in [0, T]. \tag{1.2}$$

In the special case $k = d = 1$ and when $\sigma(x) = \lambda\,x$ is linear (with $\lambda \in \mathbb{R}$), we have the explicit solution $Z_t = z_0 \exp(\lambda\,(X_t - X_0))$, which has the interesting property of being well-defined also when X is non differentiable.

For any dimensions $k, d \in \mathbb{N}$, if we assume that $\sigma(\cdot)$ is Lipschitz, classical results in the theory of ODEs guarantee that *equation (1.1)-(1.2) is well-posed for any continuously differentiable path X*, namely for any $Z_0 \in \mathbb{R}^k$ there is one and only one solution Z (with no explicit formula, in general).

Our aim is to extend such a well-posedness result to a setting where X *is continuous but not differentiable* (also in cases where $\sigma(\cdot)$ may be non-linear). Of course, to this purpose it is first necessary to provide a generalized formulation of (1.1)-(1.2) where the derivative of X does not appear.

1.2. CONTROLLED DIFFERENCE EQUATION

Let us still suppose that X is continuously differentiable. We deduce by (1.1)-(1.2) that for $0 \leqslant s \leqslant t \leqslant T$

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + \int_s^t (\sigma(Z_u) - \sigma(Z_s))\,\dot{X}_u\,\mathrm{d}u, \tag{1.3}$$

which implies that Z satisfies the following *controlled difference equation*:

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.4}$$

because $u \mapsto \sigma(Z_u)$ is continuous and $u \mapsto \dot{X}_u$ is (continuous, hence) bounded on $[0,T]$.

Remark 1.1. (UNIFORMITY) Whenever we write $o(t-s)$, as in (1.4), we always mean *uniformly for* $0 \leqslant s \leqslant t \leqslant T$, i.e.

$$\forall \varepsilon > 0\ \exists \delta > 0\colon \quad 0 \leqslant s \leqslant t \leqslant T,\ \ t - s \leq \delta \quad \text{implies} \quad |o(t-s)| \leq \varepsilon\,(t-s)\,. \tag{1.5}$$

This will be implicitly assumed in the sequel.

Let us make two simple observations.

- If X is continuously differentiable we deduced (1.4) from (1.1), but we can easily deduce (1.1) from (1.4): in other terms, the two equations (1.1) and (1.4) are *equivalent*.
- If X is *not* continuously differentiable, equation (1.4) is still *meaningful*, unlike equation (1.1) which contains explicitly $\dot{X}$.

For these reasons, henceforth we focus on the difference equation (1.4), which provides a generalized formulation of the differential equation (1.1) when X is continuous but not necessarily differentiable.

The problem is now to prove *well-posedness* for the difference equation (1.4). We are going to show that this is possible assuming a suitable *Hölder regularity* on X, but non trivial ideas are required. In this chapter we illustrate some key ideas, showing how to prove uniqueness of solutions via *a priori estimates* (existence of solutions will be studied in the next chapters). We start from a basic result, which ensures the continuity of solutions; more precise result will be obtained later.

LEMMA 1.2. (CONTINUITY OF SOLUTIONS) *Let X and σ be continuous. Then any solution Z of (1.4) is a continuous path, more precisely it satisfies*

$$|Z_t - Z_s| \leqslant C\,|X_t - X_s| + o(t-s)\,, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.6}$$

for a suitable constant $C < \infty$ which depends on Z.

Proof. Relation (1.6) follows by (1.4) with $C := \|\sigma(Z)\|_\infty = \sup_{0 \leqslant t \leqslant T} |\sigma(Z_t)|$, renaming $|o(t-s)|$ as $o(t-s)$. We only have to prove that $C < \infty$. Since σ is continuous by assumption, it is enough to show that Z is *bounded*.

Since $o(t-s)$ is uniform, see (1.5), we can fix $\bar{\delta} > 0$ such that $|o(t-s)| \leqslant 1$ for all $0 \leqslant s \leqslant t \leqslant T$ with $|t-s| \leqslant \bar{\delta}$. It follows that Z is bounded in any interval $[\bar{s}, \bar{t}]$ with $|\bar{t} - \bar{s}| \leqslant \bar{\delta}$, because by (1.4) we can bound

$$\sup_{t \in [\bar{s}, \bar{t}]} |Z_t| \leqslant |Z_{\bar{s}}| + |\sigma(Z_{\bar{s}})| \sup_{t \in [\bar{s}, \bar{t}]} |X_t - X_{\bar{s}}| + 1 < \infty.$$

We conclude that Z is bounded in the whole interval $[0, T]$, because we can write $[0, T]$ as a finite union of intervals $[\bar{s}, \bar{t}]$ with $|\bar{t} - \bar{s}| \leqslant \bar{\delta}$. □

Remark 1.3. (COUNTEREXAMPLES) The weaker requirement that (1.4) holds *for any fixed* $s \in [0, T]$ *as* $t \downarrow s$ is not enough for our purposes, since in this case Z *needs not be continuous*. An easy conterexample is the following: given any continuous path $X: [0, 2] \to \mathbb{R}$, we define $Z: [0, 2] \to \mathbb{R}$ by

$$Z_t := \begin{cases} X_t & \text{if} \quad 0 \leqslant t < 1, \\ X_t + 1 & \text{if} \quad 1 \leqslant t \leqslant 2. \end{cases}$$

Note that $Z_t - Z_s = X_t - X_s$ when either $0 \leqslant s \leqslant t < 1$ or $1 \leqslant s \leqslant t \leqslant 2$, hence Z satisfies the difference equation (1.4) with $\sigma(\cdot) \equiv 1$ *for any fixed* $s \in [0, 2)$ *as* $t \downarrow s$, but *not uniformly for* $0 \leqslant s \leqslant t \leqslant 2$, since Z is discontinuous at $t = 1$.

For another counterexample, which is even unbounded, consider

$$Z_t := \begin{cases} \dfrac{1}{1-t} & \text{if} \quad 0 \leqslant t < 1, \\ 0 & \text{if} \quad 1 \leqslant t \leqslant 2, \end{cases}$$

which satisfies (1.4) as $t \downarrow s$ for any fixed $s \in [0, 2]$, for $X_t \equiv t$ and $\sigma(z) = z^2$.

1.3. SOME USEFUL FUNCTION SPACES

For $n \geqslant 1$ we define the simplex

$$[0, T]^n_\leqslant := \{(t_1, \ldots, t_n): \quad 0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T\} \tag{1.7}$$

(note that $[0, T]^1_\leqslant = [0, T]$). We then write $C_n = C([0, T]^n_\leqslant, \mathbb{R}^k)$ as a shorthand for the space of *continuous functions from* $[0, T]^n_\leqslant$ *to* $\mathbb{R}^k$:

$$C_n := C([0, T]^n_\leqslant, \mathbb{R}^k) := \{F: [0, T]^n_\leqslant \to \mathbb{R}^k: \ F \text{ is continuous}\}. \tag{1.8}$$

We are going to work with functions of one (f_s), two (F_{st}) or three (G_{sut}) ordered variables in $[0, T]$, hence we focus on the spaces C_1, C_2, C_3.

- On the spaces C_2 and C_3 we introduce a Hölder-like structure: given any $\eta \in (0, \infty)$, we define for $F \in C_2$ and $G \in C_3$

$$\|F\|_\eta := \sup_{0 \leqslant s < t \leqslant T} \frac{|F_{st}|}{(t-s)^\eta}, \qquad \|G\|_\eta := \sup_{\substack{0 \leqslant s \leqslant u \leqslant t \leqslant T \\ s < t}} \frac{|G_{sut}|}{(t-s)^\eta}, \tag{1.9}$$

and we denote by C_2^η and C_3^η the corresponding function spaces:

$$C_2^\eta := \{F \in C_2\colon \ \|F\|_\eta < \infty\}, \qquad C_3^\eta := \{G \in C_3\colon \ \|G\|_\eta < \infty\}, \tag{1.10}$$

which are Banach spaces endowed with the norm $\|\cdot\|_\eta$ (exercise).

- On the space C_1 of continuous functions $f\colon [0,T] \to \mathbb{R}^k$ we consider the usual Hölder structure. We first introduce the *increment* δf by

$$(\delta f)_{st} := f_t - f_s, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.11}$$

and note that $\delta f \in C_2$ for any $f \in C_1$. Then, for $\alpha \in (0,1]$, we define the classical space $\mathcal{C}^\alpha = \mathcal{C}^\alpha([0,T], \mathbb{R}^k)$ of α-Hölder functions

$$\mathcal{C}^\alpha := \left\{ f\colon [0,T] \to \mathbb{R}^k\colon \quad \|\delta f\|_\alpha = \sup_{0 \le s < t \le T} \frac{|f_t - f_s|}{(t-s)^\alpha} < \infty \right\} \tag{1.12}$$

(for $\alpha = 1$ it is the space of Lipschitz functions). Note that $\|\delta f\|_\alpha$ in (1.12) is consistent with (1.11) and (1.9).

Remark 1.4. (HÖLDER SEMI-NORM) We stress that $f \mapsto \|\delta f\|_\alpha$ is a semi-norm on $\mathcal{C}^\alpha$ (it vanishes on constant functions). The standard norm on $\mathcal{C}^\alpha$ is

$$\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha, \tag{1.13}$$

where we define the standard sup norm

$$\|f\|_\infty := \sup_{t \in [0,T]} |f_t|. \tag{1.14}$$

For $f\colon [0,T] \to \mathbb{R}^k$ we can bound $\|f\|_\infty \le |f(0)| + T^\alpha \|\delta f\|_\alpha$ (see (1.39) below), hence

$$\|f\|_{\mathcal{C}^\alpha} \le |f(0)| + (1 + T^\alpha) \|\delta f\|_\alpha. \tag{1.15}$$

This explains why it is often enough to focus on the semi-norm $\|\delta f\|_\alpha$.

Remark 1.5. (HÖLDER EXPONENTS) We only consider the Hölder space $\mathcal{C}^\alpha$ for $\alpha \in (0,1]$ because for $\alpha > 1$ the only functions in $\mathcal{C}^\alpha$ are constant functions (note that $\|\delta f\|_\alpha < \infty$ for $\alpha > 1$ implies $\dot{f}_t = 0$ for every $t \in [0,T]$).

On the other hand, the spaces C_2^η and C_3^η in (1.10) are interesting *for any exponent* $\eta \in (0, \infty)$. For instance, the condition $\|F\|_\eta < \infty$ for a function $F \in C_2$ means that $|F_{st}| \leqslant C\,(t-s)^\eta$, which does not imply $F \equiv 0$ when $\eta > 1$ (unless $F = \delta f$ is the increment of some function $f \in C_1$).

In our results below we will have to assume that the non-linearity $\sigma\colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ belongs to classes of Hölder functions, in the following sense.

DEFINITION 1.6. *Let $\gamma > 0$. A function $F\colon \mathbb{R}^k \to \mathbb{R}^N$ is said to be globally γ-Hölder (or globally of class $\mathcal{C}^\gamma$) if*

- *for $\gamma \in (0,1]$ we have*

$$[F]_{\mathcal{C}^\gamma} := \sup_{x,y \in \mathbb{R}^k, x \ne y} \frac{|F(x) - F(y)|}{|x-y|^\gamma} < +\infty$$

- *for* $\gamma \in (n, n+1]$ *and* $n = \{1, 2, \ldots\}$, *F is n times continuously differentiable and*

$$[D^{(n)}F]_{\mathcal{C}^\gamma} := \sup_{x,y\in\mathbb{R}^k, x\neq y} \frac{|D^{(n)}F(x) - D^{(n)}F(y)|}{|x-y|^{\gamma-n}} < +\infty$$

where $D^{(n)}$ *is the n-fold differential of F.*

Moreover $F\colon \mathbb{R}^k \to \mathbb{R}^N$ *is said to be locally* γ*-Hölder (or locally of class* $\mathcal{C}^\gamma$*) if*

- *for* $\gamma \in (0, 1]$ *we have for all* $R > 0$

$$\sup_{\substack{x,y\in\mathbb{R}^k, x\neq y\\ |x|,|y|\leqslant R}} \frac{|F(x) - F(y)|}{|x-y|^{\gamma}} < +\infty$$

- *for* $\gamma \in (n, n+1]$ *and* $n = \{1, 2, \ldots\}$, *F is n times continuously differentiable and*

$$\sup_{\substack{x,y\in\mathbb{R}^k, x\neq y\\ |x|,|y|\leqslant R}} \frac{|D^{(n)}F(x) - D^{(n)}F(y)|}{|x-y|^{\gamma-n}} < +\infty.$$

We stress that in the previous definition we do not assume F of $D^{(n)}F$ to be bounded. The case $\gamma = 1$ corresponds to the classical *Lipschitz* condition.

1.4. LOCAL UNIQUENESS OF SOLUTIONS

We prove *uniqueness of solutions* for the controlled difference equation (1.4) when $X \in \mathcal{C}^\alpha$ *is an Hölder path of exponent* $\alpha > \frac{1}{2}$. For simplicity, we focus on the case when $\sigma\colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is a *linear* application: $\sigma \in (\mathbb{R}^k \otimes (\mathbb{R}^d)^*) \otimes (\mathbb{R}^k)^*$, and we write σZ instead of $\sigma(Z)$ (we discuss non linear $\sigma(\cdot)$ in Chapter 2).

THEOREM 1.7. (LOCAL UNIQUENESS OF SOLUTIONS, LINEAR CASE) *Fix a path* $X\colon [0, T] \to \mathbb{R}^d$ *in* $\mathcal{C}^\alpha$*, with* $\alpha \in \left]\frac{1}{2}, 1\right]$*, and a linear map* $\sigma\colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$*. If* $T > 0$ *is small enough (depending on* X, α, σ*), then for any* $z_0 \in \mathbb{R}^k$ *there is at most one path* $Z\colon [0, T] \to \mathbb{R}^k$ *with* $Z_0 = z_0$ *which solves the linear controlled difference equation (1.4), that is (recalling (1.11))*

$$\delta Z_{st} - (\sigma Z_s)\,\delta X_{st} = o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{1.16}$$

Proof. Suppose that we have two paths $Z, \bar{Z}\colon [0, T] \to \mathbb{R}^k$ satisfying (1.16) with $Z_0 = \bar{Z}_0$ and define $Y := Z - \bar{Z}$. Our goal is to show that $Y = 0$.

Let us introduce the function $R \in C_2 = C([0, T]^2_\leqslant, \mathbb{R}^k)$ defined by

$$R_{st} := \delta Y_{st} - (\sigma Y_s)\,\delta X_{st}\,, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.17}$$

and note that by (1.16) and linearity we have

$$R_{st} = o(t-s)\,. \tag{1.18}$$

Recalling (1.9), we can estimate

$$\|\delta Y\|_\alpha \leqslant |\sigma|\, \|Y\|_\infty\, \|\delta X\|_\alpha + \|R\|_\alpha\,,$$

and since $R_{st} = o(t-s) = o((t-s)^\alpha)$, we have $\|R\|_\alpha < +\infty$ and therefore $\|\delta Y\|_\alpha < +\infty$. Since $Y_0 = 0$, we can bound

$$\|Y\|_\infty \leqslant |Y_0| + \sup_{0 \leqslant t \leqslant T} |Y_t - Y_0| \leqslant T^\alpha \|\delta Y\|_\alpha .$$

Since $1 \leqslant T^\alpha (t-s)^{-\alpha}$ for $0 \leqslant s < t \leqslant T$, we can also bound

$$\|R\|_\alpha \leqslant T^\alpha \|R\|_{2\alpha} ,$$

so that

$$\|\delta Y\|_\alpha \leqslant T^\alpha (|\sigma| \, \|\delta Y\|_\alpha \, \|\delta X\|_\alpha + \|R\|_{2\alpha}).$$

Suppose we can prove that, for some constant $C = C(X, \alpha, \sigma) < \infty$,

$$\|R\|_{2\alpha} \leqslant C \, \|\delta Y\|_\alpha. \tag{1.19}$$

Then we obtain

$$\|\delta Y\|_\alpha \leqslant T^\alpha (|\sigma| \, \|\delta X\|_\alpha + C) \, \|\delta Y\|_\alpha .$$

If we fix T small enough, so that $T^\alpha (|\sigma| \, \|\delta X\|_\alpha + C) < 1$, we get $\|\delta Y\|_\alpha = 0$, hence $\delta Y \equiv 0$. This means that $Y_t = Y_s$ for all $s, t \in [0, T]$, and since $Y_0 = 0$ we obtain $Y \equiv 0$, namely our goal $Z \equiv \bar{Z}$. This completes the proof *assuming the estimate (1.19)* (where the hypothesis $\alpha > \frac{1}{2}$ will play a key role). □

To actually complete the proof of Theorem 1.7, it remains to show that the inequality (1.19) holds. This is performed in the next two sections:

- in Section 1.5 we present a fundamental estimate, the *Sewing Bound*, which applies to *any function* $R_{st} = o(t-s)$ (recall (1.18));
- in Section 1.6 we apply the Sewing Bound to R_{st} in (1.17) and we prove the desired estimate (1.19) for $\alpha > \frac{1}{2}$ (see the assumptions of Theorem 1.7).

1.5. THE SEWING BOUND

Let us fix an arbitrary function $R \in C_2 = C([0, T]^2_\leqslant, \mathbb{R}^k)$ with $R_{st} = o(t-s)$. Our goal is to bound $|R_{ab}|$ for any given $0 \leqslant a < b \leqslant T$.

We first show that we can express R_{ab} via "Riemann sums" along partitions $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of $[a, b]$. These are defined by

$$I_{\mathcal{P}}(R) := \sum_{i=1}^{\#\mathcal{P}} R_{t_{i-1} t_i} , \tag{1.20}$$

where we denote by $\#\mathcal{P} := m$ the number of intervals of the partition $\mathcal{P}$. Let us denote by $|\mathcal{P}| := \max_{1 \leqslant i \leqslant m} (t_i - t_{i-1})$ the *mesh* of $\mathcal{P}$.

LEMMA 1.8. (RIEMANN SUMS) *Given any $R \in C_2$ with $R_{st} = o(t-s)$, for any $0 \leqslant a < b \leqslant T$ and for any sequence $(\mathcal{P}_n)_{n \geqslant 0}$ of partitions of $[a, b]$ with vanishing mesh $\lim_{n \to \infty} |\mathcal{P}_n| = 0$ we have*

$$\lim_{n \to \infty} I_{\mathcal{P}_n}(R) = 0 .$$

If furthermore $\mathcal{P}_0 = \{a, b\}$ *is the trivial partition, then we can write*

$$R_{ab} = \sum_{n=0}^{\infty} (I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)), \qquad 0 \leqslant a < b \leqslant T. \tag{1.21}$$

Proof. Writing $\mathcal{P}_n = \{a = t_0^n < t_1^n < \ldots < t_{\#\mathcal{P}_n}^n = b\}$, we can estimate

$$|I_{\mathcal{P}_n}(R)| \leqslant \sum_{i=1}^{\#\mathcal{P}_n} |R_{t_{i-1}^n t_i^n}| \leqslant \left\{ \max_{j=1,\ldots,\#\mathcal{P}_n} \frac{|R_{t_{j-1}^n t_j^n}|}{(t_j^n - t_{j-1}^n)} \right\} \sum_{j=1}^{\#\mathcal{P}_n} (t_j^n - t_{j-1}^n),$$

hence $|I_{\mathcal{P}_n}(R)| \to 0$ as $n \to \infty$, because the final sum equals $b - a$ and the bracket vanishes (since $R_{st} = o(t - s)$ and $|\mathcal{P}_n| = \max_{1 \leqslant j \leqslant \#\mathcal{P}_n} (t_j^n - t_{j-1}^n) \to 0$).

We deduce relation (1.21) by the telescopic sum

$$I_{\mathcal{P}_0}(R) - I_{\mathcal{P}_N}(R) = \sum_{n=0}^{N-1} (I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)),$$

because $\lim_{N \to \infty} I_{\mathcal{P}_N}(R) = 0$ while $I_{\mathcal{P}_0}(R) = R_{ab}$ for $\mathcal{P}_0 = \{a, b\}$. □

If we remove a single point t_i from a partition $\mathcal{P} = \{t_0 < t_1 < \ldots < t_m\}$, we obtain a new partition $\mathcal{P}'$ for which, recalling (1.20), we can write

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = R_{t_{i-1} t_{i+1}} - R_{t_{i-1} t_i} - R_{t_i t_{i+1}}. \tag{1.22}$$

The expression in the RHS deserves a name: given any two-variables function $F \in C_2$, we define its increment $\delta F \in C_3$ as the three-variables function

$$\delta F_{sut} := F_{st} - F_{su} - F_{ut}, \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{1.23}$$

We can then rewrite (1.22) as

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = \delta R_{t_{i-1} t_i t_{i+1}}, \tag{1.24}$$

and recalling (1.9) we obtain the following estimate, for any $\eta > 0$:

$$|I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R)| \leqslant \|\delta R\|_\eta \, |t_{i+1} - t_{i-1}|^\eta. \tag{1.25}$$

We are now ready to state and prove the Sewing Bound.

THEOREM 1.9. (SEWING BOUND) *Given any* $R \in C_2$ *with* $R_{st} = o(t - s)$*, the following estimate holds for any* $\eta \in (1, \infty)$ *(recall (1.9)):*

$$\|R\|_\eta \leqslant K_\eta \, \|\delta R\|_\eta \qquad \textit{where} \qquad K_\eta := (1 - 2^{1-\eta})^{-1}. \tag{1.26}$$

Proof. Fix $R \in C_2$ such that $\|\delta R\|_\eta < \infty$ for some $\eta > 1$ (otherwise there is nothing to prove). Also fix $0 \leqslant a < b \leqslant T$ and consider for $n \geqslant 0$ the dyadic partitions $\mathcal{P}_n := \{t_i^n := a + \frac{i}{2^n}(b - a)\colon\ 0 \leq i \leq 2^n\}$ of $[a, b]$. Since $\mathcal{P}_0 = \{a, b\}$ is the trivial partition, we can apply (1.21) to bound

$$|R_{ab}| \leqslant \sum_{n=0}^{\infty} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)|. \tag{1.27}$$

If we remove from $\mathcal{P}_{n+1}$ all the "odd points" t_{2j+1}^{n+1}, with $0 \leq j \leq 2^n - 1$, we obtain $\mathcal{P}_n$. Then, iterating relations (1.24)-(1.25), we have

$$\begin{aligned} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)| &\leqslant \sum_{j=0}^{2^n-1} |\delta R_{t_{2j}^{n+1} t_{2j+1}^{n+1} t_{2j+2}^{n+1}}| \\ &\leqslant 2^n \, \|\delta R\|_\eta \left(\frac{2(b-a)}{2^{n+1}}\right)^\eta \\ &= 2^{-(\eta-1)n} \, \|\delta R\|_\eta \, (b-a)^\eta . \end{aligned} \tag{1.28}$$

Plugging this into (1.27), since $\sum_{n=0}^\infty 2^{-(\eta-1)n} = (1-2^{1-\eta})^{-1}$, we obtain

$$|R_{ab}| \leqslant (1-2^{1-\eta})^{-1} \, \|\delta R\|_\eta \, (b-a)^\eta, \qquad 0 \leqslant a < b \leqslant T, \tag{1.29}$$

which proves (1.26). ☐

Remark 1.10. Recalling (1.11) and (1.23), we have defined linear maps

$$C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \tag{1.30}$$

which satisfy $\delta \circ \delta = 0$. Indeed, for any $f \in C_1$ we have

$$\delta(\delta f)_{sut} = (f_t - f_s) - (f_u - f_s) - (f_t - f_u) = 0.$$

Intuitively, $\delta F \in C_3$ measures how much a function $F \in C_2$ differs from being the increment δf of some $f \in C_1$, because $\delta F \equiv 0$ *if and only if* $F = \delta f$ *for some* $f \in C_1$ (it suffices to define $f_t := F_{0t}$ and to check that $\delta f_{st} = \delta F_{0st} + F_{st} = F_{st}$).

Remark 1.11. The assumption $R_{st} = o(t-s)$ in Theorem 1.9 cannot be avoided: if $R := \delta f$ for a non constant $f \in C_1$, then $\delta R = 0$ while $\|R\|_\eta > 0$.

1.6. END OF PROOF OF UNIQUENESS

In this section, we apply the Sewing Bound (1.26) to the function R_{st} defined in (1.17), in order to prove the estimate (1.19) for $\alpha > \frac{1}{2}$.

We first determine the increment δR through a simple and instructive computation: by (1.17), since $\delta(\delta Z) = 0$ (see Remark 1.10), we have

$$\begin{aligned} \delta R_{sut} &:= R_{st} - R_{su} - R_{ut} \\ &= (Y_t - Y_s) - (Y_u - Y_s) - (Y_t - Y_u) \\ &\quad - (\sigma Y_s)\,(X_t - X_s) + (\sigma Y_s)\,(X_u - X_s) + (\sigma Y_u)\,(X_t - X_u) \\ &= [\sigma\,(Y_u - Y_s)]\,(X_t - X_u). \end{aligned} \tag{1.31}$$

Recalling (1.9), this implies

$$\|\delta R\|_{2\alpha} \leqslant |\sigma| \, \|\delta Y\|_\alpha \, \|\delta X\|_\alpha.$$

We next note that if $\alpha > \frac{1}{2}$ (as it is assumed in Theorem 1.7) we can apply the Sewing Bound (1.26) for $\eta = 2\alpha > 1$ to obtain

$$\|R\|_{2\alpha} \leqslant K_{2\alpha} \, \|\delta R\|_{2\alpha} \leqslant K_{2\alpha} \, |\sigma| \, \|\delta Y\|_\alpha \, \|\delta X\|_\alpha .$$

This is precisely our goal (1.19) with $C = C(X, \alpha, \sigma) := K_{2\alpha} |\sigma| \|\delta X\|_\alpha$.

Summarizing: thanks to the Sewing bound (1.26), we have obtained the estimate (1.19) and completed the proof of Theorem 1.7, showing uniqueness of solutions to the difference equation (1.4) for any $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$. In the next chapters we extend this approach to non-linear $\sigma(\cdot)$ and to situations where $X \in \mathcal{C}^\alpha$ with $\alpha \leqslant \frac{1}{2}$.

Remark 1.12. For later purpose, let us record the computation (1.31) withouth σ: given any (say, real) paths X and Y, if

$$A_{st} = Y_s \, \delta X_{st}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T,$$

then

$$\delta A_{sut} = -\delta Y_{su} \, \delta X_{ut}\,, \qquad \forall 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{1.32}$$

1.7. Weighted norms

We conclude this chapter defining *weighted versions* $\|\cdot\|_{\eta,\tau}$ of the norms $\|\cdot\|_\eta$ introduced in (1.9): given $F \in C_2$ and $G \in C_3$, we set for $\eta, \tau \in (0, \infty)$

$$\|F\|_{\eta,\tau} := \sup_{0 \leqslant s \leqslant t \leqslant T} \mathbb{1}_{\{0 < t - s \leqslant \tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|F_{st}|}{(t-s)^\eta}\,, \tag{1.33}$$

$$\|G\|_{\eta,\tau} := \sup_{0 \leqslant s \leqslant u \leqslant t \leqslant T} \mathbb{1}_{\{0 < t - s \leqslant \tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|G_{sut}|}{(t-s)^\eta}, \tag{1.34}$$

where C_2 and C_3 are the spaces of continuous functions from $[0,T]^2_\leqslant$ and $[0,T]^3_\leqslant$ to $\mathbb{R}^k$, see (1.8). Note that as $\tau \to \infty$ we recover the usual norms:

$$\|\cdot\|_\eta = \lim_{\tau \to \infty} \|\cdot\|_{\eta,\tau}\,. \tag{1.35}$$

Remark 1.13. (norms vs. semi-norms) While $\|\cdot\|_\eta$ is a norm, $\|\cdot\|_{\eta,\tau}$ is a norm for $\tau \geqslant T$ but *it is only a semi-norm for* $\tau < T$ (for instance, $\|F\|_{\eta,\tau} = 0$ for $F \in C_2$ implies $F_{st} = 0$ only for $t - s \leqslant \tau$: no constraint is imposed on F_{st} for $t - s > \tau$).

However, if $F = \delta f$, that is $F_{st} = f_t - f_s$ for some $f \in C_1$, we have the equivalence

$$\|\delta f\|_{\eta,\tau} \leqslant \|\delta f\|_\eta \leqslant \left(1 + \frac{T}{\tau}\right) \mathrm{e}^{\frac{T}{\tau}} \|\delta f\|_{\eta,\tau}\,. \tag{1.36}$$

The first inequality is clear. For the second one, given $0 \leqslant s < t \leqslant T$, we can write $s = t_0 < t_1 < \cdots < t_N = t$ with $t_i - t_{i-1} \leqslant \tau$ and $N \leqslant 1 + \frac{T}{\tau}$ (for instance, we can consider $t_i = s + i\,\frac{t-s}{N}$ where $N := \lceil \frac{t-s}{\tau} \rceil$); we then obtain $\delta f_{st} = \sum_{i=1}^N \delta f_{t_{i-1} t_i}$ and $|\delta f_{t_{i-1} t_i}| \leqslant \|\delta f\|_{\eta,\tau} \, \mathrm{e}^{t_i/\tau} (t_i - t_{i-1})^\eta \leqslant \|\delta f\|_{\eta,\tau} \, \mathrm{e}^{T/\tau} (t-s)^\eta$, which yields (1.36).

Remark 1.14. (from local to global) The weighted semi-norms $\|\cdot\|_{\eta,\tau}$ will be useful to transform *local* results in *global* results. Indeed, using the standard norms $\|\cdot\|_\eta$ often requires the size $T > 0$ of the time interval $[0,T]$ to be *small*, as in Theorem 1.7, which can be annoying. Using $\|\cdot\|_{\eta,\tau}$ will allow us to *keep* $T > 0$ *arbitrary*, by choosing a sufficiently small $\tau > 0$.

Recalling the supremum norm $\|f\|_\infty$ of a function $f \in C_1$, see (1.14), we define the corresponding weighted version

$$\|f\|_{\infty,\tau} := \sup_{0 \leqslant t \leqslant T} \mathrm{e}^{-\frac{t}{\tau}} |f_t| \,. \tag{1.37}$$

We stress that $\|\cdot\|_{\infty,\tau}$ *is a norm equivalent to* $\|\cdot\|_\infty$ *for any* $\tau > 0$, since

$$\|\cdot\|_{\infty,\tau} \leqslant \|\cdot\|_\infty \leqslant \mathrm{e}^{\frac{T}{\tau}} \|\cdot\|_{\infty,\tau} \,. \tag{1.38}$$

Remark 1.15. (EQUIVALENT HÖLDER NORM) It follows by (1.36) and (1.38) that $\|\cdot\|_{\infty,\tau} + \|\cdot\|_{\alpha,\tau}$ *is a norm equivalent to* $\|\cdot\|_{\mathcal{C}^\alpha} := \|\cdot\|_\infty + \|\cdot\|_\alpha$ on the space $\mathcal{C}^\alpha$ of Hölder functions, see Remark 1.4, for any $\tau > 0$.

We will often use the Hölder semi-norms $\|\delta f\|_\alpha$ and $\|\delta f\|_{\alpha,\tau}$ to bound the supremum norms $\|f\|_\infty$ and $\|f\|_{\infty,\tau}$, thanks to the following result.

LEMMA 1.16. (SUPREMUM-HÖLDER BOUND) *For any* $f \in C_1$ *and* $\eta \in (0, \infty)$

$$\|f\|_\infty \leqslant |f_0| + T^\eta \|\delta f\|_\eta \,, \tag{1.39}$$

$$\|f\|_{\infty,\tau} \leqslant |f_0| + 3\,(\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau}, \qquad \forall \tau > 0 \,. \tag{1.40}$$

Proof. Let us prove (1.39): for any $f \in C_1$ and for $t \in \,]0, T]$ we have

$$|f_t| \leqslant |f_0| + |f_t - f_0| = |f_0| + t^\eta \frac{|f_t - f_0|}{t^\eta} \leqslant |f_0| + T^\eta \|\delta f\|_\eta.$$

The proof of (1.40) is slightly more involved. If $t \in \,]0, \tau \wedge T]$, then

$$\mathrm{e}^{-\frac{t}{\tau}} |f_t| \leqslant |f_0| + t^\eta \mathrm{e}^{-\frac{t}{\tau}} \frac{|f_t - f_0|}{t^\eta} \leqslant |f_0| + (\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau},$$

which, in particular, implies (1.40) when $\tau \geqslant T$. When $\tau < T$, it remains to consider $\tau < t \leqslant T$: in this case, we define $N := \min\{n \in \mathbb{N}:\ n\tau \geq t\} \geq 2$ so that $\frac{t}{N} \leqslant \tau$. We set $t_k = k\frac{t}{N}$ for $k \geq 0$, so that $t_N = t$. Then

$$\begin{aligned} \mathrm{e}^{-\frac{t}{\tau}} |f_t| &\leqslant |f_0| + \sum_{k=1}^{N} (t_k - t_{k-1})^\eta \mathrm{e}^{-\frac{t - t_k}{\tau}} \left[\mathrm{e}^{-\frac{t_k}{\tau}} \frac{|f_{t_k} - f_{t_{k-1}}|}{(t_k - t_{k-1})^\eta} \right] \\ &\leqslant |f_0| + (\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau} \sum_{k=1}^{N} \mathrm{e}^{-\frac{t - t_k}{\tau}}. \end{aligned}$$

By definition of N we have $(N-1)\tau < t$; since $\tau < t$ we obtain $N\tau < 2t$ and therefore $\frac{t}{N\tau} \geq \frac{1}{2}$. Since $t - t_k = (N-k)\frac{t}{N}$, renaming $\ell := N - k$ we obtain

$$\sum_{k=1}^{N} \mathrm{e}^{-\frac{t-t_k}{\tau}} = \sum_{\ell=0}^{N-1} \mathrm{e}^{-\ell \frac{t}{N\tau}} = \frac{1 - \mathrm{e}^{-\frac{t}{\tau}}}{1 - \mathrm{e}^{-\frac{t}{N\tau}}} \leqslant \frac{1}{1 - \mathrm{e}^{-\frac{1}{2}}} \leqslant 3.$$

The proof is complete. □

We finally show that the Sewing Bound (1.26) still holds if we replace $\|\cdot\|_\eta$ by $\|\cdot\|_{\eta,\tau}$, for any $\tau > 0$.

Theorem 1.17. (weighted sewing bound) *Given any $R \in C_2$ with $R_{st} = o(t-s)$, the following estimate holds for any $\eta \in (1,\infty)$ and $\tau > 0$:*

$$\|R\|_{\eta,\tau} \leqslant K_\eta \, \|\delta R\|_{\eta,\tau} \qquad \text{where} \qquad K_\eta := (1-2^{1-\eta})^{-1} . \tag{1.41}$$

Proof. Given $0 \leqslant a \leqslant b \leqslant T$, let us define

$$\|\delta R\|_{\eta,[a,b]} := \sup_{\substack{s,u,t \in [a,b]: \\ s \leqslant u \leqslant t,\ s<t}} \frac{|\delta R_{sut}|}{(t-s)^\eta} . \tag{1.42}$$

Following the proof of Theorem 1.9, we can replace $\|\delta R\|_\eta$ by $\|\delta R\|_{\eta,[a,b]}$ in (1.28) and in (1.29), hence we obtain $|R_{ab}| \leqslant K_\eta \, \|\delta R\|_{\eta,[a,b]} \, (b-a)^\eta$. Then for $b-a \leqslant \tau$ we can estimate

$$e^{-\frac{b}{\tau}} \frac{|R_{ab}|}{(b-a)^\eta} \leqslant e^{-\frac{b}{\tau}} K_\eta \, \|\delta R\|_{\eta,[a,b]} \leqslant K_\eta \, \|\delta R\|_{\eta,\tau} ,$$

and (1.41) follows taking the supremum over $0 \leqslant a \leqslant b \leqslant T$ with $b-a \leqslant \tau$. □

1.8. A discrete Sewing Bound

We can prove a version of the Sewing Bound for functions $R = (R_{st})_{s<t\in\mathbb{T}}$ defined on a *finite set of points* $\mathbb{T} := \{0 = t_1 < \cdots < t_{\#\mathbb{T}}\} \subseteq \mathbb{R}_+$ (this will be useful to construct solutions to difference equations via Euler schemes, see Sections 2.6 and 3.9). The condition $R_{st} = o(t-s)$ from Theorem 1.9 is now replaced by the requirement that *R vanishes on consecutive points of $\mathbb{T}$*, i.e. $R_{t_i t_{i+1}} = 0$ for all $1 \leqslant i < \#\mathbb{T}$.

We define versions $\|\cdot\|_{\eta,\tau}^{\mathbb{T}}$ of the norms $\|\cdot\|_{\eta,\tau}$ restricted on $\mathbb{T}$ for $\tau > 0$, recall (1.33)-(1.34):

$$\|A\|_{\eta,\tau}^{\mathbb{T}} := \sup_{\substack{0 \leqslant s < t \\ s,t \in \mathbb{T}}} \mathbb{1}_{\{0<t-s\leqslant\tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|A_{st}|}{|t-s|^\eta} , \tag{1.43}$$

$$\|B\|_{\eta,\tau}^{\mathbb{T}} := \sup_{\substack{0 \leqslant s \leqslant u \leqslant t \\ s,u,t \in \mathbb{T},\ s<t}} \mathbb{1}_{\{0<t-s\leqslant\tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|B_{sut}|}{|t-s|^\eta} \tag{1.44}$$

for $A: \{(s,t) \in \mathbb{T}^2 : 0 \leqslant s < t\} \to \mathbb{R}$ and $B: \{(s,u,t) \in \mathbb{T}^3 : 0 \leqslant s \leqslant u \leqslant t, s < t\} \to \mathbb{R}$.

Theorem 1.18. (Discrete Sewing Bound) *If a function $R = (R_{st})_{s<t\in\mathbb{T}}$ vanishes on consecutive points of $\mathbb{T}$ (i.e. $R_{t_i t_{i+1}} = 0$), then for any $\eta > 1$ and $\tau > 0$ we have*

$$\|R\|_{\eta,\tau}^{\mathbb{T}} \leqslant C_\eta \, \|\delta R\|_{\eta,\tau}^{\mathbb{T}} \qquad \text{with} \qquad C_\eta := 2^\eta \sum_{n\geq 1} \frac{1}{n^\eta} = 2^\eta \zeta(\eta) < \infty . \tag{1.45}$$

Proof. We fix $s, t \in \mathbb{T}$ with $s < t$ and we start by proving that

$$|R_{st}| \leqslant C_\eta \, \|\delta R\|_\eta^{\mathbb{T}} \, (t-s)^\eta .$$

We have $s=t_k$ and $t=t_{k+m}$ and we may assume that $m\geqslant 2$ (otherwise there is nothing to prove, since for $m=1$ we have $R_{t_it_{i+1}}=0$).

