Ten lectures on rough paths
(work in progress)

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Part I

Rough Equations

Chapter 1

The Sewing Bound

The problem of interest in this book is the study of differential equations driven by *irregular functions* (more specifically: continuous but not differentiable). This will be achieved through the powerful and elegant theory of *rough paths*. A key motivation comes from stochastic differential equations driven by Brownian motion, but the goal is to develop a general theory which does not rely on probability.

This first chapter is dedicated to an elementary but fundamental tool, the *Sewing Bound*, that will be applied extensively throughout the book. It is a general Hölder-type bound for functions of two real variables that can be understood by itself, see Theorem 1.9 below. To provide motivation, we present it as a natural a priori estimate for solutions of differential equations.

Notation. We fix a time horizon $T>0$ and two dimensions $k,d\in\mathbb{N}$. We use "path" as a synonymous of "function defined on $[0,T]$" with values in $\mathbb{R}^d$. We denote by $|\cdot|$ the Euclidean norm. The space of linear maps from $\mathbb{R}^d$ to $\mathbb{R}^k$, identified by $k\times d$ real matrices, is denoted by $\mathbb{R}^k\otimes(\mathbb{R}^d)^*\simeq\mathbb{R}^{k\times d}$ and is equipped with the Hilbert-Schmidt norm $|\cdot|$ (i.e. the Euclidean norm on $\mathbb{R}^{k\times d}$). For $A\in\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ and $v\in\mathbb{R}^d$ we have $|A\,v|\leqslant|A|\,|v|$.

1.1. Controlled differential equation

Consider the following *controlled ordinary differential equation (ODE)*: given a continuously differentiable path $X\colon[0,T]\to\mathbb{R}^d$ and a continuous function $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$, we look for a differentiable path $Z\colon[0,T]\to\mathbb{R}^k$ such that

$$\dot{Z}_t=\sigma(Z_t)\,\dot{X}_t\,,\qquad t\in[0,T]. \tag{1.1}$$

By the fundamental theorem of calculus, this is equivalent to

$$Z_t=Z_0+\int_0^t\sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s\,,\qquad t\in[0,T]. \tag{1.2}$$

In the special case $k=d=1$ and when $\sigma(x)=\lambda\,x$ is linear (with $\lambda\in\mathbb{R}$), we have the explicit solution $Z_t=z_0\exp(\lambda\,(X_t-X_0))$, which has the interesting property of being well-defined also when X is non differentiable.

For any dimensions $k,d\in\mathbb{N}$, if we assume that $\sigma(\cdot)$ is Lipschitz, classical results in the theory of ODEs guarantee that *equation (1.1)-(1.2) is well-posed for any continuously differentiable path* X, namely for any $Z_0\in\mathbb{R}^k$ there is one and only one solution Z (with no explicit formula, in general).

Our aim is to extend such a well-posedness result to a setting where X *is continuous but not differentiable* (also in cases where $\sigma(\cdot)$ may be non-linear). Of course, to this purpose it is first necessary to provide a generalized formulation of (1.1)-(1.2) where the derivative of X does not appear.

1.2. Controlled difference equation

Let us still suppose that X is continuously differentiable. We deduce by (1.1)-(1.2) that for $0 \leqslant s \leqslant t \leqslant T$

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + \int_s^t (\sigma(Z_u) - \sigma(Z_s))\,\dot{X}_u\,\mathrm{d}u, \tag{1.3}$$

which implies that Z satisfies the following *controlled difference equation*:

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.4}$$

because $u \mapsto \sigma(Z_u)$ is continuous and $u \mapsto \dot{X}_u$ is (continuous, hence) bounded on $[0, T]$.

Remark 1.1. (Uniformity) Whenever we write $o(t-s)$, as in (1.4), we always mean *uniformly for* $0 \leqslant s \leqslant t \leqslant T$, i.e.

$$\forall \varepsilon > 0\ \exists \delta > 0: \quad 0 \leqslant s \leqslant t \leqslant T,\ \ t - s \leq \delta \quad \text{implies} \quad |o(t-s)| \leq \varepsilon\,(t-s)\,. \tag{1.5}$$

This will be implicitly assumed in the sequel.

Let us make two simple observations.

- If X is continuously differentiable we deduced (1.4) from (1.1), but we can easily deduce (1.1) from (1.4): in other terms, the two equations (1.1) and (1.4) are *equivalent*.
- If X is *not* continuously differentiable, equation (1.4) is still *meaningful*, unlike equation (1.1) which contains explicitly $\dot{X}$.

For these reasons, henceforth we focus on the difference equation (1.4), which provides a generalized formulation of the differential equation (1.1) when X is continuous but not necessarily differentiable.

The problem is now to prove *well-posedness* for the difference equation (1.4). We are going to show that this is possible assuming a suitable *Hölder regularity* on X, but non trivial ideas are required. In this chapter we illustrate some key ideas, showing how to prove uniqueness of solutions via *a priori estimates* (existence of solutions will be studied in the next chapters). We start from a basic result, which ensures the continuity of solutions; more precise result will be obtained later.

Lemma 1.2. (Continuity of solutions) *Let X and σ be continuous. Then any solution Z of (1.4) is a continuous path, more precisely it satisfies*

$$|Z_t - Z_s| \leqslant C\,|X_t - X_s| + o(t-s)\,, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.6}$$

for a suitable constant $C < \infty$ which depends on Z.

Proof. Relation (1.6) follows by (1.4) with $C := \|\sigma(Z)\|_\infty = \sup_{0 \leqslant t \leqslant T} |\sigma(Z_t)|$, renaming $|o(t-s)|$ as $o(t-s)$. We only have to prove that $C < \infty$. Since σ is continuous by assumption, it is enough to show that Z is *bounded*.