Consider the partition $\mathcal{P}=\{s=t_k<t_{k+1}<\ldots<t_{k+m}=t\}$ with m intervals. Note that for some index $i\in\{k+1,\ldots,k+m-1\}$ we must have $t_{i+1}-t_{i-1}\leq\frac{2\,(t-s)}{m-1}$, otherwise we would get the contradiction

$$2\,(t-s)\geq\sum_{i=k+1}^{k+m-1}(t_{i+1}-t_{i-1})>\sum_{i=k+1}^{k+m-1}\frac{2\,(t-s)}{m-1}=2\,(t-s)\,.$$

Removing the point t_i from $\mathcal{P}$ we obtain a partition $\mathcal{P}'$ with $m-1$ intervals. If we define $I_{\mathcal{P}}(R):=\sum_{i=k}^{k+m-1}R_{t_it_{i+1}}$ as in (1.20), as in (1.24) we have

$$|I_{\mathcal{P}}(R)-I_{\mathcal{P}'}(R)|=|\delta R_{t_{i-1}t_it_{i+1}}|\leqslant\frac{2^\eta\,(t-s)^\eta}{(m-1)^\eta}\sup_{\substack{s\leqslant u<v<w\leqslant t\\ u,v,w\in\mathbb{T}}}\frac{|\delta R_{uvw}|}{|w-u|^\eta}\,.$$

Iterating this argument, until we arrive at the trivial partition $\{s,t\}$, we get

$$|I_{\mathcal{P}}(R)-R_{st}|\leq C_\eta\,(t-s)^\eta\sup_{\substack{s\leqslant u<v<w\leqslant t\\ u,v,w\in\mathbb{T}}}\frac{|\delta R_{uvw}|}{|w-u|^\eta},\tag{1.46}$$

with $C_\eta:=\sum_{n\geq 1}\frac{2^\eta}{n^\eta}<\infty$ because $\eta>1$. We finally note that $I_{\mathcal{P}}(R)=0$ by the assumption $R_{t_i\,t_{i+1}}=0$. Finally if $t-s\leqslant\tau$ then $w-u\leqslant\tau$ in the supremum in (1.46) and since $\mathrm{e}^{-\frac{t}{\tau}}\leqslant\mathrm{e}^{-\frac{w}{\tau}}$ we obtain

$$\mathrm{e}^{-\frac{t}{\tau}}\,|R_{st}|\leqslant C_\eta\,(t-s)^\eta\,\|\delta R\|_{\eta,\tau}^{\mathbb{T}},$$

and the proof is complete. □

We also have an analog of Lemma 1.16. We set for $f\colon\mathbb{T}\to\mathbb{R}$ and $\tau>0$

$$\|f\|_{\infty,\tau}^{\mathbb{T}}:=\sup_{t\in\mathbb{T}}\mathrm{e}^{-\frac{t}{\tau}}\,|f_t|\,.$$

LEMMA 1.19. (DISCRETE SUPREMUM-HÖLDER BOUND) *For* $\mathbb{T}:=\{0=t_1<\cdots<t_{\#\mathbb{T}}\}\subseteq\mathbb{R}_+$ *set*

$$M:=\max_{i=2,\ldots,\#\mathbb{T}}|t_i-t_{i-1}|.$$

Then for all $f\colon\mathbb{T}\to\mathbb{R}$, $\tau\geqslant 2M$ *and* $\eta>0$

$$\|f\|_{\infty,\tau}^{\mathbb{T}}\leqslant|f_0|+5\,\tau^\eta\,\|\delta f\|_{\eta,\tau}^{\mathbb{T}}\,.\tag{1.47}$$

Proof. We define $T_0:=0$ and for $i\geqslant 1$, as long as $\mathbb{T}\cap(T_{i-1},T_{i-1}+\tau]$ is not empty, we set

$$T_i:=\max\mathbb{T}\cap(T_{i-1},T_{i-1}+\tau],\qquad i=1,\ldots,N,$$

so that $T_N=\max\mathbb{T}$. We have by construction $T_i+M>T_{i-1}+\tau$ for all $i=1,\ldots,N-1$, and since $M\leqslant\frac{\tau}{2}$

$$T_i-T_{i-1}\geqslant\tau-M\geqslant\frac{\tau}{2}\,.$$

For $i = N$ we have only $T_N > T_{N-1}$. Therefore for $i = 1, \ldots N$

$$\begin{aligned}
\mathrm{e}^{-\frac{T_i}{\tau}} |f_{T_i}| &\leqslant |f_0| + \sum_{k=1}^{i} (T_k - T_{k-1})^\eta \, \mathrm{e}^{-\frac{T_i - T_k}{\tau}} \left[\mathrm{e}^{-\frac{T_k}{\tau}} \frac{|f_{T_k} - f_{T_{k-1}}|}{(T_k - T_{k-1})^\eta} \right] \\
&\leqslant |f_0| + \tau^\eta \, \|\delta f\|_{\eta,\tau}^{\mathbb{T}} \sum_{k=1}^{i} \mathrm{e}^{-\frac{T_i - T_k}{\tau}} \\
&\leqslant |f_0| + \tau^\eta \, \|\delta f\|_{\eta,\tau}^{\mathbb{T}} \left(1 + \sum_{k=0}^{\infty} \mathrm{e}^{-\frac{k}{2}} \right) \\
&\leqslant |f_{t_0}| + 4\tau^\eta \, \|\delta f\|_{\eta,\tau}^{\mathbb{T}}.
\end{aligned}$$

Now for $t \in \mathbb{T} \setminus \{T_i\}_i$ we have $T_i < t < T_{i+1}$ for some i and then

$$\begin{aligned}
\mathrm{e}^{-\frac{t}{\tau}} |f_t| &\leqslant \mathrm{e}^{-\frac{t}{\tau}} |f_{T_i}| + (t - T_i)^\eta \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|f_t - f_{T_i}|}{(t - T_i)^\eta} \leqslant \mathrm{e}^{-\frac{T_i}{\tau}} |f_{T_i}| + \tau^\eta \, \|\delta f\|_{\eta,\tau}^{\mathbb{T}} \\
&\leqslant |f_0| + 5\tau^\eta \, \|\delta f\|_{\eta,\tau}^{\mathbb{T}}.
\end{aligned}$$

The proof is complete. □

1.9. Extra (to be completed)

We also introduce the usual supremum norm, for $F \in C_2$ and $G \in C_3$:

$$\|F\|_\infty := \sup_{0 \leqslant s \leqslant t \leqslant T} |F_{st}|, \qquad \|G\|_\infty := \sup_{0 \leqslant s \leqslant u \leqslant t \leqslant T} |G_{sut}|,$$

and a corresponding weighted version, for $\tau \in (0, \infty)$:

$$\|F\|_{\infty,\tau} := \sup_{0 \leqslant s \leqslant t \leqslant T} \mathrm{e}^{-\frac{t}{\tau}} |F_{st}|, \qquad \|G\|_{\infty,\tau} := \sup_{0 \leqslant s \leqslant u \leqslant t \leqslant T} \mathrm{e}^{-\frac{t}{\tau}} |G_{sut}|. \tag{1.48}$$

Note that

$$\lim_{\tau \to +\infty} \|F\|_{\infty,\tau} = \|F\|_\infty, \qquad \lim_{\tau \to +\infty} \|G\|_{\eta,\tau} = \|G\|_\eta, \qquad \lim_{\tau \to +\infty} \|H\|_{\eta,\tau} = \|H\|_\eta.$$

We have

$$\|F\|_{\eta,\tau} \leqslant \|G\|_{\infty,\tau} \|H\|_\eta, \qquad (F_{sut} = G_{su} H_{ut}), \tag{1.49}$$

Note that $\|\cdot\|_{\eta,\tau}$ is only a semi-norm on C_n^η if $\tau < T$; we have at least

$$\|\cdot\|_{\eta,\tau} \leqslant \|\cdot\|_\eta \leqslant e^{\frac{T}{\tau}} \left(\|\cdot\|_{\eta,\tau} + \frac{1}{\tau^\eta} \|\cdot\|_{\infty,\tau} \right). \tag{1.50}$$

However, if $\tau \geq T$ we have again equivalence of norms

$$\|\cdot\|_{\eta,\tau} \leqslant \|\cdot\|_\eta \leqslant e^{\frac{T}{\tau}} \|\cdot\|_{\eta,\tau}, \qquad \tau \geq T. \tag{1.51}$$

Chapter 2

Difference equations: the Young case

Fix a time horizon $T>0$ and two dimensions $k,d\in\mathbb{N}$. We study the following *controlled difference equation* for an unknown path $Z\colon[0,T]\to\mathbb{R}^k$:

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + o(t-s)\,, \qquad 0\leqslant s\leqslant t\leqslant T, \tag{2.1}$$

where the "driving path" $X\colon[0,T]\to\mathbb{R}^d$ and the function $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ are given, and $o(t-s)$ is *uniform* for $0\leqslant s\leqslant t\leqslant T$ (see Remark 1.1).

The difference equation (2.1) is a natural generalized formulation of the *controlled differential equation*

$$\dot{Z}_t = \sigma(Z_t)\,\dot{X}_t\,, \qquad 0\leqslant t\leqslant T. \tag{2.2}$$

Indeed, as we showed in Chapter 1 (see Section 1.2), equations (2.1) and (2.2) are *equivalent* when X is continuously differentiable and σ is continuous, but (2.1) is meaningful also when X is non differentiable.

In this chapter we prove *well-posedness for the difference equation (2.1)* when the driving path $X\in\mathcal{C}^\alpha$ is Hölder continuous in the regime $\alpha\in\left]\frac{1}{2},1\right]$, called the *Young case*. The more challenging regime $\alpha\leqslant\frac{1}{2}$, called the *rough case*, is the object of the next Chapter 3, where new ideas will be introduced.

2.1. Summary

Using the increment notation $\delta f_{st}:=f_t-f_s$ from (1.11), we rewrite (2.1) as

$$\delta Z_{st} = \sigma(Z_s)\,\delta X_{st} + o(t-s), \qquad 0\leqslant s\leqslant t\leqslant T, \tag{2.3}$$

so that a solution of (2.3) is any path $Z\colon[0,T]\to\mathbb{R}^k$ such that the *"remainder"*

$$Z^{[2]}_{st} := \delta Z_{st} - \sigma(Z_s)\,\delta X_{st} \qquad \text{satisfies} \qquad Z^{[2]}_{st} = o(t-s)\,. \tag{2.4}$$

We summarize the main results of this chapter stating *local and global existence, uniqueness of solutions and continuity of the solution map* for the difference equation (2.3) under natural assumptions on σ. We will actually prove more precise results, which yield quantitative estimates.

Theorem 2.1. (Well-posedness) *Let $X\colon[0,T]\to\mathbb{R}^d$ be of class $\mathcal{C}^\alpha$ with $\alpha\in\left]\frac{1}{2},1\right]$ and let $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$. Then we have:*

- ***local existence**: if σ is locally γ-Hölder with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$ (e.g. of class C^1), then for every $z_0\in\mathbb{R}^k$ there is a possibly shorter time horizon $T'=T'_{\alpha,X,\sigma}(z_0)\in\,]0,T]$ and a path $Z\colon[0,T']\to\mathbb{R}^k$ starting from $Z_0=z_0$ which solves (2.3) for $0\leqslant s\leqslant t\leqslant T'$;*

- ***global existence**: if σ is globally γ-Hölder with $\gamma \in \left(\frac{1}{\alpha} - 1, 1\right]$ (e.g. of class C^1 with $\|\nabla\sigma\|_\infty < \infty$), then we can take $T'_{\alpha,X,\sigma}(z_0) = T$ for any $z_0 \in \mathbb{R}^d$;*
- ***uniqueness**: if σ is of class $\mathcal{C}^\gamma$ with $\gamma \in \left(\frac{1}{\alpha}, 2\right]$ (e.g. if σ is of class C^2), then there is exactly one solution Z of (2.3) with $Z_0 = z_0$;*
- ***continuity of the solution map**: if σ is differentiable with bounded and globally $(\gamma - 1)$-Hölder gradient with $\gamma \in \left(\frac{1}{\alpha}, 2\right]$ (i.e. $\|\nabla\sigma\|_\infty < \infty$, $[\nabla\sigma]_{\mathcal{C}^{\gamma-1}} < \infty$), then the solution Z of (2.3) is a continuous function of the starting point z_0 and driving path X: the map $(z_0, X) \mapsto Z$ is continuous from $\mathbb{R}^k \times \mathcal{C}^\alpha \to \mathcal{C}^\alpha$.*

In the first part of this chapter, we give for granted the existence of solutions and we focus on their properties: we prove *a priori estimates* in Section 2.3, *uniqueness of solutions* in Section 2.4 and *continuity of the solution map* in Section 2.5. A key role is played by the Sewing Bound from Chapter 1, see Theorems 1.9 and 1.17, and its discrete version, see Theorem 1.18.

The proof of local and global *existence of solutions of (2.3)* is given at the end of this chapter, see Section 2.6, exploiting a suitable Euler scheme.

2.2. SET-UP

We collect here some notions and tools that will be used extensively.

We recall that C_1 denotes the space of continuous functions $f: [0,T] \to \mathbb{R}^k$. Similarly, C_2 and C_3 are the spaces of continuous functions of two and three ordered variables, i.e. defined on $[0,T]^2_\leqslant$ and $[0,T]^3_\leqslant$, see (1.7)-(1.8).

We are going to exploit the *weighted semi-norms* $\|\cdot\|_{\eta,\tau}$, see (1.33)-(1.34) (see also (1.9) for the original norm $\|\cdot\|_\eta$). These are useful to bound the *weighted supremum norm* $\|f\|_{\infty,\tau}$ of a function $f \in C_1$, see (1.37) and (1.40):

$$\|f\|_{\infty,\tau} \leqslant |f_0| + 3\,(\tau \wedge T)^\eta\,\|\delta f\|_{\eta,\tau}, \qquad \forall \eta, \tau > 0. \tag{2.5}$$

It follows directly from the definitions (1.33)-(1.34) that

$$\|\cdot\|_{\eta,\tau} \leqslant (\tau \wedge T)^{\eta'}\,\|\cdot\|_{\eta+\eta',\tau}\,, \qquad \forall \eta, \eta' > 0, \tag{2.6}$$

because $(t-s)^\eta \geqslant (t-s)^{\eta+\eta'}(\tau \wedge T)^{-\eta'}$ for $0 \leqslant s \leqslant t \leqslant T$ with $t - s \leqslant \tau$.

Remark 2.2. The factor $(\tau \wedge T)^{\eta'}$ in the RHS of (2.6) can be made small *by choosing τ small while keeping T fixed*. This is why we included the indicator function $\mathbb{1}_{\{0 < t-s \leqslant \tau\}}$ in the definition (1.33)-(1.34) of the norms $\|\cdot\|_{\eta,\tau}$: without this indicator function, instead of $(\tau \wedge T)^{\eta'}$ we would have $T^{\eta'}$, which is small only when T is small.

We will often work with functions $F \in C_2$ or $F \in C_3$ that are *product of two factors*, like $F_{st} = g_s\,H_{st}$ or $F_{sut} = G_{su}\,H_{ut}$. We show in the next result that the semi-norm $\|F\|_{\eta,\tau}$ can be controlled by a product of suitable norms for each factor.

Lemma 2.3. (Weighted bounds) *For any $\eta, \eta' \in (0, \infty)$ and $\tau > 0$, we have*

$$\text{if} \quad F_{st} = g_s H_{st} \quad \text{or} \quad F_{st} = g_t H_{st} \qquad \text{then} \qquad \|F\|_{\eta,\tau} \leqslant \|g\|_{\infty,\tau} \|H\|_{\eta}, \tag{2.7}$$

$$\text{if} \quad F_{sut} = G_{su} H_{ut} \qquad \text{then} \qquad \|F\|_{\eta+\eta',\tau} \leqslant \|G\|_{\eta,\tau} \|H\|_{\eta'} . \tag{2.8}$$

Proof. If $F_{st} = g_t H_{st}$, by (1.37) we can estimate $\mathrm{e}^{-t/\tau} |g_t| \leqslant \|g\|_{\infty,\tau}$ to get (2.7). If $F_{st} = g_s H_{st}$, for $s \leqslant t$ we can bound $\mathrm{e}^{-t/\tau} \leqslant \mathrm{e}^{-s/\tau}$ in the definition (1.33)-(1.34) of $\|\cdot\|_{\eta,\tau}$, hence again by (1.37) we can estimate $\mathrm{e}^{-s/\tau} |g_s| \leqslant \|g\|_{\infty,\tau}$ to get (2.7).

If $F_{sut} = G_{su} H_{ut}$, we can further bound $(t-s)^{\eta+\eta'} \geqslant (t-u)^{\eta} (u-s)^{\eta'}$ in (1.34) and then estimate $\mathrm{e}^{-s/\tau} G_{su} / (u-s)^{\eta} \leqslant \|G\|_{\eta,\tau}$, which yields (2.8). □

We stress that in the RHS of (2.7) and (2.8) *only one factor gets the weighted norm or semi-norm*, while the other factor gets the non-weigthed norm $\|\cdot\|_{\eta}$. We will sometimes need an extra weight, which can be introduced as follows.

Lemma 2.4. (Extra weight) *For any $\eta, \bar{\tau} \in (0, \infty)$ and $0 < \tau \leqslant \bar{\tau}$, we have*

$$\text{if} \quad F_{st} = g_s H_{st} \quad \text{or} \quad F_{st} = g_t H_{st} \qquad \text{then} \qquad \|F\|_{\eta,\tau} \leqslant \|g\|_{\infty,\tau} \, \mathrm{e}^{\frac{T}{\bar{\tau}}} \|H\|_{\eta,\bar{\tau}} . \tag{2.9}$$

Proof. Recall the definition (1.33)-(1.34) of $\|\cdot\|_{\eta,\tau}$ and note that for $0 \leqslant s \leqslant t \leqslant T$ we have $\mathrm{e}^{-t/\tau} |g_t| \leqslant \|g\|_{\infty,\tau}$ and $\mathrm{e}^{-s/\tau} |g_s| \leqslant \|g\|_{\infty,\tau}$ (see the proof of Lemma 2.3). Finally, for $t - s \leqslant \tau \leqslant \bar{\tau}$ we can estimate $|H_{st}| \leqslant \mathrm{e}^{T/\bar{\tau}} \mathrm{e}^{-t/\bar{\tau}} |H_{st}| \leqslant \mathrm{e}^{T/\bar{\tau}} \|H\|_{\eta,\bar{\tau}} (t-s)^{\eta}$. □

We recall that $\mathbb{R}^k \otimes (\mathbb{R}^d)^* \simeq \mathbb{R}^{k \times d}$ is the space of linear applications from $\mathbb{R}^d$ to $\mathbb{R}^k$ equipped with the Hilbert-Schmidt (Euclidean) norm $|\cdot|$. We say that a function is of class C^m if it is continuously differentiable m times. Given $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ of class C^2, that we represent by $\sigma_j^i(z)$ with $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, d\}$, we denote by $\nabla\sigma$: $\mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^k)^*$ its gradient and by $\nabla^2\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^k)^* \otimes (\mathbb{R}^k)^*$ its Hessian, represented for $i, a, b \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, d\}$ by

$$(\nabla\sigma(z))^i_{ja} = \frac{\partial \sigma^i_j}{\partial z_a}(z), \qquad (\nabla^2\sigma(z))^i_{jab} = \frac{\partial^2 \sigma^i_j}{\partial z_a \partial z_b}(z).$$

Remark 2.5. (norm of the gradient of Lipschitz functions) For a *locally Lipschitz function* $\psi: \mathbb{R}^k \to \mathbb{R}^\ell$ we can define the "norm of the gradient" at any point (even where ψ may not be differentiable):

$$|\nabla\psi(z)| := \limsup_{y \to z} \frac{|\psi(y) - \psi(z)|}{|y - z|} \in [0, \infty) .$$

Similarly, $|\nabla^2\psi(z)|$ is well defined as soon as ψ is *differentiable with locally Lipschitz gradient* $\nabla\psi$ (which is slightly less than requiring $\psi \in C^2$).

2.3. A priori estimates

In this section we prove *a priori estimates* for solutions of (2.3) assuming that σ is *globally Lipschitz*, that is $\|\nabla\sigma\|_{\infty} < \infty$ (recall Remark 2.5).

We first observe that if the driving path X is of class $\mathcal{C}^\alpha$, then any solution Z of (2.3) is also of class $\mathcal{C}^\alpha$, as soon as σ is continuous.

LEMMA 2.6. (HÖLDER REGULARITY) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha \in]0,1]$ and let σ be continuous. Then any solution Z of (2.3) is of class $\mathcal{C}^\alpha$.*

Proof. We know by Lemma 1.2 that Z is continuous, more precisely by (1.6) we have $|\delta Z_{st}| \leqslant C\,|\delta X_{st}| + o(t-s)$ with $C < \infty$. Since $|\delta X_{st}| \leqslant \|\delta X\|_\alpha (t-s)^\alpha$ and $o(t-s) = o((t-s)^\alpha)$ for any $\alpha \leqslant 1$, it follows that $Z \in \mathcal{C}^\alpha$. □

We next formulate the announced a priori estimates. It is convenient to use the weighted semi-norms $\|\cdot\|_{\eta,\tau}$ in (1.33)-(1.34) (note that the usual norms $\|\cdot\|_\eta$ in (1.9) can be recovered by letting $\tau \to \infty$).

THEOREM 2.7. (A PRIORI ESTIMATES) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2},1\right]$ and let σ be globally γ-Hölder with $\gamma \in \left(\frac{1}{\alpha}-1,1\right]$. Then, for any solution $Z\colon [0,T] \to \mathbb{R}^k$ of (2.3), the remainder $Z^{[2]}_{st} := \delta Z_{st} - \sigma(Z_s)\,\delta X_{st}$ satisfies $Z^{[2]} \in C_2^{(\gamma+1)\alpha}$, more precisely for any $\tau > 0$*

$$\|Z^{[2]}\|_{(\gamma+1)\alpha,\tau} \leqslant C_{\alpha,\gamma,X,\sigma}\,\|\delta Z\|^\gamma_{\alpha,\tau} \qquad \textit{with } C_{\alpha,\gamma,X,\sigma} := K_{(\gamma+1)\alpha}\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}\,, \tag{2.10}$$

where $K_\eta = (1-2^{1-\eta})^{-1}$. Moreover, if either T or τ is small enough, we have

$$\|\delta Z\|_{\alpha,\tau} \leqslant 1 \vee (2\,\|\delta X\|_\alpha\,|\sigma(Z_0)|) \qquad \textit{for } (\tau\wedge T)^{\alpha\gamma} \leqslant \varepsilon_{\alpha,\gamma,X,\sigma}, \tag{2.11}$$

where we define

$$\varepsilon_{\alpha,\gamma,X,\sigma} := \frac{1}{2\,(K_{(\gamma+1)\alpha}+3)\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}}\,. \tag{2.12}$$

If σ is globally Lipschitz, namely if we can take $\gamma = 1$, we can improve (2.11) to

$$\|\delta Z\|_{\alpha,\tau} \leqslant 2\,\|\delta X\|_\alpha\,|\sigma(Z_0)| \qquad \textit{for } (\tau\wedge T)^\alpha \leqslant \varepsilon_{\alpha,1,X,\sigma}\,. \tag{2.13}$$

Proof. We first prove (2.10). Since $Z^{[2]}_{st} = o(t-s)$ by definition of solution, see (2.4), *we can estimate $Z^{[2]}$ in terms of $\delta Z^{[2]}$*, by the weighted Sewing Bound (1.41). Let us compute $\delta Z^{[2]}_{sut} = Z^{[2]}_{st} - Z^{[2]}_{su} - Z^{[2]}_{ut}$: recalling (2.4) and (1.32), since $\delta \circ \delta = 0$, we have

$$\delta Z^{[2]}_{sut} = \delta\sigma(Z)_{su}\,\delta X_{ut} = (\sigma(Z_u) - \sigma(Z_s))\,(X_t - X_u)\,. \tag{2.14}$$

Since $|\sigma(z) - \sigma(\bar{z})| \leqslant [\sigma]_{\mathcal{C}^\gamma}\,|z-\bar{z}|^\gamma$ for all $z, \bar{z} \in \mathbb{R}^d$, we can bound

$$\|\delta\sigma(Z)\|_{\gamma\alpha,\tau} \leqslant [\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|^\gamma_{\alpha,\tau}, \tag{2.15}$$

hence by (2.8) we obtain

$$\|\delta Z^{[2]}\|_{(\gamma+1)\alpha,\tau} \leqslant \|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|^\gamma_{\alpha,\tau}.$$

Applying the weighted Sewing Bound (1.41), for $(\gamma+1)\alpha > 1$ we then obtain

$$\|Z^{[2]}\|_{(\gamma+1)\alpha,\tau} \leqslant K_{(\gamma+1)\alpha}\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|^\gamma_{\alpha,\tau}\,, \tag{2.16}$$

which proves (2.10).

We next prove (2.11). To simplify notation, let us set $\varepsilon := (\tau \wedge T)^\alpha$. Recalling (2.7) and (2.6), we obtain by (2.4)

$$\begin{aligned} \|\delta Z\|_{\alpha,\tau} &\leqslant \|\sigma(Z)\,\delta X\|_{\alpha,\tau} + \|Z^{[2]}\|_{\alpha,\tau} \\ &\leqslant \|\sigma(Z)\|_{\infty,\tau}\,\|\delta X\|_\alpha + \varepsilon^\gamma\,\|Z^{[2]}\|_{(\gamma+1)\alpha,\tau}\,. \end{aligned} \tag{2.17}$$

We can estimate $\|\sigma(Z)\|_{\infty,\tau}$ by (2.5) and (2.15):

$$\|\sigma(Z)\|_{\infty,\tau} \leqslant |\sigma(Z_0)| + 3\,\varepsilon^\gamma\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|_{\alpha,\tau}^\gamma\,.$$

Plugging this and (2.16) into (2.17), we get

$$\begin{aligned} \|\delta Z\|_{\alpha,\tau} &\leqslant (|\sigma(Z_0)| + 3\,\varepsilon^\gamma\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|_{\alpha,\tau}^\gamma)\,\|\delta X\|_\alpha + \\ &\qquad + \varepsilon^\gamma\,K_{(\gamma+1)\alpha}\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|_{\alpha,\tau}^\gamma \\ &= \|\delta X\|_\alpha\,|\sigma(Z_0)| + \frac{1}{2}\,\frac{\varepsilon^\gamma}{\varepsilon_{\alpha,\gamma,X,\sigma}}\,\|\delta Z\|_{\alpha,\tau}^\gamma\,, \end{aligned}$$

where $\varepsilon_{\alpha,\gamma,X,\sigma}$ is defined in (2.12). For $\varepsilon^\gamma \leqslant \varepsilon_{\alpha,\gamma,X,\sigma}$ the last term is bounded by $\frac{1}{2}\|\delta Z\|_{\alpha,\tau}^\gamma$ *which is finite* by Lemma 2.6. If $\|\delta Z\|_{\alpha,\tau} \leqslant 1$ then (2.11) holds trivially; if not, $\frac{1}{2}\|\delta Z\|_{\alpha,\tau}^\gamma \leqslant \frac{1}{2}\|\delta Z\|_{\alpha,\tau}$. Bringing this term in the LHS we obtain (2.11).

To prove (2.13), we argue as for (2.11) and since $\gamma = 1$ we obtain

$$\|\delta Z\|_{\alpha,\tau} \leqslant \|\delta X\|_\alpha\,|\sigma(Z_0)| + \frac{1}{2}\,\frac{\varepsilon}{\varepsilon_{\alpha,1,X,\sigma}}\,\|\delta Z\|_{\alpha,\tau}.$$

For $\varepsilon \leqslant \varepsilon_{\alpha,1,X,\sigma}$ the last term is bounded by $\frac{1}{2}\|\delta Z\|_{\alpha,\tau}$ *which is finite* by Lemma 2.6. Bringing this term in the LHS we obtain (2.13), and this completes the proof. □

2.4. UNIQUENESS

In this section we prove uniqueness of solutions to (2.3) assuming that σ is of class C^1 *with locally Hölder gradient* (we stress that we make no boundedness assumption on σ). This improves on Theorem 1.7, both because we allow for non-linear σ and because we do not require that the time horizon $T > 0$ is small.

We first need an elementary but fundamental estimate on the difference of increments of a function. Given $\Psi: \mathbb{R}^k \to \mathbb{R}^\ell$, we use the notation

$$C_{\Psi,R} := \sup\{|\Psi(x)|:\ \ x \in \mathbb{R}^k,\ |x| \leqslant R\}\,. \tag{2.18}$$

LEMMA 2.8. (DIFFERENCE OF INCREMENTS) *Let $\psi: \mathbb{R}^k \to \mathbb{R}^\ell$ be of class $\mathcal{C}^{1+\rho}_{\mathrm{loc}}$ for some $0 < \rho \leqslant 1$ (i.e. ψ is differentiable with $\nabla\psi$ of class $\mathcal{C}^\rho_{\mathrm{loc}}$). Then for any $R > 0$ and for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^k$ with $\max\{|x|, |y|, |\bar{x}|, |\bar{y}|\} \leqslant R$ we can estimate*

$$\begin{aligned} &|[\psi(x) - \psi(y)] - [\psi(\bar{x}) - \psi(\bar{y})]| \\ &\qquad \leqslant C_R'\,|(x-y) - (\bar{x} - \bar{y})| + C_R''\,\{|x-y|^\rho + |\bar{x} - \bar{y}|^\rho\}\,|y - \bar{y}|\,, \end{aligned} \tag{2.19}$$

where $C_R' := \sup\{|\nabla\psi(x)|:\ \ |x| \leqslant R\}$ and $C_R'' := \sup\left\{\frac{|\nabla\psi(x) - \nabla\psi(y)|}{|x-y|^\rho}:\ \ |x|, |y| \leqslant R\right\}$.

Proof. For $z, w \in \mathbb{R}^k$ we can write

$$\psi(z) - \psi(w) = \hat{\psi}(z, w)\,(z - w),$$

where $\hat{\psi}(z, w) := \int_0^1 \nabla\psi(u\,z + (1-u)\,w)\,\mathrm{d}u \in \mathbb{R}^\ell \otimes (\mathbb{R}^k)^*$, therefore

$$\begin{aligned}
[\psi(x) - \psi(y)] - [\psi(\bar{x}) - \psi(\bar{y})] &= [\psi(x) - \psi(\bar{x})] - [\psi(y) - \psi(\bar{y})] \\
&= \hat{\psi}(x, \bar{x})\,(x - \bar{x}) - \hat{\psi}(y, \bar{y})\,(y - \bar{y}) \\
&= \hat{\psi}(x, \bar{x})\,[(x - \bar{x}) - (y - \bar{y})] \\
&\quad + [\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{y})]\,(y - \bar{y})\,.
\end{aligned}$$

By definition of C'_R and C''_R we have $|\hat{\psi}(x, \bar{x})| \leqslant C'_R$ and

$$\begin{aligned}
|\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{y})| &\leqslant |\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{x})| + |\hat{\psi}(y, \bar{x}) - \hat{\psi}(y, \bar{y})| \\
&\leqslant C''_R\,\{|x - y|^\rho + |\bar{x} - \bar{y}|^\rho\},
\end{aligned}$$

hence (2.19) follows. □

We are now ready to state and prove the announced uniqueness result.

THEOREM 2.9. (UNIQUENESS) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$ and let σ be of class $\mathcal{C}^\gamma$ for some $\gamma > \frac{1}{\alpha}$ (for instance, we can take $\sigma \in \mathcal{C}^2$). Then for every $z_0 \in \mathbb{R}^k$ there exists at most one solution Z to (2.3) with $Z_0 = z_0$.*

Proof. Let Z and $\bar{Z}$ be two solutions of (2.3), i.e. they satisfy (2.4), and set

$$Y := Z - \bar{Z}\,.$$

We want to show that, for $\tau > 0$ small enough, we have

$$\|Y\|_{\infty,\tau} \leqslant 2\,|Y_0|,$$

where the weighted norm $\|\cdot\|_{\infty,\tau}$ was defined in (1.37). In particular, if we assume that $Z_0 = \bar{Z}_0$, we obtain $\|Y\|_{\infty,\tau} = 0$ and hence $Z = \bar{Z}$.

We know by (2.5) that for any $\tau > 0$

$$\|Y\|_{\infty,\tau} \leqslant |Y_0| + 3\tau^\alpha\,\|\delta Y\|_{\alpha,\tau}\,, \tag{2.20}$$

where we recall that the weighted semi-norm $\|\cdot\|_{\alpha,\tau}$ was defined in (1.33). We now define $Y^{[2]}$ as the difference between the remainders $Z^{[2]}$ and $\bar{Z}^{[2]}$ of the solutions Z and $\bar{Z}$ as defined in (2.4), that is

$$Y^{[2]}_{st} := Z^{[2]}_{st} - \bar{Z}^{[2]}_{st} = \delta Y_{st} - (\sigma(Z_s) - \sigma(\bar{Z}_s))\,\delta X_{st}\,. \tag{2.21}$$

(We are slightly abusing notation, since $Y^{[2]}$ is not the remainder of Y when σ is not linear.) By assumption $\sigma \in \mathcal{C}^\gamma$ for some $\gamma > \frac{1}{\alpha}$: renaming γ as $\gamma \wedge 2$, we may assume that $\gamma \in \left]\frac{1}{\alpha}, 2\right]$. We are going to prove the following inequalities: for any $\tau > 0$

$$\|\delta Y\|_{\alpha,\tau} \leqslant c_1\,\|Y\|_{\infty,\tau} + \tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}\,, \tag{2.22}$$

$$\|Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant c_2\,\|Y\|_{\infty,\tau} + c'_2\,\tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}\,, \tag{2.23}$$

for finite constants c_i, c_i' *that may depend on* $X, \sigma, Z, \bar{Z}$ *but not on* τ.

Let us complete the proof assuming (2.22) and (2.23). Note that $(\gamma - 1)\,\alpha > 0$ by assumption. If we fix $\tau > 0$ small, so that $c_2'\,\tau^{(\gamma-1)\alpha} < \frac{1}{2}$, from (2.23) we get $\|Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant 2\,c_2\,\|Y\|_{\infty,\tau}$ which plugged into (2.22) yields $\|\delta Y\|_{\alpha,\tau} \leqslant 2\,c_1\,\|Y\|_{\infty,\tau}$ for $\tau > 0$ small (it suffices that $2\,c_2\,\tau^{(\gamma-1)\alpha} < c_1$). Finally, plugging this into (2.20) and possibly choosing $\tau > 0$ even smaller, we obtain our goal $\|Y\|_{\infty,\tau} \leqslant 2\,|Y_0|$ which completes the proof.

It remains to prove (2.22) and (2.23). Using the notation from Lemma 2.8 we set

$$
\begin{aligned}
C_1' &:= \sup\,\{|\nabla\sigma(x)|:\ |x| \leqslant \|Z\|_\infty \vee \|\bar{Z}\|_\infty\}\,,\\
C_1'' &:= \sup\left\{\frac{|\nabla\sigma(x) - \nabla\sigma(y)|}{|x-y|^\rho}:\ |x|,|y| \leqslant \|Z\|_\infty \vee \|\bar{Z}\|_\infty\right\}.
\end{aligned}
$$

so that $|\sigma(Z_t) - \sigma(\bar{Z}_t)| \leqslant C_1'\,|Z_t - \bar{Z}_t|$ and, therefore,

$$
\|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau} \leqslant C_1'\,\|Y\|_{\infty,\tau}\,. \tag{2.24}
$$

We now exploit (2.21) to estimate $\|\delta Y\|_{\alpha,\tau}$: applying (2.7) we obtain

$$
\begin{aligned}
\|\delta Y\|_{\alpha,\tau} &\leqslant \|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau}\,\|\delta X\|_\alpha + \|Y^{[2]}\|_{\alpha,\tau}\\
&\leqslant C_1'\,\|Y\|_{\infty,\tau}\,\|\delta X\|_\alpha + \tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}\,,
\end{aligned} \tag{2.25}
$$

where we note that $\|Y^{[2]}\|_{\alpha,\tau} \leqslant \tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}$ by (2.6). We have shown that (2.22) holds with $c_1 = C_1'\,\|\delta X\|_\alpha$.

We finally prove (2.23). Since $Y_{st}^{[2]} = o(t-s)$, see (2.21) and (2.4), we bound $Z^{[2]}$ by its increment $\delta Z^{[2]}$ through the weighted Sewing Bound (1.41):

$$
\|Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant K_{\gamma\alpha}\,\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}\,, \tag{2.26}
$$

hence we focus on $\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}$. By (2.21) and (1.32), since $\delta\circ\delta = 0$, we have

$$
\delta Y_{sut}^{[2]} = (\delta\sigma(Z)_{su} - \delta\sigma(\bar{Z})_{su})\,\delta X_{ut}\,. \tag{2.27}
$$

Applying the estimate (2.19) for $x = Z_u$, $y = Z_s$, $\bar{x} = \bar{Z}_u$, $\bar{y} = \bar{Z}_s$, we can write

$$
\begin{aligned}
|\delta\sigma(Z)_{su} - \delta\sigma(\bar{Z})_{su}| &\leqslant C_1'\,|\delta Z_{su} - \delta\bar{Z}_{su}| + C_1''\,\{|\delta Z_{su}|^{\gamma-1} + |\delta\bar{Z}_{su}|^{\gamma-1}\}|Z_s - \bar{Z}_s|\\
&= C_1'\,|\delta Y_{su}| + C_1''\,\{|\delta Z_{su}|^{\gamma-1} + |\delta\bar{Z}_{su}|^{\gamma-1}\}\,|Y_s|.
\end{aligned} \tag{2.28}
$$

hence by (2.7) we get

$$
\begin{aligned}
\|\delta\sigma(Z) - \delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau} \leqslant\ & C_1'\,\|\delta Y\|_{(\gamma-1)\alpha,\tau} +\\
&+ C_1''\,\{\|\delta Z\|_\alpha^{\gamma-1} + \|\delta\bar{Z}\|_\alpha^{\gamma-1}\}\,\|Y\|_{\infty,\tau}.
\end{aligned} \tag{2.29}
$$

If we take $\tau \leqslant 1$ we can bound $\|\delta Y\|_{(\gamma-1)\alpha,\tau} \leqslant \|\delta Y\|_{\alpha,\tau}$ by (2.6) (recall that we are assuming $\gamma \leqslant 2$). Then by (2.27) we obtain, recalling (2.8),

$$
\|\delta Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant \|\delta X\|_\alpha\,\|\delta\sigma(Z) - \delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau} \leqslant \tilde{c}_1\,(\|\delta Y\|_{\alpha,\tau} + \|Y\|_{\infty,\tau})\,,
$$

for a suitable (explicit) constant $\tilde{c}_1 = \tilde{c}_1(\sigma, Z, \bar{Z}, X)$. Applying (2.22), we obtain

$$
\|\delta Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant (c_1 + 1)\,\tilde{c}_1\,\|Y\|_{\infty,\tau} + \tilde{c}_1\,\tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau},
$$

which plugged into (2.26) shows that (2.23) holds. The proof is complete. □

We conclude with an example of (2.19).

Example 2.10. If $\sigma: \mathbb{R} \to \mathbb{R}$ is $\sigma(x) = x^2$, then we have

$$\begin{aligned}
&(\sigma(x) - \sigma(y)) - (\sigma(\bar{x}) - \sigma(\bar{y})) \\
&\quad = (x^2 - y^2) - (\bar{x}^2 - \bar{y}^2) = (x^2 - \bar{x}^2) - (y^2 - \bar{y}^2) \\
&\quad = (x - \bar{x})\,(x + \bar{x}) - (y - \bar{y})\,(y + \bar{y}) \\
&\quad = [(x - \bar{x}) - (y - \bar{y})]\,(y + \bar{y}) + (x - \bar{x})\,[(x + \bar{x}) - (y + \bar{y})] \\
&\quad = [(x - \bar{x}) - (y - \bar{y})]\,(y + \bar{y}) + (x - \bar{x})\,[(x - y) + (\bar{x} - \bar{y})],
\end{aligned}$$

where in the second last equality we have summed and subtracted $(y - \bar{y})\,(x + \bar{x})$. If we use this formula for $x = Z_t$, $y = Z_s$ and $\bar{x} = \bar{Z}_t$, $\bar{y} = \bar{Z}_s$, then we obtain

$$\delta(Z^2 - \bar{Z}^2)_{st} = \delta(Z - \bar{Z})_{st}\,(Z_s + \bar{Z}_s) + (Z_t - \bar{Z}_t)\,[\delta Z_{st} + \delta Z_{st}],$$

which is in the spirit of (2.19) with $\rho = 1$. It follows that

$$\|\delta(Z^2 - \bar{Z}^2)\|_\alpha \leqslant 2\,\|\bar{Z}\|_\infty\,\|\delta(Z - \bar{Z})\|_\alpha + \|Z - \bar{Z}\|_\infty\,[\|\delta Z\|_\alpha + \|\delta\bar{Z}\|_\alpha],$$

which is the form that (2.29) takes in this particular case.

2.5. CONTINUITY OF THE SOLUTION MAP

In this section we assume that σ is *globally Lipschitz* and of class C^1 with a *globally γ-Hölder gradient*, i.e. $\|\nabla\sigma\|_\infty < \infty$ and $[\nabla\sigma]_{\mathcal{C}^\gamma} < \infty$, with $\gamma > \frac{1}{\alpha}$. Under these assumptions, we have *global existence and uniqueness* of solutions $Z: [0, T] \to \mathbb{R}^k$ to (2.3) for any time horizon $T > 0$, for any starting point $Z_0 \in \mathbb{R}^k$ and for any driving path X of class $\mathcal{C}^\alpha$ with $\frac{1}{2} < \alpha \leqslant 1$ (as we will prove in Section 2.6).

We can thus consider the *solution map*:

$$\begin{aligned}
\Phi:\ \mathbb{R}^k \times \mathcal{C}^\alpha &\longrightarrow \mathcal{C}^\alpha \\
(Z_0, X) &\longmapsto Z := \begin{cases} \text{unique solution of (2.3) for } t \in [0, T] \\ \text{starting from } Z_0 \end{cases} .
\end{aligned} \tag{2.30}$$

We prove in this section that this map is *continuous*, in fact *locally Lipschitz*.

Remark 2.11. The continuity of the solution map is a highly non-trivial property. Indeed, when X is of class C^1, note that Z solves the equation

$$Z_t = Z_0 + \int_0^t \sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s\,, \tag{2.31}$$

which is based on the derivative $\dot{X}$ of X. We instead consider driving paths $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$ which are continuous but may be non-differentiable.

We shall see in the next chapters that the continuity of the solution map holds also in more complex situations such as $X \in \mathcal{C}^\alpha$ with $\alpha \leqslant \frac{1}{2}$, which cover the case when X is a Brownian motion and Z is the solution to a SDE.

Before stating the continuity of the solution map, we recall that the space $\mathcal{C}^\alpha$ is equipped with the norm $\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha$, see Remark 1.4, but *an equivalent norm is* $\|f\|_{\infty,\tau} + \|\delta f\|_{\alpha,\tau}$ *for any choice of the weight* $\tau > 0$, see Remark 1.15.