Since $o(t-s)$ is uniform, see (1.5), we can fix $\bar{\delta} > 0$ such that $|o(t-s)| \leqslant 1$ for all $0 \leqslant s \leqslant t \leqslant T$ with $|t-s| \leqslant \bar{\delta}$. It follows that Z is bounded in any interval $[\bar{s}, \bar{t}]$ with $|\bar{t} - \bar{s}| \leqslant \bar{\delta}$, because by (1.4) we can bound

$$\sup_{t \in [\bar{s}, \bar{t}]} |Z_t| \leqslant |Z_{\bar{s}}| + |\sigma(Z_{\bar{s}})| \sup_{t \in [\bar{s}, \bar{t}]} |X_t - X_{\bar{s}}| + 1 < \infty.$$

We conclude that Z is bounded in the whole interval $[0, T]$, because we can write $[0, T]$ as a finite union of intervals $[\bar{s}, \bar{t}]$ with $|\bar{t} - \bar{s}| \leqslant \bar{\delta}$. □

Remark 1.3. (Counterexamples) The weaker requirement that (1.4) holds *for any fixed* $s \in [0, T]$ *as* $t \downarrow s$ is not enough for our purposes, since in this case Z *needs not be continuous*. An easy conterexample is the following: given any continuous path $X: [0, 2] \to \mathbb{R}$, we define $Z: [0, 2] \to \mathbb{R}$ by

$$Z_t := \begin{cases} X_t & \text{if} \quad 0 \leqslant t < 1, \\ X_t + 1 & \text{if} \quad 1 \leqslant t \leqslant 2. \end{cases}$$

Note that $Z_t - Z_s = X_t - X_s$ when either $0 \leqslant s \leqslant t < 1$ or $1 \leqslant s \leqslant t \leqslant 2$, hence Z satisfies the difference equation (1.4) with $\sigma(\cdot) \equiv 1$ *for any fixed* $s \in [0, 2)$ *as* $t \downarrow s$, but *not uniformly for* $0 \leqslant s \leqslant t \leqslant 2$, since Z is discontinuous at $t = 1$.

For another counterexample, which is even unbounded, consider

$$Z_t := \begin{cases} \dfrac{1}{1-t} & \text{if} \quad 0 \leqslant t < 1, \\ 0 & \text{if} \quad 1 \leqslant t \leqslant 2, \end{cases}$$

which satisfies (1.4) as $t \downarrow s$ for any fixed $s \in [0, 2]$, for $X_t \equiv t$ and $\sigma(z) = z^2$.

1.3. Some useful function spaces

For $n \geqslant 1$ we define the simplex

$$[0, T]^n_\leqslant := \{(t_1, \ldots, t_n): \quad 0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T\} \tag{1.7}$$

(note that $[0, T]^1_\leqslant = [0, T]$). We then write $C_n = C([0, T]^n_\leqslant, \mathbb{R}^k)$ as a shorthand for the space of *continuous functions from* $[0, T]^n_\leqslant$ *to* $\mathbb{R}^k$:

$$C_n := C([0, T]^n_\leqslant, \mathbb{R}^k) := \{F: [0, T]^n_\leqslant \to \mathbb{R}^k : \ F \text{ is continuous}\}. \tag{1.8}$$

We are going to work with functions of one (f_s), two (F_{st}) or three (G_{sut}) ordered variables in $[0, T]$, hence we focus on the spaces C_1, C_2, C_3.

- On the spaces C_2 and C_3 we introduce a Hölder-like structure: given any $\eta \in (0, \infty)$, we define for $F \in C_2$ and $G \in C_3$

$$\|F\|_\eta := \sup_{0 \leqslant s < t \leqslant T} \frac{|F_{st}|}{(t-s)^\eta}, \qquad \|G\|_\eta := \sup_{\substack{0 \leqslant s \leqslant u \leqslant t \leqslant T \\ s < t}} \frac{|G_{sut}|}{(t-s)^\eta}, \tag{1.9}$$

and we denote by C_2^η and C_3^η the corresponding function spaces:

$$C_2^\eta := \{F \in C_2\colon \ \|F\|_\eta < \infty\}, \qquad C_3^\eta := \{G \in C_3\colon \ \|G\|_\eta < \infty\}, \tag{1.10}$$

which are Banach spaces endowed with the norm $\|\cdot\|_\eta$ (exercise).

- On the space C_1 of continuous functions $f\colon [0,T] \to \mathbb{R}^k$ we consider the usual Hölder structure. We first introduce the *increment* δf by

$$(\delta f)_{st} := f_t - f_s, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.11}$$

and note that $\delta f \in C_2$ for any $f \in C_1$. Then, for $\alpha \in (0,1]$, we define the classical space $\mathcal{C}^\alpha = \mathcal{C}^\alpha([0,T], \mathbb{R}^k)$ of α-Hölder functions

$$\mathcal{C}^\alpha := \left\{ f\colon [0,T] \to \mathbb{R}^k\colon \quad \|\delta f\|_\alpha = \sup_{0 \le s < t \le T} \frac{|f_t - f_s|}{(t-s)^\alpha} < \infty \right\} \tag{1.12}$$

(for $\alpha = 1$ it is the space of Lipschitz functions). Note that $\|\delta f\|_\alpha$ in (1.12) is consistent with (1.11) and (1.9).