Theorem 2.12. (Continuity of the solution map) *Let σ be globally Lipschitz with a globally $(\gamma-1)$-Hölder gradient: $\|\nabla\sigma\|_\infty < \infty$ and $[\nabla\sigma]_{\mathcal{C}^{\gamma-1}} < \infty$, with $\gamma \in \left(\frac{1}{\alpha}, 2\right]$. Then, for any $T > 0$ and $\alpha \in \left]\frac{1}{2}, 1\right]$, the solution map $(Z_0, X) \mapsto Z$ in (2.30) is locally Lipschitz.*

More explicitly, given $M_0, M, D < \infty$, if we assume that

$$\max\{\|\nabla\sigma\|_\infty, [\nabla\sigma]_{\mathcal{C}^{\gamma-1}}\} \leqslant D,$$

and we consider starting points $Z_0, \bar{Z}_0 \in \mathbb{R}^d$ and driving paths X, $\bar{X} \in \mathcal{C}^\alpha$ with

$$\max\{|\sigma(Z_0)|, |\sigma(\bar{Z}_0)|\} \leqslant M_0, \qquad \max\{\|\delta X\|_\alpha, \|\delta\bar{X}\|_\alpha\} \leqslant M, \tag{2.32}$$

then the corresponding solutions $Z = (Z_s)_{s\in[0,T]}$, $\bar{Z} = (\bar{Z}_s)_{s\in[0,T]}$ of (2.3) satisfy

$$\|Z - \bar{Z}\|_{\infty,\tau} + \|\delta Z - \delta\bar{Z}\|_{\alpha,\tau} \leqslant \mathfrak{C}_M |Z_0 - \bar{Z}_0| + 6 M_0 \|\delta X - \delta\bar{X}\|_\alpha, \tag{2.33}$$

provided $0 < \tau \wedge T \leqslant \hat{\tau}$ for a suitable $\hat{\tau} = \hat{\tau}_{\alpha,\gamma,T,D,M_0,M} > 0$, where we set

$$\mathfrak{C}_M := 2(\|\nabla\sigma\|_\infty M + 1) \leqslant 2(DM + 1).$$

Proof. Let us define the constant

$$\mathfrak{c}_M := \|\nabla\sigma\|_\infty M \leqslant DM. \tag{2.34}$$

We fix two solutions Z and $\bar{Z}$ of (2.3) with respective driving paths X and $\bar{X}$. If we define $Y := Z - \bar{Z}$, we can rewrite our goal (2.33) as

$$\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau} \leqslant 6 M_0 \|\delta X - \delta\bar{X}\|_\alpha + 2(\mathfrak{c}_M + 1)|Y_0|. \tag{2.35}$$

Let us introduce the shorthand

$$\varepsilon := (\tau \wedge T)^\alpha$$

and let us agree that, whenever we write *for ε small enough* we mean *for $0 < \varepsilon \leqslant \varepsilon_0$ for a suitable $\varepsilon_0 > 0$ which depends on α, T, M_0, M, D*. By (2.5), *for ε small enough*,

$$\|Y\|_{\infty,\tau} \leqslant |Y_0| + \varepsilon\|\delta Y\|_{\alpha,\tau} \leqslant |Y_0| + \frac{1}{5}\|\delta Y\|_{\alpha,\tau}, \tag{2.36}$$

hence to prove (2.35) we can focus on $\|\delta Y\|_{\alpha,\tau}$.

Recalling (2.4), let us define $Y^{[2]} := Z^{[2]} - \bar{Z}^{[2]}$. We are going to establish the following two relations, *for ε small enough*:

$$\frac{4}{5}\|\delta Y\|_{\alpha,\tau} \leqslant 2 M_0 \|\delta X - \delta\bar{X}\|_\alpha + \mathfrak{c}_M |Y_0| + \|Y^{[2]}\|_{\alpha,\tau}, \tag{2.37}$$

$$\|Y^{[2]}\|_{\alpha,\tau} \leqslant M_0 \|\delta X - \delta\bar{X}\|_\alpha + \frac{1}{2}|Y_0| + \frac{1}{5}\|\delta Y\|_{\alpha,\tau}. \tag{2.38}$$

Plugging (2.38) into (2.37) and applying (2.36), we obtain (2.35).

It remains to prove (2.37) and (2.38). We record some useful bounds. Let us set

$$\bar{\varepsilon} = \bar{\varepsilon}_{\alpha,D,M} := \frac{1}{2\,(K_{2\alpha}+3)\,D\,M}. \tag{2.39}$$

We exploit the a priori estimate (2.13) from Theorem 2.7: by (2.32), we have

$$\text{for } \varepsilon = (\tau\wedge T)^\alpha \leqslant \bar{\varepsilon}: \qquad \max\{\|\delta Z\|_{\alpha,\tau}, \|\delta\bar{Z}\|_{\alpha,\tau}\} \leqslant 2\,M_0\,M, \tag{2.40}$$

therefore

$$\|\delta\sigma(Z)\|_{\alpha,\tau} \leqslant \|\nabla\sigma\|_\infty\,\|\delta Z\|_{\alpha,\tau} \leqslant 2\,\|\nabla\sigma\|_\infty\,M_0\,M = 2\,M_0\,\mathfrak{c}_M, \tag{2.41}$$

and applying (2.5) and (2.32) we get, *for ε small enough*,

$$\|\sigma(Z)\|_{\infty,\tau} \leqslant |\sigma(Z_0)| + 3\varepsilon\,\|\delta\sigma(Z)\|_{\alpha,\tau} \leqslant M_0\,(1+6\,\mathfrak{c}_M\,\varepsilon) \leqslant 2\,M_0 \ . \tag{2.42}$$

We can now prove (2.37). Defining $Y^{[2]} := Z^{[2]} - \bar{Z}^{[2]}$, we obtain from (2.4)

$$\begin{aligned} \delta Y_{st} &= \delta Z_{st} - \delta\bar{Z}_{st} = \sigma(Z_s)\,\delta X_{st} - \sigma(\bar{Z}_s)\,\delta\bar{X}_{st} + Y^{[2]}_{st} \\ &= \sigma(Z_s)\,(\delta X - \delta\bar{X})_{st} + (\sigma(Z_s) - \sigma(\bar{Z}_s))\,\delta\bar{X}_{st} + Y^{[2]}_{st}, \end{aligned}$$

hence by (2.7) we can bound

$$\begin{aligned} \|\delta Y\|_{\alpha,\tau} \leqslant\ & \|\sigma(Z)\|_{\infty,\tau}\,\|\delta X - \delta\bar{X}\|_\alpha \\ & + \|\delta\bar{X}\|_\alpha\,\|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau} + \|Y^{[2]}\|_{\alpha,\tau}\,. \end{aligned} \tag{2.43}$$

Let us look at the second term in the RHS of (2.43): by (2.5)

$$\begin{aligned} \|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau} &\leqslant \|\nabla\sigma\|_\infty\,\|Z - \bar{Z}\|_{\infty,\tau} \\ &\leqslant \|\nabla\sigma\|_\infty\,(|Y_0| + 3\varepsilon\,\|\delta Y\|_{\alpha,\tau}). \end{aligned} \tag{2.44}$$

Hence by (2.32) and (2.34) we get, *for ε small enough*,

$$\|\delta\bar{X}\|_\alpha\,\|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau} \leqslant \mathfrak{c}_M\,|Y_0| + \frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}\,. \tag{2.45}$$

Plugging this into (2.43) we then obtain, by (2.42),

$$\frac{4}{5}\,\|\delta Y\|_{\alpha,\tau} \leqslant 2\,M_0\,\|\delta X - \delta\bar{X}\|_\alpha + \mathfrak{c}_M\,|Y_0| + \|Y^{[2]}\|_{\alpha,\tau}, \tag{2.46}$$

which proves (2.37).

We finally prove (2.38). Since $Y^{[2]}_{st} = Z^{[2]}_{st} - \bar{Z}^{[2]}_{st} = o(t-s)$, see (2.4), the weighted Sewing Bound (1.41) and (2.6) give

$$\|Y^{[2]}\|_{\alpha,\tau} \leqslant \varepsilon^{\gamma-1}\,\|Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta Y^{[2]}\|_{\gamma\alpha,\tau} \ . \tag{2.47}$$

To estimate $\delta Y^{[2]} = \delta Z^{[2]} - \delta\,\bar{Z}^{[2]}$, note that by (2.4) and (1.32) we can write

$$\delta Y^{[2]}_{sut} = \delta\sigma(Z)_{su}\,(\delta X - \delta\bar{X})_{ut} + (\delta\sigma(Z) - \delta\sigma(\bar{Z}))_{su}\,\delta\bar{X}_{ut}\,, \tag{2.48}$$

hence by (2.8)

$$\|\delta Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant \|\delta\sigma(Z)\|_{(\gamma-1)\alpha,\tau}\,\|\delta X - \delta\bar{X}\|_\alpha + \|\delta\bar{X}\|_\alpha\,\|\delta\sigma(Z) - \delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\,. \tag{2.49}$$

The first term is easy to control: by (2.41), *for ε small enough*,

$$K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta\sigma(Z)\|_{(\gamma-1)\alpha,\tau}\,\|\delta X-\delta\bar{X}\|_{\alpha}\leqslant M_0\,\|\delta X-\delta\bar{X}\|_{\alpha}\,. \tag{2.50}$$

Let us now focus on the second term. By (2.19) we have, see also (2.28),

$$|\delta\sigma(Z)_{su}-\delta\sigma(\bar{Z})_{su}|\leqslant\|\nabla\sigma\|_{\infty}\,|\delta Y_{su}|+[\nabla\sigma]_{\mathcal{C}^{\gamma-1}}\{|\delta Z_{su}|^{\gamma-1}+|\delta\bar{Z}_{su}|^{\gamma-1}\}\,|Y_s|\,.$$

We apply (2.9) for $H=\delta Z$, $g=Y$ and $\bar{\tau}=(\bar{\varepsilon})^{1/\alpha}$ from (2.39):

$$\begin{aligned}\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau} &\leqslant \|\nabla\sigma\|_{\infty}\,\|\delta Y\|_{(\gamma-1)\alpha,\tau}+\\ &\quad+[\nabla\sigma]_{\mathcal{C}^{\gamma-1}}\,\mathrm{e}^{\frac{T}{\bar{\tau}}}\,(\|\delta Z\|_{\alpha,\bar{\tau}}^{\gamma-1}+\|\delta\bar{Z}\|_{\alpha,\bar{\tau}}^{\gamma-1})\|Y\|_{\infty,\tau}\\ &\leqslant D\,\|\delta Y\|_{\alpha,\tau}+2\,(2M_0\,M)^{\gamma-1}\,\mathrm{e}^{\frac{T}{\bar{\tau}}}D\,\|Y\|_{\infty,\tau},\end{aligned} \tag{2.51}$$

where we applied (2.40). Hence by (2.51), recalling (2.32), *for ε small enough* we obtain

$$K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta\bar{X}\|_{\alpha}\,\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\leqslant\frac{1}{10}\,\|\delta Y\|_{\alpha,\tau}+\frac{1}{2}\|Y\|_{\infty,\tau}, \tag{2.52}$$

and since $\|Y\|_{\infty,\tau}\leqslant|Y_0|+\frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}$, see (2.36), we obtain

$$K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta\bar{X}\|_{\alpha}\,\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\leqslant\frac{1}{2}|Y_0|+\frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}.$$

Finally, plugging this bound and (2.50) into (2.49) and (2.47), we obtain

$$\|Y^{[2]}\|_{\alpha,\tau}\leqslant M_0\,\|\delta X-\delta\bar{X}\|_{\alpha}+\frac{1}{2}|Y_0|+\frac{1}{5}\|\delta Y\|_{\alpha,\tau},$$

which proves (2.38) and completes the proof. □

Remark 2.13. An explicit choice for $\hat{\tau}$ in Theorem 2.12 is

$$\hat{\tau}^{\alpha}:=\frac{\mathrm{e}^{-\frac{T}{\bar{\tau}}}}{10\,(K_{2\alpha}+3)\,(1+M_0)\,(1+D\,(M+M^2))}, \tag{2.53}$$

with $\bar{\tau}=\bar{\tau}_{\alpha,D,M}$ defined in (2.39). This is obtained by tracking all the points in the proof of Theorem 2.12 where $\varepsilon=(\tau\wedge T)^{\alpha}$ was assumed to be *small enough*: see Section 2.8 for the details.

2.6. Euler scheme and local/global existence

In this section we discuss *global existence of solutions*, under the assumption that σ is globally γ-Hölder with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$, i.e. $[\sigma]_{\mathcal{C}^{\gamma}}<\infty$ (again with no boundedness assumption on σ). We also state a result of *local existence of solutions* for equation (2.3), where we only assume that σ is *locally γ-Hölder* with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$ (with no boundedness assumption on σ).

We fix $X:[0,T]\to\mathbb{R}^d$ of class $\mathcal{C}^{\alpha}$ with $\alpha\in\left]\frac{1}{2},1\right]$ and a starting point $z_0\in\mathbb{R}^k$. We split the proof in two parts: we first assume that $\sigma:\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ is globally γ-Hölder, then we consider the case when σ is locally γ-Hölder.

First part: globally Hölder case.

We consider a finite set $\mathbb{T}=\{0=t_1<\cdots<t_{\#\mathbb{T}}\}\subset\mathbb{R}_+$ and we define an approximate solution $Z=Z^{\mathbb{T}}=(Z_t)_{t\in\mathbb{T}}$ through the *Euler scheme*

$$Z_0:=z_0,\qquad Z_{t_{i+1}}:=Z_{t_i}+\sigma(Z_{t_i})\,\delta X_{t_i,t_{i+1}}\qquad\text{for } 1\leqslant i\leqslant\#\mathbb{T}-1. \tag{2.54}$$

Let us define the "remainder"

$$R_{st}:=\delta Z_{st}-\sigma(Z_s)\,\delta X_{st}\qquad\text{for } s<t\in\mathbb{T}. \tag{2.55}$$

We assume that σ is *globally γ-Hölder,* namely $[\sigma]_{\mathcal{C}^\gamma}<\infty$, with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$. We set

$$\hat{\varepsilon}_{\alpha,\gamma,X,\sigma}:=\frac{1}{2\,(C_{(\gamma+1)\alpha}+5)\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}}, \tag{2.56}$$

where the constant C_η is defined in (1.45). We prove the following *a priori estimates* on the Euler scheme (2.54), which are analogous to those in Theorem 2.7.

LEMMA 2.14. *If σ is globally γ-Hölder, namely $[\sigma]_{\mathcal{C}^\gamma}<\infty$, with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$, then*

$$\|R\|_{(\gamma+1)\alpha}^{\mathbb{T}}\leqslant C_{(\gamma+1)\alpha}\,[\sigma]_{\mathcal{C}^\gamma}\,(\|\delta Z\|_\alpha^{\mathbb{T}})^\gamma\,\|\delta X\|_\alpha, \tag{2.57}$$

$$\textit{and for } \tau^{\gamma\alpha}\leqslant\hat{\varepsilon}_{\alpha,\gamma,X,\sigma}:\qquad \|\delta Z\|_\alpha^{\mathbb{T}}\leqslant 1\vee(2\,|\sigma(z_0)|\,\|\delta X\|_\alpha)\,. \tag{2.58}$$

Proof. Since $\delta R_{sut}=(\sigma(Z_s)-\sigma(Z_u))\,\delta X_{ut}$, recall (1.32), and since $R_{t_it_{i+1}}=0$ by (2.54), we can apply the discrete Sewing Bound (1.45) with $\eta=(\gamma+1)\alpha>1$ to get

$$\|R\|_{(\gamma+1)\alpha,\tau}^{\mathbb{T}}\leqslant C_{(\gamma+1)\alpha}\,\|\delta R\|_{(\gamma+1)\alpha,\tau}^{\mathbb{T}}\leqslant C_{(\gamma+1)\alpha}\,[\sigma]_{\mathcal{C}^\gamma}\,(\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma\,\|\delta X\|_\alpha. \tag{2.59}$$

We have proved (2.57).

We next prove (2.58). Recalling (2.55) we can bound, by (2.6) for $\|\cdot\|_{\gamma\alpha,\mathbb{T}_n}$,

$$\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}}\leqslant\|\sigma(Z)\|_{\infty,\tau}^{\mathbb{T}}\,\|\delta X\|_\alpha+\tau^{\gamma\alpha}\,\|R\|_{(\gamma+1)\alpha,\tau}^{\mathbb{T}}\,.$$

By (1.47)

$$\|\sigma(Z)\|_{\infty,\tau}^{\mathbb{T}}\leqslant|\sigma(z_0)|+5\,\tau^{\gamma\alpha}\|\delta\sigma(Z)\|_{\gamma\alpha,\tau}^{\mathbb{T}}\leqslant|\sigma(z_0)|+5\,\tau^{\gamma\alpha}\,[\sigma]_{\mathcal{C}^\gamma}\,(\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma\,.$$

We thus obtain, combining the previous bounds,

$$\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}}\leqslant|\sigma(z_0)|\,\|\delta X\|_\alpha+\{\tau^{\gamma\alpha}\,(C_{\gamma\alpha}+5)\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta X\|_\alpha\}\,(\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma\,.$$

Now if $\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}}\leqslant 1$ then (2.58) is proved, otherwise $(\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma\leqslant\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}}$ and then for τ as in (2.56) the term in brackets is less than $\frac{1}{2}$ and we obtain (2.58). □

We can now prove the following

THEOREM 2.15. (GLOBAL EXISTENCE) *Let X be of class $\mathcal{C}^\alpha$, with $\alpha\in\left]\frac{1}{2},1\right]$, and let σ be globally γ-Hölder with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$, i.e. $[\sigma]_{\mathcal{C}^\gamma}<\infty$. For every $z_0\in\mathbb{R}^k$, with no restriction on $T>0$, there exists a solution $(Z_t)_{t\in[0,T]}$ of (2.3) with $Z_0=z_0$.*

Proof. Given $n \in \mathbb{N}$, we construct an approximate solution $Z^n = (Z_t^n)_{t \in \mathbb{T}_n}$ of (2.3) defined in the discrete set of times $\mathbb{T}_n := (\{i2^{-n}\colon\ i = 0, 1, \ldots\} \cap [0, T]) \cup \{T\}$ through the *Euler scheme* (2.54).

$$Z_0^n := z_0, \qquad Z_{t_{i+1}}^n := Z_{t_i}^n + \sigma(Z_{t_i}^n)\, \delta X_{t_i, t_{i+1}} \qquad \text{for } t_i, t_{i+1} \in \mathbb{T}_n\,. \tag{2.60}$$

Let us define the "remainder"

$$R_{st}^n := \delta Z_{st}^n - \sigma(Z_s^n)\, \delta X_{st} \qquad \text{for } s < t \in \mathbb{T}_n\,. \tag{2.61}$$

We fix $T > 0$ such that

We extend Z^n by linear interpolation to a continuous function defined on $[0, T]$, still denoted by Z^n. Given two points $t_i \leqslant s < t \leqslant t_{i+1}$ inside the same interval $[t_i, t_{i+1}]$ of the partition $\mathbb{T}_n$, since $\delta Z_{st}^n = \frac{t - s}{t_{i+1} - t_i}\, \delta Z_{t_i t_{i+1}}^n$, we can bound for $\alpha \in (0, 1]$

$$\frac{|\delta Z_{st}^n|}{(t - s)^\alpha} = \left(\frac{t - s}{t_{i+1} - t_i}\right)^{1 - \alpha} \frac{|\delta Z_{t_i t_{i+1}}^n|}{(t_{i+1} - t_i)^\alpha} \leqslant \frac{|\delta Z_{t_i t_{i+1}}^n|}{(t_{i+1} - t_i)^\alpha}\,.$$

Given two points $s < t$ in different intervals, say $t_i \leqslant s \leqslant t_{i+1} \leqslant t_j \leqslant t \leqslant t_{j+1}$ for some $i < j$, by the triangle inequality we can bound $|\delta Z_{st}^n| \leqslant |\delta Z_{s t_{i+1}}^n| + |\delta Z_{t_{i+1} t_j}^n| + |\delta Z_{t_j t}^n|$. Recalling (1.9) and (1.43), we then obtain $\|\cdot\|_\alpha \leqslant 3\, \|\cdot\|_\alpha^{\mathbb{T}_n}$, hence by (2.58) we get

$$\|\delta Z^n\|_{\alpha, \tau} \leqslant 3 \vee (6\, |\sigma(z_0)|\, \|\delta X\|_\alpha)\,. \tag{2.62}$$

The family $(Z^n)_{n \in \mathbb{N}}$ is *equi-continuous* by (2.62) and *equi-bounded*, since $Z_0^n = z_0$ for all $n \in \mathbb{N}$, hence by the Arzelà-Ascoli Theorem it is *compact* in the space $C([0, T], \mathbb{R}^k)$. Let us denote by $Z\colon [0, T] \to \mathbb{R}^k$ any limit point. Plugging (2.58) into (2.57), by (2.61) we can write

$$\text{if } T^\alpha \leqslant \hat{\varepsilon}_{\alpha, X, \sigma}\colon \qquad |\delta Z_{st}^n - \sigma(Z_s^n)\, \delta X_{st}| \leqslant c(z_0)\, (t - s)^{2\alpha} \qquad \forall s < t \in \mathbb{T}_n\,, \tag{2.63}$$

where $c(z_0) := C_{(\gamma + 1)\alpha}\, [\sigma]_{\mathcal{C}^\gamma}\, (3 \vee (6\, |\sigma(z_0)|\, \|\delta X\|_\alpha))^\gamma\, \|\delta X\|_\alpha$. Letting $n \to \infty$ and observing that $\mathbb{T}_n \subseteq \mathbb{T}_{n+1}$, we see that (2.63) still holds with Z^n replaced by Z and $\mathbb{T}_n$ replaced by the set $\mathbb{T} := \bigcup_{\ell \in \mathbb{N}} \mathbb{T}_{2^\ell} = \left(\left\{\frac{i}{2^n}:\ i, n \in \mathbb{N}\right\} \cap [0, T]\right) \cup \{T\}$ of dyadic rationals:

$$\text{if } T^\alpha \leqslant \hat{\varepsilon}_{\alpha, X, \sigma}\colon \qquad |\delta Z_{st} - \sigma(Z_s)\, \delta X_{st}| \leqslant c(z_0)\, (t - s)^{2\alpha} \qquad \forall s < t \in \mathbb{T}.$$

Since $\mathbb{T}$ is dense in $[0, T]$ and Z is continuous, this bound extends to all $0 \leqslant s < t \leqslant T$, which shows that Z is a solution of (2.3). This completes the proof. □

Second part: locally Lipschitz case.

We now assume that σ is *locally γ-Hölder* and we fix $z_0 \in \mathbb{R}^k$. We also fix $T > 0$ such that $T \leqslant \tilde{\varepsilon}_{\alpha, X, \sigma}(z_0)$, see (2.64), and we prove that there exists a solution $Z\colon [0, T] \to \mathbb{R}^k$ of (2.3) with $Z_0 = z_0$.

Theorem 2.16. (Local existence) *Let X be of class $\mathcal{C}^\alpha$, with $\alpha \in \left]\frac{1}{2}, 1\right]$, and let σ be locally Lipschitz (e.g. of class C^1). For any $z_0 \in \mathbb{R}^k$ and for $T > 0$ small enough, i.e.*

$$T^\alpha \leqslant \tilde{\varepsilon}_{\alpha, X, \sigma}(z_0) := \frac{1}{2}\, \frac{1}{(C_{2\alpha} + 3)\, \|\delta X\|_\alpha\, \{1 + \sup_{|z - z_0| \leqslant |\sigma(z_0)|} |\nabla \sigma(z)|\}}\,, \tag{2.64}$$

there exists a solution $(Z_t)_{t\in[0,T]}$ *of (2.3) with* $Z_0=z_0$.

Let $\tilde{\sigma}$ be a *globally* γ*-Hölder* function (depending on z_0) such that

$$\tilde{\sigma}(z)=\sigma(z)\ \ \forall|z-z_0|\leqslant\sigma(z_0) \qquad \text{and} \qquad [\tilde{\sigma}]_{C^\gamma}=\sup_{|z-z_0|\leqslant\sigma(z_0)}|\nabla\sigma(z)|\,. \tag{2.65}$$

Since $T\leqslant\tilde{\varepsilon}_{\alpha,X,\sigma}(z_0)\leqslant\hat{\varepsilon}_{\alpha,X,\sigma}$, see (2.64) and (2.56), by the first part of the proof *there exists a solution* Z *of (2.3) with* $\tilde{\sigma}$ *in place of* σ and $Z_0=z_0$. We will prove that

$$|Z_t-z_0|\leqslant\sigma(z_0)\ \text{ for all }\ t\in[0,T]\,, \tag{2.66}$$

therefore $\tilde{\sigma}(Z_t)=\sigma(Z_t)$ for all $t\in[0,T]$, see (2.65). This means that Z is a solution of the original (2.3) with σ, which completes the proof of Theorem 2.16.

To prove (2.66), we apply the a priori estimate (2.13) with $\tau=\infty$: we note that $T\leqslant\tilde{\varepsilon}_{\alpha,X,\sigma}(z_0)\leqslant\varepsilon_{\alpha,X,\sigma}$ (see (2.64) and (2.12), and note that $C_{2\alpha}\geqslant K_{2\alpha}$), therefore

$$\|\delta Z\|_\alpha\leqslant 2\,\|\delta X\|_\alpha\,|\sigma(z_0)|,$$

because $\tilde{\sigma}(z_0)=\sigma(z_0)$. Then for every $t\in[0,T]$ we can bound

$$|Z_t-z_0|\leqslant T^\alpha\,\|\delta Z\|_\alpha\leqslant 2\,T^\alpha\,\|\delta X\|_\alpha\,|\sigma(z_0)|\leqslant|\sigma(z_0)|,$$

where the last inequality holds because $T^\alpha\leqslant\tilde{\varepsilon}_{\alpha,X,\sigma}(z_0)\leqslant(2\,\|\delta X\|_\alpha)^{-1}$, see (2.64). This completes the proof of (2.66).

2.7. Error estimate in the Euler scheme

We suppose in this section that σ is of class C^2 with $\|\nabla\sigma\|_\infty+\|\nabla^2\sigma\|_\infty<+\infty$.

Theorem 2.17. *The Euler scheme converges at speed* $n^{2\alpha-1}$.

Proof. Let us set $z_i:=\partial y_i/\partial y_0$, where $(y_i)_{i\geqslant0}$ is defined by (2.60). Then

$$z_{i+1}=z_i+\nabla\sigma(y_i)\,z_i\,\delta X_{t_it_{i+1}}, \qquad i\geqslant0.$$

This shows that the pair $(y_i,z_i)_{i\geqslant0}$ satisfies a recurrence which is similar to (2.60) with a map Σ of class C^1 and therefore we can apply the above results to obtain that $|z_i|\leqslant$ const. In particular the map $y_0\to y_k$ is Lipschitz-continuous, uniformly over $k\geqslant0$.

Let us call, for $k\geqslant0$, $(z_\ell^{(k)})_{\ell\geqslant k}$ as the sequence which satisfies (2.60) but has initial value $z_k^{(k)}=y(t_k)$. Since $(y(t))_{t\geqslant0}$ is a solution to (2.4), we have

$$|z_{k+1}^{(k)}-y(t_{k+1})|\lesssim n^{-2\alpha}.$$

Since the map $y_0\to y_k$ is Lipschitz-continuous uniformly over $k\geqslant0$, we have

$$|z_\ell^{(k)}-z_\ell^{(k+1)}|\lesssim|z_{k+1}^{(k)}-y(t_{k+1})|\lesssim n^{-2\alpha}, \qquad \ell\geqslant k+1.$$

Therefore

$$|y_\ell-y(t_\ell)|=|z_\ell^{(0)}-z_\ell^{(\ell)}|\leqslant\sum_{k=0}^{\ell-1}|z_\ell^{(k)}-z_\ell^{(k+1)}|\lesssim\frac{\ell}{n^{2\alpha}}=\frac{t_\ell}{n^{2\alpha-1}}\to0$$

as t_ℓ is bounded and $n \to \infty$. □

2.8. EXTRA: A VALUE FOR $\hat{\tau}$

We can give an explicit expression for $\hat{\tau} = \hat{\tau}_{M_0,M,T}$ in Theorem 2.12, by tracking all the points in the proof where τ *is small enough*, namely:

- for (2.36) we need $\tau^\alpha \leqslant \frac{1}{15}$;
- for (2.40) we need $\tau^\alpha \leqslant (\hat{\rho}_M)^\alpha := (2\,(K_{2\alpha} + 3)\,\mathfrak{c}_M)^{-1}$;
- for (2.42) we need $\tau^\alpha \leqslant (6\,\mathfrak{c}_M)^{-1}$, for (2.45) we need $\tau^\alpha \leqslant (15\,\mathfrak{c}_M)^{-1}$;
- for (2.50) we need $\tau^{(\gamma-1)\alpha} \leqslant (2\,K_{\gamma\alpha}\,\mathfrak{c}_M)^{-1}$;
- for (2.52) we need $\tau^{(\gamma-1)\alpha} \leqslant (10\,K_{\gamma\alpha}\,\mathfrak{c}_M)^{-1}$ (first term in the RHS) and also $\tau^{(\gamma-1)\alpha} \leqslant \left(K_{\gamma\alpha}\,\mathrm{e}^{\frac{T}{\hat{\rho}_M}}\,M_0\,M^2\,\|\nabla^2\sigma\|_\infty\right)^{-1}$ (second term in the RHS).

Since $\mathfrak{c}_M = M\,\|\nabla\sigma\|_\infty$, see (2.34), it is easy to check that all these constraints are satisfied for $0 < \tau \leqslant \hat{\tau}$ given by formula (2.53) in Remark 2.13.

Chapter 3

Difference equations: the rough case

We have so far considered the difference equation (2.3), that is

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.1}$$

where X is given, Z is the unknown and $\sigma(\cdot)$ is sufficiently regular. This is a generalization of the differential equation $\dot{Z}_t = \sigma(Z_t)\,\dot{X}_t$ which is meaningful for non smooth X, as we showed in Chapter 2, where we proved *well-posedness* in the so-called *Young case*, i.e. assuming that $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$.

However, the restriction $\alpha > \frac{1}{2}$ is a substantial limitation: in particular, we cannot take $X = B$ as a typical path of Brownian motion, which is in $\mathcal{C}^\alpha$ only for $\alpha < \frac{1}{2}$. For this reason, we show in this chapter how to *enrich* the difference equation (3.1) and prove *well-posedness when* $X \in \mathcal{C}^\alpha$ *with* $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$, called the *rough case*. This will be applied to Brownian motion in the next Chapter 4, in order to obtain a robust formulation of classical *stochastic differential equations*.

Remark 3.1. (Young vs. Rough case) The restriction $\alpha > \frac{1}{2}$ for the study of the difference equation (3.1) has a substantial reason, namely *there is no solution to (3.1) for general* $X \in \mathcal{C}^\alpha$ *with* $\alpha \leqslant \frac{1}{2}$. Indeed, taking the "increment" δ of both sides of (3.1) and recalling (1.23) and (1.32), we obtain

$$(\sigma(Z_u) - \sigma(Z_s))\,(X_t - X_u) = o(t-s) \qquad \text{for } 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{3.2}$$

If $X \in \mathcal{C}^\alpha$, for any $\alpha \in (0,1]$, then we know from Lemma 2.6 that $Z \in \mathcal{C}^\alpha$, but not better in general (e.g. when $\sigma(\cdot) \equiv c$ is constant we have $Z = c\,X$), hence the LHS of (3.2) is $\lesssim (u-s)^\alpha\,(t-u)^\alpha \lesssim (t-s)^{2\alpha}$, but not better in general. This shows that the restriction $\alpha > \frac{1}{2}$ is generally necessary for (3.1) to have solutions.

3.1. Enhanced Taylor expansion

We fix $d, k \in \mathbb{N}$, a time horizon $T > 0$ and a sufficiently regular function $\sigma \colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$. Our goal is to give a meaning to the integral equation

$$Z_t = Z_0 + \int_0^t \sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s, \qquad 0 \leqslant t \leqslant T, \tag{3.3}$$

where $Z \colon [0,T] \to \mathbb{R}^k$ is the unknown and $X \colon [0,T] \to \mathbb{R}^d$ is a non smooth path, more precisely $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$.

The difference equation (3.1) is no longer enough, for the crucial reason that typically *it admits no solutions for* $\alpha \leqslant \frac{1}{2}$, see Remark 3.1. We are going to solve this problem by *enriching the RHS of (3.1)* in a suitable, but non canonical way: this leads to the key notion of *rough path* which is central in this book.

To provide motivation, suppose for the moment that X is continuously differentiable, so that (3.3) is meaningful. As we saw in (1.3), an integration yields for $s<t$

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + \int_s^t (\sigma(Z_u) - \sigma(Z_s))\,\dot{X}_u\,\mathrm{d}u\,. \tag{3.4}$$

In Chapter 1 we observed that the integral is $o(t-s)$, which leads to the difference equation (3.1). More precisely, the integral is $O((t-s)^2)$ if $X \in C^1$ and σ is locally Lipschitz (note that $Z \in C^1$). The idea is now to go further, expanding the integral to get a more accurate local description, with a better remainder $O((t-s)^3)$.

To this purpose, we assume that σ is differentiable and we introduce the key function $\sigma_2\colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*$ by

$$\sigma_2(z) := \nabla\sigma(z)\,\sigma(z), \qquad \text{i.e.} \qquad [\sigma_2(z)]^i_{j\ell} := \sum_{a=1}^k \frac{\partial \sigma^i_j}{\partial z_a}(z)\,\sigma^a_\ell(z)\,. \tag{3.5}$$

Since $\frac{\mathrm{d}}{\mathrm{d}r}\sigma(Z_r) = \nabla\sigma(Z_r)\,\dot{Z}_r = \sigma_2(Z_r)\,\dot{X}_r$ by (3.3), we can write for $s<u$

$$\begin{aligned} \sigma(Z_u) - \sigma(Z_s) &= \int_s^u \sigma_2(Z_r)\,\dot{X}_r\,\mathrm{d}r \\ &= \sigma_2(Z_s)\,(X_u - X_s) + \int_s^u (\sigma_2(Z_r) - \sigma_2(Z_s))\,\dot{X}_r\,\mathrm{d}r, \end{aligned} \tag{3.6}$$

where for $z \in \mathbb{R}^d$ and $a \in \mathbb{R}^d$ we define $\sigma_2(z)\,a \in \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ by

$$[\sigma_2(z)\,a]^i_j = \sum_{\ell=1}^d [\sigma_2(z)]^i_{j\ell}\,a^\ell.$$

If we assume that σ_2 is locally Lipschitz, then the last integral in (3.6) is $O((u-s)^2)$ (recall that $X \in C^1$). Plugging this into (3.4), we then obtain

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + \sigma_2(Z_s)\int_s^t (X_u - X_s) \otimes \dot{X}_u\,\mathrm{d}u + O((t-s)^3), \tag{3.7}$$

where now for $z \in \mathbb{R}^d$ and $B \in \mathbb{R}^d \otimes \mathbb{R}^d$ we define $\sigma_2(z)\,B \in \mathbb{R}^k$ by

$$[\sigma_2(z)\,B]^i = \sum_{\ell,m=1}^d [\sigma_2(z)]^i_{\ell m}\,B^{m\ell}. \tag{3.8}$$

Let us rewrite the integral in the right-hand side of (3.7) more conveniently. To this purpose we introduce the shorthands

$$\mathbb{X}^1_{st} := X_t - X_s\,, \qquad \mathbb{X}^2_{st} := \int_s^t (X_r - X_s) \otimes \dot{X}_r\,\mathrm{d}r, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.9}$$

so that $\mathbb{X}^1\colon [0,T]^2_{\leqslant} \to \mathbb{R}^d$ and $\mathbb{X}^2\colon [0,T]^2_{\leqslant} \to \mathbb{R}^d \otimes \mathbb{R}^d$, see (1.7). More explicitly:

$$(\mathbb{X}^2_{st})^{ij} := \int_s^t (X^i_r - X^i_s)\, \dot{X}^j_r \,\mathrm{d}r, \qquad i, j \in \{1, \ldots, d\}.$$

We can thus rewrite (3.7), replacing $O((t-s)^3)$ by $o(t-s)$, in the compact form

$$Z_t - Z_s = \sigma(Z_s)\, \mathbb{X}^1_{st} + \sigma_2(Z_s)\, \mathbb{X}^2_{st} + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.10}$$

where for the product $\sigma_2(Z_s)\, \mathbb{X}^2_{st}$ we use the contraction rule (3.8).

We have obtained an *enhanced Taylor expansion*: comparing with (3.1), we added a "second order term" containing $\mathbb{X}^2_{st}$. The idea is to take this new difference equation, that we call *rough difference equation*, as a generalized formulation of the differential equation (3.3), just as we did in Chapter 1 (see Section 1.2). However, there is a problem: the term $\mathbb{X}^2_{st}$ *depends on the derivative* $\dot{X}$, see (3.9), so it is not clearly defined for a non-differentiable X.

To overcome this problem, we will *assign* a suitable function $\mathbb{X}^2 = (\mathbb{X}^2_{st})_{0 \leqslant s \leqslant t \leqslant T}$ playing the role of the integral (3.9) when X is not differentiable: this leads to the notion of *rough paths*, defined in the next section and studied in depth in Chapter 7. We will show in this chapter that rough paths are the key to a robust solution theory of rough difference equations when X of class $\mathcal{C}^\alpha$ with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$.

3.2. Rough paths

Let us fix a path $X\colon [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$ with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. Motivated by the previous section, we are going to reformulate the ill-posed integral equation (3.3) as the difference equation (3.10), which contains $\mathbb{X}^1$ and $\mathbb{X}^2$.

We can certainly define $\mathbb{X}^1_{st} := X_t - X_s$ as in (3.9), but there is no canonical definition of $\mathbb{X}^2_{st} = \int_s^t (X_r - X_s) \otimes \dot{X}_r \,\mathrm{d}r$, since X may not be differentiable. We therefore *assign* a function $\mathbb{X}^2_{st}$ which satisfies *suitable properties*. Note that when X is continuously differentiable the function $\mathbb{X}^2$ in (3.9) satisfies:

- an algebraic identity known as *Chen's relation*: for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

  $$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut} = (X_u - X_s) \otimes (X_t - X_u), \tag{3.11}$$

  which follows from (3.9) noting that

  $$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \int_u^t (X_r - X_s) \otimes \dot{X}_r \,\mathrm{d}r = (X_u - X_s) \otimes (X_t - X_u);$$

- the analytic bounds

  $$|\mathbb{X}^1_{st}| \lesssim |t-s|, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^2, \tag{3.12}$$

  which follow from the fact that $\dot{X}$ is bounded.

The algebraic relation (3.11) is still meaningful for non-differentiable X, while the analytic bounds (3.12) can naturally be adapted to the case of Hölder paths $X \in \mathcal{C}^\alpha$ by changing the exponents $1, 2$ to $\alpha, 2\alpha$. This leads to the following key definition.

DEFINITION 3.2. (ROUGH PATHS) *Fix* $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$, $d \in \mathbb{N}$ *and a path* $X: [0,T] \to \mathbb{R}^d$ *of class* $\mathcal{C}^\alpha$. *An* ***α-rough path over X*** *is a pair* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *where the functions* $\mathbb{X}^1: [0,T]^2_\leqslant \to \mathbb{R}^d$ *and* $\mathbb{X}^2: [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ *satisfy, for* $0 \leqslant s \leqslant u \leqslant t \leqslant T$:

- *the algebraic relations*

$$\mathbb{X}^1_{st} = X_t - X_s\,, \qquad \delta\mathbb{X}^2_{sut} := \mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}\,, \tag{3.13}$$

  where the second identity is called Chen's relation;

- *the analytic bounds*

$$|\mathbb{X}^1_{st}| \lesssim |t-s|^\alpha\,, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}\,. \tag{3.14}$$

We call $\mathcal{R}_{\alpha,d}(X)$ *the set of d-dimensional* α*-rough paths* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *over* X *and* $\mathcal{R}_{\alpha,d} = \bigcup_{X \in \mathcal{C}^\alpha} \mathcal{R}_{\alpha,d}(X)$ *the set of all d-dimensional* α*-rough paths.*

When X is of class C^1, the choice (3.9) yields by (3.11)-(3.12) a α-rough path for any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ which we call the *canonical rough path*, see Section 7.7 below.

When $X = B$ is Brownian motion, the theory of stochastic integration provides a natural candidate for $\mathbb{X}^2$, in fact *multiple candidates* (think of Ito vs. Stratonovich integration), as we discuss in Chapter 4 below. Incidentally, this makes it clear that the construction of $\mathbb{X}^2$ is in general *non canonical*, i.e. there are multiple choices of $\mathbb{X}^2$ for a given path X. *This is a strength of the theory of rough paths*, since it allows to treat different non equivalent forms of integration.

Remark 3.3. The existence of rough paths over any given path X (i.e. the fact that $\mathcal{R}_{\alpha,d}(X) \neq \emptyset$) is a non trivial fact, which will be proved in Chapter 7.

Remark 3.4. ($\mathbb{X}^2$ AS A "PATH") The two-parameters function $\mathbb{X}^2_{st}$ is determined by the one-parameter function

$$\mathbb{I}_t := \mathbb{X}^2_{0t} + X_0 \otimes (X_t - X_0)\,, \tag{3.15}$$

which intuitively describes the integral $\int_0^t X_r \otimes \dot{X}_r \, \mathrm{d}r$. Indeed, we can write

$$\mathbb{X}^2_{st} = \mathbb{I}_t - \mathbb{I}_s - X_s \otimes (X_t - X_s)\,, \tag{3.16}$$

since $\mathbb{X}^2_{st} = \mathbb{X}^2_{0t} - \mathbb{X}^2_{0s} - (X_s - X_0) \otimes (X_t - X_s)$ by Chen's relation (3.13).

Vice versa, given a function $\mathbb{I}: [0,T] \to \mathbb{R}^d$, if we *define* $\mathbb{X}^2$ by (3.16), then Chen's relation (3.13) is automatically satisfied (recall (1.32)). In order to satisfy the analytic bound in (3.14), we must require that

$$|\mathbb{I}_t - \mathbb{I}_s - X_s \otimes (X_t - X_s)| \lesssim (t-s)^{2\alpha}\,, \tag{3.17}$$

which is a natural estimate if $\mathbb{I}_t - \mathbb{I}_s$ should describe "$= \int_s^t X_r \otimes \dot{X}_r \, \mathrm{d}r$".

Summarizing: given any path $X: [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$, *it is equivalent to assign* $\mathbb{X}^2: [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ *satisfying (3.13)-(3.14) or to assign* $\mathbb{I}: [0,T] \to \mathbb{R}^d$ *satisfying (3.17)*, the correspondence being given by (3.15)-(3.16).