Remark 1.4. (HÖLDER SEMI-NORM) We stress that $f \mapsto \|\delta f\|_\alpha$ is a semi-norm on $\mathcal{C}^\alpha$ (it vanishes on constant functions). The standard norm on $\mathcal{C}^\alpha$ is

$$\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha, \tag{1.13}$$

where we define the standard sup norm

$$\|f\|_\infty := \sup_{t \in [0,T]} |f_t|. \tag{1.14}$$

For $f\colon [0,T] \to \mathbb{R}^k$ we can bound $\|f\|_\infty \le |f(0)| + T^\alpha \|\delta f\|_\alpha$ (see (1.39) below), hence

$$\|f\|_{\mathcal{C}^\alpha} \le |f(0)| + (1 + T^\alpha) \|\delta f\|_\alpha. \tag{1.15}$$

This explains why it is often enough to focus on the semi-norm $\|\delta f\|_\alpha$.

Remark 1.5. (HÖLDER EXPONENTS) We only consider the Hölder space $\mathcal{C}^\alpha$ for $\alpha \in (0,1]$ because for $\alpha > 1$ the only functions in $\mathcal{C}^\alpha$ are constant functions (note that $\|\delta f\|_\alpha < \infty$ for $\alpha > 1$ implies $\dot{f}_t = 0$ for every $t \in [0,T]$).

On the other hand, the spaces C_2^η and C_3^η in (1.10) are interesting *for any exponent* $\eta \in (0,\infty)$. For instance, the condition $\|F\|_\eta < \infty$ for a function $F \in C_2$ means that $|F_{st}| \leqslant C\,(t-s)^\eta$, which does not imply $F \equiv 0$ when $\eta > 1$ (unless $F = \delta f$ is the increment of some function $f \in C_1$).

In our results below we will have to assume that the non-linearity $\sigma\colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ belongs to classes of Hölder functions, in the following sense.

DEFINITION 1.6. *Let $\gamma > 0$. A function $F\colon \mathbb{R}^k \to \mathbb{R}^N$ is said to be globally γ-Hölder (or globally of class $\mathcal{C}^\gamma$) if*

- *for $\gamma \in (0,1]$ we have*

$$[F]_{\mathcal{C}^\gamma} := \sup_{x,y \in \mathbb{R}^k, x \neq y} \frac{|F(x) - F(y)|}{|x-y|^\gamma} < +\infty$$

- *for* $\gamma \in (n, n+1]$ *and* $n = \{1, 2, \ldots\}$*, F is n times continuously differentiable and*

$$[D^{(n)}F]_{\mathcal{C}^\gamma} := \sup_{x,y \in \mathbb{R}^k, x \neq y} \frac{|D^{(n)}F(x) - D^{(n)}F(y)|}{|x-y|^{\gamma-n}} < +\infty$$

where $D^{(n)}$ *is the n-fold differential of F.*

Moreover $F: \mathbb{R}^k \to \mathbb{R}^N$ *is said to be locally* γ*-Hölder (or locally of class* $\mathcal{C}^\gamma$*) if*

- *for* $\gamma \in (0, 1]$ *we have for all* $R > 0$

$$\sup_{\substack{x,y \in \mathbb{R}^k, x \neq y \\ |x|,|y| \leqslant R}} \frac{|F(x) - F(y)|}{|x-y|^\gamma} < +\infty$$

- *for* $\gamma \in (n, n+1]$ *and* $n = \{1, 2, \ldots\}$*, F is n times continuously differentiable and*

$$\sup_{\substack{x,y \in \mathbb{R}^k, x \neq y \\ |x|,|y| \leqslant R}} \frac{|D^{(n)}F(x) - D^{(n)}F(y)|}{|x-y|^{\gamma-n}} < +\infty.$$

We stress that in the previous definition we do not assume F of $D^{(n)}F$ to be bounded. The case $\gamma = 1$ corresponds to the classical *Lipschitz* condition.

1.4. LOCAL UNIQUENESS OF SOLUTIONS

We prove *uniqueness of solutions* for the controlled difference equation (1.4) when $X \in \mathcal{C}^\alpha$ *is an Hölder path of exponent* $\alpha > \frac{1}{2}$. For simplicity, we focus on the case when $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is a *linear* application: $\sigma \in (\mathbb{R}^k \otimes (\mathbb{R}^d)^*) \otimes (\mathbb{R}^k)^*$, and we write σZ instead of $\sigma(Z)$ (we discuss non linear $\sigma(\cdot)$ in Chapter 2).

THEOREM 1.7. (LOCAL UNIQUENESS OF SOLUTIONS, LINEAR CASE) *Fix a path* $X: [0, T] \to \mathbb{R}^d$ *in* $\mathcal{C}^\alpha$*, with* $\alpha \in \left]\frac{1}{2}, 1\right]$*, and a linear map* $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$*. If* $T > 0$ *is small enough (depending on* X, α, σ*), then for any* $z_0 \in \mathbb{R}^k$ *there is at most one path* $Z: [0, T] \to \mathbb{R}^k$ *with* $Z_0 = z_0$ *which solves the linear controlled difference equation (1.4), that is (recalling (1.11))*

$$\delta Z_{st} - (\sigma Z_s)\, \delta X_{st} = o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{1.16}$$

Proof. Suppose that we have two paths $Z, \bar{Z}: [0, T] \to \mathbb{R}^k$ satisfying (1.16) with $Z_0 = \bar{Z}_0$ and define $Y := Z - \bar{Z}$. Our goal is to show that $Y = 0$.