3.3. Rough difference equations

Given a time horizon $T>0$ and two dimensions $d,k\in\mathbb{N}$, let us fix:

- a path $X\colon[0,T]\to\mathbb{R}^d$ of class $\mathcal{C}^\alpha$ with $\alpha\in\left]\frac{1}{3},\frac{1}{2}\right]$;
- an α-rough path $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ over X, see Definition 3.2;
- a *differentiable* function $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$, which lets us define the function

$$\sigma_2\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*\otimes(\mathbb{R}^d)^* \qquad (\text{see } (3.5))\,.$$

Motivated by the previous discussions, see in particular (3.10), we study in this chapter the following *rough difference equation* for an unknown path $Z\colon[0,T]\to\mathbb{R}^k$:

$$\delta Z_{st}=\sigma(Z_s)\,\mathbb{X}^1_{st}+\sigma_2(Z_s)\,\mathbb{X}^2_{st}+o(t-s), \qquad 0\leqslant s\leqslant t\leqslant T, \tag{3.18}$$

where we recall the increment notation $\delta Z_{st}:=Z_t-Z_s$ and the contraction rule (3.8), and we stress that $o(t-s)$ is *uniform* for $0\leqslant s\leqslant t\leqslant T$, see Remark 1.1. In analogy with (2.3)-(2.4), a solution of (3.18) is a path $Z\colon[0,T]\to\mathbb{R}^k$ such that

$$Z^{[3]}_{st}:=\delta Z_{st}-\sigma(Z_s)\,\mathbb{X}^1_{st}-\sigma_2(Z_s)\,\mathbb{X}^2_{st}=o(t-s)\,. \tag{3.19}$$

We stress that the rough difference equation (3.18) is a generalization of the integral equation (3.3), as we show in the next result.

Proposition 3.5. *If X and σ are of class C^1 and σ_2 is locally Lipschitz (e.g. if σ is of class C^2), then any solution Z to the integral equation (3.3) satisfies the difference equation (3.18) for the canonical rough path $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ in (3.9).*

Proof. If $X\in C^1$, then $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ defined in (3.9) is an α-rough path over X for any $\alpha\in\left]\frac{1}{3},\frac{1}{2}\right]$, as we showed in (3.11)-(3.12). Given a solution Z of (3.3), if σ_2 is locally Lipschitz we derived the Taylor expansion (3.10), hence (3.18) holds. ☐

We now state *local and global existence, uniqueness of solutions and continuity of the solution map* for the rough difference equation (3.18) under natural assumptions on σ and σ_2, summarizing the main results of this chapter. We refer to the next sections for more precise and quantitative results.

To be completed.

Proposition 3.6. *Let $z_0\in\mathbb{R}^d$. We suppose that σ and σ_2 are of class C^1 and globally Lipschitz, namely $\|\nabla\sigma\|_\infty+\|\nabla\sigma_2\|_\infty<+\infty$. Let $D:=\max\{1,\|\nabla\sigma\|_\infty,\|\nabla\sigma_2\|_\infty\}$ and $M>0$.*

There exists $T_{M,D,\alpha}>0$ such that, for all $T\in(0,T_{M,D,\alpha})$ and $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)\in\mathcal{R}_{\alpha,d}$ such that $\|\mathbb{X}^1\|_\alpha+\|\mathbb{X}^2\|_{2\alpha}\leqslant M$, there exists a solution Z to (3.19) on the interval $[0,T]$ such that $Z_0=z_0$ and

$$\|Z\|_\alpha\leqslant 15\,M(|\sigma(z_0)|+|\sigma_2(z_0)|). \tag{3.20}$$

The proof of this Proposition, based on a discretization argument, is postponed to section 3.9 below.

We are going to use the Sewing Bound (1.26), its weighted version (1.41) and its discrete formulation (1.45).

3.4. SET-UP

We recall that the *weighted semi-norms* $\|\cdot\|_{\eta,\tau}$ are defined in (1.33)-(1.34). We are going to use the various properties that we recalled in Section 2.2, see in particular (2.5), (2.6) and (2.7)-(2.8), as well as the natural generalization

$$\text{if} \quad F_{sut} = G_{su} H_{ut} \quad \text{then} \qquad \|F\|_{3\eta,\tau} \begin{cases} \leqslant \|G\|_{2\eta,\tau} \, \|H\|_{\eta}, \\ \leqslant \|G\|_{\eta,\tau} \, \|H\|_{2\eta}. \end{cases} \tag{3.21}$$

In all these bounds, whenever there is a product, *only one factor gets the weighted semi-norm, while the other factor gets the ordinary semi-norm*. We sometimes need to introduce an additional weight, which is possible applying (2.9).

In Chapter 2 a key tool to study the Young difference equation (2.4) was the estimate on the "difference of increments" in Lemma 2.8. This tool is still crucial in this chapter, but we will need an additional ingredient that we now present.

LEMMA 3.7. (TAYLOR IDENTITY) *Let $z_1, z_2 \in \mathbb{R}^k$ and $x \in \mathbb{R}^d$. If $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is of class C^1, defining $\sigma_2: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*$ by (3.5) and setting $\delta z_{12} := z_2 - z_1$, we have the identities*

$$\begin{aligned} &\sigma(z_2) - \sigma(z_1) - \sigma_2(z_1)\, x \\ &\quad = \nabla\sigma(z_1)(\delta z_{12} - \sigma(z_1)\, x) + \int_0^1 [(\nabla\sigma(z_1 + r\,\delta z_{12}) - \nabla\sigma(z_1))\,\delta z_{12}]\, \mathrm{d}r, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \sigma(z_2) - \sigma(z_1) - \sigma_2(z_1)\, x \;=\; & \int_0^1 [(\sigma_2(z_1 + r\,\delta z_{12}) - \sigma_2(z_1))\, x]\, \mathrm{d}r \\ & + \int_0^1 [\nabla\sigma(z_1 + r\,\delta z_{12})\,(\delta z_{12} - \sigma(z_1)\, x)]\, \mathrm{d}r \\ & - \int_0^1 \nabla\sigma(z_1 + r\delta z_{12}) \left(\int_0^r [\nabla\sigma(z_1 + v\,\delta z_{12})\,\delta z_{12}\, x]\, \mathrm{d}v \right) \mathrm{d}r. \end{aligned} \tag{3.23}$$

Proof. The first formula is based on elementary manipulations and on the fact that

$$\sigma(z_2) - \sigma(z_1) = \int_0^1 [\nabla\sigma(z_1 + r\,\delta z_{12})\,\delta z_{12}]\, \mathrm{d}r.$$

For the second formula, setting $\delta z := \delta z_{12}$ for short, we similarly write

$$\begin{aligned} \sigma(z_2) - \sigma(z_1) \;&=\; \int_0^1 [\nabla\sigma(z_1 + r\,\delta z)\,\delta z]\, \mathrm{d}r \\ &=\; \int_0^1 [\nabla\sigma(z_1 + r\,\delta z)\,(\delta z - \sigma(z_1)\, x)]\, \mathrm{d}r + \underbrace{\int_0^1 [\nabla\sigma(z_1 + r\,\delta z)\,\sigma(z_1)\, x]\, \mathrm{d}r}_{A} \end{aligned}$$

and then, recalling the definition (3.5) of σ_2,

$$A = \int_0^1 [\sigma_2(z_1 + r\,\delta z)\,x]\,\mathrm{d}r - \underbrace{\int_0^1 [\nabla\sigma(z_1 + r\,\delta z)\,(\sigma(z_1 + r\,\delta z) - \sigma(z_1))\,x]\,\mathrm{d}r}_{B}.$$

Finally

$$B = \int_0^1 \nabla\sigma(z_1 + r\,\delta z)\left(\int_0^r [\nabla\sigma(z_1 + v\,\delta z)\,\delta z\,x]\,\mathrm{d}v\right)\mathrm{d}r$$

from which (3.23) follows easily. □

We will see below that (3.22) is useful for the comparison between *two solutions*, as in the proofs of uniqueness (Theorem 3.10) and continuity of the solution map (Theorem 3.11), while (3.23) is well suited for a priori estimates on a *single solution* (Theorem 3.9) or on a discretization scheme (Lemma 3.13).

3.5. A priori estimates

In this section we prove *a priori estimates* for solutions of the rough difference equation (3.18) for *globally Lipschitz* σ and σ_2, i.e. $\|\nabla\sigma\|_\infty < \infty$ and $\|\nabla\sigma_2\|_\infty < \infty$. A sufficient condition is that σ, $\nabla\sigma$, $\nabla^2\sigma$ are bounded, see (3.5), but it is interesting that *boundedness of* σ *is not necessary* (think of the case of linear σ).

Given a solution Z of (3.18), we define the "remainders" $Z^{[3]}$ and $Z^{[2]}$ by

$$Z_{st}^{[3]} = \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}_{st}^1 - \sigma_2(Z_s)\,\mathbb{X}_{st}^2\,, \qquad Z_{st}^{[2]} = \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}_{st}^1\,. \tag{3.24}$$

Let us first show, by easy arguments, that any solution Z of (3.18) has the same Hölder regularity $\mathcal{C}^\alpha$ of the driving path X (in analogy with Lemmas 1.2 and 2.6), and that the "level 2 remainder" $Z_{st}^{[2]}$ is in $C_2^{2\alpha}$, that is $|Z_{st}^{[2]}| \lesssim (t-s)^{2\alpha}$.

Lemma 3.8. (Hölder regularity) *Let σ be of class C^1 and let Z be a solution of (3.18). There is a constant $C = C(Z) < \infty$ such that*

$$\begin{cases} |Z_{st}^{[2]}| \leqslant C\,|\mathbb{X}_{st}^2| + o(t-s), \\ |\delta Z_{st}| \leqslant C\,(|\mathbb{X}_{st}^1| + |\mathbb{X}_{st}^2|) + o(t-s), \end{cases} \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{3.25}$$

In particular, if $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ is an α-rough path, then $Z^{[2]} \in C_2^{2\alpha}$ and Z is of class $\mathcal{C}^\alpha$.

Proof. If $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ is an α-rough path, then by the first bound in (3.25) we have $|Z_{st}^{[2]}| \lesssim (t-s)^{2\alpha} + o(t-s) \lesssim (t-s)^{2\alpha}$, that is $Z^{[2]} \in C_2^{2\alpha}$. Similarly, the second bound in (3.25) gives $|\delta Z_{st}| \lesssim (t-s)^\alpha + (t-s)^{2\alpha} + o(t-s) \lesssim (t-s)^\alpha$, that is Z is of class $\mathcal{C}^\alpha$.

It remains to prove (3.25). This follows by (3.18) with $C := \sup_{0 \leqslant s \leqslant T} \{|\sigma(Z_s)| + |\sigma_2(Z_s)|\}$, so we need to show that $C < \infty$. Since σ and σ_2 are continuous (because σ is of class C^1), it is enough to prove that Z is bounded: $\sup_{0 \leqslant t \leqslant T} |Z_t| < \infty$.

Arguing as in the proof of Lemma 1.2, we fix $\bar{\delta} > 0$ such that $|o(t-s)| \leqslant 1$ for all $0 \leqslant s \leqslant t \leqslant T$ with $|t-s| \leqslant \bar{\delta}$. Since $[0,T]$ is a finite union of intervals $[\bar{s}, \bar{t}]$ with $\bar{t} - \bar{s} \leqslant \bar{\delta}$, we may focus on one such interval: by (3.18) we can bound

$$\sup_{t \in [\bar{s},\bar{t}]} |Z_t| \leqslant |Z_{\bar{s}}| + |\sigma(Z_{\bar{s}})| \sup_{t \in [\bar{s},\bar{t}]} |\mathbb{X}_{st}^1| + |\sigma_2(Z_{\bar{s}})| \sup_{t \in [\bar{s},\bar{t}]} |\mathbb{X}_{st}^2| + 1 < \infty.$$

This completes the proof that $\sup_{0\leqslant t\leqslant T}|Z_t|<\infty$. □

We next get to our main a priori estimates, showing in particular that the "level 3 remainder" $Z_{st}^{[3]}$ is in $C_2^{3\alpha}$, that is $|Z_{st}^{[3]}|\lesssim|t-s|^{3\alpha}$. Let us first record a useful computation: recalling (1.23) and (1.32), by $\delta\circ\delta=0$ and (3.13), we have

$$\begin{aligned}\delta Z_{sut}^{[3]} &= Z_{st}^{[3]}-Z_{su}^{[3]}-Z_{ut}^{[3]}\\ &= \underbrace{(\sigma(Z_u)-\sigma(Z_s)-\sigma_2(Z_s)\,\mathbb{X}_{su}^1)}_{B_{su}}\mathbb{X}_{ut}^1+(\sigma_2(Z_u)-\sigma_2(Z_s))\,\mathbb{X}_{ut}^2\,. \end{aligned} \tag{3.26}$$

THEOREM 3.9. (ROUGH A PRIORI ESTIMATES) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha\in\left]\frac{1}{3},\frac{1}{2}\right]$ and let $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ be an α-rough path over X. Let σ and σ_2 be globally Lipschitz.*

For any solution Z of (3.18), recalling the "remainders" $Z^{[3]}$ and $Z^{[2]}$ from (3.24), we have $Z^{[3]}\in C_2^{3\alpha}$: more precisely, for any $\tau>0$,

$$\|Z^{[3]}\|_{3\alpha,\tau}\leqslant K_{3\alpha}\,c'_{\alpha,\mathbb{X},\sigma}\,(\|\delta Z\|_{\alpha,\tau}+\|Z^{[2]}\|_{2\alpha,\tau})\,, \tag{3.27}$$

where we recall that $K_{3\alpha}=(1-2^{1-3\alpha})^{-1}$ and we define the constant

$$c'_{\alpha,\mathbb{X},\sigma}:=\|\nabla\sigma\|_\infty\,\|\mathbb{X}^1\|_\alpha+\|\nabla\sigma_2\|_\infty\,\|\mathbb{X}^2\|_{2\alpha}+(\|\nabla\sigma\|_\infty^2+\|\nabla\sigma_2\|_\infty)\,\|\mathbb{X}^1\|_\alpha^2\,. \tag{3.28}$$

Moreover, if either T or τ is small enough, we have

$$\begin{aligned}\|\delta Z\|_{\alpha,\tau}+\|Z^{[2]}\|_{2\alpha,\tau} &\leqslant 2\,(\sigma(Z_0)\,\|\mathbb{X}^1\|_\alpha+\sigma_2(Z_0)\,\|\mathbb{X}^2\|_{2\alpha})\\ &\quad \textit{for } (T\wedge\tau)^\alpha\leqslant\varepsilon'_{\alpha,\mathbb{X},\sigma}\,,\end{aligned} \tag{3.29}$$

where we set

$$\varepsilon'_{\alpha,\mathbb{X},\sigma}:=\frac{1}{4\,(K_{3\alpha}+3)\,(c'_{\alpha,\mathbb{X},\sigma}+1)}\,. \tag{3.30}$$

Proof. Let us prove (3.27). Since $3\alpha>1$ and $Z_{st}^{[3]}=o(t-s)$, see (3.19), we can apply the weighted Sewing Bound (1.41) which gives $\|Z^{[3]}\|_{3\alpha,\tau}\leqslant K_{3\alpha}\,\|\delta Z^{[3]}\|_{3\alpha,\tau}$. It remains to estimate $\delta Z^{[3]}$ from (3.26): applying (3.21) we can write

$$\|\delta Z^{[3]}\|_{3\alpha,\tau}\leqslant\|B\|_{2\alpha,\tau}\,\|\mathbb{X}^1\|_\alpha+\|\delta\sigma_2(Z)\|_{\alpha,\tau}\,\|\mathbb{X}^2\|_{2\alpha}\,. \tag{3.31}$$

We now focus on B_{su} from (3.26): by (3.23) we have

$$\begin{aligned}B_{su} &= \int_0^1[(\sigma_2(Z_s+u\,\delta Z_{su})-\sigma_2(Z_s))\,\mathbb{X}_{su}^1]\,\mathrm{d}u+\int_0^1[\nabla\sigma(Z_s+u\,\delta Z_{su})\,Z_{su}^{[2]}]\,\mathrm{d}u\\ &\quad -\int_0^1\nabla\sigma(Z_s+u\,\delta Z_{su})\left(\int_0^u[\nabla\sigma(Z_s+v\,\delta Z_{su})\,\delta Z_{su}\,\mathbb{X}_{su}^1]\,\mathrm{d}v\right)\mathrm{d}u\,,\end{aligned}$$

so that, by (2.8),

$$\|B\|_{2\alpha,\tau}\leqslant(\|\nabla\sigma_2\|_\infty+\|\nabla\sigma\|_\infty^2)\,\|\mathbb{X}^1\|_\alpha\,\|\delta Z\|_{\alpha,\tau}+\|\nabla\sigma\|_\infty\,\|Z^{[2]}\|_{2\alpha,\tau}\,. \tag{3.32}$$

We can plug this estimate into (3.31), together with the elementary bound

$$\|\delta\sigma_2(Z)\|_{\alpha,\tau}\leqslant\|\nabla\sigma_2\|_\infty\,\|\delta Z\|_{\alpha,\tau}\,. \tag{3.33}$$

Recalling that $\|Z^{[3]}\|_{3\alpha,\tau} \leqslant K_{3\alpha} \|\delta Z^{[3]}\|_{3\alpha,\tau}$, we have proved (3.27)-(3.28).

We next prove (3.29), for which we need to estimate $Z^{[2]}$ and δZ. Writing $Z_{st}^{[2]} = \sigma_2(Z_s)\,\mathbb{X}_{st}^2 + Z_{st}^{[3]}$ and setting $\varepsilon := (\tau \wedge T)^\alpha$ for short, we can bound by (2.6) and (2.7)

$$\|Z^{[2]}\|_{2\alpha,\tau} \leqslant \|\sigma_2(Z)\|_{\infty,\tau} \|\mathbb{X}^2\|_{2\alpha} + \varepsilon \|Z^{[3]}\|_{3\alpha,\tau} .$$

By (2.5) we have $\|\sigma_2(Z)\|_{\infty,\tau} \leqslant \sigma_2(Z_0) + 3\varepsilon \|\delta\sigma_2(Z)\|_{\alpha,\tau}$ and we can bound $\|\delta\sigma_2(Z)\|_{\alpha,\tau}$ by (3.33). Applying (3.27) and recalling (3.28), we then obtain

$$\begin{aligned} \|Z^{[2]}\|_{2\alpha,\tau} &\leqslant \sigma_2(Z_0) \|\mathbb{X}^2\|_{2\alpha} + \varepsilon (K_{3\alpha} + 3)\, c'_{\alpha,\mathbb{X},\sigma} (\|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau}) \\ &\leqslant \sigma_2(Z_0) \|\mathbb{X}^2\|_{2\alpha} + \frac{1}{4} \frac{\varepsilon}{\varepsilon'_{\alpha,\mathbb{X},\sigma}} (\|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau}) , \end{aligned} \tag{3.34}$$

where we recall that $\varepsilon'_{\alpha,\mathbb{X},\sigma}$ is defined in (3.30).

Similarly, writing $\delta Z_{st} = \sigma(Z_s)\,\mathbb{X}_{st}^1 + Z_{st}^{[2]}$ we can bound, by (2.6) and (2.7),

$$\|\delta Z\|_{\alpha,\tau} \leqslant \|\sigma(Z)\|_{\infty,\tau} \|\mathbb{X}^1\|_\alpha + \varepsilon \|Z^{[2]}\|_{2\alpha,\tau},$$

and since $\|\sigma(Z)\|_{\infty,\tau} \leqslant \sigma(Z_0) + 3\,\varepsilon \|\delta\sigma(Z)\|_{\alpha,\tau} \leqslant \sigma(Z_0) + 3\,\varepsilon \|\nabla\sigma\|_\infty \|\delta Z\|_{\alpha,\tau}$ we get, recalling (3.28),

$$\begin{aligned} \|\delta Z\|_{\alpha,\tau} &\leqslant \sigma(Z_0) \|\mathbb{X}^1\|_\alpha + 3\,\varepsilon\, c'_{\alpha,\mathbb{X},\sigma} \|\delta Z\|_{\alpha,\tau} + \varepsilon \|Z^{[2]}\|_{2\alpha,\tau} \\ &\leqslant \sigma(Z_0) \|\mathbb{X}^1\|_\alpha + \frac{1}{4} \frac{\varepsilon}{\varepsilon'_{\alpha,\mathbb{X},\sigma}} \|\delta Z\|_{\alpha,\tau} + \varepsilon \|Z^{[2]}\|_{2\alpha,\tau} . \end{aligned} \tag{3.35}$$

Finally, for $\varepsilon \leqslant \varepsilon'_{\alpha,\mathbb{X},\sigma}$ (hence $\varepsilon \leqslant \frac{1}{4}$, see (3.28)), by (3.34) and (3.35) we obtain

$$\|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau} \leqslant \sigma(Z_0) \|\mathbb{X}^1\|_\alpha + \sigma_2(Z_0) \|\mathbb{X}^2\|_{2\alpha} + \frac{1}{2} (\|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau}) .$$

Since $\|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau} < \infty$ by Lemma 3.8, we have proved (3.29). □

3.6. UNIQUENESS

In this section we prove uniqueness of solutions of (3.18) under the assumption that $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is *of class* C^γ *with* $\gamma > \frac{1}{\alpha}$ (e.g. it suffices that σ is of class C^3). This implies that σ_2 from (3.5) is of class C^1 with locally $(\gamma - 2)$-Hölder gradient $\nabla\sigma_2$. We stress that σ and σ_2 are *not* required to be bounded.

THEOREM 3.10. (UNIQUENESS) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$, let $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ be an α-rough path over X, and let σ be of class C^γ with $\gamma > \frac{1}{\alpha}$ (e.g. if σ is of class C^3). Then for every $z_0 \in \mathbb{R}^k$ there exists at most one solution Z to (3.18) such that $Z_0 = z_0$.*

Proof. Let us fix two solutions $Z, \bar{Z}$ of (3.18) and define their difference

$$Y := Z - \bar{Z}.$$

Our goal is to show that, for $\tau > 0$ small, we have $\|Y\|_{\infty,\tau} \leqslant 2\,|Y_0|$. In particular, if $Z_0 = \bar{Z}_0$, then $Y_0 = 0$ and therefore $\|Y\|_{\infty,\tau} = 0$, i.e. $Z = \bar{Z}$, which completes the proof.

We know by (2.5) that

$$\|Y\|_{\infty,\tau} \leqslant |Y_0| + 3\,\tau^{\alpha}\,\|\delta Y\|_{\alpha,\tau}\,. \tag{3.36}$$

With some abuse of notation, we denote by $Y_{st}^{[2]} := Z_{st}^{[2]} - \bar{Z}_{st}^{[2]}$ and $Y_{st}^{[3]} := Z_{st}^{[3]} - \bar{Z}_{st}^{[3]}$ the "differences of remainders", recall (3.24), so that we can write

$$\delta Y_{st} = (\sigma(Z_s) - \sigma(\bar{Z}_s))\,\mathbb{X}_{st}^1 + Y_{st}^{[2]}, \tag{3.37}$$

$$Y_{st}^{[2]} = (\sigma_2(Z_s) - \sigma_2(\bar{Z}_s))\,\mathbb{X}_{st}^2 + Y_{st}^{[3]}. \tag{3.38}$$

We are going to show that, for $\tau > 0$ small enough, the following bounds hold:

$$\|\delta Y\|_{\alpha,\tau} \leqslant c_1\,\|Y\|_{\infty,\tau} + \tau^{\alpha}\,\|Y^{[2]}\|_{2\alpha,\tau}\,, \tag{3.39}$$

$$\|Y^{[2]}\|_{2\alpha,\tau} \leqslant c_2\,\|Y\|_{\infty,\tau} + \tau^{(\gamma-2)\alpha}\,\|Y^{[3]}\|_{\gamma\alpha,\tau}\,, \tag{3.40}$$

$$\|Y^{[3]}\|_{\gamma\alpha,\tau} \leqslant c_3\,\|Y\|_{\infty,\tau} + c_3'\,\tau^{(\gamma-2)\alpha}\,\|Y^{[3]}\|_{\gamma\alpha,\tau}\,, \tag{3.41}$$

for suitable constants c_i, c_i' that may depend on $Z, \bar{Z}, \mathbb{X}^1, \mathbb{X}^2, \sigma$, *but not on* τ.

We can easily complete the proof, assuming (3.39)-(3.41): if we fix $\tau > 0$ small enough so that $c_3'\,\tau^{(\gamma-2)\alpha} < \frac{1}{2}$, by (3.41) we have $\|Y^{[3]}\|_{\gamma\alpha,\tau} \leqslant 2\,c_3\,\|Y\|_{\infty,\tau}$; plugging this into (3.40) and taking $\tau > 0$ small, we obtain $\|Y^{[2]}\|_{2\alpha,\tau} \leqslant 2\,c_2\,\|Y\|_{\infty,\tau}$, which plugged into (3.39) yields $\|\delta Y\|_{\alpha,\tau} \leqslant 2\,c_1\,\|Y\|_{\infty,\tau}$, for $\tau > 0$ is small enough. Finally, by (3.36) we obtain, for $\tau > 0$ small, our goal $\|Y\|_{\infty,\tau} \leqslant 2\,|Y_0|$.

It remains to prove (3.39)-(3.41). Recalling (2.18), let us define the constants

$$C_1' := C_{\nabla\sigma, \|Z\|_\infty \vee \|\bar{Z}\|_\infty}, \qquad C_1'' := C_{\nabla^2\sigma, \|Z\|_\infty \vee \|\bar{Z}\|_\infty}, \qquad C_2' := C_{\nabla\sigma_2, \|Z\|_\infty \vee \|\bar{Z}\|_\infty},$$

$$C_1''' := \sup\left\{\frac{|\nabla^2\sigma(x) - \nabla^2\sigma(y)|}{|x-y|^{\gamma-2}}\colon\ |x|, |y| \leqslant \|Z\|_\infty \vee \|\bar{Z}\|_\infty\right\},$$

$$C_2'' := \sup\left\{\frac{|\nabla\sigma_2(x) - \nabla\sigma_2(y)|}{|x-y|^{\gamma-2}}\colon\ |x|, |y| \leqslant \|Z\|_\infty \vee \|\bar{Z}\|_\infty\right\}.$$

(Note that $\|Z\|_\infty, \|\bar{Z}\|_\infty < \infty$ because $Z, \bar{Z}$ are continuous, see Lemma 3.8.)

We can prove (3.39) and (3.40) arguing as in the proof of Theorem 2.9, see (2.24) and (2.25). Indeed, from (3.37) we can bound, by (2.6) and (2.7),

$$\begin{aligned} \|\delta Y\|_{\alpha,\tau} &\leqslant \|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau}\,\|\mathbb{X}^1\|_{\alpha} + \tau^{\alpha}\,\|Y^{[2]}\|_{2\alpha,\tau} \\ &\leqslant C_1'\,\|Y\|_{\infty,\tau}\,\|\mathbb{X}^1\|_{\alpha} + \tau^{\alpha}\,\|Y^{[2]}\|_{2\alpha,\tau}\,, \end{aligned} \tag{3.42}$$

because $|\sigma(Z_t) - \sigma(\bar{Z}_t)| \leqslant C_1'\,|Z_t - \bar{Z}_t|$, hence (3.39) holds with $c_1 = C_1'\,\|\mathbb{X}^1\|_{\alpha}$. Similarly, by (3.38) we can bound

$$\begin{aligned} \|Y^{[2]}\|_{2\alpha,\tau} &\leqslant \|\sigma_2(Z) - \sigma_2(\bar{Z})\|_{\infty,\tau}\,\|\mathbb{X}^2\|_{2\alpha} + \tau^{(\gamma-2)\alpha}\,\|Y^{[3]}\|_{\gamma\alpha,\tau} \\ &\leqslant C_2'\,\|Y\|_{\infty,\tau}\,\|\mathbb{X}^2\|_{2\alpha} + \tau^{(\gamma-2)\alpha}\,\|Y^{[3]}\|_{\gamma\alpha,\tau}\,, \end{aligned} \tag{3.43}$$

because $|\sigma_2(Z_t) - \sigma_2(\bar{Z}_t)| \leqslant C_2'\,|Z_t - \bar{Z}_t|$, hence also (3.40) holds with $c_2 = C_2'\,\|\mathbb{X}^2\|_{2\alpha}$.

We finally prove (3.41). Since $Y_{st}^{[3]} = Z_{st}^{[3]} - \bar{Z}_{st}^{[3]} = o(t-s)$, see (3.19), we can bound $Z^{[3]}$ by its increment $\delta Z^{[3]}$ through the weighted Sewing Bound (1.41):

$$\|Y^{[3]}\|_{\gamma\alpha,\tau} \leqslant K_{\gamma\alpha} \|\delta Y^{[3]}\|_{\gamma\alpha,\tau} . \tag{3.44}$$

We are going to prove the following estimate:

$$\|\delta Y^{[3]}\|_{\gamma\alpha,\tau} \leqslant \tilde{c}_3 \|Y\|_{\infty,\tau} + \tilde{c}_3' \|\delta Y\|_{\alpha,\tau} + \tilde{c}_3'' \|Y^{[2]}\|_{2\alpha,\tau} , \tag{3.45}$$

for suitable constants $\tilde{c}_3, \tilde{c}_3'', \tilde{c}_3''$ that depend on $Z, \bar{Z}, \mathbb{X}^1, \mathbb{X}^2, \sigma$, *but not on* τ. Plugging the estimates (3.39) and (3.40) (that we already proved) for $\|\delta Y\|_{\alpha,\tau}$ and $\|Y^{[2]}\|_{2\alpha,\tau}$, we obtain (3.41) for suitable (explicit) constants c_3, c_3'.

Let us then prove (3.45). Recalling (3.26), for $0 \leqslant s \leqslant u \leqslant t \leqslant T$ we can write

$$\delta Y_{sut}^{[3]} = (B_{su} - \bar{B}_{su}) \mathbb{X}_{ut}^1 + (\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z}))_{su} \mathbb{X}_{ut}^2 ,$$

where $B_{su} := \sigma(Z_u) - \sigma(Z_s) - \sigma_2(Z_s) \mathbb{X}_{su}^1$ and similarly for $\bar{B}_{su}$, hence by (3.21)

$$\|\delta Y^{[3]}\|_{\gamma\alpha,\tau} \leqslant \|B - \bar{B}\|_{(\gamma-1)\alpha,\tau} \|\mathbb{X}\|_{\alpha} + \|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{(\gamma-2)\alpha,\tau} \|\mathbb{X}^2\|_{2\alpha} . \tag{3.46}$$

To obtain (3.45) we need to show that $\|B - \bar{B}\|_{(\gamma-1)\alpha,\tau}$ and $\|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{(\gamma-2)\alpha,\tau}$ can be bounded by *linear combinations of* $\|Y\|_{\infty,\tau}$, $\|\delta Y\|_{\alpha,\tau}$ *and* $\|Y^{[2]}\|_{2\alpha,\tau}$.

We start from $\|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{(\gamma-2)\alpha,\tau}$, which can be bounded as in (2.29):

$$\|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{(\gamma-2)\alpha,\tau} \leqslant C_2' \|\delta Y\|_{\alpha,\tau} + C_2'' \{\|\delta Z\|_{\alpha}^{\gamma-1} + \|\delta\bar{Z}\|_{\alpha}^{\gamma-1}\} \|Y\|_{\infty,\tau}.$$

We next focus on $\|B - \bar{B}\|_{(\gamma-1)\alpha,\tau}$, which we are going to estimate by the following explicit linear combination of $\|Y\|_{\infty,\tau}$, $\|\delta Y\|_{\alpha,\tau}$ and $\|Y^{[2]}\|_{2\alpha,\tau}$:

$$\begin{aligned} \|B - \bar{B}\|_{(\gamma-1)\alpha,\tau} \leqslant\ & C_1'' \|Y\|_{\infty,\tau} \|Z^{[2]}\|_{2\alpha} + C_1' \|Y^{[2]}\|_{2\alpha,\tau} \\ & + C_1'' \|\delta Y\|_{\alpha,\tau} \|\delta Z\|_{\alpha} + 2 C_1''' \|Y\|_{\infty,\tau} \|\delta Z\|_{\alpha}^2 \\ & + C_1'' \|\delta\bar{Z}\|_{\alpha} \|\delta Y\|_{\alpha,\tau}, \end{aligned} \tag{3.47}$$

which completes the proof of (3.45) when plugged into (3.46).

It only remains to prove (3.47). Recalling (3.24), it follows by (3.22) that

$$\begin{aligned} B_{su} &:= \sigma(Z_u) - \sigma(Z_s) - \sigma_2(Z_s) \mathbb{X}_{su}^1 \\ &= \nabla\sigma(Z_s) Z_{su}^{[2]} + \int_0^1 \underbrace{(\nabla\sigma(Z_u + r\,\delta Z_{su}) - \nabla\sigma(Z_u))}_{F_{su}} \delta Z_{su} \,\mathrm{d}r, \end{aligned}$$

and likewise for $\bar{B}_{su}$ (with $\bar{F}_{su}$ defined similarly), therefore

$$|B_{su} - \bar{B}_{su}| \leqslant |\nabla\sigma(Z_s) Z_{su}^{[2]} - \nabla\sigma(\bar{Z}_s) \bar{Z}_{su}^{[2]}| + \int_0^1 |F_{su} \delta Z_{su} - \bar{F}_{su} \delta\bar{Z}_{su}| \,\mathrm{d}r . \tag{3.48}$$

By the elementary estimate $|a\,b - \bar{a}\,\bar{b}| = |a\,b - \bar{a}\,b + \bar{a}\,b - \bar{a}\,\bar{b}| \leqslant |a - \bar{a}|\,|b| + |\bar{a}|\,|b - \bar{b}|$, that we apply repeatedly, we can bound

$$\begin{aligned} |\nabla\sigma(Z_s) Z_{su}^{[2]} - \nabla\sigma(\bar{Z}_s) \bar{Z}_{su}^{[2]}| &\leqslant |\nabla\sigma(Z_s) - \nabla\sigma(\bar{Z}_s)|\,|Z_{su}^{[2]}| + |\nabla\sigma(\bar{Z}_s)|\,|Z_{su}^{[2]} - \bar{Z}_{su}^{[2]}| \\ &\leqslant C_1'' |Y_s|\,|Z_{su}^{[2]}| + C_1' |Y_{su}^{[2]}|, \end{aligned}$$

and note that by (2.7) we obtain the first line in the RHS of (3.47).

To complete the proof of (3.47), we look at the second term in the RHS of (3.48):

$$\begin{aligned}|F_{su}\,\delta Z_{su}-\bar{F}_{su}\,\delta\bar{Z}_{su}| &\leqslant |F_{su}-\bar{F}_{su}|\,|\delta Z_{su}|+|\bar{F}_{su}|\,|\delta Z_{su}-\delta\bar{Z}_{su}| \\ &\leqslant |F_{su}-\bar{F}_{su}|\,|\delta Z_{su}|+C_1''\,r\,|\delta\bar{Z}_{su}|\,|\delta Y_{su}|,\end{aligned} \tag{3.49}$$

because $|\bar{F}_{su}|\leqslant C_1''\,r\,|\delta\bar{Z}_{su}|$. We then see, applying (2.8), that the last term in (3.49) produces the third line in (3.47). Finally, by (2.19) we estimate

$$\begin{aligned}|F_{su}-\bar{F}_{su}| &= |(\nabla\sigma(Z_u+r\,\delta Z_{su})-\nabla\sigma(Z_u))-(\nabla\sigma(\bar{Z}_u+r\,\delta\bar{Z}_{su})-\nabla\sigma(\bar{Z}_u))| \\ &\leqslant C_1''\,r\,|\delta Y_{su}|+C_1'''\,\{|r\,\delta Z_{su}|^{\gamma-2}+|r\,\delta Z_{su}|^{\gamma-2}\}\,|Y_s|\,.\end{aligned}$$

We obtain by (2.7) for $0\leqslant r\leqslant 1$

$$\|F-\bar{F}\|_{(\gamma-2)\alpha,\tau}\leqslant C_1''\,\|\delta Y\|_{\alpha,\tau}+2\,C_1'''\,\|Y\|_{\infty,\tau}\,\|\delta Z\|_{\alpha}^{\gamma-2}\,.$$

Applying again (2.8), we finally see that the first term in (3.49) yields the second line in (3.47), which completes the proof. □

3.7. CONTINUITY OF THE SOLUTION MAP

In this section we assume that σ has bounded first, second and third derivatives, while σ_2 has bounded first and second derivatives:

$$\|\nabla\sigma\|_\infty,\|\nabla^2\sigma\|_\infty,\|\nabla^3\sigma\|_\infty<\infty,\qquad \|\nabla\sigma_2\|_\infty,\|\nabla^2\sigma_2\|_\infty<\infty. \tag{3.50}$$

(We stress that no boundedness assumption is made on σ and σ_2.) Under these assumptions, given any time horizon $T>0$, any starting point $Z_0\in\mathbb{R}^k$ and any α-rough path $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ with $\frac{1}{3}<\alpha\leqslant\frac{1}{2}$, we have *global existence and uniqueness* of solutions $Z\colon[0,T]\to\mathbb{R}^k$ to (3.18) (as we will prove in Theorem 3.12).

Denoting by $\mathcal{R}_{\alpha,d}$ the space of d-dimensional α-rough paths $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$, that we endow with the norm $\|\mathbb{X}^1\|_\alpha+\|\mathbb{X}^2\|_{2\alpha}$ we can thus consider the *solution map*:

$$\begin{aligned}\Phi\colon\ \mathbb{R}^k\times\mathcal{R}_{\alpha,d} &\longrightarrow \mathcal{C}^\alpha \\ (Z_0,\mathbb{X}) &\longmapsto Z:=\begin{cases}\text{unique solution of (3.18) for } t\in[0,T] \\ \text{starting from } Z_0\end{cases}.\end{aligned} \tag{3.51}$$

We prove the highly non-trivial result that this map is *locally Lipschitz*. In the space $\mathcal{C}^\alpha$ of Hölder functions we work with the weighted norm $\|f\|_{\infty,\tau}+\|\delta f\|_{\alpha,\tau}$, which is equivalent to the usual norm $\|f\|_{\mathcal{C}^\alpha}:=\|f\|_\infty+\|\delta f\|_\alpha$, see Remark 1.15.

THEOREM 3.11. (CONTINUITY OF THE SOLUTION MAP) *Let σ and σ_2 satisfy (3.50) (with no boundedness assumption on the functions σ and σ_2). Then, for any $T>0$ and $\alpha\in\left]\frac{1}{3},\frac{1}{2}\right]$, the solution map $(Z_0,\mathbb{X})\mapsto Z$ in (3.51) is locally Lipschitz.*

More explicitly, given any $M_0,M,D<\infty$, if we assume that

$$\max\{\|\nabla\sigma\|_\infty,\|\nabla^2\sigma\|_\infty,\|\nabla^3\sigma\|_\infty,\|\nabla\sigma_2\|_\infty,\|\nabla^2\sigma_2\|_\infty\}\leqslant D, \tag{3.52}$$

and we consider starting points $Z_0, \bar{Z}_0 \in \mathbb{R}^d$ *and rough paths* $\mathbb{X}, \bar{\mathbb{X}} \in \mathcal{C}^\alpha$ *with*

$$\max\{|\sigma(Z_0)|, |\sigma_2(Z_0)|, |\sigma(\bar{Z}_0)|, |\sigma_2(\bar{Z}_0)|\} \leqslant M_0, \tag{3.53}$$

$$\max\{\|\mathbb{X}^1\|_\alpha, \|\mathbb{X}^2\|_{2\alpha}, \|\bar{\mathbb{X}}^1\|_\alpha, \|\bar{\mathbb{X}}^2\|_{2\alpha}\} \leqslant M, \tag{3.54}$$

then the corresponding solutions $Z = (Z_s)_{s\in[0,T]}$, $\bar{Z} = (\bar{Z}_s)_{s\in[0,T]}$ *of (3.18) satisfy*

$$\begin{aligned}&\|Z - \bar{Z}\|_{\infty,\tau} + \|\delta Z - \delta\bar{Z}\|_{\alpha,\tau} + \|Z^{[2]} - \bar{Z}^{[2]}\|_{2\alpha,\tau}\\ &\quad\leqslant \mathfrak{C}'_M\, |Z_0 - \bar{Z}_0| + 30\, M_0\, (\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}).\end{aligned} \tag{3.55}$$

provided τ *satisfies* $0 < \tau \wedge T \leqslant \hat{\tau}'$ *for a suitable* $\hat{\tau}' = \hat{\tau}'_{\alpha,T,D,M_0,M} > 0$*, where we set*

$$\mathfrak{C}'_M := 16\,\{(\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty)\, M + 1\} \leqslant 32\,(D\,M + 1).$$

Proof. It is convenient to define the constant

$$\mathfrak{c}'_M := (\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty)\, M \leqslant 2\, D\, M. \tag{3.56}$$

Let Z and $\bar{Z}$ be two solutions of (3.18) with respective routh paths $\mathbb{X}$ and $\bar{\mathbb{X}}$. Defining $Y := Z - \bar{Z}$ and $Y^{[2]} := Z^{[2]} - \bar{Z}^{[2]}$, see (3.24), we rewrite our goal (3.55) as

$$\begin{aligned}\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau} + \|Y^{[2]}\|_{2\alpha,\tau} \;\leqslant\;& 16\,(\mathfrak{c}'_M + 1)\, |Y_0|\\ &+ 30\, M_0\, (\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}).\end{aligned} \tag{3.57}$$

Throughout the proof we use the shorthand

$$\varepsilon := (\tau \wedge T)^\alpha \tag{3.58}$$

and we write *for* ε *small enough* to mean *for all* $0 < \varepsilon < \varepsilon_0$*, for a suitable* ε_0 *depending on* α, T, M_0, M, D. We claim that the following estimates hold for δY and $Y^{[2]}$:

$$\|\delta Y\|_{\alpha,\tau} \leqslant \mathfrak{c}'_M\, \|Y\|_{\infty,\tau} + 2\, M_0\, \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \varepsilon\, \|Y^{[2]}\|_{2\alpha,\tau}, \tag{3.59}$$

$$\|Y^{[2]}\|_{2\alpha,\tau} \leqslant \mathfrak{c}'_M\, \|Y\|_{\infty,\tau} + 2\, M_0\, \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} + \varepsilon\, \|Y^{[3]}\|_{3\alpha,\tau}, \tag{3.60}$$

and, moreover, *for* ε *small enough* the following estimate holds for $Y^{[3]} := Z^{[3]} - \bar{Z}^{[3]}$:

$$\varepsilon\, \|Y^{[3]}\|_{3\alpha,\tau} \leqslant \|Y\|_{\infty,\tau} + M_0\, (\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}) + \|\delta Y\|_{\alpha,\tau} + \frac{1}{4}\, \|Y^{[2]}\|_{\alpha,\tau}. \tag{3.61}$$

It is now elementary (but tedious) to deduce our goal (3.57). Plugging (3.61) into (3.60) we obtain $\|Y^{[2]}\|_{2\alpha,\tau} \leqslant (\cdots) + \frac{1}{4}\|Y^{[2]}\|_{2\alpha,\tau}$ which yields $\|Y^{[2]}\|_{2\alpha,\tau} \leqslant \frac{4}{3}(\ldots)$ (since $\|Y^{[2]}\|_{2\alpha,\tau} < \infty$ by Lemma 3.8). Making $(\ldots)$ explicit, we get

$$\|Y^{[2]}\|_{2\alpha,\tau} \leqslant 2\,(\mathfrak{c}'_M + 1)\, \|Y\|_{\infty,\tau} + 4\, M_0\, (\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}) + 2\, \|\delta Y\|_{\alpha,\tau} \tag{3.62}$$

which plugged into (3.59) yields, *for* ε *small enough* (it suffices that $\varepsilon \leqslant \frac{1}{4}$),

$$\|\delta Y\|_{\alpha,\tau} \leqslant 3\,(\mathfrak{c}'_M + 1)\, \|Y\|_{\infty,\tau} + 6\, M_0\, (\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}), \tag{3.63}$$

and looking back at (3.62) we obtain

$$\|Y^{[2]}\|_{2\alpha,\tau} \leqslant 8\,(\mathfrak{c}'_M + 1)\,\|Y\|_{\infty,\tau} + 16\,M_0\,(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}), \tag{3.64}$$

so that, overall,

$$\begin{aligned}\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau} + \|Y^{[2]}\|_{2\alpha,\tau} \;\leqslant\; & 12\,(\mathfrak{c}'_M + 1)\,\|Y\|_{\infty,\tau} \\ & + 22\,M_0\,(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha})\,.\end{aligned} \tag{3.65}$$

It only remains to make $\|Y\|_{\infty,\tau}$ explicit. Since $\|Y\|_{\infty,\tau} \leqslant |Y_0| + 3\,\varepsilon\,\|\delta Y\|_{\alpha,\tau}$ by (2.5), *for ε small enough* (more precisely for $\varepsilon \leqslant \frac{1}{36\,(\mathfrak{c}'_M + 1)}$) we can bound

$$(\mathfrak{c}'_M + 1)\,\|Y\|_{\infty,\tau} \leqslant (\mathfrak{c}'_M + 1)\,|Y_0| + \frac{1}{12}\,\|\delta Y\|_{\alpha,\tau}, \tag{3.66}$$

which inserted into (3.63) yields

$$\|\delta Y\|_{\alpha,\tau} \leqslant 4\,(\mathfrak{c}'_M + 1)\,|Y_0| + 8\,M_0\,(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}).$$

Plugging this into (3.66), and then (3.66) into (3.65), we obtain our goal (3.57).

It remains to prove (3.59), (3.60) and (3.61). We first state some useful bounds that will be used repeatedly. Recalling (3.52) and (3.28)-(3.30), let us define

$$\bar{\tau} = \bar{\tau}_{\alpha,D,M} := \frac{1}{\{4\,(K_{3\alpha} + 3)\,(2\,(D^2 + D)\,(M^2 + M) + 1)\}^{1/\alpha}}\,, \tag{3.67}$$

By the a priori estimate (3.29) we can then bound

$$\text{for } \varepsilon = (\tau \wedge T)^\alpha \leqslant \bar{\tau}^\alpha: \qquad \|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau} \leqslant 4\,M_0\,M\,, \tag{3.68}$$

hence

$$\max\{\|\delta\sigma(Z)\|_{\alpha,\tau}, \|\delta\sigma_2(Z)\|_{\alpha,\tau}\} \leqslant \max\{\|\nabla\sigma\|_\infty, \|\nabla\sigma_2\|_\infty\}\,\|\delta Z\|_{\alpha,\tau} \leqslant 4\,M_0\,\mathfrak{c}'_M, \tag{3.69}$$

which implies that, by (2.5) and *for ε small enough*,

$$\max\{\|\sigma(Z)\|_{\infty,\tau}\,, \|\sigma_2(Z)\|_{\infty,\tau}\} \leqslant M_0 + 3\,\varepsilon\,4\,M_0\,\mathfrak{c}'_M \leqslant 2\,M_0.$$

We record the following simple bound, for any Lipschitz function f,

$$\|f(Z) - f(\bar{Z})\|_{\infty,\tau} \leqslant \|\nabla f\|_\infty\,\|Z - \bar{Z}\|_{\infty,\tau} = \|\nabla f\|_\infty\,\|Y\|_{\infty,\tau}. \tag{3.70}$$

We will also use a number of times the elementary estimate, for $a, b, \bar{a}, \bar{b} \in \mathbb{R}$,

$$|a\,b - \bar{a}\,\bar{b}| = |a\,b - a\,\bar{b} + a\,\bar{b} - \bar{a}\,\bar{b}| \leqslant |a|\,|b - \bar{b}| + |\bar{b}|\,|a - \bar{a}|\,. \tag{3.71}$$

We can now prove (3.59). Since $\delta Y_{st} = \delta Z_{st} - \delta\bar{Z}_{st} = \sigma(Z_s)\,\mathbb{X}^1_{st} - \sigma(\bar{Z}_s)\,\bar{\mathbb{X}}^1_{st} + Y^{[2]}_{st}$, see (3.24) for Z and $\bar{Z}$, by (2.7) and (3.53)-(3.54) we get, applying (3.71),

$$\begin{aligned}\|\delta Y\|_{\alpha,\tau} \;\leqslant\; & \|\sigma(Z)\|_{\infty,\tau}\,\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau}\,\|\bar{\mathbb{X}}^1\|_\alpha + \|Y^{[2]}\|_{\alpha,\tau} \\ \leqslant\; & 2\,M_0\,\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau}\,M + \varepsilon\,\|Y^{[2]}\|_{2\alpha,\tau}\,,\end{aligned}$$

because $\|Y^{[2]}\|_{\alpha,\tau} \leqslant \varepsilon\,\|Y^{[2]}\|_{2\alpha,\tau}$ by (2.6) (recall the definition (3.58) of ε). Applying (3.70) with $f = \sigma$ and recalling $\mathfrak{c}'_M$ from (3.56), we obtain (3.59).

The proof of (3.60) is similar. Since $Z^{[3]}_{st} = Z^{[2]}_{st} - \sigma_2(Z_s)\,\mathbb{X}^2_{st}$ and similarly for $\bar{Z}^{[3]}$, see (3.24), we can write $Y^{[2]}_{st} = Z^{[2]} - \bar{Z}^{[2]} = \sigma_2(Z_s)\,\mathbb{X}^2_{st} - \sigma_2(\bar{Z}_s)\,\bar{\mathbb{X}}^2_{st} + Y^{[3]}_{st}$, therefore

$$\begin{aligned}\|Y^{[2]}\|_{2\alpha,\tau} &\leqslant \|\sigma_2(Z)\|_{\infty,\tau}\,\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} + \|\sigma_2(Z) - \sigma_2(\bar{Z})\|_{\infty,\tau}\,\|\bar{\mathbb{X}}^2\|_{2\alpha} + \|Y^{[3]}\|_{2\alpha,\tau} \\ &\leqslant 2\,M_0\,\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} + \|\sigma_2(Z) - \sigma_2(\bar{Z})\|_{\infty,\tau}\,M + \varepsilon\,\|Y^{[3]}\|_{3\alpha,\tau}\,,\end{aligned}$$

since $\|Y^{[3]}\|_{2\alpha,\tau} \leqslant \varepsilon\,\|Y^{[3]}\|_{3\alpha,\tau}$ by (2.6). Applying (3.70) for $f = \sigma_2$ we obtain (3.60).

We finally prove (3.61). Since $Y^{[3]}_{st} = Z^{[3]}_{st} - \bar{Z}^{[3]}_{st} = o(t-s)$, see (3.19), the weighted Sewing Bound (1.41) yields

$$\|Y^{[3]}\|_{3\alpha,\tau} \leqslant K_{3\alpha}\,\|\delta Y^{[3]}\|_{3\alpha,\tau}\,, \tag{3.72}$$

hence we can focus on $\delta Y^{[3]} = \delta Z^{[3]} - \delta \bar{Z}^{[3]}$. Let us recall (3.26): for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\delta Z^{[3]}_{sut} = \underbrace{(\sigma(Z_u) - \sigma(Z_s) - \sigma_2(Z_s)\,\mathbb{X}^1_{su})}_{B_{su}}\,\mathbb{X}^1_{ut} + \delta\sigma_2(Z)_{su}\,\mathbb{X}^2_{ut}\,,$$

and analogously for $\delta\bar{Z}^{[3]}$ and $\bar{B}_{su}$, therefore by (3.71) and (3.21) we obtain

$$\begin{aligned}\|\delta Y^{[3]}\|_{3\alpha,\tau} &\leqslant \|B\|_{2\alpha,\tau}\,\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_{\alpha} + \|B - \bar{B}\|_{2\alpha,\tau}\,\|\bar{\mathbb{X}}^1\|_{\alpha,\tau} \\ &\quad + \|\delta\sigma_2(Z)\|_{\alpha,\tau}\,\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} + \|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{\alpha,\tau}\,\|\bar{\mathbb{X}}^2\|_{2\alpha}\,.\end{aligned} \tag{3.73}$$

It remains to estimate the four terms in the RHS: in view of (3.72), relation (3.61) is proved if we show that, *for ε small enough*,

$$\varepsilon\,K_{3\alpha}\,\|B\|_{2\alpha,\tau}\,\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_{\alpha} \leqslant M_0\,\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_{\alpha}\,, \tag{3.74}$$

$$\varepsilon\,K_{3\alpha}\,\|B - \bar{B}\|_{2\alpha,\tau}\,\|\bar{\mathbb{X}}^1\|_{\alpha,\tau} \leqslant \frac{1}{2}(\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau}) + \frac{1}{4}\|Y^{[2]}\|_{2\alpha,\tau}, \tag{3.75}$$

$$\varepsilon\,K_{3\alpha}\,\|\delta\sigma_2(Z)\|_{\alpha,\tau}\,\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} \leqslant M_0\,\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}\,, \tag{3.76}$$

$$\varepsilon\,K_{3\alpha}\,\|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{\alpha,\tau}\,\|\bar{\mathbb{X}}^2\|_{2\alpha} \leqslant \frac{1}{2}(\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau})\,. \tag{3.77}$$

We first deal with (3.76) and (3.77), then we focus on (3.74) and (3.75).

Proving (3.76) is very simple: since $\|\delta\sigma_2(Z)\|_{\alpha,\tau} \leqslant 4\,M_0\,\mathfrak{c}'_M$ by (3.69), we see that (3.76) holds *for ε small enough*. To prove (3.77), note that by (2.51) we have

$$\|\delta\sigma(Z) - \delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau} \leqslant \|\nabla\sigma\|_{\infty}\,\|\delta Y\|_{\alpha,\tau} + 4\,M_0\,M\,[\sigma]_{\mathcal{C}^{\gamma-1}}\,\|Y\|_{\infty,\tau}\,.$$

Applying (3.54) and (3.68) we obtain

$$\|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{\alpha,\tau}\,\|\bar{\mathbb{X}}^2\|_{2\alpha} \leqslant \|\nabla\sigma_2\|_{\infty}\,M\,\|\delta Y\|_{\alpha,\tau} + \mathrm{e}^{\frac{T}{\tau}}\,\|\nabla^2\sigma_2\|_{\infty}\,8\,M_0\,M^2\,\|Y\|_{\infty,\tau},$$

which shows that (3.77) holds *for ε small enough*.

Let us now prove (3.74). By (3.22) we have, for $0 \leqslant s \leqslant t \leqslant T$,

$$B_{st} = \underbrace{\nabla\sigma(Z_s)\,Z^{[2]}_{st}}_{E_{st}} + \underbrace{\int_0^1 [(\nabla\sigma(Z_s + r\,\delta Z_{st}) - \nabla\sigma(Z_s))\,\delta Z_{st}]\,\mathrm{d}r}_{F_{st}} \tag{3.78}$$

and similarly for $\bar{E}_{st}$ and $\bar{F}_{st}$. In particular, recalling (3.68), we get

$$\begin{aligned}\|B\|_{2\alpha,\tau} &\leqslant \|\nabla\sigma\|_\infty \|Z^{[2]}\|_{2\alpha,\tau} + \|\nabla^2\sigma\|_\infty \|\delta Z\|^2_{\alpha,\tau} \\ &\leqslant \|\nabla\sigma\|_\infty 4\, M_0\, M + \|\nabla^2\sigma\|_\infty (4\, M_0\, M)^2,\end{aligned}$$

hence we see that (3.74) holds *for ε small enough*.

We finally prove (3.75), which is a bit tedious. In view of (3.78), we first consider

$$E_{st} - \bar{E}_{st} = (\nabla\sigma(Z_s) - \nabla\sigma(\bar{Z}_s))\, Z^{[2]}_{st} + \nabla\sigma(\bar{Z}_s)\, (Z^{[2]}_{st} - \bar{Z}^{[2]}_{st}) \,.$$

Applying (2.9) with $H = Z^{[2]}$ and $\bar{\tau}$ from (3.67), we obtain

$$\|E - \bar{E}\|_{2\alpha,\tau} \leqslant \|\nabla\sigma(Z) - \nabla\sigma(\bar{Z})\|_{\infty,\tau}\, e^{\frac{T}{\bar{\tau}}}\, \|Z^{[2]}\|_{2\alpha,\bar{\tau}} + \|\nabla\sigma\|_\infty \|Y^{[2]}\|_{2\alpha,\tau} \,.$$

By (3.70) with $f = \nabla\sigma$ and the a priori estimate (3.68) we obtain

$$\|E - \bar{E}\|_{2\alpha,\tau} \leqslant \|\nabla^2\sigma\|_\infty \|Y\|_{\infty,\tau}\, e^{\frac{T}{\bar{\tau}}}\, 4\, M_0\, M + \|\nabla\sigma\|_\infty \|Y^{[2]}\|_{2\alpha,\tau} \,. \tag{3.79}$$

We then consider $F_{st} - \bar{F}_{st}$. By (2.19), for $0 \leqslant r \leqslant 1$ we can estimate

$$\begin{aligned}&|(\nabla\sigma(Z_s + r\,\delta Z_{st}) - \nabla\sigma(Z_s)) - (\nabla\sigma(\bar{Z}_s + r\delta\bar{Z}_{st}) - \nabla\sigma(\bar{Z}_s))|\, |\delta Z_{st}| \\ &\leqslant \|\nabla^2\sigma\|_\infty |\delta Y_{st}|\, |\delta Z_{st}| + \|\nabla^3\sigma\|_\infty \max_{0\leqslant u\leqslant 1} \{(1-u)\, |Y_s| + u\, |Y_t|\}\, |\delta Z_{st}|^2,\end{aligned}$$

as well as

$$|\nabla\sigma(Z_s + r\,\delta Z_{st}) - \nabla\sigma(Z_s)|\, |\delta Z_{st} - \delta\bar{Z}_{st}| \leqslant \|\nabla^2\sigma\|_\infty |\delta Z_{st}|\, |\delta Y_{st}| \,.$$

We can then estimate $F_{st} - \bar{F}_{st}$ from (3.78) as in (3.71): applying (2.9) twice with $H = \delta Z$ and $H = (\delta Z)^2$, always with $\bar{\tau}$ from (3.67), and recalling (3.68), we obtain

$$\begin{aligned}\|F - \bar{F}\|_{2\alpha,\tau} &\leqslant 2\, \|\nabla^2\sigma\|_\infty \|\delta Y\|_{\alpha,\tau}\, e^{\frac{T}{\bar{\tau}}}\, \|\delta Z\|_{\alpha,\bar{\tau}} + \|\nabla^3\sigma\|_\infty \|Y\|_{\infty,\tau}\, e^{\frac{T}{\bar{\tau}}}\, \|\delta Z\|^2_{\alpha,\bar{\tau}} \\ &\leqslant e^{\frac{T}{\bar{\tau}}} \{8\, M_0\, M\, \|\nabla^2\sigma\|_\infty \|\delta Y\|_{\alpha,\tau} + (4\, M_0\, M)^2 \|\nabla^3\sigma\|_\infty \|Y\|_{\infty,\tau}\} \,.\end{aligned} \tag{3.80}$$

Since $\|B - \bar{B}\|_{2\alpha,\tau} \leqslant \|E - \bar{E}\|_{2\alpha,\tau} + \|F - \bar{F}\|_{2\alpha,\tau}$ in view of (3.78), we see by (3.79) and (3.80) that (3.75) holds *for ε small enough*. The proof is complete. □

3.8. GLOBAL EXISTENCE AND UNIQUENESS

Let us suppose that $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is of class C^3 with $\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty < +\infty$.

THEOREM 3.12. *Let $\alpha > \frac{1}{3}$. If $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is of class C^3 with $\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty < +\infty$ then for every $z_0 \in \mathbb{R}^k$ and $T > 0$ there is a unique solution $(Z_t)_{t\in[0,T]}$ to (3.19) such that $Z_0 = z_0$.*

Proof. By Theorem 3.10 we have at most one solution. We now construct a solution on an arbitrary finite interval $[0, T]$, arguing as in the proof of Theorem 2.15. We define $\Lambda \subseteq [0, T]$ as the set of all s such that there is a solution $(Z_t)_{t\in[0,s]}$ to (3.19). By Proposition 3.6, Λ is an open subset of $[0, T]$ and contains 0. By the a priori estimates of Theorem 3.9, Λ is a closed subset of $[0, T]$. Therefore $\Lambda = [0, T]$. □

3.9. Milstein scheme and local existence

In this section we prove the local existence result of Proposition 3.6, under the assumption that σ, σ_2 are of class C^1 and uniformly Lipschitz. To construct a solution to (3.10), we set $t_i := \frac{i}{n}$, $i \geqslant 0$, and for a given $y_0 \in \mathbb{R}^k$

$$y_{t_{i+1}} = y_{t_i} + \sigma(y_{t_i})\, \mathbb{X}^1_{t_i t_{i+1}} + \sigma_2(y_{t_i})\, \mathbb{X}^2_{t_i t_{i+1}}, \qquad i \geqslant 0.$$

We set $D := \max\{1, \|\nabla \sigma\|_\infty, \|\nabla \sigma_2\|_\infty\}$, $\mathbb{T} := \{t_i : t_i \leqslant T\}$ and

$$\begin{aligned} \delta y_{t_i t_j} &:= y_{t_j} - y_{t_i}, \\ \|\delta y\|^{\mathbb{T}}_\alpha &:= \sup_{0 < i < j \leqslant nT} \frac{|y_{t_j} - y_{t_i}|}{|t_j - t_i|^\alpha}, \\ A_{t_i t_j} &:= \sigma(y_{t_i})\, \mathbb{X}^1_{t_i t_j} + \sigma_2(y_{t_i})\, \mathbb{X}^2_{t_i t_j}. \end{aligned}$$

The main technical estimate is the following

Lemma 3.13. *Let $M > 0$. There exists $T_{M,D,\alpha} > 0$ such that, for all $T \in (0, T_{M,D,\alpha})$ and $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{\alpha,d}$ such that $\|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} \leqslant M$, we have*

$$\begin{aligned} \|\delta y\|^{\mathbb{T}}_\alpha &\leqslant 5M(|\sigma(y_0)| + |\sigma_2(y_0)|), \\ \|\delta y - A\|^{\mathbb{T}}_{3\alpha} &\lesssim_{M,D,\alpha} (|\sigma(y_0)| + |\sigma_2(y_0)|). \end{aligned}$$

Proof. Let us set $R_{t_i t_j} := \delta y_{t_i t_j} - A_{t_i t_j}$. By the definitions, $R_{t_i t_{i+1}} = 0$. Then we can apply the discrete Sewing bound (Theorem 1.18) to R on $\mathbb{T} := \left\{\frac{i}{n} : i \leqslant nT\right\}$ and we obtain

$$\|R\|^{\mathbb{T}}_{3\alpha} \leqslant C_{3\alpha} \|\delta R\|^{\mathbb{T}}_{3\alpha}, \qquad C_{3\alpha} = 2^{3\alpha} \sum_{n \geqslant 1} \frac{1}{n^{3\alpha}}.$$

Now, analogously to (3.26), since $\delta R = -\delta A$,

$$\delta R_{t_i t_j t_k} = -\underbrace{(\sigma(y_{t_j}) - \sigma(y_{t_i}) - \sigma_2(y_{t_i})\, \mathbb{X}^1_{t_i t_j})}_{B_{ij}} \mathbb{X}^1_{t_j t_k} - \underbrace{(\sigma_2(y_{t_i}) - \sigma_2(y_{t_j}))}_{C_{ij}}\, \mathbb{X}^2_{t_j t_k},$$

so that

$$\|\delta R\|^{\mathbb{T}}_{3\alpha} \leqslant M(\|B\|^{\mathbb{T}}_{2\alpha} + \|C\|^{\mathbb{T}}_\alpha).$$

We set

$$H_{t_i t_j} := \delta y_{t_i t_j} - \sigma(y_{t_i})\, \mathbb{X}^1_{t_i t_j},$$

and by (3.23) we obtain

$$\begin{aligned} &B_{t_i t_j} = \sigma(y_{t_j}) - \sigma(y_{t_i}) - \sigma_2(y_{t_i})\, \mathbb{X}^1_{t_i t_j} = \\ &= \underbrace{\int_0^1 (\sigma_2(y_{t_i} + u\,\delta y_{t_i t_j}) - \sigma_2(y_{t_i}))\, \mathbb{X}^1_{t_i t_j} \mathrm{d}u}_{E_{ij}} + \underbrace{\int_0^1 \nabla \sigma(y_{t_i} + u\,\delta y_{t_i t_j})\, \mathrm{d}u\, H_{t_i t_j}}_{F_{ij}} \\ &- \underbrace{\int_0^1 \nabla \sigma(y_{t_i} + u\,\delta y_{t_i t_j})(\sigma(y_{t_i} + u\,\delta y_{t_i t_j}) - \sigma(y_{t_i}))\mathbb{X}^1_{t_i t_j} \mathrm{d}u}_{G_{ij}}. \end{aligned}$$

First

$$\|E\|_{2\alpha}^{\mathbb{T}} \leqslant \|\nabla\sigma_2\|_\infty \|\delta y\|_\alpha^{\mathbb{T}} \|\mathbb{X}^1\|_\alpha \leqslant DM\|\delta y\|_\alpha^{\mathbb{T}}.$$

Similarly

$$\|G\|_{2\alpha}^{\mathbb{T}} \leqslant \|\nabla\sigma\|_\infty^2 \|\delta y\|_\alpha^{\mathbb{T}} \|\mathbb{X}^1\|_\alpha \leqslant D^2 M\|\delta y\|_\alpha^{\mathbb{T}}.$$

By the definition of $R_{t_i t_j}$

$$\begin{aligned} |H_{t_i t_j}| &\leqslant |R_{t_i t_j}| + |\sigma_2(y_{t_i})\,\mathbb{X}^2_{t_i t_j}| \\ &\leqslant [T^\alpha \|R\|_{3\alpha}^{\mathbb{T}} + (|\sigma_2(y_0)| + T^\alpha \|\nabla\sigma_2\|_\infty \|\delta y\|_\alpha^{\mathbb{T}})\|\mathbb{X}^2\|_{2\alpha}]\,|t_j - t_i|^{2\alpha} \\ &\leqslant (T^\alpha \|R\|_{3\alpha}^{\mathbb{T}} + M|\sigma_2(y_0)| + T^\alpha DM\|\delta y\|_\alpha^{\mathbb{T}})|t_j - t_i|^{2\alpha}. \end{aligned}$$

Therefore

$$\begin{aligned} \|F\|_{2\alpha}^{\mathbb{T}} &\leqslant D\|H\|_{2\alpha}^{\mathbb{T}} \\ &\leqslant D(T^\alpha \|R\|_{3\alpha}^{\mathbb{T}} + M|\sigma_2(y_0)| + T^\alpha DM\|\delta y\|_\alpha^{\mathbb{T}}). \end{aligned}$$

Finally

$$\begin{aligned} \|B\|_{2\alpha}^{\mathbb{T}} &\leqslant \|E\|_{2\alpha}^{\mathbb{T}} + \|F\|_{2\alpha}^{\mathbb{T}} + \|G\|_{2\alpha}^{\mathbb{T}} \\ &\leqslant D\,[M|\sigma_2(y_0)| + T^\alpha \|R\|_{3\alpha}^{\mathbb{T}} + DM(2 + T^\alpha)\|\delta y\|_\alpha^{\mathbb{T}}]\,. \end{aligned}$$

Analogously

$$\|C\|_{2\alpha}^{\mathbb{T}} \leqslant D\|\delta y\|_\alpha^{\mathbb{T}}.$$

Therefore

$$\|R\|_{3\alpha}^{\mathbb{T}} \leqslant C_{3\alpha} DM(M|\sigma_2(y_0)| + T^\alpha \|R\|_{3\alpha}^{\mathbb{T}} + [1 + DM(2 + T^\alpha)]\|\delta y\|_\alpha^{\mathbb{T}}).$$

If $T^\alpha C_{3\alpha} DM \leqslant \frac{1}{2}$ then

$$\|R\|_{3\alpha}^{\mathbb{T}} \leqslant 2C_{3\alpha} DM(M|\sigma_2(y_0)| + [1 + DM(2 + T^\alpha)]\|\delta y\|_\alpha^{\mathbb{T}}). \tag{3.81}$$

We set

$$L(y) := 2C_{3\alpha} DM(M|\sigma_2(y_0)| + [1 + DM(2 + T^\alpha)]\|\delta y\|_\alpha^{\mathbb{T}})$$

Now we obtain by (3.81)

$$\begin{aligned} \|\delta y\|_\alpha^{\mathbb{T}} &\leqslant \|R\|_\alpha^{\mathbb{T}} + \|A\|_\alpha^{\mathbb{T}} \\ &\leqslant T^{2\alpha} L(y) + (|\sigma(y_0)| + |\sigma_2(y_0)| + 2DT^\alpha \|\delta y\|_\alpha^{\mathbb{T}})M. \end{aligned}$$

If we assume also that $2DMT^\alpha \leqslant \frac{1}{2}$, we obtain

$$\|\delta y\|_\alpha \leqslant 2\,T^{2\alpha} L(y) + 2M(|\sigma(y_0)| + |\sigma_2(y_0)|).$$

By the definition of $L(y)$, if furthermore $2C_{3\alpha} DM[1 + DM(2 + T^\alpha)]\,T^{2\alpha} \leqslant \frac{1}{2}$, we obtain finally

$$\begin{aligned} \|\delta y\|_\alpha^{\mathbb{T}} &\leqslant 5M(|\sigma(y_0)| + |\sigma_2(y_0)|)\,, \\ L(y) &\leqslant 12C_{3\alpha} DM^2[1 + DM(2 + T^\alpha)](|\sigma(y_0)| + |\sigma_2(y_0)|) =: K, \end{aligned}$$

and by (3.81)

$$\|\delta y - A\|_{3\alpha}^{\mathbb{T}} \leqslant K.$$

The proof is complete. □

Proof of Proposition 3.6. Arguing as in Theorem 2.16 we obtain the result of local existence for equation (3.19) of Proposition 3.6. □

Chapter 4
Stochastic Differential Equations

In this chapter we connect the *rough difference equations (RDE)* discussed in the previous chapter, see (3.18), with the classical *stochastic differential equations (SDE)* $\mathrm{dY}_t = \sigma(Y_t)\,\mathrm{d}B_t$ driven by a Brownian motion B. Indeed, both RDE and SDE are ways to make sense of the ill-posed differential equation $\dot{Y}_t = \sigma(Y_t)\,\dot{B}_t$.

We fix a time horizon $T > 0$ and two dimensions $k, d \in \mathbb{N}$. Let $B = (B_t)_{t\in[0,T]}$ be a d-dimensional Brownian motion (with continuous paths) relative to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We fix a sufficiently regular function $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ and we consider a solution $Y = (Y_t)_{t\in[0,T]}$ of the SDE

$$\mathrm{dY}_t = \sigma(Y_t)\,\mathrm{d}B_t \qquad \text{i.e.} \qquad Y_t = Y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s\,, \qquad t \geqslant 0, \tag{4.1}$$

where the stochastic integral is in the Ito sense. *We always fix a version of Y with continuous paths* (we recall that the Ito integral is a continuous local martingale).

We want to show that Y solves a *rough difference equation* driven by the *rough path* $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ (see Definition 3.2) defined by

$$\mathbb{B}^1_{st} := B_t - B_s, \qquad \mathbb{B}^2_{st} := \int_s^t (B_r - B_s) \otimes \mathrm{d}B_r, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{4.2}$$

where the stochastic integral is in the Ito sense. More explicitly, for $i, j \in \{1, \ldots, d\}$

$$(\mathbb{B}^1_{st})^i := B^i_t - B^i_s\,, \qquad (\mathbb{B}^2_{st})^{ij} := \int_s^t (B^i_r - B^i_s)\,\mathrm{d}B^j_r\,, \tag{4.3}$$

where we write $B_t = (B^1_t, \ldots, B^d_t)$, so that $\mathbb{B}^1: [0,T]^2_{\leqslant} \to \mathbb{R}^d$ and $\mathbb{B}^2: [0,T]^2_{\leqslant} \to \mathbb{R}^d \otimes \mathbb{R}^d$.

Our first main result is that $(\mathbb{B}^1, \mathbb{B}^2)$ is indeed a rough path over B.

Theorem 4.1. (Ito rough path) *Almost surely, $\mathbb{B} := (\mathbb{B}^1, \mathbb{B}^2)$ is an α-rough path over B (see Definition 3.2) for any $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right[$.*

Our second main result is that, under suitable assumptions, the solution Y of the SDE (4.1) solves the RDE (3.18) driven by the Ito rough path $\mathbb{X} = \mathbb{B}$.

Theorem 4.2. (SDE & RDE) *If $\sigma(\cdot)$ is of class C^2, then almost surely a solution $Y = (Y_t)_{t\in[0,T]}$ of the SDE (4.1) is also a solution of the RDE*

$$\delta Y_{st} = \sigma(Y_s)\,\mathbb{B}^1_{st} + \sigma_2(Y_s)\,\mathbb{B}^2_{st} + o(t - s), \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{4.4}$$

(We recall that $\sigma_2(\cdot) := \nabla\sigma(\cdot)\,\sigma(\cdot)$ is defined in (3.5).)

If $\sigma(\cdot)$ is of class C^3 and, furthermore, $\sigma(\cdot)$ and $\sigma_2(\cdot)$ are globally Lipschitz, i.e. $\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty < \infty$, then almost surely both the SDE (4.1) and the RDE (4.4) admit a unique solution $Y = (Y_t)_{t\in[0,T]}$ and these solutions coincide.

The key tool we exploit in this chapter is a *local expansion of stochastic integrals*, see Theorem 4.3 in the next Section 4.1. The proofs of Theorems 4.1 and 4.2 are direct consequences of this result, see Section 4.2.

In Sections 4.3 and 4.4 we discuss useful generalizations of the SDE (4.1), where we add a drift and we allow for stochastic integration in the Stratonovich sense, which leads to generalized versions of Theorems 4.1 and 4.2.

In Section 4.5 we present the celebrated result by Wong-Zakai on the limit of solutions of the SDE (4.1) with a regularized Brownian motion (via convolution).

Finally, Section 4.6 is devoted to a far-reaching generalization of Kolmogorov's continuity criterion, which leads to the proof of Theorem 4.3 in Section 4.7.

NOTATION. *Throughout this chapter we write $f_{st} \lesssim g_{st}$ to mean that $f_{st} \leqslant C\, g_{st}$ for all $0 \leqslant s \leqslant t \leqslant T$, where $C < \infty$ is a suitable random constant.*

4.1. LOCAL EXPANSION OF STOCHASTIC INTEGRALS

We recall that $B = (B_t)_{t\in[0,T]}$ is a d-dimensional Brownian motion. Let $h = (h_t)_{t\in[0,T]}$ be a stochastic process with values in $\mathbb{R}^k \otimes (\mathbb{R}^d)^*$. We assume that h is adapted and has continuous paths, in particular $\int_0^T |h_s|^2\, \mathrm{d}s < \infty$, hence the Itô integral

$$I_t := I_0 + \int_0^t h_r\, \mathrm{d}B_r \tag{4.5}$$

is well-defined as a local martingale. It is a classical result that the stochastic process $I = (I_t)_{t\in[0,T]}$ admits a version with continuous paths, which we always fix.

We now state the main technical result of this chapter, proved in Section 4.7 below, which connects the regularity of h to the regularity of I.

THEOREM 4.3. (LOCAL EXPANSION OF STOCHASTIC INTEGRALS) *Let $h = (h_t)_{t\in[0,T]}$ be adapted with continuous paths. Fix any $\alpha \in \left]0, \frac{1}{2}\right[$ and recall $(\mathbb{B}^1, \mathbb{B}^2)$ from (4.2).*

1. *Almost surely I is of class $\mathcal{C}^\alpha$, i.e.*

$$|I_t - I_s| \lesssim (t-s)^\alpha, \qquad \forall 0 \leqslant s \leqslant t \leqslant T. \tag{4.6}$$

   (We recall that the implicit constant in the relation $\lesssim$ is random.)

2. *Assume that, almost surely, $|\delta h_{sr}| \lesssim (r-s)^\beta$ for some $\beta \in]0,1]$ (i.e. h is of class $\mathcal{C}^\beta$). Then, almost surely,*

$$|\delta I_{st} - h_s\, \mathbb{B}^1_{st}| = \left| \int_s^t \delta h_{sr}\, \mathrm{d}B_r \right| \lesssim (t-s)^{\alpha+\beta}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T. \tag{4.7}$$

3. *Assume that, almost surely, $|\delta h_{sr} - \tilde{h}_s\, \mathbb{B}^1_{sr}| \lesssim (r-s)^{\alpha+\gamma}$ for some $\gamma \in]0,1]$, where $\tilde{h} = (\tilde{h}_t)_{t\in[0,T]}$ is an adapted process of class $\mathcal{C}^\gamma$. Then, almost surely,*

$$\begin{aligned} |\delta I_{st} - h_s\, \mathbb{B}^1_{st} - \tilde{h}_s\, \mathbb{B}^2_{st}| &= \left| \int_s^t (\delta h_{sr} - \tilde{h}_s\, \mathbb{B}^1_{sr})\, \mathrm{d}B_r \right| \\ &\lesssim (t-s)^{2\alpha+\gamma}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T. \end{aligned} \tag{4.8}$$

4.2. Brownian rough path and SDE

In this section we exploit Theorem 4.3 to prove Theorems 4.1 and 4.2.

Proof. (of Theorem 4.1) We need to verify that $\mathbb{B}=(\mathbb{B}^1,\mathbb{B}^2)$ satisfies the Chen relation (3.13) and the analytic bounds (3.14).

The Chen relation $\delta\mathbb{B}^2_{sut}=\mathbb{B}^1_{su}\otimes\mathbb{B}^1_{ut}$ for $0\leqslant s\leqslant u\leqslant t\leqslant T$ holds by (4.3):

$$\begin{aligned}\delta(\mathbb{B}^2)^{ij}_{sut} &= (\mathbb{B}^2)^{ij}_{st}-(\mathbb{B}^2)^{ij}_{su}-(\mathbb{B}^2)^{ij}_{ut}\\ &= \int_s^t (B^i_r-B^i_s)\,\mathrm{d}B^j_r-\int_s^u (B^i_r-B^i_s)\,\mathrm{d}B^j_r-\int_u^t (B^i_r-B^i_u)\,\mathrm{d}B^j_r\\ &= \int_u^t (B^i_u-B^i_s)\,\mathrm{d}B^j_r=(B^i_u-B^i_s)\int_u^t 1\,\mathrm{d}B^j_r=(B^i_u-B^i_s)(B^j_t-B^j_u),\end{aligned}$$

by the properties of the Itô integral and the fact that the times $s\leqslant u\leqslant t$ are ordered.

The first analytic bound $|\mathbb{B}^1_{st}|\lesssim|t-s|^\alpha$ for $\alpha\in\left]0,\frac{1}{2}\right[$ is a well-known almost sure property of Brownian motion, which also follows from Theorem 4.3, applying (4.6) with $h\equiv 1$. Finally, the second analytic bound $|\mathbb{B}^2_{st}|\lesssim|t-s|^{2\alpha}$ is also a consequence of Theorem 4.3: it suffices to apply (4.7) with $h_s:=B_s$ and $\beta=\alpha$. □

Proof. (Theorem 4.2) We first prove the second part of the statement.

- When σ is globally Lipschitz ($\|\nabla\sigma\|_\infty<+\infty$), it is a classical result that for the SDE (4.1) there is existence of strong solutions and pathwise uniqueness.
- When σ is of class C^3, by Theorem 3.10 there is uniqueness of solutions for the RDE (3.19), and if both σ and σ_2 are globally Lipschitz ($\|\nabla\sigma\|_\infty<+\infty$ and $\|\nabla\sigma_2\|_\infty<+\infty$) there is also existence of solutions, by Theorem 3.12.

Therefore we only need to prove the first part of the statement: we assume that σ is of class C^2 and we show that given a solution $Y=(Y_t)_{t\in[0,T]}$ of the SDE (4.1), almost surely Y is also a solution to the RDE (4.4).

Since Y is solution to (4.1), recalling (4.2) we can write

$$\begin{aligned}\delta Y_{st}-\sigma(Y_s)\,\mathbb{B}^1_{st}-\sigma_2(Y_s)\,\mathbb{B}^2_{st} &= \int_s^t(\sigma(Y_r)-\sigma(Y_s))\,\mathrm{d}B_r-\sigma_2(Y_s)\int_s^t(B_r-B_s)\,\mathrm{d}B_r\\ &= \int_s^t(\delta\sigma(Y)_{sr}-\sigma_2(Y_s)\,\mathbb{B}^1_{sr})\,\mathrm{d}B_r\,.\end{aligned}$$

Let us fix $\alpha\in\left]0,\frac{1}{2}\right[$. We prove below that, almost surely,

$$|\delta\sigma(Y)_{st}-\sigma_2(Y_s)\mathbb{B}^1_{st}|\lesssim(t-s)^{2\alpha},\qquad\forall 0\leqslant s\leqslant t\leqslant T. \tag{4.9}$$

This means that the assumptions of part 3 of Theorem 4.3 are satisfied by $h_r=\sigma(Y_r)$ and $\tilde{h}_r=\sigma_2(Y_r)$ with $\gamma=\alpha$: applying (4.8) we then obtain, almost surely,

$$|\delta Y_{st}-\sigma(Y_s)\,\mathbb{B}^1_{st}-\sigma_2(Y_s)\,\mathbb{B}^2_{st}|\lesssim(t-s)^{3\alpha}.$$

If we fix $\alpha>\frac{1}{3}$, this shows that Y is indeed a solution of the RDE (4.4).

It remains to prove (4.9). By Itô's formula and (4.1) we have, for $0 \leqslant s < t \leqslant T$,

$$\begin{aligned}
\sigma(Y_t) &= \sigma(Y_s) + \int_s^t \sum_{a=1}^k \partial_a \sigma(Y_r)\, \mathrm{d}Y_r^a + \frac{1}{2}\int_s^t \sum_{a,b=1}^k \partial_{ab}\sigma(Y_r)\, \mathrm{d}\langle Y^a, Y^b\rangle_r \\
&= \sigma(Y_s) + \int_s^t \sum_{a=1}^k \partial_a \sigma(Y_r) \sum_{c=1}^d \sigma_c^a(Y_r)\, \mathrm{d}B_r^c + \\
&\quad + \int_s^t \underbrace{\frac{1}{2}\sum_{a,b=1}^k \sum_{c=1}^d \partial_{ab}\sigma\,(Y_r)\,\sigma_c^a(Y_r)\,\sigma_c^b(Y_r)}_{p(Y_r)}\, \mathrm{d}r \\
&= \sigma(Y_s) + \int_s^t \sigma_2(Y_r)\, \mathrm{d}B_r + \int_s^t p(Y_r)\, \mathrm{d}r,
\end{aligned} \tag{4.10}$$

therefore

$$\delta\sigma(Y)_{st} - \sigma_2(Y_s)\,\mathbb{B}_{st}^1 = \int_s^t (\sigma_2(Y_r) - \sigma_2(Y_s))\, \mathrm{d}B_r + \int_s^t p(Y_r)\, \mathrm{d}r.$$

To prove (4.9), we show that both integrals in the RHS are $O((t-s)^{2\alpha})$.