Let us introduce the function $R \in C_2 = C([0, T]^2_{\leqslant}, \mathbb{R}^k)$ defined by

$$R_{st} := \delta Y_{st} - (\sigma Y_s)\, \delta X_{st}, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.17}$$

and note that by (1.16) and linearity we have

$$R_{st} = o(t-s). \tag{1.18}$$

Recalling (1.9), we can estimate

$$\|\delta Y\|_\alpha \leqslant |\sigma|\, \|Y\|_\infty\, \|\delta X\|_\alpha + \|R\|_\alpha,$$

and since $R_{st} = o(t-s) = o((t-s)^\alpha)$, we have $\|R\|_\alpha < +\infty$ and therefore $\|\delta Y\|_\alpha < +\infty$. Since $Y_0 = 0$, we can bound

$$\|Y\|_\infty \leqslant |Y_0| + \sup_{0 \leqslant t \leqslant T} |Y_t - Y_0| \leqslant T^\alpha \, \|\delta Y\|_\alpha \,.$$

Since $1 \leqslant T^\alpha \, (t-s)^{-\alpha}$ for $0 \leqslant s < t \leqslant T$, we can also bound

$$\|R\|_\alpha \leqslant T^\alpha \, \|R\|_{2\alpha} \,,$$

so that

$$\|\delta Y\|_\alpha \leqslant T^\alpha \, (|\sigma| \, \|\delta Y\|_\alpha \, \|\delta X\|_\alpha + \|R\|_{2\alpha}).$$

Suppose we can prove that, for some constant $C = C(X, \alpha, \sigma) < \infty$,

$$\|R\|_{2\alpha} \leqslant C \, \|\delta Y\|_\alpha. \tag{1.19}$$

Then we obtain

$$\|\delta Y\|_\alpha \leqslant T^\alpha \, (|\sigma| \, \|\delta X\|_\alpha + C) \, \|\delta Y\|_\alpha \,.$$

If we fix T small enough, so that $T^\alpha \, (|\sigma| \, \|\delta X\|_\alpha + C) < 1$, we get $\|\delta Y\|_\alpha = 0$, hence $\delta Y \equiv 0$. This means that $Y_t = Y_s$ for all $s, t \in [0, T]$, and since $Y_0 = 0$ we obtain $Y \equiv 0$, namely our goal $Z \equiv \bar{Z}$. This completes the proof *assuming the estimate (1.19)* (where the hypothesis $\alpha > \frac{1}{2}$ will play a key role). □

To actually complete the proof of Theorem 1.7, it remains to show that the inequality (1.19) holds. This is performed in the next two sections:

- in Section 1.5 we present a fundamental estimate, the *Sewing Bound*, which applies to *any function* $R_{st} = o(t-s)$ (recall (1.18));
- in Section 1.6 we apply the Sewing Bound to R_{st} in (1.17) and we prove the desired estimate (1.19) for $\alpha > \frac{1}{2}$ (see the assumptions of Theorem 1.7).

1.5. THE SEWING BOUND

Let us fix an arbitrary function $R \in C_2 = C([0, T]^2_\leqslant, \mathbb{R}^k)$ with $R_{st} = o(t-s)$. Our goal is to bound $|R_{ab}|$ for any given $0 \leqslant a < b \leqslant T$.

We first show that we can express R_{ab} via "Riemann sums" along partitions $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of $[a, b]$. These are defined by

$$I_{\mathcal{P}}(R) := \sum_{i=1}^{\#\mathcal{P}} R_{t_{i-1} t_i} \,, \tag{1.20}$$

where we denote by $\#\mathcal{P} := m$ the number of intervals of the partition $\mathcal{P}$. Let us denote by $|\mathcal{P}| := \max_{1 \leqslant i \leqslant m} (t_i - t_{i-1})$ the *mesh* of $\mathcal{P}$.

LEMMA 1.8. (RIEMANN SUMS) *Given any $R \in C_2$ with $R_{st} = o(t-s)$, for any $0 \leqslant a < b \leqslant T$ and for any sequence $(\mathcal{P}_n)_{n \geqslant 0}$ of partitions of $[a, b]$ with vanishing mesh $\lim_{n \to \infty} |\mathcal{P}_n| = 0$ we have*

$$\lim_{n \to \infty} I_{\mathcal{P}_n}(R) = 0 \,.$$

If furthermore $\mathcal{P}_0 = \{a, b\}$ is the trivial partition, then we can write

$$R_{ab} = \sum_{n=0}^{\infty} (I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)), \qquad 0 \leqslant a < b \leqslant T. \tag{1.21}$$

Proof. Writing $\mathcal{P}_n = \{a = t_0^n < t_1^n < \ldots < t_{\#\mathcal{P}_n}^n = b\}$, we can estimate

$$|I_{\mathcal{P}_n}(R)| \leqslant \sum_{i=1}^{\#\mathcal{P}_n} |R_{t_{i-1}^n t_i^n}| \leqslant \left\{ \max_{j=1,\ldots,\#\mathcal{P}_n} \frac{|R_{t_{j-1}^n t_j^n}|}{(t_j^n - t_{j-1}^n)} \right\} \sum_{j=1}^{\#\mathcal{P}_n} (t_j^n - t_{j-1}^n),$$

hence $|I_{\mathcal{P}_n}(R)| \to 0$ as $n \to \infty$, because the final sum equals $b - a$ and the bracket vanishes (since $R_{st} = o(t - s)$ and $|\mathcal{P}_n| = \max_{1 \leqslant j \leqslant \#\mathcal{P}_n} (t_j^n - t_{j-1}^n) \to 0$).