- Since σ is of class C^2 and Y has continuous paths, the random function $r \mapsto p(Y_r)$ is continuous, hence bounded for $r \in [0, T]$, therefore

$$\left|\int_s^t p(Y_r)\, \mathrm{d}r\right| \lesssim (t-s) \lesssim (t-s)^{2\alpha}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T.$$

- Almost surely Y is of class $\mathcal{C}^\alpha$, thanks to (4.6) from Theorem 4.3 and (4.1). Since σ_2 is of class C^1, hence locally Lipschitz, $r \mapsto \sigma_2(Y_r)$ is of class $\mathcal{C}^\alpha$ too. Applying (4.7) from Theorem 4.3 we then obtain, almost surely,

$$\left|\int_s^t (\sigma_2(Y_r) - \sigma_2(Y_s))\, \mathrm{d}B_r\right| \lesssim (t-s)^{2\alpha}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T.$$

This completes the proof. □

4.3. SDE with a drift

It is natural to consider the SDE (4.1) with a non-zero drift term:

$$\mathrm{d}Y_t = b(Y_t)\, \mathrm{d}t + \sigma(Y_t)\, \mathrm{d}B_t \qquad \text{i.e.}$$

$$Y_t = Y_0 + \int_0^t b(Y_s)\, \mathrm{d}s + \int_0^t \sigma(Y_s)\, \mathrm{d}B_s, \qquad t \geqslant 0, \tag{4.11}$$

where $b\colon \mathbb{R}^k \to \mathbb{R}^k$ and $\sigma\colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ are given and we recall that $B = (B_t)_{t \geqslant 0}$ is a d-dimensional Brownian motion. We can generalize Theorem 4.2 as follows.

THEOREM 4.4. (SDE & RDE WITH DRIFT) *If $\sigma(\cdot)$ is of class C^2 and $b(\cdot)$ is continuous, then almost surely a solution $Y = (Y_t)_{t \in [0,T]}$ of the SDE (4.11) is also a solution of the RDE*

$$\delta Y_{st} = b(Y_s)\,(t-s) + \sigma(Y_s)\,\mathbb{B}_{st}^1 + \sigma_2(Y_s)\,\mathbb{B}_{st}^2 + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{4.12}$$

If $\sigma(\cdot)$ and $b(\cdot)$ are of class C^3 and, furthermore, $\sigma(\cdot)$, $\sigma_2(\cdot)$ and $b(\cdot)$ are globally Lipschitz, i.e. $\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty + \|\nabla b\|_\infty < \infty$, almost surely the SDE (4.11) and the RDE (4.12) have a unique solution $Y = (Y_t)_{t\in[0,T]}$ and these solutions coincide.

Proof. We cast the generalized SDE (4.11) in the "usual framework" by adding a component to the driving noise B, i.e. we define $\tilde{B}\colon [0,T] \to \mathbb{R}^d \times \mathbb{R}$ by

$$\tilde{B}_t := (B_t, t) = (B_t^1, \ldots, B_t^d, t), \qquad t \in [0,T],$$

and accordingly we define $\tilde{\sigma}\colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^{d+1})^*$ by

$$\tilde{\sigma}(\cdot)\,\tilde{b} := \sigma(\cdot)\,b + b(\cdot)\,t \qquad \text{for} \qquad \tilde{b} = (b,t) \in \mathbb{R}^d \times \mathbb{R},$$

that is $\tilde{\sigma}(\cdot)^i_j = \sigma(\cdot)^i_j\,\mathbb{1}_{\{j\leqslant d\}} + b(\cdot)^i\,\mathbb{1}_{\{j=d+1\}}$. We can then rewrite the SDE (4.11) as

$$\mathrm{d}Y_t = \tilde{\sigma}(Y_t)\,\mathrm{d}\tilde{B}_t \qquad \text{i.e.} \qquad Y_t = Y_0 + \int_0^t \tilde{\sigma}(Y_s)\,\mathrm{d}\tilde{B}_s, \qquad t \geqslant 0. \tag{4.13}$$

We next extend the Ito rough path $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ from (4.2), defining

$$\tilde{\mathbb{B}}^1_{st} := \tilde{B}_t - \tilde{B}_s = \begin{pmatrix} \mathbb{B}^1_{st} \\ t-s \end{pmatrix}, \tag{4.14}$$

$$\tilde{\mathbb{B}}^2_{st} := \int_s^t (\tilde{B}_r - \tilde{B}_s) \otimes \mathrm{d}\tilde{B}_r = \begin{pmatrix} \mathbb{B}^2_{st} & \int_s^t (B_r - B_s)\,\mathrm{d}r \\ \int_s^t (r-s)\,\mathrm{d}B_r & \int_s^t (r-s)\,\mathrm{d}r = \frac{(t-s)^2}{2} \end{pmatrix}. \tag{4.15}$$

One can show that $\tilde{\mathbb{B}} = (\tilde{\mathbb{B}}^1, \tilde{\mathbb{B}}^2)$ *is a rough path over* $\tilde{B}$, following closely the proof of Theorem 4.1. Indeed, if we fix $\alpha \in \left]0, \frac{1}{2}\right[$, we have almost surely $B \in \mathcal{C}^\alpha$, hence

$$\left|\int_s^t (B_r - B_s)\,\mathrm{d}r\right| \lesssim (t-s)^{\alpha+1}, \qquad \left|\int_s^t (r-s)\,\mathrm{d}B_r\right| \lesssim (t-s)^{\alpha+1}. \tag{4.16}$$

We can now write the RDE which generalizes (4.4):

$$\delta Y_{st} = \tilde{\sigma}(Y_s)\,\tilde{\mathbb{B}}^1_{st} + \tilde{\sigma}_2(Y_s)\,\tilde{\mathbb{B}}^2_{st} + o(t-s)\,. \tag{4.17}$$

Interestingly, plugging the definitions of $\tilde{\mathbb{B}}$ and $\tilde{\sigma}$ into (4.17) *we do not obtain (4.12)*, because the components of $\tilde{\mathbb{B}}^2_{st}$ other than $\mathbb{B}^2_{st}$ are missing in (4.12), see (4.15). The point is that these components can be absorbed in the reminder $o(t-s)$, see (4.16), hence *the RDE (4.17) and (4.12) are fully equivalent*.

To complete the proof, we are left with comparing the SDE (4.13) with the RDE (4.17). This can be done following the very same arguments as in the proof of Theorem 4.2. The details are left to the reader. □

Remark 4.5. The strategy of adding the drift term as an additional component of the driving noise, as in the proof of Theorem 4.4, suffers from a technical limitation, namely *we are forced to use the same regularity exponent α for all components* due to Definition 3.2 of rough paths. This prevents us from exploiting the additional regularity of the drift term: for instance, in the second part of Theorem 4.4, the assumption that $b(\cdot)$ is of class C^3 could be removed, because the "driving noise" t is smooth and the classical theory of ordinary differential equations applies.

A natural solution would be to generalize Definition 3.2, allowing rough paths to have a different regularity exponent for each component. The key results can be generalized to this setting, but for simplicity we refrain from pursuing this path.

4.4. Itô versus Stratonovich

We recall that $B=(B_t)_{t\in[0,T]}$ is a Brownian motion in $\mathbb{R}^d$. Given the Itô rough path $\mathbb{B}=(\mathbb{B}^1,\mathbb{B}^2)$ over B constructed in Theorem 4.2, see (4.2), we can define a new rough path $\bar{\mathbb{B}}=(\bar{\mathbb{B}}^1,\bar{\mathbb{B}}^2)$ over B, called the *Stratonovich rough path*, given by

$$\bar{\mathbb{B}}^1_{st}:=\mathbb{B}^1_{st}, \qquad \bar{\mathbb{B}}^2_{st}:=\mathbb{B}^2_{st}+\frac{t-s}{2}\,\mathrm{Id}_{\mathbb{R}^d}, \qquad \forall 0\leqslant s\leqslant t\leqslant T,$$

that is $(\bar{\mathbb{B}}^2_{st})^{ij}:=(\mathbb{B}^2_{st})^{ij}+\frac{t-s}{2}\,\mathbb{1}_{\{i=j\}}$ for $i,j\in\{1,\ldots,d\}$. The fact that $\bar{\mathbb{B}}$ is indeed an α-rough path over B, for any $\alpha\in\left]\frac{1}{3},\frac{1}{2}\right[$, is a direct consequence of Theorem 4.1 (note that $\bar{\mathbb{B}}^2_{st}=\mathbb{B}^2_{st}+\delta f_{st}$ with $f_t=\frac{t}{2}\,\mathrm{Id}_{\mathbb{R}^d}$, hence $\delta\bar{\mathbb{B}}^2=\delta\mathbb{B}^2$ because $\delta^2=0$).

Remark 4.6. (Stratonovich integral) If $X,Y\colon[0,T]\to\mathbb{R}$ are continuous semimartingales, the Stratonovich integral of X with respecto to Y is defined by

$$\int_0^t X_s\circ \mathrm{d}Y_s:=\int_0^t X_s\,\mathrm{d}Y_s+\frac{1}{2}\langle X,Y\rangle_t, \qquad t\in[0,T], \tag{4.18}$$

where $\int_0^t X_s\,\mathrm{d}Y_s$ is the Itô integral and $\langle\cdot,\cdot\rangle$ is the quadratic covariation. For Brownian motion B on $\mathbb{R}^d$ we have $\langle B^i,B^j\rangle_t=t\,\mathbb{1}_{\{i=j\}}$, hence it is easy to check by (4.2) that

$$\bar{\mathbb{B}}^2_{st}:=\int_s^t \bar{\mathbb{B}}^1_{sr}\otimes\circ\mathrm{d}B_r, \qquad 0\leqslant s\leqslant t\leqslant T. \tag{4.19}$$

This explains why we call $\bar{\mathbb{B}}=(\bar{\mathbb{B}}^1,\bar{\mathbb{B}}^2)$ the Stratonovich rough path.

Let us consider now the Stratonovich version of the SDE (4.11):

$$\mathrm{d}Y_t=b(Y_t)\,\mathrm{d}t+\sigma(Y_t)\circ\mathrm{d}B_t \qquad \text{i.e.}$$

$$Y_t=Y_0+\int_0^t b(Y_s)\,\mathrm{d}s+\int_0^t\sigma(Y_s)\circ\mathrm{d}B_s, \qquad t\geqslant 0, \tag{4.20}$$

where $b\colon\mathbb{R}^k\to\mathbb{R}^k$ and $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ are given. This equation can be recast in the Itô form by the conversion rule (4.18): since the martingale part of $(\sigma(Y_t))_{t\geqslant 0}$ is $(\int_0^t\sigma_2(Y_s)\mathrm{d}B_s)_{t\geqslant 0}$ by the Itô formula, see (4.10), we obtain

$$Y_t=Y_0+\int_0^t\left(b(Y_s)+\frac{1}{2}\mathrm{Tr}_{\mathbb{R}^d}[\sigma_2(Y_s)]\right)\mathrm{d}s+\int_0^t\sigma(Y_s)\,\mathrm{d}B_s, \qquad t\geqslant 0.$$

This is precisely the SDE (4.11) with a different drift $\hat{b}(\cdot):=b(\cdot)+\frac{1}{2}\mathrm{Tr}_{\mathbb{R}^d}[\sigma_2(\cdot)]$.

As an immediate corollary of Theorem 4.4, we obtain the following result.

THEOREM 4.7. (STRATONOVICH SDE & RDE) *f $\sigma(\cdot)$ is of class C^2 and $b(\cdot)$ is continuous, then almost surely a solution $Y=(Y_t)_{t\in[0,T]}$ of the Stratonovich SDE (4.20) is also a solution of the following RDE, for $0\leqslant s\leqslant t\leqslant T$:*

$$\begin{aligned}\delta Y_{st} &= b(Y_s)\,(t-s)+\sigma(Y_s)\,\bar{\mathbb{B}}^1_{st}+\sigma_2(Y_s)\,\bar{\mathbb{B}}^2_{st}+o(t-s) \\ &= \left(b(Y_s)+\frac{1}{2}\,\mathrm{Tr}_{\mathbb{R}^d}[\sigma_2(Y_s)]\right)(t-s)+\sigma(Y_s)\,\mathbb{B}^1_{st}+\sigma_2(Y_s)\,\mathbb{B}^2_{st}+o(t-s).\end{aligned} \tag{4.21}$$

If $\sigma(\cdot)$, $\sigma_2(\cdot)$, $b(\cdot)$ are of class C^3 and, furthermore, $\sigma(\cdot)$, $\sigma_2(\cdot)$, $b(\cdot)$ are globally Lipschitz, i.e. $\|\nabla\sigma\|_\infty+\|\nabla\sigma_2\|_\infty+\|\nabla b\|_\infty<\infty$, almost surely the SDE (4.20) and the RDE (4.21) have a unique solution $Y=(Y_t)_{t\in[0,T]}$ and these solutions coincide.

In conclusion, if the coefficients $b(\cdot)$ and $\sigma(\cdot)$ are sufficiently regular, the Itô equation (4.11) can be reintepreted as the RDE

$$\delta Y_{st}=b(Y_s)\,(t-s)+\sigma(Y_s)\,\mathbb{B}^1_{st}+\sigma_2(Y_s)\,\mathbb{B}^2_{st}+o(t-s),\qquad 0\leqslant s\leqslant t\leqslant T,$$

while the Stratonovich equation (4.20) can be reintepreted as the RDE

$$\delta Y_{st}=b(Y_s)\,(t-s)+\sigma(Y_s)\,\bar{\mathbb{B}}^1_{st}+\sigma_2(Y_s)\,\bar{\mathbb{B}}^2_{st}+o(t-s),\qquad 0\leqslant s\leqslant t\leqslant T.$$

In other words, rough paths allow to describe the Itô and the Stratonovich SDEs as *the same equation* where only the second level of the rough path has been changed. This shows that, in a sense, the *relevant noise* for a SDE is not only the Brownian path $(B_t)_{t\geqslant 0}$, but rather the rough path $\mathbb{B}$ or $\bar{\mathbb{B}}$.

4.5. WONG-ZAKAI

In this section we want to show the following application of the previous results. We consider a family $(\rho_\varepsilon)_{\varepsilon>0}$ of (even, compactly supported) mollifiers on $\mathbb{R}$, namely ρ: $\mathbb{R}\to[0,\infty)$ is smooth and even, has compact support, satisfies $\int_{\mathbb{R}}\rho(x)\,\mathrm{d}x=1$ and we set

$$\rho_\varepsilon(x):=\frac{1}{\varepsilon}\,\rho\left(\frac{x}{\varepsilon}\right),\qquad \varepsilon>0, x\in\mathbb{R}.$$

We consider a d-dimensional two-sided Brownian motion $(B_t)_{t\in\mathbb{R}}$, namely a Gaussian centered process with values in $\mathbb{R}^d$ such that

$$B_0=0,\qquad \mathbb{E}[B^i_s\,B^j_t]=\mathbb{1}_{(i=j)}\,\mathbb{1}_{(st\geqslant 0)}\,(|s|\wedge|t|),$$

which is equivalent to say that $(B_t)_{t\geqslant 0}$ and $(B_{-t})_{t\geqslant 0}$ are two independent d-dimensional Brownian motions.

We consider the following problem: we define the regularization of $(B_t)_{t\geqslant 0}$ defined by

$$B^\varepsilon_t:=(\rho_\varepsilon*B)_t=\int_{\mathbb{R}}\rho_\varepsilon(t-s)\,B_s\,\mathrm{d}s,\qquad t\geqslant 0.$$

We want now to consider the integral equation (3.3) controlled by B^ε, namely

$$Z^\varepsilon_t=Z_0+\int_0^t\sigma(Z^\varepsilon_s)\,\dot{B}^\varepsilon_s\,\mathrm{d}s,\qquad 0\leqslant t\leqslant T. \tag{4.22}$$

It is well known that $(B^\varepsilon_t)_{t\geqslant 0}$ converges to $(B_t)_{t\geqslant 0}$: then we want to understand whether $(Z^\varepsilon_t)_{t\geqslant 0}$ also converges, and especially to which limit.

This question has a very natural answer in the context of rough paths. We define the *canonical rough path* over B^ε (see section 7.7 below for more on this notion):

$$\mathbb{B}^{\varepsilon,1}_{st} := B^\varepsilon_t - B^\varepsilon_s, \qquad \mathbb{B}^{\varepsilon,2}_{st} := \int_s^t \mathbb{B}^{\varepsilon,1}_{su} \otimes \dot{B}^\varepsilon_s \,\mathrm{d}s, \qquad 0 \leqslant s \leqslant t.$$

We suppose now that $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is of class C^3, with $\|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty + \|\nabla\sigma_2\|_\infty + \|\nabla^2\sigma_2\|_\infty < +\infty$, as in Section 3.7. Then we can prove the following result.

Theorem 4.8. *A.s. $\mathbb{B}^\varepsilon$ converges to the Stratonovich rough path $\bar{\mathbb{B}}$, namely for any $\alpha < \frac{1}{2}$*

$$\lim_{\varepsilon\downarrow 0} (\|\mathbb{B}^{\varepsilon,1} - \bar{\mathbb{B}}^1\|_\alpha + \|\mathbb{B}^{\varepsilon,2} - \bar{\mathbb{B}}^2\|_{2\alpha}) = 0. \tag{4.23}$$

Moreover let $(Z^\varepsilon_t)_{t\in[0,T]}$ be the solution to the controlled equation

$$Z^\varepsilon_t = Z_0 + \int_0^t \sigma(Z^\varepsilon_s)\, \dot{B}^\varepsilon_s \,\mathrm{d}s, \qquad t \geqslant 0.$$

Then for all $\alpha \in \left(0, \frac{1}{2}\right)$ a.s. $Z^\varepsilon \to Z$ in $C^\alpha([0,T];\mathbb{R}^k)$ as $\varepsilon\downarrow 0$, where Z is the unique solution to the Stratonovich SDE

$$Z_t = Z_0 + \int_0^t \sigma(Z_s) \circ \mathrm{d}B_s = Z_0 + \int_0^t \sigma(Z_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \mathrm{Tr}_{\mathbb{R}^d}[\sigma_2(Z_s)]\,\mathrm{d}s.$$

Proof. Fix $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Let $\mathbb{B}^\varepsilon$ be the canonical smooth rough path associated with B^ε as in (3.9). Suppose we have proved that $\mathbb{B}^\varepsilon$ converges to $\bar{\mathbb{B}}$ as in (4.23). By Proposition 3.5, the solution Z^ε to the controlled equation (4.22) is equal to the (unique by Theorem 3.10) solution to the rough finite difference equation (3.19) associated with the α-rough path $\mathbb{B}^\varepsilon$. In the notation (3.51), we have $Z^\varepsilon = \Phi(Z_0, \mathbb{B}^\varepsilon)$, and by Theorem 4.7 we have $Z = \Phi(Z_0, \bar{\mathbb{B}})$. By the continuity result Theorem 3.11 we obtain that $Z^\varepsilon = \Phi(Z_0, \mathbb{B}^\varepsilon) \to \Phi(Z_0, \bar{\mathbb{B}}) = Z$ a.s. as $\varepsilon\downarrow 0$.

It remains now to prove (4.23). We consider $i, j \in \{1, \ldots, d\}$ with $i \neq j$ and we set $(X, Y) := (B^i, B^j)$. Let Q be a real-valued random variable with density ρ, so that

$$R_\varepsilon(t) := \int_{-\infty}^t \rho_\varepsilon(u)\,\mathrm{d}u = \mathbb{P}(\varepsilon Q \leqslant t), \qquad t \in \mathbb{R}.$$

Setting $(X^\varepsilon_t, Y^\varepsilon_t) := ((\rho_\varepsilon * X)_t, (\rho_\varepsilon * Y)_t) - ((\rho_\varepsilon * X)_0, (\rho_\varepsilon * Y)_0)$, we have for $0 \le s \le t$

$$\begin{aligned} \delta X^\varepsilon_{st} &:= \int (\rho_\varepsilon(t-v) - \rho_\varepsilon(s-v))\, X_v \,\mathrm{d}v = \\ &= \int (R_\varepsilon(t-v) - R_\varepsilon(s-v))\,\mathrm{d}X_v, \\ \dot{Y}_\varepsilon(t) &:= \int (\rho_\varepsilon)'(t-w)\; Y_w \,\mathrm{d}w = \int \rho_\varepsilon(t-w)\,\mathrm{d}Y_w. \end{aligned}$$

We want to show first that $\|\delta X^\varepsilon - \delta X\|_\alpha \to 0$ a.s. for any $\alpha < \frac{1}{2}$. We have

$$\begin{aligned}\delta X^\varepsilon_{st} - \delta X_{st} &= \int (R_\varepsilon(t-v) - R_\varepsilon(s-v) - \mathbb{1}_{(s\leqslant v\leqslant t)})\,\mathrm{d}X_v\\ &= \int (\mathbb{P}(s\leqslant \varepsilon Q + v\leqslant t) - \mathbb{1}_{(s\leqslant v\leqslant t)})\,\mathrm{d}X_v\end{aligned}$$

and setting $\delta := t - s \geqslant 0$

$$\begin{aligned}\mathbb{E}[(\delta X^\varepsilon_{st} - \delta X_{st})^2] &= \int (\mathbb{P}(s\leqslant \varepsilon Q + v\leqslant t) - \mathbb{1}_{(s\leqslant v\leqslant t)})^2\,\mathrm{d}v\\ &= \delta\int \left(\mathbb{E}\left[\mathbb{1}_{(0\leqslant \frac{\varepsilon}{\delta}Q + v\leqslant 1)} - \mathbb{1}_{(0\leqslant v\leqslant 1)}\right]\right)^2\mathrm{d}v\\ &\leqslant \delta\int \mathbb{E}\left[\left(\mathbb{1}_{(0\leqslant \frac{\varepsilon}{\delta}Q + v\leqslant 1)} - \mathbb{1}_{(0\leqslant v\leqslant 1)}\right)^2\right]\mathrm{d}v\\ &= \delta\,\mathbb{E}\left[\left|\left([0,1] - \frac{\varepsilon}{\delta}Q\right)\triangle[0,1]\right|\right],\end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure and $\triangle$ the symmetric difference between the two sets. Now we have for $y \in \mathbb{R}$

$$|([0,1] - y)\triangle[0,1]| \leqslant 2(1\wedge |y|)$$

and therefore

$$\begin{aligned}\mathbb{E}[(\delta X^\varepsilon_{st} - \delta X_{st})^2] &\leqslant 2\delta\,\mathbb{E}\left[1\wedge\left(\frac{\varepsilon}{\delta}|Q|\right)\right]\\ &\leqslant C_\kappa\,\delta^{1-\kappa}\varepsilon^\kappa, \qquad C_\kappa := 2\sup_{\lambda>0}\lambda^{-\kappa}\mathbb{E}[1\wedge(\lambda|Q|)] < +\infty.\end{aligned}$$

Now we prove that $\|\mathbb{B}^{\varepsilon,2} - \bar{\mathbb{B}}^2\|_{2\alpha} \to 0$ a.s. for all $\alpha < \frac{1}{2}$. We define for $0 \leq s \leq t$ the processes

$$\begin{aligned}L_{st} &:= \int_s^t \delta X_{su}\,\mathrm{d}Y_u = \int_s^t \mathrm{d}Y_w \int_s^w \mathrm{d}X_v,\\ L^\varepsilon_{st} &:= \int_s^t \delta X^\varepsilon_{su}\,\dot{Y}^\varepsilon_u\,\mathrm{d}u =\\ &= \int_s^t \mathrm{d}u \int \rho_\varepsilon(u-w)\,\mathrm{d}Y_w \int (R_\varepsilon(u-v) - R_\varepsilon(s-v))\,\mathrm{d}X_v\\ &= \int \mathrm{d}Y_w \int \mathrm{d}X_v \int_s^t \rho_\varepsilon(u-w)\,(R_\varepsilon(u-v) - R_\varepsilon(s-v))\,\mathrm{d}u.\end{aligned}$$

We want to show that $L^\varepsilon \to L$ in an appropriate sense as $\varepsilon \to 0$, namely

$$\lim_{\varepsilon\downarrow 0}\sup_{s,t\in[0,T],s\neq t}\frac{|L^\varepsilon_{st} - L_{st}|}{|t-s|^{2\alpha}} = 0.$$

We start by showing that $\mathbb{E}((L^{\varepsilon}_{st}-L_{st})^2)\to 0$ as $\varepsilon\to 0$. We have

$$\begin{aligned} L^{\varepsilon}_{st}-L_{st} &= \iint g(v,w)\,\mathrm{d}X_v\,\mathrm{d}Y_w,\\ g(v,w) &:= \int_s^t \rho_{\varepsilon}(u-w)\,(R_{\varepsilon}(u-v)-R_{\varepsilon}(s-v))\,\mathrm{d}u-\mathbb{1}_{(s\le v\le w\le t)}\\ &= \mathbb{P}(s\le\varepsilon Q_1+v\le\varepsilon Q_2+w\leqslant t)-\mathbb{1}_{(s\le v\le w\le t)} \end{aligned}$$

where (Q_1,Q_2) is an independent pair such that Q_i has density ρ. Setting $\delta:=t-s$

$$\begin{aligned} \mathbb{E}((L^{\varepsilon}_{st}-L_{st})^2) &= \int g^2(v,w)\,\mathrm{d}v\,\mathrm{d}w\\ &= \iint \mathrm{d}v\,\mathrm{d}w(\mathbb{P}(s\le\varepsilon Q_1+v\le\varepsilon Q_2+w\leqslant t)-\mathbb{1}_{(s\le v\le w\le t)})^2\\ &= \delta^2\iint \mathrm{d}v\,\mathrm{d}w\left(\mathbb{P}\Big(0\le\frac{\varepsilon}{\delta}Q_1+v\le\frac{\varepsilon}{\delta}Q_2+w\leqslant 1\Big)-\mathbb{1}_{(0\le v\le w\le 1)}\right)^2\\ &= \delta^2\iint \mathrm{d}v\,\mathrm{d}w\,(\mathbb{E}[\mathbb{1}_{T-\frac{\varepsilon}{\delta}(Q_1,Q_2)}(v,w)-\mathbb{1}_T(v,w)])^2 \end{aligned}$$

where $T:=\{0\le v\le w\le 1\}$. Now we obtain

$$\begin{aligned} \mathbb{E}((L^{\varepsilon}_{st}-L_{st})^2) &\leqslant \delta^2\iint \mathrm{d}v\,\mathrm{d}w\,\mathbb{E}[(\mathbb{1}_{T-\frac{\varepsilon}{\delta}(Q_1,Q_2)}(v,w)-\mathbb{1}_T(v,w))^2]\\ &= \delta^2\iint \mathrm{d}v\,\mathrm{d}w\,\mathbb{E}[\mathbb{1}_{(T-\frac{\varepsilon}{\delta}(Q_1,Q_2))\triangle T}(v,w)]\\ &= \delta^2\,\mathbb{E}\Big[\Big|\Big(T-\frac{\varepsilon}{\delta}(Q_1,Q_2)\Big)\triangle T\Big|\Big], \end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^2$. Now for all $y\in\mathbb{R}^2$, the set $(T-y)\triangle T$ is included in the set

$$\{z\in\mathbb{R}^2:\operatorname{dist}(z,\partial T)\leqslant|y|\}$$

where ∂T is the boundary of T. Since the length of ∂T is $2+\sqrt{2}\leqslant 4$, the area of $\{z\in\mathbb{R}^2:\operatorname{dist}(z,\partial T)\leqslant|y|\}$ is bounded above by $8|y|$. At the same time the same area is at most the sum of the areas of the two triangles $T-y$ and T, namely 1. Therefore for $x\geqslant 0$

$$f(x):=\mathbb{E}[|(T-x(Q_1,Q_2))\triangle T|]\leqslant\mathbb{E}[1\wedge(8x|(Q_1,Q_2)|)]$$

and then for any $\kappa>0$

$$\mathbb{E}((L^{\varepsilon}_{st}-L_{st})^2)=\delta^2 f(\varepsilon/\delta)\leqslant C_{\kappa}\,\delta^{2-\kappa}\varepsilon^{\kappa},$$

where $C_{\kappa}:=\sup_{\lambda>0}\lambda^{-\kappa}f(\lambda)<+\infty$.

Since for any $1<p<\infty$ the L^p and the L^2 norms are equivalent on a homogeneous Wiener chaos, and $L^{\varepsilon}_{st}-L_{st}$ belongs to such a space of order 2, we obtain that for any $p>1$

$$\mathbb{E}(|L^{\varepsilon}_{st}-L_{st}|^p)\le C_{p,\kappa}\,\delta^{p\left(1-\frac{\kappa}{2}\right)}\,\varepsilon^{p\kappa}.$$

Therefore if we set $A_{st} := L^{\varepsilon}_{st} - L_{st}$ in (4.25), we obtain $Q_{2\alpha} < +\infty$ a.s for any $\alpha < \frac{1}{2}$ (take $p \geqslant 1, \kappa > 0$ such that $2\,\alpha < 1 - \frac{\kappa}{2} - \frac{1}{p}$).

Now we estimate the constant $K_{2\alpha,\alpha,\alpha}$ in (4.26): since

$$\delta\, A_{sut} = \delta\, X^{\varepsilon}_{su}\, \delta\, Y^{\varepsilon}_{ut} - \delta\, X_{su}\, \delta\, Y_{ut} = \delta\, X^{\varepsilon}_{su}\, (\delta\, Y^{\varepsilon}_{ut} - \delta\, Y_{ut}) + \delta\, Y_{ut}\, (\delta\, X^{\varepsilon}_{su} - \delta\, X_{su})$$

therefore

$$K_{2\alpha,\alpha,\alpha} \leq \|X\|_{\alpha} \|Y^{\varepsilon} - Y\|_{\alpha} + \|X^{\varepsilon} - X\|_{\alpha} \|Y\|_{\alpha}.$$

We conclude by (4.27). □

4.6. A refined Kolmogorov criterion

In this section we prepare the ground for the proof of Lemmas 4.13 and 4.14 in Section 4.7 below, which are the main technical tools in the proof of Theorem 4.3. We suppose without loss of generality that $T = 1$, namely our processes are defined on the interval $[0, 1]$. Define the set $\mathbb{D}$ of dyadic points in $[0, 1]$ by

$$\mathbb{D} := \bigcup_{k \geq 0} D_k, \qquad \text{where} \qquad D_k := \left\{ d^k_i := \tfrac{i}{2^k} \right\}_{0 \leqslant i \leqslant 2^k}. \tag{4.24}$$

We equip $\mathbb{D}$ with a *directed graph structure*: given $d, \tilde{d} \in \mathbb{D}$, we write $d \to \tilde{d}$ if and only if $d = d^k_i$ and $\tilde{d} = d^k_{i+1}$, for some $k \geq 0$ and $0 \leqslant i \leqslant 2^k - 1$. More explicitly, $d \to \tilde{d}$ if and only if the point $\tilde{d}$ is consecutive to d in some layer D_k of $\mathbb{D}$.

Remarkably, in order to prove relation (4.39), *it is enough to have a suitable control on $R_{d,\tilde{d}}$ for consecutive points $d \to \tilde{d}$* (together with a global control on δR). This is the heart of the Kolmogorov continuity criterion, but we stress that it is a deterministic statement.

Theorem 4.9. (Kolmogorov criterion: deterministic part) *Given a function $A: \mathbb{D}^2_{<} \to \mathbb{R}$, let $0 < \alpha < \gamma$. Define*

$$Q_{\gamma} := \sup_{d, \tilde{d} \in \mathbb{D}: d \to \tilde{d}} \frac{|A_{d,\tilde{d}}|}{|\tilde{d} - d|^{\gamma}}, \tag{4.25}$$

$$K_{\alpha,\gamma} := \sup_{\substack{0 \leqslant s < u < t \leqslant 1\\ s,u,t \in \mathbb{D}}} \frac{|\delta A_{s,u,t}|}{\min\,(u - s, t - u)^{\alpha} |t - s|^{\gamma - \alpha}}. \tag{4.26}$$

Then there is a constant $C_{\alpha,\gamma} < \infty$ such that

$$|A_{st}| \leqslant C_{\alpha,\gamma} (Q_{\gamma} + K_{\alpha,\gamma}) |t - s|^{\gamma}, \quad \forall (s, t) \in \mathbb{D}^2_{<}. \tag{4.27}$$

A key tool for Theorem 4.9 is the next result, proved at this end of this section, which ensures the existence of suitable *short paths* in the graph $\mathbb{D}$.

Lemma 4.10. (Dyadic paths) *For any $s, t \in \mathbb{D}$ with $s < t$, there are integers n, $m \geq 1$ and a path of $(m + n + 1)$ points in $\mathbb{D}$ which leads from s to t, labelled as follows:*

$$s = s_m < \ldots < s_1 < s_0 = t_0 < t_1 < \ldots < t_n = t, \tag{4.28}$$

with the property that for all $i\in\{0,\ldots,m-1\}$ *and* $j\in\{0,\ldots,n-1\}$

$$s_{i+1}\to s_i,\quad t_j\to t_{j+1};\quad |s_i-s_{i+1}|<\frac{|t-s|}{2^i},\quad |t_{j+1}-t_j|<\frac{|t-s|}{2^j}. \tag{4.29}$$

Proof of Theorem 4.9. Fix $s,t\in\mathbb{D}$ with $s<t$. We use Lemma 4.10 with the same notation. By the definition of δA, we write

$$A_{st}=A_{st_0}+A_{t_0t}+\delta A_{s,t_0,t}\,.$$

In the case $m\geq 2$, we can develop A_{st_0} as follows (recall that $s=s_m$ and $s_0=t_0$):

$$A_{st_0}\;=\;\sum_{i=0}^{m-1}A_{s_{i+1}s_i}+\sum_{i=0}^{m-2}\delta A_{s,s_{i+1},s_i}\,.$$

Similarly, when $n\geq 2$, we develop

$$A_{t_0t}=\sum_{j=0}^{n-1}A_{t_jt_{j+1}}+\sum_{j=0}^{n-2}\delta A_{t_j,t_{j+1},t},$$

so that

$$\begin{aligned}A_{st}\;=\;&\underbrace{\sum_{i=0}^{m-1}A_{s_{i+1}s_i}+\sum_{j=0}^{n-1}A_{t_jt_{j+1}}}_{\Xi_1}+\\ &\underbrace{+\delta A_{s,t_0,t}+\sum_{i=0}^{m-2}\delta A_{s,s_{i+1},s_i}+\sum_{j=0}^{n-2}\delta A_{t_j,t_{j+1},t}}_{\Xi_2}.\end{aligned} \tag{4.30}$$

By the definition of Q_γ, for any $d\to\tilde d$ we can bound

$$|A_{d\tilde d}|\leqslant Q_\gamma|\tilde d-d|^\gamma.$$

By Lemma 4.10, this bound applies to any couple (s_{i+1},s_i) and (t_j,t_{j+1}). Then we can estimate Ξ_1 in (4.30) as follows, exploiting the bounds in (4.29):

$$\begin{aligned}&Q_\gamma\left\{\sum_{i=0}^{m-1}|s_i-s_{i+1}|^\gamma+\sum_{j=0}^{n-1}|t_{j+1}-t_j|^\gamma\right\}\leqslant\\ &\leqslant Q_\gamma\left\{\sum_{i=0}^{\infty}(2^{-i})^\gamma+\sum_{j=0}^{\infty}(2^{-j})^\gamma\right\}|t-s|^\gamma=\\ &=Q_\gamma\left\{\frac{2}{1-2^{-\gamma}}\right\}|t-s|^\gamma,\end{aligned}$$

which agrees with (4.27). On the other hand, thanks to (4.26) and (4.29),

$$|\delta A_{s,s_{i+1},s_i}|\leqslant K_{\alpha,\gamma}\left(\frac{|t-s|}{2^i}\right)^\alpha|t-s|^{\gamma-\alpha}=K_{\alpha,\gamma}\,2^{-i\alpha}\,|t-s|^\gamma$$

and similarly for $\delta A_{t_j,t_{j+1},t}$, so that the term Ξ_2 can be bounded above by

$$K_{\alpha,\gamma}|t-s|^\gamma\left(1+\sum_{i=0}^{m-2}2^{-i\alpha}+\sum_{j=0}^{n-2}2^{-j\alpha}\right)\leqslant K_{\alpha,\gamma}|t-s|^\gamma\left(1+\frac{2}{1-2^{-\alpha}}\right).$$

This completes the proof of (4.27). □

As a simple consequence of Theorem 4.9, we show that suitable moment conditions ensure the finiteness of the constant Q_γ in (4.25), as in the classical Kolmogorov criterion.

PROPOSITION 4.11. (KOLMOGOROV CRITERION: PROBABILISTIC PART) *Let $A = (A_{st})_{(s,t)\in\mathbb{D}^2_<}$ be a stochastic process which satisfies the following bound, for some γ_0, $p, c \in (0,\infty)$:*

$$\mathbb{E}[|A_{st}|^p] \leqslant c|t-s|^{p\gamma_0}, \qquad \forall (s,t) \in \mathbb{D}^2_<.$$

Then, for any value of γ such that

$$\gamma < \gamma_0 - \tfrac{1}{p}, \tag{4.31}$$

the random variable $Q_\gamma = Q_\gamma(A)$ defined in (4.25) is in L^p:

$$\mathbb{E}[|Q_\gamma|^p] < \infty.$$

In particular, $Q_\gamma < \infty$ a.s..

Proof. By definition of Q_γ in (4.25), bounding the supremum with a sum we can write

$$|Q_\gamma|^p \leqslant \sum_{d,\tilde{d}\in\mathbb{D}: d\to\tilde{d}} \left(\frac{|A_{d,\tilde{d}}|}{|\tilde{d}-d|^\gamma}\right)^p = \sum_{k\geq 0}\sum_{i=0}^{2^k-1} \frac{|A_{d_i^k d_{i+1}^k}|^p}{|d_{i+1}^k - d_i^k|^{p\gamma}}.$$

Let us write $\gamma = \gamma_0 - \frac{1+\epsilon}{p}$, for some $\epsilon > 0$. Since $d_{i+1}^k - d_i^k = \frac{1}{2^k}$ we have

$$\begin{aligned}
\mathbb{E}[|Q_\gamma|^p] &\leqslant \sum_{k\geq 0}\sum_{i=0}^{2^k-1} c|d_{i+1}^k - d_i^k|^{p(\gamma_0-\gamma)} \\
&\leqslant \sum_{k\geq 0}\sum_{i=0}^{2^k-1} \frac{c}{2^{(1+\epsilon)k}} = \sum_{k\geq 0}\frac{c}{2^{\epsilon k}} = \frac{c}{1-2^{-\epsilon}} < \infty.
\end{aligned}$$

The proof is complete. □

Remark 4.12. Given a stochastic process $(X_t)_{t\in\mathbb{D}}$ defined on dyadic times, if we apply Theorem 4.9 and Proposition 4.11 to $(A_{st} := \delta X_{st} = X_t - X_s)_{(s,t)\in\mathbb{D}^2_<}$ we obtain the classical Kolmogorov continuity criterion. Note that in this case $K_{\rho,\sigma} = 0$ because $\delta A = 0$.

Proof of Lemma 4.10. We refer to Figure 4.1 for a graphical representation. Given $s, t \in \mathbb{D}$ with $s < t$, since $0 < t - s \leqslant 1$, we can define $\ell \geq 1$ as the unique integer such that

$$\frac{1}{2^\ell} < t - s \leqslant \frac{1}{2^{\ell-1}}. \tag{4.32}$$

We now take the smallest $k \in \{0, \ldots, 2^\ell - 1\}$ for which $d_k^\ell > s$ and define

$$s_0 := t_0 := d_k^\ell.$$

The definition of k guarantees that $d_k^\ell < t$, because if $d_k^\ell \geqslant t$ then $\frac{k}{2^\ell} - s \geqslant t - s > \frac{1}{2^\ell}$ and this would violate the minimality of k.

Note that $0 < d_k^\ell - s \leqslant d_k^\ell - d_{k-1}^\ell = \frac{1}{2^\ell}$ and $0 < t - d_k^\ell < t - s$, by (4.32), therefore

$$0 < s_0 - s < \frac{1}{2^{\ell-1}}, \qquad 0 < t - t_0 < \frac{1}{2^{\ell-1}}. \tag{4.33}$$

Since both $s_0 - s \in \mathbb{D}$ and $t - t_0 \in \mathbb{D}$, for suitable integers $m \geq 1$ and $n \geq 1$ we have

$$s_0 - s = \frac{1}{2^{q_1}} + \frac{1}{2^{q_2}} + \ldots + \frac{1}{2^{q_m}}, \qquad t - t_0 = \frac{1}{2^{r_1}} + \frac{1}{2^{r_2}} + \ldots + \frac{1}{2^{r_n}},$$

where $q_m > q_{m-1} > \ldots > q_1 \geq \ell$ and $r_n > \ldots > r_1 \geq \ell$. We can thus write

$$\begin{aligned} s =& s_0 - \frac{1}{2^{q_1}} - \frac{1}{2^{q_2}} - \ldots - \frac{1}{2^{q_m}}, \\ t =& t_0 + \frac{1}{2^{r_1}} + \frac{1}{2^{r_2}} + \ldots + \frac{1}{2^{r_n}}. \end{aligned}$$

We can finally define

$$\begin{aligned} s_i :=& s_0 - \frac{1}{2^{q_1}} - \frac{1}{2^{q_2}} - \ldots - \frac{1}{2^{q_i}} \quad \text{for } i = 1, \ldots, m, \\ t_j :=& t_0 + \frac{1}{2^{r_1}} + \frac{1}{2^{r_2}} + \ldots + \frac{1}{2^{r_j}} \quad \text{for } j = 1, \ldots, n. \end{aligned}$$

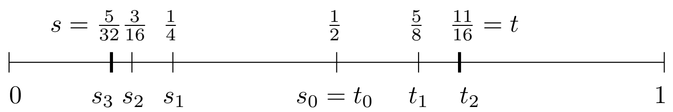

Figure 4.1. An instance of Lemma 4.10 with $s = \frac{5}{32}$ and $t = \frac{11}{16}$. Note that $\ell = 1$ (because $\frac{1}{2^1} < |t - s| = \frac{17}{32} \leq \frac{1}{2^0}$, cf. (4.32)) and $s_0 = t_0 = \frac{1}{2}$. The points $t_1, \ldots, t_n$ are built iteratively: first take the largest $\frac{1}{2^{r_1}}$ (i.e. the smallest r_1) such that $t_1 := t_0 + \frac{1}{2^{r_1}} \leq t$; if $t_1 < t$, then take the largest $\frac{1}{2^{r_2}}$ such that $t_2 := t_1 + \frac{1}{2^{r_2}} \leq t$; and so on, until $t_n = t$. Similarly for $s_1, \ldots, s_m$.