We deduce relation (1.21) by the telescopic sum

$$I_{\mathcal{P}_0}(R) - I_{\mathcal{P}_N}(R) = \sum_{n=0}^{N-1} (I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)),$$

because $\lim_{N \to \infty} I_{\mathcal{P}_N}(R) = 0$ while $I_{\mathcal{P}_0}(R) = R_{ab}$ for $\mathcal{P}_0 = \{a, b\}$. □

If we remove a single point t_i from a partition $\mathcal{P} = \{t_0 < t_1 < \ldots < t_m\}$, we obtain a new partition $\mathcal{P}'$ for which, recalling (1.20), we can write

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = R_{t_{i-1} t_{i+1}} - R_{t_{i-1} t_i} - R_{t_i t_{i+1}}. \tag{1.22}$$

The expression in the RHS deserves a name: given any two-variables function $F \in C_2$, we define its increment $\delta F \in C_3$ as the three-variables function

$$\delta F_{sut} := F_{st} - F_{su} - F_{ut}, \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{1.23}$$

We can then rewrite (1.22) as

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = \delta R_{t_{i-1} t_i t_{i+1}}, \tag{1.24}$$

and recalling (1.9) we obtain the following estimate, for any $\eta > 0$:

$$|I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R)| \leqslant \|\delta R\|_\eta \, |t_{i+1} - t_{i-1}|^\eta. \tag{1.25}$$

We are now ready to state and prove the Sewing Bound.

Theorem 1.9. (Sewing Bound) *Given any $R \in C_2$ with $R_{st} = o(t - s)$, the following estimate holds for any $\eta \in (1, \infty)$ (recall (1.9)):*

$$\|R\|_\eta \leqslant K_\eta \, \|\delta R\|_\eta \qquad \textit{where} \qquad K_\eta := (1 - 2^{1-\eta})^{-1}. \tag{1.26}$$

Proof. Fix $R \in C_2$ such that $\|\delta R\|_\eta < \infty$ for some $\eta > 1$ (otherwise there is nothing to prove). Also fix $0 \leqslant a < b \leqslant T$ and consider for $n \geqslant 0$ the dyadic partitions $\mathcal{P}_n := \{t_i^n := a + \frac{i}{2^n}(b - a) \colon 0 \leq i \leq 2^n\}$ of $[a, b]$. Since $\mathcal{P}_0 = \{a, b\}$ is the trivial partition, we can apply (1.21) to bound

$$|R_{ab}| \leqslant \sum_{n=0}^{\infty} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)|. \tag{1.27}$$

If we remove from $\mathcal{P}_{n+1}$ all the "odd points" t^{n+1}_{2j+1}, with $0 \leq j \leq 2^n - 1$, we obtain $\mathcal{P}_n$. Then, iterating relations (1.24)-(1.25), we have

$$\begin{aligned} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)| &\leqslant \sum_{j=0}^{2^n-1} |\delta R_{t^{n+1}_{2j} t^{n+1}_{2j+1} t^{n+1}_{2j+2}}| \\ &\leqslant 2^n \, \|\delta R\|_\eta \left(\frac{2(b-a)}{2^{n+1}}\right)^\eta \\ &= 2^{-(\eta-1)n} \, \|\delta R\|_\eta \, (b-a)^\eta . \end{aligned} \tag{1.28}$$

Plugging this into (1.27), since $\sum_{n=0}^\infty 2^{-(\eta-1)n} = (1 - 2^{1-\eta})^{-1}$, we obtain

$$|R_{ab}| \leqslant (1 - 2^{1-\eta})^{-1} \, \|\delta R\|_\eta \, (b-a)^\eta, \qquad 0 \leqslant a < b \leqslant T, \tag{1.29}$$

which proves (1.26). □

Remark 1.10. Recalling (1.11) and (1.23), we have defined linear maps

$$C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \tag{1.30}$$

which satisfy $\delta \circ \delta = 0$. Indeed, for any $f \in C_1$ we have

$$\delta(\delta f)_{sut} = (f_t - f_s) - (f_u - f_s) - (f_t - f_u) = 0.$$

Intuitively, $\delta F \in C_3$ measures how much a function $F \in C_2$ differs from being the increment δf of some $f \in C_1$, because $\delta F \equiv 0$ *if and only if* $F = \delta f$ *for some* $f \in C_1$ (it suffices to define $f_t := F_{0t}$ and to check that $\delta f_{st} = \delta F_{0st} + F_{st} = F_{st}$).

Remark 1.11. The assumption $R_{st} = o(t-s)$ in Theorem 1.9 cannot be avoided: if $R := \delta f$ for a non constant $f \in C_1$, then $\delta R = 0$ while $\|R\|_\eta > 0$.

1.6. END OF PROOF OF UNIQUENESS

In this section, we apply the Sewing Bound (1.26) to the function R_{st} defined in (1.17), in order to prove the estimate (1.19) for $\alpha > \frac{1}{2}$.

We first determine the increment δR through a simple and instructive computation: by (1.17), since $\delta(\delta Z) = 0$ (see Remark 1.10), we have

$$\begin{aligned} \delta R_{sut} &:= R_{st} - R_{su} - R_{ut} \\ &= (Y_t - Y_s) - (Y_u - Y_s) - (Y_t - Y_u) \\ &\quad -(\sigma Y_s)(X_t - X_s) + (\sigma Y_s)(X_u - X_s) + (\sigma Y_u)(X_t - X_u) \\ &= [\sigma (Y_u - Y_s)] (X_t - X_u). \end{aligned} \tag{1.31}$$

Recalling (1.9), this implies

$$\|\delta R\|_{2\alpha} \leqslant |\sigma| \, \|\delta Y\|_\alpha \, \|\delta X\|_\alpha.$$

We next note that if $\alpha > \frac{1}{2}$ (as it is assumed in Theorem 1.7) we can apply the Sewing Bound (1.26) for $\eta = 2\alpha > 1$ to obtain

$$\|R\|_{2\alpha} \leqslant K_{2\alpha} \, \|\delta R\|_{2\alpha} \leqslant K_{2\alpha} \, |\sigma| \, \|\delta Y\|_\alpha \, \|\delta X\|_\alpha .$$

This is precisely our goal (1.19) with $C = C(X, \alpha, \sigma) := K_{2\alpha} |\sigma| \|\delta X\|_\alpha$.

Summarizing: thanks to the Sewing bound (1.26), we have obtained the estimate (1.19) and completed the proof of Theorem 1.7, showing uniqueness of solutions to the difference equation (1.4) for any $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$. In the next chapters we extend this approach to non-linear $\sigma(\cdot)$ and to situations where $X \in \mathcal{C}^\alpha$ with $\alpha \leqslant \frac{1}{2}$.

Remark 1.12. For later purpose, let us record the computation (1.31) withouth σ: given any (say, real) paths X and Y, if

$$A_{st} = Y_s \, \delta X_{st}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T,$$

then

$$\delta A_{sut} = -\delta Y_{su} \, \delta X_{ut}, \qquad \forall 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{1.32}$$

1.7. WEIGHTED NORMS

We conclude this chapter defining *weighted versions* $\|\cdot\|_{\eta,\tau}$ of the norms $\|\cdot\|_\eta$ introduced in (1.9): given $F \in C_2$ and $G \in C_3$, we set for $\eta, \tau \in (0, \infty)$

$$\|F\|_{\eta,\tau} := \sup_{0 \leqslant s \leqslant t \leqslant T} \mathbb{1}_{\{0 < t-s \leqslant \tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|F_{st}|}{(t-s)^\eta}, \tag{1.33}$$

$$\|G\|_{\eta,\tau} := \sup_{0 \leqslant s \leqslant u \leqslant t \leqslant T} \mathbb{1}_{\{0 < t-s \leqslant \tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|G_{sut}|}{(t-s)^\eta}, \tag{1.34}$$

where C_2 and C_3 are the spaces of continuous functions from $[0,T]^2_\leqslant$ and $[0,T]^3_\leqslant$ to $\mathbb{R}^k$, see (1.8). Note that as $\tau \to \infty$ we recover the usual norms:

$$\|\cdot\|_\eta = \lim_{\tau \to \infty} \|\cdot\|_{\eta,\tau}. \tag{1.35}$$

Remark 1.13. (NORMS VS. SEMI-NORMS) While $\|\cdot\|_\eta$ is a norm, $\|\cdot\|_{\eta,\tau}$ is a norm for $\tau \geqslant T$ but *it is only a semi-norm for* $\tau < T$ (for instance, $\|F\|_{\eta,\tau} = 0$ for $F \in C_2$ implies $F_{st} = 0$ only for $t - s \leqslant \tau$: no constraint is imposed on F_{st} for $t - s > \tau$).

However, if $F = \delta f$, that is $F_{st} = f_t - f_s$ for some $f \in C_1$, we have the equivalence

$$\|\delta f\|_{\eta,\tau} \leqslant \|\delta f\|_\eta \leqslant \left(1 + \frac{T}{\tau}\right) \mathrm{e}^{\frac{T}{\tau}} \|\delta f\|_{\eta,\tau}. \tag{1.36}$$

The first inequality is clear. For the second one, given $0 \leqslant s < t \leqslant T$, we can write $s = t_0 < t_1 < \cdots < t_N = t$ with $t_i - t_{i-1} \leqslant \tau$ and $N \leqslant 1 + \frac{T}{\tau}$ (for instance, we can consider $t_i = s + i \, \frac{t-s}{N}$ where $N := \lceil \frac{t-s}{\tau} \rceil$); we then obtain $\delta f_{st} = \sum_{i=1}^N \delta f_{t_{i-1} t_i}$ and $|\delta f_{t_{i-1} t_i}| \leqslant \|\delta f\|_{\eta,\tau} \, \mathrm{e}^{t_i/\tau} (t_i - t_{i-1})^\eta \leqslant \|\delta f\|_{\eta,\tau} \, \mathrm{e}^{T/\tau} (t-s)^\eta$, which yields (1.36).

Remark 1.14. (FROM LOCAL TO GLOBAL) The weighted semi-norms $\|\cdot\|_{\eta,\tau}$ will be useful to transform *local* results in *global* results. Indeed, using the standard norms $\|\cdot\|_\eta$ often requires the size $T > 0$ of the time interval $[0,T]$ to be *small*, as in Theorem 1.7, which can be annoying. Using $\|\cdot\|_{\eta,\tau}$ will allow us to *keep* $T > 0$ *arbitrary*, by choosing a sufficiently small $\tau > 0$.

Recalling the supremum norm $\|f\|_\infty$ of a function $f\in C_1$, see (1.14), we define the corresponding weighted version

$$\|f\|_{\infty,\tau} := \sup_{0\leqslant t\leqslant T} \mathrm{e}^{-\frac{t}{\tau}}|f_t|\,. \tag{1.37}$$

We stress that $\|\cdot\|_{\infty,\tau}$ *is a norm equivalent to* $\|\cdot\|_\infty$ *for any* $\tau>0$, since

$$\|\cdot\|_{\infty,\tau}\leqslant\|\cdot\|_\infty\leqslant\mathrm{e}^{\frac{T}{\tau}}\|\cdot\|_{\infty,\tau}\,. \tag{1.38}$$

Remark 1.15. (EQUIVALENT HÖLDER NORM) It follows by (1.36) and (1.38) that $\|\cdot\|_{\infty,\tau}+\|\cdot\|_{\alpha,\tau}$ *is a norm equivalent to* $\|\cdot\|_{\mathcal{C}^\alpha}:=\|\cdot\|_\infty+\|\cdot\|_\alpha$ on the space $\mathcal{C}^\alpha$ of Hölder functions, see Remark 1.4, for any $\tau>0$.