Since q_i and r_j are strictly increasing integers with $q_1 \geq \ell$ and $r_1 \geq \ell$, we have the bounds $q_i \geq \ell + (i-1)$ and $r_j \geq \ell + (j-1)$, for all $i \in \{0, \ldots, m-1\}$ and $j \in \{0, \ldots, n-1\}$, hence

$$\begin{aligned} |s_i - s_{i+1}| &= \frac{1}{2^{q_{i+1}}} \leqslant \frac{1}{2^i} \frac{1}{2^\ell} < \frac{|t-s|}{2^i}, \\ |t_{j+1} - t_j| &= \frac{1}{2^{q_{j+1}}} \leqslant \frac{1}{2^j} \frac{1}{2^\ell} < \frac{|t-s|}{2^j}. \end{aligned}$$

having used (4.32). This proves the bounds in (4.29).

We note that, for any integer $r \geq \ell$, we have the inclusion $D_\ell \subseteq D_r$. Then, given any $x \in D_\ell$, we have that $x \in D_r$, hence $x \to x + 2^{-r}$. Since $t_0 = d_k^\ell \in D_\ell$ and $r_1 \geq \ell$, this shows that $t_0 \to t_1 = t_0 + 2^{-r_1}$. Proceeding inductively, we have $t_j \to t_{j+1} = t_j + 2^{-r_{j+1}}$. A similar argument applies to the points s_i and completes the proof of (4.29). □

4.7. Proof of Theorem 4.3

In this section we prove the three assertions of Theorem 4.3.

Proof of the first assertion of Theorem 4.3. We want to prove that for any $\alpha \in \left(0, \frac{1}{2}\right)$, a.s. I is α-Hölder continuous, namely there is an a.s. finite random constant C such that

$$|\delta I_{st}| \leqslant C|t-s|^{\alpha}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T. \tag{4.34}$$

First observation: if the claim holds under the stronger assumption $|h| \leqslant c$ almost surely, for some deterministic $c < \infty$, then we can deduce the general result by localization. Indeed, if we only assume that $\sup_{[0,T]} |h| < \infty$ a.s., we can define for $n \in \mathbb{N}$ the stopping times

$$\tau_n := \inf\{t \in [0,T]: |h_t| > n\}.$$

Let us define

$$h_s^{(n)} := h_{s \wedge \tau_n}, \qquad I_t^{(n)} := \int_0^t h_s^{(n)} \mathrm{d}B_s.$$

Note that $\sup_{[0,T]} |h^{(n)}| \leqslant n$ by the definition of τ_n. Then

$$|\delta I_{st}^{(n)}| \leqslant C^{(n)}|t-s|^{\alpha}, \qquad \forall 0 \leqslant s < t \leqslant T, \tag{4.35}$$

for a suitable a.s. finite random constant $C^{(n)}$. Let us define the events

$$A_n := \{\tau_n = \infty\} = \{\sup_{[0,T]} |h| \leqslant n\}$$

and note that $h = h^{(n)}$ on A_n. By the locality property of the stochastic integral, $I = I^{(n)}$ a.s. on A_n[4.1].

Note that $A := \bigcup_{n \in \mathbb{N}} A_n = \{\sup_{[0,T]} |h| < \infty\}$, hence $\mathbb{P}(A) = 1$. If we define $C := C^{(n)}$ on $A_n \setminus A_{n-1}$ (with $A_0 := \emptyset$) and $C := \infty$ on A^c, we have $C < \infty$ a.s. and relation (4.6) holds.

Second observation: if relation (4.34) holds for all s, t in a (deterministic) dense subset $\mathbb{D} \subseteq [0,T]$, then it holds for all $s, t \in [0,T]$, because δI_{st} is a continuous function of (s,t).

In conclusion, the proof is reduced to showing (4.34) only for $s, t \in \mathbb{D}$, under the assumption that $\sup_{[0,T]} |h| \leqslant c < \infty$ almost surely. Suppose that this is the case and set $A_{st} := \delta I_{st}$, $0 \leqslant s \leqslant t \leqslant T$. Here $\delta A = 0$ and therefore the constant $K_{\alpha,\gamma}$ in (4.26) is equal to zero for any $0 < \alpha < \gamma$. It remains to estimate Q_α using Proposition 4.11.

By the BDG inequality of Proposition 4.15, for any $p \geqslant 2$

$$\mathbb{E}[|\delta I_{st}|^p] \leqslant c_p \, \mathbb{E}\left[\left(\int_s^t h_u^2 \, \mathrm{d}u\right)^{\frac{p}{2}}\right] \leqslant C_p |t-s|^{\frac{p}{2}}.$$

Then Proposition 4.11 applies with $\gamma_0 = \frac{1}{2}$ and any $\alpha = \gamma_0 - \frac{1}{p} \in \left(0, \frac{1}{2}\right)$ for p sufficiently large. By Theorem 4.9, we obtain (4.34) and the proof is complete. □

For $0 \leqslant s \leqslant t \leqslant T$ we define the (random) continuous function

$$R_{st} := I_t - I_s - h_s\,(B_t - B_s) = \int_s^t \delta h_{sr} \, \mathrm{d}B_r. \tag{4.36}$$

4.1. We mean that $I^{(n)}$ and I are indistinguishable on A_n: for a.e. $\omega \in A_n$ one has $I_t^{(n)}(\omega) = I_t(\omega)$ for all $t \in [0,1]$ (we recall that we always fix continuous versions of the stochastic integrals).

We recall that a.s. $B \in \mathcal{C}^\beta$ for every $\beta < \frac{1}{2}$.

Proof of the second assertion of Theorem 4.3. Let $\beta < \frac{1}{2}$. We want to show that, if a.s. $h \in \mathcal{C}^\alpha$, for some $\alpha \in (0, \beta]$, then, for any , there is an a.s. finite random constant C such that

$$|R_{st}| \leqslant C|t-s|^{\alpha+\beta}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T. \tag{4.37}$$

First observation: if the claim holds under the stronger assumption $\|\delta h\|_\alpha \leqslant c$ almost surely, for some deterministic $c < \infty$, then we can deduce the general result by localization. Indeed, if we only assume that $\|\delta h\|_\alpha < \infty$ a.s., we can define for $n \in \mathbb{N}$ the stopping times

$$\tau_n := \inf\{t \in [0,1] : \|\delta h\|_{\alpha,[0,t]} > n\},$$

where $\|\delta h\|_{\alpha,[0,t]}$ is the Hölder semi-norm of h restricted to $[0,t]$ (equivalently, the Hölder semi-norm of $s \mapsto h_{s \wedge t}$ on the whole interval $s \in [0,1]$). Let us define

$$h_s^{(n)} := h_{s \wedge \tau_n}, \qquad I_t^{(n)} := \int_0^t h_s^{(n)} \mathrm{d}B_s, \qquad R_{st}^{(n)} := I_t^{(n)} - I_s^{(n)} - h_s^{(n)}(B_t - B_s).$$

Note that $\|\delta h^{(n)}\|_\alpha \leqslant n$, by definition of τ_n. (Indeed, $\|\delta h\|_{\alpha,[0,t]} \leqslant n$ for all $t < \tau_n$, which means that $|h(r) - h(s)| \leqslant n|r-s|^\alpha$ for all $r, s \in [0, \tau_n)$; then, by continuity, $|h(r) - h(s)| \leqslant n|r-s|^\alpha$ for all $r, s \in [0, \tau_n]$, which means that $\|\delta h\|_{\alpha,[0,\tau_n]} = \|\delta h^{(n)}\|_\alpha \leqslant n$). Then

$$|R_{st}^{(n)}| \leqslant C^{(n)}|t-s|^{\alpha+\beta}, \qquad \forall 0 \leqslant s < t \leqslant T, \tag{4.38}$$

for a suitable a.s. finite random constant $C^{(n)}$. Let us define the events

$$A_n := \{\tau_n = \infty\} = \{\|\delta h\|_\alpha \leqslant n\}$$

and note that $h = h^{(n)}$ on A_n. By the locality property of the stochastic integral, $I = I^{(n)}$ a.s. on A_n,[4.2] hence also $R = R^{(n)}$ a.s. on A_n. Redefining $C^{(n)} = \infty$ on the exceptional set $\{R = R^{(n)}\}^c$, we get by (4.38)

$$\text{on the event } A_n: \qquad |R_{st}| \leqslant C^{(n)}|t-s|^{\alpha+\beta}, \qquad \forall 0 \leqslant s < t \leqslant T.$$

Note that $A := \bigcup_{n \in \mathbb{N}} A_n = \{\|\delta h\|_\alpha < \infty\}$, hence $\mathbb{P}(A) = 1$. If we define $C := C^{(n)}$ on $A_n \setminus A_{n-1}$ (with $A_0 := \emptyset$) and $C := \infty$ on A^c, we have $C < \infty$ a.s. and relation (4.7) holds.

Second observation: if relation (4.37) holds for all s, t in a (deterministic) dense subset $\mathbb{D} \subseteq [0,1]$, then it holds for all $s, t \in [0,1]$, because R_{st} is a continuous function of (s,t).

In conclusion, the proof is reduced to showing (4.37) only for $s, t \in \mathbb{D}$, under the assumption that $\|\delta h\|_\alpha \leqslant c < \infty$. This technical result is formulated in the separate Lemma 4.13. □

4.2. We mean that $I^{(n)}$ and I are indistinguishable on A_n: for a.e. $\omega \in A_n$ one has $I_t^{(n)}(\omega) = I_t(\omega)$ for all $t \in [0,1]$ (we recall that we always fix continuous versions of the stochastic integrals).

LEMMA 4.13. *Let* $0<\alpha\leqslant\beta<\frac{1}{2}$. *Assume that* $\mathbb{E}[\|\delta h\|_\alpha^p]<\infty$ *for all* $p>0$. *Then there is an a.s. finite random constant* C *such that*

$$|R_{st}|\leqslant C|t-s|^{\alpha+\beta}, \qquad \forall s,t\in\mathbb{D} \quad \text{with } s\leqslant t. \tag{4.39}$$

Equivalently, a.s. $R\in C_2^{\alpha+\beta}$.

Proof. We apply Theorem 4.9 to the (random) function $A(s,t)=R_{st}$, with $\gamma=\eta=\alpha+\beta$, $N=1$, $\rho=\alpha$ and p large enough (to be fixed later). Then relation (4.27) yields (4.39). It remains to show that a.s. $Q_{\alpha+\beta}<\infty$ and $K_{\alpha,\alpha+\beta}<\infty$.

We recall that R_{st} is defined in (4.36). In particular, for $s<u<t$

$$\delta R_{sut}=R_{st}-R_{su}-R_{ut}=(h_u-h_s)(B_t-B_u).$$

Then by (4.26), a.s.

$$K_{\alpha,\alpha+\beta}(R)\leqslant\|\delta h\|_\alpha\|\delta B\|_\beta \sup_{0\leqslant s<u<t\leqslant 1}\frac{|u-s|^\alpha|t-u|^\beta}{\min(u-s,t-u)^\alpha|t-s|^\beta}.$$

By our assumption that $\|\delta h\|_\alpha\in L^p$ and by the fact that B is a Brownian motion, it only remains to show that the constant defined by the supremum is bounded above by 1. The constant is in fact easily seen to be equal to

$$\sup_{a,b>0,a+b=1}\frac{a^\alpha b^\beta}{\min(a,b)^\alpha}=\sup_{a,b>0,a+b=1}\left(\frac{ab}{\min(a,b)}\right)^\alpha b^{\beta-\alpha}\leqslant 1.$$

We want now to estimate $Q_{\alpha+\beta}(R)$. We note that, for fixed $s<t$, we have $R_{st}=\int_s^t(h_u-h_s)\,\mathrm{d}B_u$ a.s.. By the Burkholder-Davies-Gundy inequality, see Proposition 4.15, for any $p>2$ there is a universal constant c_p such that

$$\begin{aligned}\mathbb{E}[|R_{st}|^p] &\leqslant c_p\,\mathbb{E}\left[\left(\int_s^t(h_u-h_s)^2\,\mathrm{d}u\right)^{\frac{p}{2}}\right]\\ &\leqslant c_p\,\mathbb{E}\left[\|\delta h\|_\alpha^p\left(\int_s^t(u-s)^{2\alpha}\,\mathrm{d}u\right)^{\frac{p}{2}}\right]\\ &\leqslant c_p\,\mathbb{E}[\|\delta h\|_\alpha^p]\,(t-s)^{p\left(\alpha+\frac{1}{2}\right)}.\end{aligned}$$

By Proposition 4.11, we have $Q_\gamma<\infty$ a.s. for any $\gamma<\alpha+\frac{1}{2}-\frac{1}{p}$. Plugging $\gamma=\alpha+\beta$ we get $\beta<\frac{1}{2}-\frac{1}{p}$, which is satisfied for p large enough, since $\beta<\frac{1}{2}$. □

Next, we suppose that there exists another adapted process $h^1=(h_t^1)_{t\in[0,T]}$ with values in $\mathbb{R}^k\otimes(\mathbb{R}^d)$ such that a.s.

$$|\delta h_{st}-h_s^1\,\mathbb{B}_{st}^1|\lesssim|t-s|^{2\alpha}.$$

Then we define

$$\begin{aligned}\hat{R}_{st} &:= R_{st}-h_s^1\,\mathbb{B}_{st}^2=\delta I_{st}-h_s\,\mathbb{B}_{st}^1-h_s^1\,\mathbb{B}_{st}^2\\ &= \int_s^t(\delta h_{sr}-h_s^1\,\mathbb{B}_{sr}^1)\,\mathrm{d}B_r,\end{aligned} \tag{4.40}$$

where $\mathbb{B}^2$ is defined in (4.2). Then the third assertion of Theorem 4.3 follows with the same localisation argument as for the second one and from the following

LEMMA 4.14. *Assume that* $\mathbb{E}[\|\delta h^1\|_\alpha^p + \|\delta h - h^1\mathbb{B}^1\|_{2\alpha}^p] < \infty$*, for some* $\alpha \in (0, \frac{1}{2})$ *and for all* $p > 0$*. Then there is an a.s. finite random constant* C *such that*

$$|\hat{R}_{st}| \leqslant C|t-s|^{3\alpha}, \qquad \forall s,t \in \mathbb{D} \quad \text{with } s \leqslant t. \tag{4.41}$$

Equivalently, a.s. $\hat{R} \in C_2^{3\alpha}$.

Proof. We set $\gamma = \eta = 3\alpha$, $N = 2$, $\rho_1 = 2\alpha$, $\rho_2 = \alpha$. Then

$$\delta \hat{R}_{sut} = (\delta h_{su} - h_s^1 \,\mathbb{B}_{su}^1)\,\mathbb{B}_{ut}^1 + \delta h_{su}^1 \,\mathbb{B}_{ut}^2,$$

which implies that a.s. $K_{\alpha,3\alpha}(\hat{R}) < +\infty$. Indeed

$$\begin{aligned} K_{\alpha,3\alpha}(\hat{R}) &\leqslant \|\delta h - h^1\mathbb{B}^1\|_{2\alpha}\|\mathbb{B}^1\|_\alpha \sup_{0\leqslant s<u<t\leqslant 1} \frac{|u-s|^{2\alpha}|t-u|^\alpha}{\min\,(u-s,t-u)^\alpha |t-s|^{2\alpha}} \\ &\quad + \|\delta h^1\|_\alpha \|\mathbb{B}^2\|_{2\alpha} \sup_{0\leqslant s<u<t\leqslant 1} \frac{|u-s|^\alpha |t-u|^{2\alpha}}{\min\,(u-s,t-u)^\alpha |t-s|^{2\alpha}}. \end{aligned}$$

We note that both suprema are equal to

$$\left(\sup_{a,b>0,a+b=1} \frac{a b^2}{\min\,(a,b)}\right)^\alpha \leqslant 1.$$

Now by (4.40)

$$\begin{aligned} \mathbb{E}[|\hat{R}_{st}|^p] &\leqslant \mathbb{E}\left[\left(\int_s^t (\delta h_{su} - h_s^1 \,\mathbb{B}_{su}^1)^2 \,\mathrm{d}u\right)^{\frac{p}{2}}\right] \\ &\leqslant c_p \,\mathbb{E}\left[\|\delta h - h^1\mathbb{B}^1\|_{2\alpha}^p \left(\int_s^t (u-s)^{4\alpha}\,\mathrm{d}u\right)^{\frac{p}{2}}\right] \\ &\leqslant c_p \,\mathbb{E}[\|\delta h - h^1\mathbb{B}^1\|_{2\alpha}^p]\,(t-s)^{p\left(2\alpha + \frac{1}{2}\right)}. \end{aligned}$$

By Proposition 4.11, we have $Q_\gamma < \infty$ a.s. for any $\gamma < 2\alpha + \frac{1}{2} - \frac{1}{p}$. Plugging $\gamma = 3\alpha$ we get $\alpha < \frac{1}{2} - \frac{1}{p}$, which is satisfied for p large enough, since $\alpha < \frac{1}{2}$. □

Finally, we give a proof of (half of) Burkholder-Davies-Gundy inequality for $p \geq 2$.

PROPOSITION 4.15. *For all* $p \geq 2$ *there is a constant* $c_p < \infty$ *such that for all* $0 \leqslant s < t \leqslant T$

$$\mathbb{E}\left[\left(\int_s^t y_u \,\mathrm{d}B_u\right)^p\right] \leqslant c_p \,\mathbb{E}\left[\left(\int_s^t y_u^2 \,\mathrm{d}u\right)^{\frac{p}{2}}\right]$$

for any progressively measurable process such that $\int_0^1 y_u^2 \,\mathrm{d}u < \infty$*,* $\mathbb{P}$*-a.s..*

Proof. To simplify notation we set $s = 0$ and $m_t := \int_0^t y_u \,\mathrm{d}B_u$.

In a first time we make the additional assumptions that $\mathbb{E}[\int_0^1 y_u^2\,\mathrm{d}u] < \infty$ and m is bounded by some deterministic constant. By the Itô formula applied to m_t, we get

$$\mathrm{d}|m_t|^p = p|m_t|^{p-1}\mathrm{sgn}(m_t)y_t\,\mathrm{d}B_t + \frac{p(p-1)}{2}|m_t|^{p-2}y_t^2\,\mathrm{d}t.$$

In general $(\int_0^t |m_u|^{p-1}\mathrm{sgn}(m_u)y_u\,\mathrm{d}B_u)_t$ is a local martingale, but under our additional assumptions it is a true martingale with zero expectation, because $\mathbb{E}[\int_0^1 |m_u|^{2(p-1)}\,y_u^2\mathrm{d}u] < \infty$ (recall that m is bounded). Consequently

$$\mathbb{E}[|m_t|^p] = \frac{p(p-1)}{2}\,\mathbb{E}\bigg[\int_0^t |m_u|^{p-2}y_u^2\,\mathrm{d}u\bigg].$$

If we set $|\bar{m}_t| := \sup_{u\leqslant t}|m_u|$, we obtain by Hölder

$$\begin{aligned}\mathbb{E}[|m_t|^p] &\leqslant \frac{p(p-1)}{2}\mathbb{E}\bigg[|\bar{m}_t|^{p-2}\int_0^t y_u^2\,\mathrm{d}u\bigg]\\ &\leqslant \frac{p(p-1)}{2}\,\mathbb{E}[|\bar{m}_t|^p]^{1-\frac{2}{p}}\mathbb{E}\bigg[\bigg(\int_0^t y_u^2\,\mathrm{d}u\bigg)^{\frac{p}{2}}\bigg]^{\frac{2}{p}}. \qquad (4.42)\end{aligned}$$

Since $(|m_t|)_{t\geq 0}$ is submartingale bounded in L^p with continuous trajectories, by Doob L^p inequality we have: $\mathbb{E}[|\bar{m}_t|^p] \leqslant (\frac{p}{p-1})^p\,\mathbb{E}[|m_t|^p]$. Plugging the above in (4.42) we conclude:

$$\mathbb{E}\bigg[\bigg|\int_0^t y_u\,\mathrm{d}B_u|^p\bigg] \leqslant c_p\mathbb{E}\bigg[\bigg(\int_0^t y_u^2\,\mathrm{d}u\bigg)^{p/2}\bigg].$$

As far as the general case is concerned, let us define

$$\tau^n = \inf\{t \geq 0\colon |m_t| > n\} \wedge \inf\bigg\{t \geq 0\colon \int_0^t y_u^2\,\mathrm{d}u > n\bigg\}$$

Note that τ^n is a non decreasing sequence of stopping times, with $\tau^n = \infty$ for n large enough, $\mathbb{P}$-a.s.. We denote $y_t^n := y\,\mathbb{1}_{[0,\tau^n]}(t)$ and $m_t^n := \int_0^t y_u^n\mathrm{d}B_u$. By construction, y^n and m^n satisfy our additional assumptions. Since $m_t^n = m_{t\wedge\tau^n}$ a.s., we have

$$\begin{aligned}\mathbb{E}\bigg[\bigg|\int_0^{t\wedge\tau^n} y_u\,\mathrm{d}B_u|^p\bigg] &\leqslant c_p\,\mathbb{E}\bigg[\bigg(\int_0^t y_u^2\,\mathbf{1}_{[0,\tau^n]}(u)\,\mathrm{d}u\bigg)^{p/2}\bigg]\\ &\leqslant c_p\,\mathbb{E}\bigg[\bigg(\int_0^t y_u^2\,\mathrm{d}u\bigg)^{p/2}\bigg].\end{aligned}$$

Finally we notice that by Fatou's Lemma

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\int_s^t y_u\,\mathrm{d}B_u\bigg)^p\bigg] &= \mathbb{E}\bigg[\liminf_{n\to\infty}\bigg|\int_s^{t\wedge\tau^n} y_u\,\mathrm{d}B_u|^p\bigg]\\ &\leqslant \liminf_{n\to\infty}\mathbb{E}\bigg[\bigg|\int_s^{t\wedge\tau^n} y_u\,\mathrm{d}B_u|^p\bigg]\\ &\leqslant c_p\,\mathbb{E}\bigg[\bigg(\int_s^t y_u^2\,\mathrm{d}u\bigg)^{p/2}\bigg].\end{aligned}$$

The proof is complete. □

Part II

Rough Integration

Chapter 5

The Sewing Lemma

We fix throughout the chapter a time horizon $T>0$ and two continuous functions $X,Y\colon[0,T]\to\mathbb{R}$. In this setting the *integral*

$$\int_0^T Y_r\,\mathrm{d}X_r \tag{5.1}$$

can be defined as $\int_0^T Y_r\,\dot{X}_r\,\mathrm{d}r$ if X is differentiable or, more generally, as a Lebesgue integral if X is of bounded variation, so that $\mathrm{d}X$ is a signed measure. The key question we want to address is: *how to define the integral when X does not have such regularity?* This is an example of a more general problem: given a distribution (generalized function) $\dot{X}$ and a non-smooth function Y, how to define their product $Y\dot{X}$?

A motivation is given by $X=B$ with $(B_t)_{t\geq 0}$ a Brownian motion. In this special case, one can use probability theory to answer the question and define the integral in (5.1), but one sees that there are several possible definitions: for example Itô, Stratonovich, etc.

In this book, we are going to present the alternative answer provided by the theory of Rough Paths, originally introduced by Terry Lyons. This theory yields a robust construction of the integral in (5.1) and sheds a new "pathwise" light on stochastic integration.

The approach we follow is based on the *Sewing Lemma*, to which this chapter is devoted. In particular, we will show in Chapter 6 that the integral in (5.1) has a canonical definition (*Young integral*) when Y and X are Hölder continuous, under a constraint on their Hölder exponents. Going beyond this constraint requires Rough Paths, which will be studied in Chapter 7.

5.1. Local approximation

If X is of class C^1, we can define the integral function

$$I_t := \int_0^t Y_r\,\dot{X}_r\,\mathrm{dr}, \qquad t\in[0,T].$$

Then we have $I_0=0$ and for $0\leqslant s\leqslant t\leqslant T$

$$I_t - I_s - Y_s\,(X_t - X_s) = \int_s^t (Y_r - Y_s)\,\dot{X}_r\,\mathrm{d}r = o(t-s) \tag{5.2}$$

as $t-s\to 0$, because $\dot{X}$ is bounded and $|Y_r-Y_s|=o(1)$ as $|r-s|\to 0$. Thus the integral function I_t satisfies

$$I_0=0, \qquad I_t-I_s=Y_s(X_t-X_s)+o(t-s), \qquad 0\leqslant s\leqslant t\leqslant T. \tag{5.3}$$

Remarkably, *the relation (5.3) characterizes* $(I_t)_{t\in[0,T]}$. Indeed, if I^1 and I^2 satisfy (5.3) with the same functions X,Y, their difference $\Delta:=I^1-I^2$ satisfies

$$|\Delta_t-\Delta_s|=o(t-s), \qquad 0\leqslant s\leqslant t\leqslant T,$$

which implies $\frac{\mathrm{d}}{\mathrm{d}t}\Delta_t\equiv 0$ and then $\Delta_t=\Delta_0=I_0^1-I_0^2=0$ by (5.3). This simple result deserves to be stated in a separate

LEMMA 5.1. *Given any pair of functions* $X,Y\colon[0,T]\to\mathbb{R}$*, there can be at most one function* $I\colon[0,T]\to\mathbb{R}$ *satisfying (5.3).*

The formulation (5.3) is interesting also because the derivative $\dot{X}$ of X does not appear. Therefore, if we can find a function $I\colon[0,T]\to\mathbb{R}$ which satisfies (5.3), such a function is *unique* and we can take it as a *definition* of the integral (5.1).

We will see in Section 6.1 that this program can be accomplished when X and Y satisfy suitable Hölder regularity assumptions. In order to get there, in the next sections we will look at a more general problem.

5.2. A GENERAL PROBLEM

Let us generalise the problem (5.3). We define $A\colon[0,T]^2_{\leqslant}\to\mathbb{R}$ by setting for $0\leqslant s\leqslant t\leqslant T$

$$A_{st}:=Y_s(X_t-X_s)\ . \tag{5.4}$$

We can then decouple (5.3) in two relations:

$$I_0=0, \qquad I_t-I_s=A_{st}+R_{st}, \qquad 0\leqslant s\leqslant t\leqslant T, \tag{5.5}$$

$$R\colon[0,T]^2_{\leqslant}\to\mathbb{R}, \qquad R_{st}=o(t-s)\,. \tag{5.6}$$

The general problem is, given a continuous $A\colon[0,T]^2_{\leqslant}\to\mathbb{R}$, to find a pair of functions (I,R) satisfying (5.5)-(5.6). We call

- $A\colon[0,T]^2_{\leqslant}\to\mathbb{R}$ the *germ*,
- $I\colon[0,T]\to\mathbb{R}$ the *integral*,
- $R\colon[0,T]^2_{\leqslant}\to\mathbb{R}$ the *remainder*.

We are going to present conditions which allow to solve this problem.

Note that *we always have uniqueness*. Indeed, given (I^1,R^1) and (I^2,R^2) which solve (5.5)-(5.6) for the same A, by the same arguments which lead to Lemma 5.1 we have $\frac{\mathrm{d}}{\mathrm{d}t}(I_t^1-I_t^2)\equiv 0$, hence $I^1=I^2$ and then $R^1=R^2$ by (5.5). We record this as

LEMMA 5.2. *Given any germ* A*, there can be at most one pair of functions* (I,R) *satisfying (5.5)-(5.6).*

5.3. An algebraic look

We first focus on relation (5.5) alone. For a fixed germ A, this equation has infinitely many solutions (I, R), because given *any* I we can simply *define* R so as to fulfill (5.5). Interestingly, all solutions admit an algebraic characterization in terms of R alone.

Lemma 5.3. *Fix a function $A \in C_2$.*

1. *If a pair $(I, R) \in C_1 \times C_2$ satisfies (5.5), then R satisfies*

$$(\delta R)_{sut} = -(\delta A)_{sut}, \qquad \forall 0 \leqslant s \leqslant u \leqslant t \leqslant T\,. \tag{5.7}$$

2. *Viceversa, given any function $R \in C_2$ which satisfies (5.7), if we set $I_t := A_{0t} + R_{0t}$, the pair $(I, R) \in C_1 \times C_2$ satisfies (5.5).*

Proof. Relation (5.5) clearly implies (5.7), simply because $\delta(\delta I) = 0$. Viceversa, given R satisfying (5.7), we can define $L_{st} := A_{st} + R_{st}$ so that

$$L_{st} - L_{su} - L_{ut} = 0.$$

Applying this formula to $(s', u', t') = (0, s, t)$, we obtain that $I_t := L_{0t}$ satisfies

$$I_t - I_s = L_{0t} - L_{0s} = L_{st} = A_{st} + R_{st}$$

and the proof is complete because $I_0 := L_{00} = A_{00} + R_{00} = 0$, which follows by (5.7) for $s = u = 0$. □

We can now rephrase Lemma 5.3 as follows.

Proposition 5.4. *Fix $A \in C_2$. Finding a pair $(I, R) \in C_1 \times C_2$ satisfying (5.5) is equivalent to finding $R \in C_2$ such that*

$$\delta R_{sut} = -\delta A_{sut}\,, \qquad \forall\, 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{5.8}$$

5.4. Enters analysis: the Sewing Lemma

So far we have analyzed (5.5). We now let (5.6) enter the game, i.e. we look for a pair of functions $(I, R) \in C_1 \times C_2$ which fulfills (5.5)-(5.6), given a (general) germ $A \in C_2$.

We stress that condition (5.6) is essential to ensure *uniqueness*: without it, equation (5.5) admits infinitely many solutions, as discussed before Lemma 5.3. When we couple (5.5) with (5.6), uniqueness is guaranteed by Lemma 5.2, but *existence* is no longer obvious. This is what we now focus on.

We start with a simple necessary condition.

Lemma 5.5. *For (5.5)-(5.6) to admit a solution, it is necessary that the germ A satisfies*

$$|\delta A_{sut}| = o(t - s), \qquad \text{for } 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{5.9}$$

Proof. If (5.5) admits a solution, by Proposition 5.4 we have $|\delta A_{sut}| = |\delta R_{sut}|$. If furthermore R satisfies (5.6), we must have for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$|\delta R_{sut}| \leq |R_{st}| + |R_{su}| + |R_{ut}| = o(t-s) + o(u-s) + o(t-u) = o(t-s)\,. \qquad \square$$

Remark 5.6. Choosing $u = s$ in (5.9) we obtain that $-A_{ss} = o(t-s)$, which means that $A_{ss} = 0$. Therefore a necessary condition for (5.5)-(5.6) to admit a solution is that A *vanishes on the diagonal of* $[0,T]^2_{\leqslant}$.

Remarkably, the necessary condition in Lemma 5.5 is close to being sufficient: it is enough to upgrade $o(t-s)$ in $O((t-s)^\eta)$ for some $\eta > 1$. This is the content of the celebrated *Sewing Lemma*, which we next present.

We have seen in the Sewing bound (Theorem 1.9) that any $R \in C_2$ such that $R_{st} = o(t-s)$ for $0 \leqslant s \leqslant t \leqslant T$ satisfies an a priori estimate $\|R\|_\eta \leqslant K_\eta \|\delta R\|_\eta$ for any $\eta > 1$. Of course, this estimate is only interesting if $\|\delta R\|_\eta < \infty$ for some $\eta > 1$. This property, that we call *coherence*, is at the heart of the celebrated Sewing Lemma (Gubinelli [2], Feyel-de La Pradelle [1]), as it provides a sufficient condition on the germ A for the solution of (5.5)-(5.6).

DEFINITION 5.7. (COHERENCE) *A germ* $A \in C_2$ *is called* coherent *if, for some* $\eta > 1$*, it satisfies* $\delta A \in C_3^\eta$*, i.e.* $\|\delta A\|_\eta < \infty$*. More explicitly:*

$$\exists \eta \in (1,\infty): \qquad |\delta A_{sut}| \lesssim |t-s|^\eta\,, \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{5.10}$$

THEOREM 5.8. (SEWING LEMMA) *For any coherent germ* $A \in C_2$ *there exists a (unique) function* $I\colon [0,T] \to \mathbb{R}$ *such that* $|A_{st} - \delta I_{st}| = o(t-s)$*; equivalently, there exists a unique pair* $(I, R) \in C_1 \times C_2$ *such that*

$$I_0 = 0, \qquad I_t - I_s = A_{st} + R_{st} \qquad \text{with} \qquad R_{st} = o(t-s)\,. \tag{5.11}$$

- *The "remainder"* $R_{st} := \delta I_{st} - A_{st}$ *satisfies the Sewing Bound:*

$$\|R\|_\eta \leq K_\eta\,\|\delta A\|_\eta \qquad \text{where} \qquad K_\eta := (1 - 2^{1-\eta})^{-1}\,. \tag{5.12}$$

- *The* integral $I \in C_1$ *is the limit of* Riemann sums *of the germ:*

$$I_t := \lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{\#\mathcal{P}-1} A_{t_i t_{i+1}} \tag{5.13}$$

  along arbitrary partitions $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_k = t\}$ *of* $[0,t]$ *with vanishing mesh* $|\mathcal{P}| := \max_{i=0,\ldots,k-1} |t_{i+1} - t_i| \to 0$ *(we set* $\#\mathcal{P} := k$*).*

The Sewing Lemma is a cornerstone of the theory of *Rough Paths*, to be introduced in Chapter 7. We will already see in Chapter 6 an interesting application to *Young integrals*. The (instructive) proof of Theorem 5.8 is postponed to Section 5.6.

Remark 5.9. For a fixed partition $\mathcal{P}$ of $[0,t]$ we have, by $\delta I_{st} = A_{st} + R_{st}$,

$$I_t = \sum_{i=0}^{\#\mathcal{P}-1} A_{t_i t_{i+1}} + \sum_{i=0}^{\#\mathcal{P}-1} R_{t_i t_{i+1}}.$$

Therefore, (5.13) is equivalent to

$$\lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{\#\mathcal{P}-1} R_{t_i t_{i+1}} = 0$$

which is the reason why one wants the remainder R to be small close to the diagonal. The information $R_{st} = o(t-s)$ is not enough in general to obtain the existence of (I, R), while the stronger estimate $|R_{st}| \lesssim |t-s|^\eta$ is sufficient.

5.5. THE SEWING MAP

Given a coherent germ A, by Theorem 5.8 we can find an integral I and a remainder R which solve (5.5)-(5.6). We now look closer at the remainder R.

LEMMA 5.10. *In the setting of Theorem 5.8,* the remainder R is a function of δA*: given two coherent germs A, A' with $\delta A = \delta A'$, the corresponding remainders R, R' coincide. Moreover, the map $\delta A \mapsto R$ is linear.*

Proof. By Proposition 5.4 we have $\delta(R - R') = \delta(A' - A) = 0$, hence $R - R' = \delta f$ for some $f \in C_1$ (see Remark 1.10). Both $|R_{st}|$ and $|R'_{st}|$ are $o(|t-s|)$ by (5.6), hence $|f_t - f_s| = o(|t-s|)$. Then f must be constant by Lemma 5.1 and therefore $R = R'$. Linearity of the map $\delta A \mapsto R$ is easy. □

Since R is a function of δA, we introduce a specific notation for this map:

$$R = -\Lambda(\delta A)$$

where the minus sign is for later convenience.

Let us describe more precisely this map Λ. Throughout the following discussion, we fix arbitrarily $\eta \in (1, \infty)$.

- *Domain.* The map Λ is defined on δA for coherent germs A, see Definition 5.7. The domain of Λ is then $C_3^\eta \cap \delta C_2$, where we denote by $\delta C_2 \subseteq C_3$ the image of the space C_2 under the operator δ in (1.23).
- *Codomain.* The map Λ sends δA to $-R$, and we have $|R_{st}| \lesssim |t-s|^\eta$, see (5.12). A natural choice of codomain for Λ is then C_2^η.
- *Characterization.* In view of Proposition 5.4 and Lemma 5.2, the function $-R = \Lambda(\delta A)$ is characterized by the properties

  $$\delta(-R) = \delta A\,, \qquad |R_{st}| = o(t-s)\,.$$

  The second condition is already enforced by our choice C_2^η of codomain for Λ, which yields $|R_{st}| \lesssim |t-s|^\eta$ (with $\eta > 1$). The first relation can be rewritten as $\delta(\Lambda(B)) = B$ for all B in the domain of Λ, that is $\delta \circ \Lambda$ *is the identity map*.

In conclusion, we have proved the following result.

THEOREM 5.11. (SEWING MAP) *Let $\eta \in (1, \infty)$. There exists a unique map*

$$\Lambda\colon C_3^\eta \cap \delta C_2 \longrightarrow C_2^\eta,$$

called the Sewing Map, *such that* $\delta \circ \Lambda = \mathrm{id}$ *is the identity on* $C_3^\eta \cap \delta C_2$.

- *The map* Λ *is linear and satisfies*

$$\|\Lambda(B)\|_\eta \leqslant K_\eta \|B\|_\eta \qquad \forall B \in C_3^\eta \cap \delta C_2\,, \tag{5.14}$$

where K_η *is the same constant as in (5.12).*

- *Given a coherent germ* $A \in C_2$, *i.e. such that* $\delta A \in C_3^\eta$, *the unique solution* (I, R) *of (5.5)-(5.6) is* $R := -\Lambda(\delta A)$ *and* $I_t := A_{0t} + R_{0t}$.

5.6. Proof of the Sewing Lemma

We prove the Sewing Lemma, i.e. Theorem 5.8.

Proof. We fix a germ $A \in C_2$ with $\|\delta A\|_\eta < \infty$ for some $\eta > 1$ (we do *not* require $A_{ab} = o(b-a)$). Our goal is to build a function $I: [0,T] \to \mathbb{R}$ such that $|\delta I_{st} - A_{st}| = o(t-s)$. Uniqueness of I follows by Lemma 5.2, while the bound (5.12) follows by the Sewing Bound (1.26) applied to $R_{st} := \delta I_{st} - A_{st}$ (note that $\delta R = -\delta A$, because $\delta \circ \delta = 0$).

We fix $0 \leqslant s < t \leqslant T$. Given a partition $\mathcal{P} = \{s = t_0 < t_1 < \ldots < t_m = t\}$ of $[s,t]$, let us define $I_{\mathcal{P}}(A) := \sum_{i=0}^{m-1} A_{t_i t_{i+1}}$ as in (1.20). The following bound holds:

$$|I_{\mathcal{P}}(A) - A_{st}| \leq C_\eta \, \|\delta A\|_\eta \, (t-s)^\eta \qquad \text{with} \qquad C_\eta := \sum_{n \geq 1} \frac{2^\eta}{n^\eta} < \infty, \tag{5.15}$$

as we showed in the proof of Theorem 1.18, see (1.46), which applies to any function $A = (A_{s,t})$. Similarly, if $\mathcal{Q} \supseteq \mathcal{P}$ is another partition of $[s,t]$,

$$\begin{aligned}
|I_{\mathcal{Q}}(A) - I_{\mathcal{P}}(A)| &\leqslant \sum_{i=0}^{\#\mathcal{P}-1} |I_{\mathcal{Q} \cap [t_i, t_{i+1}]}(A) - A_{t_i t_{i+1}}| \\
&\leqslant C_\eta \, \|\delta A\|_\eta \sum_{i=0}^{\#\mathcal{P}-1} (t_{i+1} - t_i)^\eta \\
&\leqslant C_\eta \, \|\delta A\|_\eta \, |\mathcal{P}|^{\eta-1} \sum_{i=0}^{\#\mathcal{P}-1} (t_{i+1} - t_i) \\
&\leqslant C_\eta \, \|\delta A\|_\eta T \, |\mathcal{P}|^{\eta-1}
\end{aligned}$$

where we recall that $|\mathcal{P}| := \max_i (t_{i+1} - t_i)$. Finally, if $\mathcal{P}$ and $\mathcal{P}'$ are *arbitrary partitions*, setting $\mathcal{Q} := \mathcal{P} \cup \mathcal{P}'$ and applying the triangle inequality yields

$$|I_{\mathcal{P}'}(A) - I_{\mathcal{P}}(A)| \leq C_\eta \, \|\delta A\|_\eta T \, (|\mathcal{P}|^{\eta-1} + |\mathcal{P}'|^{\eta-1})\,.$$

This shows that *the family* $I_{\mathcal{P}}(A)$ *is Cauchy as* $|\mathcal{P}| \to 0$ (for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $|\mathcal{P}|, |\mathcal{P}'| \leq \delta_\epsilon$ implies $|I_{\mathcal{P}'}(A) - I_{\mathcal{P}}(A)| \leq \epsilon$), hence it admits a limit as $|\mathcal{P}| \to 0$, that we call J_{st}.

We now *define* $I_t := J_{0t}$. We claim that

$$I_t - I_s = J_{st} \qquad \text{for all} \quad 0 \leq s < t \leq T\,.$$

Indeed, if we consider partitions $\mathcal{P}'$ on $[0, s]$ and $\mathcal{P}$ of $[s, t]$, then $\mathcal{P}'' := \mathcal{P} \cup \mathcal{P}'$ is a partition of $[0, t]$ such that $I_{\mathcal{P}''}(A) - I_{\mathcal{P}'}(A) = I_{\mathcal{P}}(A)$, and taking the limit of vanishing mesh we get $J_{0t} - J_{0s} = J_{st}$, that is the claim.

Finally, taking the limit of relation (5.15), since $I_{\mathcal{P}}(A) \to J_{st} = I_t - I_s$, we obtain our goal $|\delta I_{st} - A_{st}| \lesssim (t - s)^{\eta} = o(t - s)$. This completes the proof, since (5.13) holds by construction. □

Remark 5.12. Taking the limit of (5.15) gives

$$|R_{st}| \leq C_{\eta} \, \|\delta A\|_{\eta} \, |t - s|^{\eta}, \qquad R_{st} := \delta I_{st} - A_{st}, \qquad 0 \leqslant s < t \leqslant T,$$

which is the bound (5.12) with K_{η} replaced by the worse constant C_{η}. This is because the estimate (5.15) holds for arbitrary partitions.

Chapter 6

The Young integral

We can now come back to the problem that we discussed at the beginning of Chapter 5: given two continuous functions $X, Y: [0,T] \to \mathbb{R}$, how can we give a meaning to the integral $I_t = \int_0^t Y \,\mathrm{d}X$ for $t \in [0,T]$?

A natural answer, recall (5.3), is to look for a function $I: [0,T] \to \mathbb{R}$ satisfying

$$I_0 = 0, \qquad I_t - I_s = Y_s (X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T\,. \tag{6.1}$$

As an application of the *Sewing Lemma* (Theorem 5.8), we can show that such a function I exists (and is necessarily unique) when X and Y are Hölder functions of exponents $\alpha, \beta \in\,]0,1]$ such that $\alpha + \beta > 1$. This leads to the notion of *Young integral*, to which this chapter is devoted.

Going beyond this setting, in order to treat the case $\alpha + \beta \leq 1$, will require the notion of *Rough Paths*, that we discuss in Chapter 7.

6.1. Construction of the Young integral

As we did in Chapter 5, it is convenient to rewrite (6.1) as follows: we look for a function $I: [0,T] \to \mathbb{R}$ satisfying

$$I_0 = 0, \qquad I_t - I_s = A_{st} + R_{st} \qquad \text{with} \qquad R_{st} = o(t-s), \tag{6.2}$$

where the *germ* $A: [0,T]^2_{\leqslant} \to \mathbb{R}$ is defined by

$$A_{st} = Y_s\, \delta X_{st} = Y_s\, (X_t - X_s)\,. \tag{6.3}$$

This is the framework of the *Sewing Lemma*, see Theorem 5.8, for which we need to fulfill the *coherence condition* (5.10), that is $\|\delta A\|_\eta < \infty$ for some $\eta > 1$ (we use the norms introduced in (1.9)). Recalling that

$$\delta A_{sut} := A_{st} - A_{su} - A_{ut} = -\delta Y_{su}\, \delta X_{ut}\,,$$

see (1.32), we can write for any $\alpha, \beta \in\,]0,1]$

$$|\delta A_{sut}| = |Y_u - Y_s|\, |X_t - X_u| \quad \Longrightarrow \quad \|\delta A\|_{\alpha+\beta} \leqslant \|\delta X\|_\alpha\, \|\delta Y\|_\beta\,. \tag{6.4}$$

As a consequence, it is natural to assume that $\|\delta X\|_\alpha < \infty$ and $\|\delta Y\|_\beta < \infty$ for α, $\beta \in\,]0,1]$ such that $\alpha + \beta > 1$.

We can now give a consistent definition of the integral $I_t = \int_0^t Y \,\mathrm{d}X$, known as *Young integral*, when X and Y are suitable Hölder functions.

THEOREM 6.1. (YOUNG INTEGRAL) *Fix* $\alpha, \beta \in]0,1]$ *with* $\alpha + \beta > 1$. *For every* $(X, Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ *there is a (necessarily unique) function* $I: [0,T] \to \mathbb{R}$ *which satisfies (6.1), i.e.*

$$I_0 = 0, \qquad I_t - I_s = Y_s (X_t - X_s) + o(t-s)\,. \tag{6.5}$$

The functon I*, called the* Young integral*, is also denoted by* $I_t =: \int_0^t Y \mathrm{d}X$.

The remainder $R_{st} := I_t - I_s - Y_s (X_t - X_s)$ *satisfies the bound*

$$\|R\|_{\alpha+\beta} \leq K_{\alpha+\beta} \, \|\delta X\|_\alpha \, \|\delta Y\|_\beta \,, \tag{6.6}$$

where $K_\eta := (1 - 2^{1-\eta})^{-1}$*, see (5.12). This yields* $I \in \mathcal{C}^\alpha$*, more precisely*

$$\|\delta I\|_\alpha \leq (\|Y\|_\infty + K_{\alpha+\beta} T^\beta \, \|\delta Y\|_\beta) \, \|\delta X\|_\alpha \,. \tag{6.7}$$

The Young integral $I = (I_t)_{t \in [0,T]}$*, as a function of* (X, Y)*, is a continuous bilinear map* $I: \mathcal{C}^\alpha \times \mathcal{C}^\beta \to \mathcal{C}^\alpha$.

Proof. Recalling (6.2)-(6.4), we have $\|\delta A\|_{\alpha+\beta} \leqslant \|\delta X\|_\alpha \|\delta Y\|_\beta < \infty$, that is $\delta A \in C_3^\eta$ with $\eta = \alpha + \beta > 1$, where the spaces C_k^η were defined in (1.10). By the *Sewing Lemma*, see Theorem 5.8, there exists a (unique) function I which satisfies (5.11) and (5.12), hence (6.5) and (6.6) hold.

In order to prove (6.7), we note that

$$\begin{aligned} \|\delta I\|_\alpha &\leqslant \|A\|_\alpha + \|R\|_\alpha \leqslant \|Y\|_\infty \, \|\delta X\|_\alpha + T^\beta \, \|R\|_{\alpha+\beta} \\ &\leqslant \|Y\|_\infty \, \|\delta X\|_\alpha + T^\beta K_{\alpha+\beta} \, \|\delta X\|_\alpha \, \|\delta Y\|_\beta \,. \end{aligned}$$

Recalling Remark 1.4, in particular (1.15), this bound implies that I is a continuous function of (X, Y), as a map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^\alpha$.

We finally prove that the map $(X, Y) \mapsto I$ is bilinear: given $X, X' \in \mathcal{C}^\alpha$ and a fixed $Y \in \mathcal{C}^\beta$, if I satisfies (6.5) for (X, Y) and I' satisfies (6.5) for (X', Y), then for any $a, b \in \mathbb{R}$ the function $\hat{I}_t := a\, I_t + b\, I_t'$ satisfies (6.5) for $(\hat{X} := a\, X + b\, X', Y)$. Linearity with respect to Y is proved similarly. □

Remark 6.2. The setting of Theorem 6.1 provides a natural example of a germ $A_{st} := Y_s \, \delta X_{st}$ which is *not* in C_2^η for any $\eta > 1$ (excluding the trivial case when $Y \equiv 0$ on the intervals where X is not constant, hence $A \equiv 0$), but it satisfies $\delta A \in C_3^\eta$ with $\eta = \alpha + \beta > 1$.

Remark 6.3. (BEYOND YOUNG) It is natural to wonder what happens in Theorem 6.1 for $(X, Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ with $\alpha + \beta \leq 1$. In this case, *there might be no solution to (5.5)-(5.6)*, because the necessary condition (5.9) in Lemma 5.5 can fail. For a simple example, consider $X_t = t^\alpha$ and $Y_t = t^\beta$ for $t \in [0,T]$ and note that for $s = 0$ and $u = \frac{t}{2}$ we have by (1.32)

$$|\delta A_{sut}| = |\delta A_{0\frac{t}{2}t}| = \left|\delta Y_{0\frac{t}{2}}\right| \left|\delta X_{\frac{t}{2}t}\right| = \left(\frac{t}{2}\right)^\beta \left(t^\alpha - \left(\frac{t}{2}\right)^\alpha\right) \gtrsim t^{\alpha+\beta}, \tag{6.8}$$

which is not $o(t-s) = o(t)$ when $\alpha + \beta \leq 1$.

In order to define a notion of integral $I_t = \int_0^t Y_s \, \mathrm{d}X_s$ when $(X, Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ with $\alpha + \beta \leq 1$, we need to relax condition (5.3), see Definition 7.1 below. This will lead to the notion of *Rough Paths*, described in Chapter 7.

6.2. Integral formulation of Young equations

In this section we explain why we call (2.4) a *Young* equation. In fact, we can interpret the finite difference equation (2.4) as an *integral equation*, using the Young integral of section 6.1.

Proposition 6.4. *Let $Z \in \mathcal{C}^\alpha([0, T]; \mathbb{R}^k)$ with $\alpha > \frac{1}{2}$. Then Z satisfies (2.4) if and only if*

$$Z_t = Z_0 + \int_0^t \sigma(Z_s) \, \mathrm{d}X_s, \qquad t \in [0, T], \tag{6.9}$$

where the integral is in the Young sense.

Proof. We consider the germ $A_{st} := \sigma(Z_s) \, \delta X_{st}$, $0 \leqslant s \leqslant t \leqslant T$. By (6.4)

$$|\delta A_{sut}| = |\sigma(Z_u) - \sigma(Z_s)||X_t - X_u| \quad \Longrightarrow \quad \|\delta A\|_{2\alpha} \leqslant \|\nabla \sigma\|_\infty \|\delta X\|_\alpha \|\delta Z\|_\alpha .$$

Therefore we obtain that (2.4) is equivalent to (6.5) above. □

In the case $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, this argument does not work and the Young integral is not adapted, since the germ $A_{st} := \sigma(Z_s) \, \delta X_{st}$ has the property $\delta A \in C_3^{2\alpha}$ with $2\alpha \leqslant 1$, so that the Sewing Lemma can not be applied. However the equation (3.19) suggests *another* germ:

$$A_{st} := \sigma(Z_s) \, \mathbb{X}_{st}^1 + \sigma_2(Z_s) \, \mathbb{X}_{st}^2, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Note that $A = \delta Z - Z^{[3]}$, in the notation (3.19). Then by (3.27) we know that $\delta A \in C_3^{3\alpha}$. Therefore we can interpret the formula

$$\delta Z = A - \Lambda(\delta A)$$

as

$$Z_t = Z_0 + \int_0^t \sigma(Z_s) \, \mathrm{d}\mathbb{X}_s, \qquad 0 \leqslant t \leqslant T,$$

which for the moment is only a notation that will be made more precise in chapter 9.

6.3. Local existence via contraction

As an application of the estimates on the Young integral of Theorem 6.1, we want to give a local existence result for equation (2.4) which does not rely on compactness and which can be therefore used also in infinite dimension.

Let $Z_0 \in \mathbb{R}^k$ and $X \in \mathcal{C}^\alpha([0, T]; \mathbb{R}^d)$ be given, $\sigma: \mathbb{R}^k \to \mathbb{R}^d \otimes (\mathbb{R}^d)^*$ smooth and the unknown $Z: [0, T] \to \mathbb{R}^k$ is such that $\sigma(Z) \in \mathcal{C}^\alpha$ and $2\alpha > 1$, so that the right-hand side of (6.9) can be interpreted as a Young integral. We want now to show the following

THEOREM 6.5. (CONTRACTION FOR YOUNG DIFFERENTIAL EQUATIONS) *Let* σ: $\mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ *be of class* C^2 *with* $\nabla\sigma$ *and* $\nabla^2\sigma$ *bounded. Let* $\alpha \in]\frac{1}{2}, 1]$ *and* $X \in \mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ *fixed. It* $T > 0$ *is small enough, then for any* $Z_0 \in \mathbb{R}$ *there exists a unique* $Z \in \mathcal{C}^\alpha([0,T];\mathbb{R}^k)$ *which satisfies (6.9).*

Proof. For all $f \in \mathcal{C}^\alpha([0,T];\mathbb{R}^k)$ we have

$$|\sigma(f_t) - \sigma(f_s)| \leqslant \|\nabla\sigma\|_\infty |f_t - f_s|$$

so that

$$\|\delta\sigma(f)\|_\alpha \leqslant \|\nabla\sigma\|_\infty \|\delta f\|_\alpha.$$

By (6.7) with $\alpha = \beta$ we obtain for all $f \in \mathcal{C}^\alpha$ satisfying (6.9)

$$\|\delta f\|_\alpha \leqslant (|\sigma(f_0)| + (1 + K_{2\alpha}) T^\alpha \|\nabla\sigma\|_\infty \|\delta f\|_\alpha) \|\delta X\|_\alpha$$

since

$$\|\sigma(f)\|_\infty \leqslant |\sigma(f_0)| + T^\alpha \|\delta\sigma(f)\|_\alpha.$$

Therefore, if T satisfies

$$T^\alpha \leqslant \frac{1}{2} \frac{1}{(1 + K_{2\alpha}) \|\nabla\sigma\|_\infty \|\delta X\|_\alpha}$$

then we have the following a priori estimate on solutions to (6.9)

$$\|\delta Z\|_\alpha \leqslant 2|\sigma(Z_0)| \, \|\delta X\|_\alpha.$$

We fix such T and we set $\mathcal{C}^\alpha(Z_0) := \{f \in \mathcal{C}^\alpha : f_0 = Z_0, \|\delta f\|_\alpha \leqslant 2|\sigma(Z_0)| \, \|\delta X\|_\alpha\}$. Then we define $\Lambda: \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ given by

$$\Lambda(f) := h, \qquad h_t := Z_0 + \int_0^t \sigma(f_s) \, \mathrm{d}X_s, \qquad t \in [0,T].$$

It is easy to see, arguing as above, that Λ acts on $\mathcal{C}^\alpha(Z_0)$, namely $\Lambda: \mathcal{C}^\alpha(Z_0) \to \mathcal{C}^\alpha(Z_0)$. Note that the map $\mathcal{C}^\alpha(Z_0) \times \mathcal{C}^\alpha(Z_0) \ni (a,b) \mapsto \|\delta a - \delta b\|_\alpha$ defines a distance on $\mathcal{C}^\alpha(Z_0)$ which induces the same topology as $\|\cdot\|_{\mathcal{C}^\alpha}$. We want to show that Λ is a contraction for this distance if T is small enough. By (6.7) we have for $\alpha = \beta$

$$\begin{aligned}\|\delta\Lambda(a) - \delta\Lambda(b)\|_\alpha &\leqslant (\|\sigma(a) - \sigma(b)\|_\infty + K_{2\alpha} T^\alpha \|\delta\sigma(a) - \delta\sigma(b)\|_\alpha) \|\delta X\|_\alpha. \\ &\leqslant T^\alpha (1 + K_{2\alpha}) \|\delta X\|_\alpha \|\delta\sigma(a) - \delta\sigma(b)\|_\alpha.\end{aligned}$$

We now need to estimate $\|\delta\sigma(a) - \delta\sigma(b)\|_\alpha$. By Lemma 2.8

$$\|\delta\sigma(a) - \delta\sigma(b)\|_\alpha \leqslant \|\nabla\sigma\|_\infty \|\delta a - \delta b\|_\alpha + \|\nabla^2\sigma\|_\infty (\|\delta a\|_\alpha + \|\delta b\|_\alpha) \|a - b\|_\infty.$$

Since, as usual, $\|a - b\|_\infty \leqslant T^\alpha \|\delta a - \delta b\|_\alpha$, we obtain

$$\|\delta\sigma(a) - \delta\sigma(b)\|_\alpha \leqslant (\|\nabla\sigma\|_\infty + T^\alpha \|\nabla^2\sigma\|_\infty (\|\delta a\|_\alpha + \|\delta b\|_\alpha)) \|\delta a - \delta b\|_\alpha. \tag{6.10}$$

Therefore, for all $a, b \in \mathcal{C}^\alpha(Z_0)$

$$\|\delta\Lambda(a) - \delta\Lambda(b)\|_\alpha \leqslant C_T \|\delta a - \delta b\|_\alpha,$$

where $C_T := T^\alpha (1 + K_{2\alpha}) \|\delta X\|_\alpha (\|\nabla\sigma\|_\infty + T^\alpha \|\nabla^2\sigma\|_\infty 4|\sigma(Z_0)| \, \|\delta X\|_\alpha)$. It is now enough to consider T small enough so that $C_T < 1$. □

6.4. Properties of the Young integral

The Young integral $\int_0^t Y\,\mathrm{d}X$, defined in Theorem 6.1, shares many properties with the classical Riemann-Lebesgue integral, that we now discuss.

A elementary but useful observation is that $\int_0^t Y\,\mathrm{d}X$ is a linear function of Y (for fixed X) and a linear function of X (for fixed Y), by bilinearity.

For an interval $[s,t]\subset[0,T]$ we will use the notation

$$I_t - I_s =: \int_s^t Y\,\mathrm{d}X\,.$$

If the integrand $Y_u = c$ is constant for all $u\in[s,t]$, then $\int_s^t Y\,\mathrm{d}X = c\,(X_t - X_s)$, which follows directly from (6.5). As a corollary, we obtain the following useful formula for the remainder.

Lemma 6.6. *Let $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$ for $\alpha,\beta\in]0,1]$ with $\alpha+\beta>1$ and let $I_t := \int_0^t Y_u\,\mathrm{d}X_u$ be the Young integral, see Theorem 6.1. Then the remainder*

$$R_{st} := I_t - I_s - Y_s\,(X_t - X_s), \qquad 0\leqslant s\leqslant t\leqslant T,$$

admits the explicit formula

$$R_{st} = \int_s^t (Y_u - Y_s)\,\mathrm{d}X_u, \qquad 0\leqslant s\leqslant t\leqslant T, \tag{6.11}$$

where the right hand side is a Young integral.

Proof. By linearity and the basic property mentioned above, we obtain

$$\int_s^t (Y_u - Y_s)\,\mathrm{d}X_u = \int_s^t Y_u\,\mathrm{d}X_u - \int_s^t Y_s\,\mathrm{d}X_u = I_t - I_s - Y_s\,(X_t - X_s) = R_{st}\,. \qquad \square$$

An important property is *integration by parts*, which follows by the uniqueness of the solution for the problem (5.5)-(5.6), recall Lemma 5.2.

Proposition 6.7. (Integration by parts) *Fix $\alpha,\beta\in\,]0,1]$ with $\alpha+\beta>1$. For all $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$ the Young integral satisfies*

$$\int_0^t X\,\mathrm{d}Y + \int_0^t Y\,\mathrm{d}X = X_t Y_t - X_0 Y_0\,. \tag{6.12}$$

Proof. Let us set $I'_t := \int_0^t X\,\mathrm{d}Y + \int_0^t Y\,\mathrm{d}X$. By the property (6.5) we have

$$I'_t - I'_s = \underbrace{Y_s(X_t - X_s) + X_s(Y_t - Y_s)}_{A_{st}} + o(t-s)\,.$$

Next we set $I''_t := X_t\,Y_t - X_0\,Y_0$ and note that, by direct computation,

$$I''_t - I''_s = \underbrace{Y_s(X_t - X_s) + X_s(Y_t - Y_s)}_{A_{st}} + \underbrace{(X_t - X_s)(Y_t - Y_s)}_{R_{st}}\,,$$

where $|R_{st}| \leqslant \|\delta X\|_\alpha \|\delta Y\|_\beta |t-s|^{\alpha+\beta} = o(t-s)$. By Lemma 5.2, for any germ A, there can be at most one function I which satisfies $\delta I_{st} = A_{st} + o(t-s)$ (5.5)-(5.6), hence $I' = I''$. □

We next discuss the *chain rule*.

PROPOSITION 6.8. (CHAIN RULE) *Let $X \in \mathcal{C}^\alpha$ with $\alpha \in]\frac{1}{2}, 1]$. Let $\varphi: \mathbb{R} \to \mathbb{R}$ be differentiable with $\varphi' \in \mathcal{C}^\gamma(\mathbb{R})$, for $\gamma \in]0,1]$ such that $\gamma > 1/(1+\alpha)$ (a sufficient condition is that $\varphi \in C^2$). Then $\varphi'(X) = \varphi' \circ X \in \mathcal{C}^{\alpha\gamma}$ and*

$$\varphi(X_t) - \varphi(X_0) = \int_0^t \varphi'(X)\, \mathrm{d}X\,, \tag{6.13}$$

where the right hand side is a Young integral.

Proof. It is easy to see that $\varphi'(X) \in \mathcal{C}^{\alpha\gamma}$, which implies that $\int_0^t \varphi'(X)\, \mathrm{d}X$ is well-defined as a Young integral, since $\alpha + \alpha\gamma > 1$. By definition (6.5) of the Young integral, proving (6.13) amounts to showing that

$$|\varphi(X_t) - \varphi(X_s) - \varphi'(X_s)\ (X_t - X_s)| \lesssim |t-s|^{\alpha+\alpha\gamma}$$

By the classical Lagrange theorem, if say $X_t > X_s$, then

$$\varphi(X_t) - \varphi(X_s) - \varphi'(X_s)(X_t - X_s) = (\varphi'(\xi) - \varphi'(X_s))(X_t - X_s)$$

with $\xi \in]X_s, X_t[$. Since $\varphi' \in \mathcal{C}^\gamma$ and $X \in \mathcal{C}^\alpha$, it follows that

$$|\varphi(X_t) - \varphi(X_s) - \varphi'(X_s)\,(X_t - X_s)| \lesssim |X_t - X_s|^{\gamma+1} \lesssim |t-s|^{\alpha+\alpha\gamma}$$

which completes the proof. □

More generally, we have

COROLLARY 6.9. *In the same setting of Proposition 6.8, for all $s \leqslant t$*

$$\varphi(X_t) - \varphi(X_s) = \varphi'(X_s)(X_t - X_s) + \int_s^t (\varphi'(X_r) - \varphi'(X_s))\, \mathrm{d}X_r\,. \tag{6.14}$$

Proof. It is enough to note that, by (6.13),

$$\begin{aligned}\varphi(X_t) - \varphi(X_s) &= \int_s^t \varphi'(X_r)\, \mathrm{d}X_r \\ &= \varphi'(X_s)(X_t - X_s) + \int_s^t (\varphi'(X_r) - \varphi'(X_s))\, \mathrm{d}X_r\,,\end{aligned}$$

where all integrals are in the Young sense. □

In particular, for $X \in \mathcal{C}^\alpha$ with $\alpha > \frac{1}{2}$, we have

$$\frac{X_t^2}{2} - \frac{X_s^2}{2} = X_s(X_t - X_s) + \int_s^t (X_r - X_s)\, \mathrm{d}X_r, \tag{6.15}$$

which can be rewritten as follows:

$$\int_s^t (X_r - X_s)\,\mathrm{d}X_r = \frac{X_t^2}{2} - \frac{X_s^2}{2} - X_s\,(X_t - X_s) = \frac{(X_t - X_s)^2}{2}\,. \tag{6.16}$$

6.5. MORE ON HÖLDER SPACES

We discuss further properties of the Hölder spaces $\mathcal{C}^\alpha$ for $\alpha \in (0,1)$ (excluding the case $\alpha = 1$ of Lipschitz functions). These will be useful in the next Section 6.6, when we discuss the uniqueness of the Young integral.

Let us denote by C^∞ the space of infinitely differentiable functions. We note that $C^\infty \subset \mathcal{C}^\alpha$ for every $\alpha \in (0,1)$, but C^∞ *is not dense in* $\mathcal{C}^\alpha$.

THEOREM 6.10. *For any $\alpha \in (0,1)$, the closure of C^∞ in $\mathcal{C}^\alpha$ is the subset $\mathcal{C}_0^\alpha$ defined by*

$$\mathcal{C}_0^\alpha \ := \{f: [0,T] \to \mathbb{R}: \quad |f(t) - f(s)| = o(t-s) \text{ uniformly as } |t-s| \to 0\}\,.$$

Remark 6.11. Note that $f \in \mathcal{C}_0^\alpha$ if and only if

$$\forall \epsilon > 0 \quad \exists \delta_\epsilon > 0: \qquad |f(t) - f(s)| \le \epsilon |t-s|^\alpha \qquad \text{for } |t-s| \le \delta_\epsilon\,, \tag{6.17}$$

which implies (exercise) that $C^1 \subset \mathcal{C}_0^\alpha \subset \mathcal{C}^\alpha$ for $\alpha \in (0,1)$. It follows that the closure of C^1 in $\mathcal{C}^\alpha$ is again $\mathcal{C}_0^\alpha$, simply because $C^\infty \subset C^1 \subset \mathcal{C}_0^\alpha$.

Exercise 6.1. Prove that $C^1 \subset \mathcal{C}_0^\alpha$ and $\mathcal{C}_0^\alpha \subset \mathcal{C}^\alpha$ for $\alpha \in (0,1)$ (inclusions are strict).

We stress that the subset $\mathcal{C}_0^\alpha$ is strictly included in $\mathcal{C}^\alpha$, but what is left out is not so large, in the following sense.

Exercise 6.2. Prove that $\mathcal{C}^{\alpha'} \subset \mathcal{C}_0^\alpha$ for $0 < \alpha < \alpha' < 1$ (the inclusion is strict).

The proof of Theorem 6.10, which we defer to Section 6.7, is based on the following classical approximation result (also proved in Section 6.7).

LEMMA 6.12. *For any continuous $f: [0,T] \to \mathbb{R}$ there is a sequence $f_n \in C^\infty$ such that $\|f_n - f\|_\infty \to 0$. One can take f_n with the same modulus of continuity as f, in the following sense: given an arbitrary function $h(\cdot)$,*

$$\begin{array}{rll} \textit{if} & |f(t) - f(s)| \le h(t-s) & \forall s,t \in [0,T]\,, \\ \textit{then} & |f_n(t) - f_n(s)| \le h(t-s) & \forall s,t \in [0,T], \quad \forall n \in \mathbb{N}\,. \end{array} \tag{6.18}$$

It follows that $\|\delta f_n\|_\alpha \le \|\delta f\|_\alpha$ for all $n \in \mathbb{N}$ and $\alpha \in (0,1)$.

Remark 6.13. Lemma 6.12 holds with no change for functions $f: [0,T] \to R$, where R is an arbitrary Banach space. One only needs a notion of integral $\int_0^T f_s\,\mathrm{d}s$ when f is continuous, and for this one can take the Riemann integral, i.e. the limit of Riemann sums $\sum_i f(t_i)(t_{i+1} - t_i)$ along partitions (t_i) of $[0,T]$ with vanishing mesh $\max_i |t_{i+1} - t_i| \to 0$ (one can check that such Riemann sums form a Cauchy family). This integral satisfies the key usual properties: $f \mapsto \int_0^T f_s\,\mathrm{d}s$ is linear, $|\int_0^T f_s\,\mathrm{d}s| \le \int_0^T |f_s|\,\mathrm{d}s$ and $\int_0^T f_s'\,\mathrm{d}s = f_T - f_0$.

6.6. Uniqueness of the Young integral

Throughout this section we denote by I_t^{Young} the *Young integral* $I_t = \int_0^t Y \, \mathrm{d}X$ built in Theorem 6.1. We want to compare it with the *classical integral*

$$I_t^{\text{classical}} := \int_0^t Y_u \, \dot{X}_u \, \mathrm{d}u$$

which is defined for continuous Y and continuously differentiable $X \in C^1$.

We remarked in (5.2)-(5.3) that $I_t^{\text{classical}}$ satisfies property (6.5), therefore $I_t^{\text{classical}}$ coincides with I_t^{Young} when $(X, Y) \in C^1 \times \mathcal{C}^\beta$, for any $\beta \in]0, 1]$. In other terms, the *Young integral is an extension of the classical integral*.

We can be more precise: by Theorem 6.1, for $\alpha, \beta \in]0, 1]$ with $\alpha + \beta > 1$, the Young integral $I^{\text{Young}} = (I_t^{\text{Young}})_{t \in [0,T]}$ is a continuous bilinear map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^\alpha$. This means that I^{Young} is a *continuous* extension of the classical integral $I^{\text{classical}}$ defined on $C^1 \times \mathcal{C}^\beta$. It would be tempting to state that it is the *unique* continuous extension, but *this is not true*, because $C^1 \subset \mathcal{C}^\alpha$ is not dense in $\mathcal{C}^a$ (see Theorem 6.10 and Remark 6.11).

Interestingly, it is possible to characterize the Young integral as the unique continuous extension of $I^{\text{classical}}$, if we let the exponent α vary. Given $\bar{\alpha} \in]0, 1[$, we define the space

$$\mathcal{C}^{>\bar{\alpha}} := \bigcup_{\alpha \in]\bar{\alpha}, 1]} \mathcal{C}^\alpha$$

and we agree that $f_n \to f$ in $\mathcal{C}^{>\bar{\alpha}}$ if and only if $f_n \to f$ in $\mathcal{C}^\alpha$ for some $\alpha > \bar{\alpha}$. The basic observation is that C^1 *is dense in* $\mathcal{C}^{>\bar{\alpha}}$: for any $f \in \mathcal{C}^{>\bar{\alpha}}$ we can find a sequence $f_n \in C^1$ such that $f_n \to f$ in $\mathcal{C}^{>\bar{\alpha}}$.[6.1]

If we fix $\bar{\alpha} = 1 - \beta$, for $\beta \in]0, 1]$, the Young integral $I^{\text{Young}} = (I_t^{\text{Young}})_{t \in [0,T]}$ is a continuous map from $\mathcal{C}^{>(1-\beta)} \times \mathcal{C}^\beta$ to $\mathcal{C}^{>(1-\beta)}$, by Theorem 6.1.

These observations yield immediately the following result.

Proposition 6.14. (Characterization of the Young integral, I) *Fix any $\beta \in]0, 1]$. The Young integral $I^{\text{Young}} = (I_t^{\text{Young}})_{t \in [0,T]}$, viewed as a map from $\mathcal{C}^{>(1-\beta)} \times \mathcal{C}^\beta$ to $\mathcal{C}^{>(1-\beta)}$, is the unique continuous extension of the classical integral $I^{\text{classical}} = (I_t^{\text{classical}})_{t \in [0,T]}$ defined on $C^1 \times \mathcal{C}^\beta$.*

Explicitly, I^{Young} is the unique map $I: \mathcal{C}^{>(1-\beta)} \times \mathcal{C}^\beta \to \mathcal{C}^{>(1-\beta)}$ such that:

- $I_t = I_t^{\text{classical}} = \int_0^t Y_u \, \dot{X}_u \, \mathrm{d}u$ *for* $X \in C^1$;
- *if* $(X_n, Y_n) \to (X, Y)$ *in* $\mathcal{C}^\alpha \times \mathcal{C}^\beta$, *for some* $\alpha > 1 - \beta$, *then we have the convergence* $I(X_n, Y_n) \to I(X, Y)$ *in* $\mathcal{C}^{\alpha'}$ *for some* $\alpha' > 1 - \beta$.

Alternatively, we can characterize the Young integral as the unique continuous extension of the classical integral on $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ for fixed α, provided we consider a weaker notion of convergence on $\mathcal{C}^\alpha$.

6.1. If $f \in \mathcal{C}^\alpha$ with $\alpha > \bar{\alpha}$, by Exercise 6.2 we have $f \in \mathcal{C}_0^{\alpha'}$ for any $\alpha' \in]\bar{\alpha}, \alpha[$, then by Theorem 6.10 we can find $f_n \in C^\infty$ such that $f_n \to f$ in $\mathcal{C}^{\alpha'}$, hence $f_n \to f$ in $\mathcal{C}^{>\bar{\alpha}}$.

Definition 6.15. *Fix* $\alpha \in]0,1]$. *Given* $f_n, f: [0,T] \to \mathbb{R}$, *with* $n \in \mathbb{N}$, *we write*

$$f_n \rightsquigarrow_\alpha f \qquad \Longleftrightarrow \qquad \|f_n - f\|_\infty \to 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|\delta f_n\|_\alpha < \infty \,. \tag{6.19}$$

In other terms, $f_n \rightsquigarrow_\alpha f$ *if and only if* $f_n \to f$ *in the sup-norm and, moreover, the sequence* f_n *is bounded in* $\mathcal{C}^\alpha$.

We leave it as an exercise to check some basic properties.

Exercise 6.3. Fix $\alpha \in]0,1]$ and let $f_n, f: [0,T] \to \mathbb{R}$, with $n \in \mathbb{N}$. Prove the following.

1. If $f_n \rightsquigarrow_\alpha f$, then $f \in \mathcal{C}^\alpha$; more precisely $\|\delta f\|_\alpha \le \sup_{n \in \mathbb{N}} \|\delta f_n\|_\alpha < \infty$.
2. If $f_n \rightsquigarrow_\alpha f$, then $f_n \to f$ in $\mathcal{C}^{\alpha'}$ for any $\alpha' < \alpha$, but not necessarily $f_n \to f$ in $\mathcal{C}^\alpha$.
3. If $f_n \rightsquigarrow_\alpha f$ and $\varphi: \mathbb{R} \to \mathbb{R}$ is Lipschitz, then $\varphi(f_n) \rightsquigarrow_\alpha \varphi(f)$.
4. In the definition (6.19) of $f_n \rightsquigarrow_\alpha f$, the uniform convergence $\|f_n - f\|_\infty \to 0$ can be replaced by pointwise convergence: $f_n(t) \to f(t)$ for every $t \in [0,T]$.

We can now provide the following characterization of the Young integral.

Theorem 6.16. (Characterization of the Young integral, II) *Fix* α, $\beta \in]0,1]$ *with* $\alpha + \beta > 1$. *The Young integral* $I^{\text{Young}} = (I_t^{\text{Young}})_{t \in [0,T]}$ *is the unique map* $I: \mathcal{C}^\alpha \times \mathcal{C}^\beta \to \mathcal{C}^\alpha$ *such that:*

1. $I_t = I_t^{\text{classical}} = \int_0^t Y_u \, \dot{X}_u \, \mathrm{d}u$ *for* $X \in C^1$;
2. *if* $X_n \rightsquigarrow_\alpha X$ *and* $Y_n \rightsquigarrow_\beta Y$, *we have* $I(X_n, Y_n) \rightsquigarrow_\alpha I(X,Y)$.

Proof. We already know that the Young integral I^{Young} satisfies property 1. Let us show that it also satisfies property 2: given $X_n \rightsquigarrow_\alpha X$ and $Y_n \rightsquigarrow_\beta Y$, we need to prove that

$$I^{\text{Young}}(X_n, Y_n) \rightsquigarrow_\alpha I^{\text{Young}}(X,Y) \,. \tag{6.20}$$

Let us fix $\alpha' < \alpha$, $\beta' < \beta$ such that we still have $\alpha' + \beta' > 1$. We know by Exercise 6.3 that $X_n \to X$ in $\mathcal{C}^{\alpha'}$ and $Y_n \to Y$ in $\mathcal{C}^{\beta'}$. Since the Young integral is a continuous bilinear operator $I^{\text{Young}}: \mathcal{C}^{\alpha'} \times \mathcal{C}^{\beta'} \to \mathcal{C}^{\beta'}$, we have the convergence $I^{\text{Young}}(X_n, Y_n) \to I^{\text{Young}}(X,Y)$ in $\mathcal{C}^{\alpha'}$, which implies

$$\|I^{\text{Young}}(X_n, Y_n) - I^{\text{Young}}(X,Y)\|_\infty \to 0 \,.$$

To prove (6.20), it remains to observe that, by (6.7),

$$\sup_n \|I^{\text{Young}}(X_n, Y_n)\|_\alpha \leqslant \sup_n \left(\|Y_n\|_\infty + K_{\alpha+\beta} T^\alpha \, \|\delta Y_n\|_\beta \right) \|X_n\|_\alpha < \infty \,.$$

We next consider an operator $I: \mathcal{C}^\alpha \times \mathcal{C}^\beta \to \mathcal{C}^\alpha$ which satisfies properties 1 and 2 and we show that it must coincide with the Young integral I^{Young}. Given $X \in C^\alpha$ and $Y \in C^\beta$, by Lemma 6.12 we can construct a sequence $(X_n) \subset C^1$ with $\|X_n - X\|_\infty \to 0$ and $\|X_n\|_\alpha \leqslant \|X\|_\alpha$. By property 2 we have $I(X_n, Y) \rightsquigarrow_\alpha I(X,Y)$ and $I^{\text{Young}}(X_n, Y) \rightsquigarrow_\alpha I^{\text{Young}}(X,Y)$, which implies pointwise convergence: for any $t \in [0,T]$

$$I_t(X,Y) = \lim_n I_t(X_n, Y) \qquad \text{and} \qquad I_t^{\text{Young}}(X,Y) = \lim_n I_t^{\text{Young}}(X_n, Y) \,.$$

By property 1 we have $I_t(X_n, Y) = I_t^{\text{Young}}(X_n, Y)$ for any n, hence

$$I_t(X, Y) = I_t^{\text{Young}}(X, Y) \qquad \forall t \in [0, T],$$

which completes the proof. □

6.7. Two technical proofs

We give here the proof of Theorem 6.10 and Lemma 6.12.

Proof of Lemma 6.12. We extend $f: [0, T] \to \mathbb{R}$ to a function defined on the whole real line, by setting $f(t) = f(0)$ for $t < 0$ and $f(t) = f(T)$ for $t > T$.

Let us fix a C^∞ function $\varphi: \mathbb{R} \to \mathbb{R}$ supported in $[-1, 1]$ with unit integral: $\int_{\mathbb{R}} \varphi(u)\,\mathrm{d}u = 1$. Note that $\varphi_n(t) := n\varphi(nt)$ is supported in $\left[-\frac{1}{n}, \frac{1}{n}\right]$ and also has unit integral: $\int_{\mathbb{R}} \varphi_n(u)\,\mathrm{d}u = 1$. We then define $f_n = \varphi_n * f$, that is

$$f_n(t) := \int_{\mathbb{R}} \varphi_n(t-u)\, f(u)\,\mathrm{d}u.$$

It is a classical result that $f_n \in C^\infty$ (we can differentiate inside the integral by dominated convergence, since f is bounded).

We next write

$$f_n(t) = \int_{\mathbb{R}} \varphi_n(u)\, f(t-u)\,\mathrm{d}u = \int_{\mathbb{R}} \varphi(v)\, f\Big(t - \frac{v}{n}\Big)\,\mathrm{d}v,$$

which implies $\|f_n - f\|_\infty \le \sup_{t\in\mathbb{R}, |u|\le 1} \left| f\left(t - \frac{v}{n}\right) - f(t)\right|$ (since φ has unit integral), hence $\|f_n - f\|_\infty \to \infty$. Property (6.18) is also directly checked. □

Proof of Theorem 6.10. First we show that $\mathcal{C}_0^\alpha$ is closed in $\mathcal{C}^\alpha$: given f_n in $\mathcal{C}_0^\alpha$ and $f \in \mathcal{C}^\alpha$ such that $\|f_n - f\|_\alpha \to 0$, we need to show that $f \in \mathcal{C}_0^\alpha$, that is (6.17) holds. For $s < t$ and $n \in \mathbb{N}$ we can write, by the triangle inequality,

$$\frac{|f(t) - f(s)|}{(t-s)^\alpha} \le \|\delta f - \delta f_n\|_\alpha + \frac{|f_n(t) - f_n(s)|}{(t-s)^\alpha}. \tag{6.21}$$

Fix $n = \bar{n}_\epsilon$ such that $\|\delta f_{\bar{n}_\epsilon} - \delta f\|_\alpha < \frac{\epsilon}{2}$. Since $f_{\bar{n}_\epsilon} \in \mathcal{C}_0^\alpha$, by (6.17) we can fix $\delta_\epsilon > 0$ such that for $|t - s| \le \delta$ the last term in (6.21) is $\le \frac{\epsilon}{2}$ and we are done.

It remains to show that, for any $f \in \mathcal{C}_0^\alpha$, there is a sequence $f_n \in C^\infty$ such that $\|f_n - f\|_\infty + \|\delta f_n - \delta f\|_\alpha \to 0$ (recall Remark 1.4). We define $f_n \in C^\infty$ as in Lemma 6.12, so we only need to show that $\|\delta f_n - \delta f\|_\alpha \to 0$.

Since $f \in \mathcal{C}_0^\alpha$, property (6.17) holds. The same property holds replacing for f_n, uniformly for $n \in \mathbb{N}$, thanks to relation (6.18). This means that for any $\epsilon > 0$, for all $0 \leqslant s < t \leqslant T$ with $|t - s| \leqslant \delta_\epsilon$, and for any $n \in \mathbb{N}$, we can write

$$\frac{|(f_n - f)(t) - (f_n - f)(s)|}{(t-s)^\alpha} \leqslant \frac{|f_n(t) - f_n(s)|}{(t-s)^\alpha} + \frac{|f(t) - f(s)|}{(t-s)^\alpha} \leqslant 2\,\epsilon.$$

If we fix $\bar{n}_\epsilon > 0$ such that $\|f_n - f\|_\infty \leqslant \epsilon\,(\delta_\epsilon)^\alpha$ for all $n \ge \bar{n}_\epsilon$, for $|t - s| > \delta_\epsilon$ we get

$$\frac{|(f_n - f)(t) - (f_n - f)(s)|}{(t-s)^\alpha} \leqslant \frac{2\|f_n - f\|_\infty}{(\delta_\epsilon)^\alpha} \leqslant \epsilon.$$

Altogether, the previous relations show that $\|\delta f_n - \delta f\|_\alpha \leqslant 2\epsilon$ for $n \geq \bar{n}_\epsilon$. This implies that $\|\delta f_n - \delta f\|_\alpha \to 0$. □