We will often use the Hölder semi-norms $\|\delta f\|_\alpha$ and $\|\delta f\|_{\alpha,\tau}$ to bound the supremum norms $\|f\|_\infty$ and $\|f\|_{\infty,\tau}$, thanks to the following result.

LEMMA 1.16. (SUPREMUM-HÖLDER BOUND) *For any* $f\in C_1$ *and* $\eta\in(0,\infty)$

$$\|f\|_\infty\leqslant|f_0|+T^\eta\,\|\delta f\|_\eta\,, \tag{1.39}$$

$$\|f\|_{\infty,\tau}\leqslant|f_0|+3\,(\tau\wedge T)^\eta\,\|\delta f\|_{\eta,\tau},\qquad\forall\tau>0\,. \tag{1.40}$$

Proof. Let us prove (1.39): for any $f\in C_1$ and for $t\in\,]0,T]$ we have

$$|f_t|\leqslant|f_0|+|f_t-f_0|=|f_0|+t^\eta\,\frac{|f_t-f_0|}{t^\eta}\leqslant|f_0|+T^\eta\,\|\delta f\|_\eta.$$

The proof of (1.40) is slightly more involved. If $t\in\,]0,\tau\wedge T]$, then

$$\mathrm{e}^{-\frac{t}{\tau}}|f_t|\leqslant|f_0|+t^\eta\,\mathrm{e}^{-\frac{t}{\tau}}\,\frac{|f_t-f_0|}{t^\eta}\leqslant|f_0|+(\tau\wedge T)^\eta\,\|\delta f\|_{\eta,\tau},$$

which, in particular, implies (1.40) when $\tau\geqslant T$. When $\tau<T$, it remains to consider $\tau<t\leqslant T$: in this case, we define $N:=\min\{n\in\mathbb{N}:\ n\tau\geq t\}\geq 2$ so that $\frac{t}{N}\leqslant\tau$. We set $t_k=k\,\frac{t}{N}$ for $k\geq 0$, so that $t_N=t$. Then

$$\begin{aligned}\mathrm{e}^{-\frac{t}{\tau}}|f_t|&\leqslant|f_0|+\sum_{k=1}^N\,(t_k-t_{k-1})^\eta\,\mathrm{e}^{-\frac{t-t_k}{\tau}}\left[\mathrm{e}^{-\frac{t_k}{\tau}}\,\frac{|f_{t_k}-f_{t_{k-1}}|}{(t_k-t_{k-1})^\eta}\right]\\&\leqslant|f_0|+(\tau\wedge T)^\eta\,\|\delta f\|_{\eta,\tau}\sum_{k=1}^N\,\mathrm{e}^{-\frac{t-t_k}{\tau}}.\end{aligned}$$

By definition of N we have $(N-1)\tau<t$; since $\tau<t$ we obtain $N\tau<2t$ and therefore $\frac{t}{N\tau}\geq\frac{1}{2}$. Since $t-t_k=(N-k)\frac{t}{N}$, renaming $\ell:=N-k$ we obtain

$$\sum_{k=1}^N\,\mathrm{e}^{-\frac{t-t_k}{\tau}}=\sum_{\ell=0}^{N-1}\,\mathrm{e}^{-\ell\frac{t}{N\tau}}=\frac{1-\mathrm{e}^{-\frac{t}{\tau}}}{1-\mathrm{e}^{-\frac{t}{N\tau}}}\leqslant\frac{1}{1-\mathrm{e}^{-\frac{1}{2}}}\leqslant 3.$$

The proof is complete. □

We finally show that the Sewing Bound (1.26) still holds if we replace $\|\cdot\|_\eta$ by $\|\cdot\|_{\eta,\tau}$, for any $\tau > 0$.

Theorem 1.17. (weighted sewing bound) *Given any $R \in C_2$ with $R_{st} = o(t-s)$, the following estimate holds for any $\eta \in (1,\infty)$ and $\tau > 0$:*

$$\|R\|_{\eta,\tau} \leqslant K_\eta \, \|\delta R\|_{\eta,\tau} \qquad \textit{where} \qquad K_\eta := (1-2^{1-\eta})^{-1} . \tag{1.41}$$

Proof. Given $0 \leqslant a \leqslant b \leqslant T$, let us define

$$\|\delta R\|_{\eta,[a,b]} := \sup_{\substack{s,u,t\in[a,b]:\\ s\leqslant u\leqslant t,\ s<t}} \frac{|\delta R_{sut}|}{(t-s)^\eta} . \tag{1.42}$$

Following the proof of Theorem 1.9, we can replace $\|\delta R\|_\eta$ by $\|\delta R\|_{\eta,[a,b]}$ in (1.28) and in (1.29), hence we obtain $|R_{ab}| \leqslant K_\eta \, \|\delta R\|_{\eta,[a,b]} \, (b-a)^\eta$. Then for $b - a \leqslant \tau$ we can estimate

$$e^{-\frac{b}{\tau}} \frac{|R_{ab}|}{(b-a)^\eta} \leqslant e^{-\frac{b}{\tau}} K_\eta \, \|\delta R\|_{\eta,[a,b]} \leqslant K_\eta \, \|\delta R\|_{\eta,\tau} ,$$

and (1.41) follows taking the supremum over $0 \leqslant a \leqslant b \leqslant T$ with $b - a \leqslant \tau$. □

1.8. A discrete Sewing Bound

We can prove a version of the Sewing Bound for functions $R = (R_{st})_{s<t\in\mathbb{T}}$ defined on a *finite set of points* $\mathbb{T} := \{0 = t_1 < \cdots < t_{\#\mathbb{T}}\} \subseteq \mathbb{R}_+$ (this will be useful to construct solutions to difference equations via Euler schemes, see Sections 2.6 and 3.9). The condition $R_{st} = o(t-s)$ from Theorem 1.9 is now replaced by the requirement that *R vanishes on consecutive points of $\mathbb{T}$*, i.e. $R_{t_i t_{i+1}} = 0$ for all $1 \leqslant i < \#\mathbb{T}$.

We define versions $\|\cdot\|_{\eta,\tau}^{\mathbb{T}}$ of the norms $\|\cdot\|_{\eta,\tau}$ restricted on $\mathbb{T}$ for $\tau > 0$, recall (1.33)-(1.34):

$$\|A\|_{\eta,\tau}^{\mathbb{T}} := \sup_{\substack{0\leqslant s<t\\ s,t\in\mathbb{T}}} \mathbb{1}_{\{0<t-s\leqslant\tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|A_{st}|}{|t-s|^\eta} , \tag{1.43}$$

$$\|B\|_{\eta,\tau}^{\mathbb{T}} := \sup_{\substack{0\leqslant s\leqslant u\leqslant t\\ s,u,t\in\mathbb{T},\ s<t}} \mathbb{1}_{\{0<t-s\leqslant\tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|B_{sut}|}{|t-s|^\eta} \tag{1.44}$$

for $A: \{(s,t) \in \mathbb{T}^2 : 0 \leqslant s < t\} \to \mathbb{R}$ and $B: \{(s,u,t) \in \mathbb{T}^3 : 0 \leqslant s \leqslant u \leqslant t, s < t\} \to \mathbb{R}$.

Theorem 1.18. (Discrete Sewing Bound) *If a function $R = (R_{st})_{s<t\in\mathbb{T}}$ vanishes on consecutive points of $\mathbb{T}$ (i.e. $R_{t_i t_{i+1}} = 0$), then for any $\eta > 1$ and $\tau > 0$ we have*

$$\|R\|_{\eta,\tau}^{\mathbb{T}} \leqslant C_\eta \, \|\delta R\|_{\eta,\tau}^{\mathbb{T}} \qquad \textit{with} \qquad C_\eta := 2^\eta \sum_{n\geq 1} \frac{1}{n^\eta} = 2^\eta \zeta(\eta) < \infty . \tag{1.45}$$

Proof. We fix $s, t \in \mathbb{T}$ with $s < t$ and we start by proving that

$$|R_{st}| \leqslant C_\eta \, \|\delta R\|_\eta^{\mathbb{T}} \, (t-s)^\eta .$$

We have $s=t_k$ and $t=t_{k+m}$ and we may assume that $m\geqslant 2$ (otherwise there is nothing to prove, since for $m=1$ we have $R_{t_i t_{i+1}}=0$).

Consider the partition $\mathcal{P}=\{s=t_k<t_{k+1}<\ldots<t_{k+m}=t\}$ with m intervals. Note that for some index $i\in\{k+1,\ldots,k+m-1\}$ we must have $t_{i+1}-t_{i-1}\leq\frac{2\,(t-s)}{m-1}$, otherwise we would get the contradiction

$$2\,(t-s)\geq\sum_{i=k+1}^{k+m-1}(t_{i+1}-t_{i-1})>\sum_{i=k+1}^{k+m-1}\frac{2\,(t-s)}{m-1}=2\,(t-s)\,.$$

Removing the point t_i from $\mathcal{P}$ we obtain a partition $\mathcal{P}'$ with $m-1$ intervals. If we define $I_{\mathcal{P}}(R):=\sum_{i=k}^{k+m-1}R_{t_i t_{i+1}}$ as in (1.20), as in (1.24) we have

$$|I_{\mathcal{P}}(R)-I_{\mathcal{P}'}(R)|=|\delta R_{t_{i-1}t_i t_{i+1}}|\leqslant\frac{2^\eta\,(t-s)^\eta}{(m-1)^\eta}\sup_{\substack{s\leqslant u<v<w\leqslant t\\ u,v,w\in\mathbb{T}}}\frac{|\delta R_{uvw}|}{|w-u|^\eta}\,.$$

Iterating this argument, until we arrive at the trivial partition $\{s,t\}$, we get

$$|I_{\mathcal{P}}(R)-R_{st}|\leq C_\eta\,(t-s)^\eta\sup_{\substack{s\leqslant u<v<w\leqslant t\\ u,v,w\in\mathbb{T}}}\frac{|\delta R_{uvw}|}{|w-u|^\eta},\tag{1.46}$$

with $C_\eta:=\sum_{n\geq 1}\frac{2^\eta}{n^\eta}<\infty$ because $\eta>1$. We finally note that $I_{\mathcal{P}}(R)=0$ by the assumption $R_{t_i t_{i+1}}=0$. Finally if $t-s\leqslant\tau$ then $w-u\leqslant\tau$ in the supremum in (1.46) and since $\mathrm{e}^{-\frac{t}{\tau}}\leqslant\mathrm{e}^{-\frac{w}{\tau}}$ we obtain

$$\mathrm{e}^{-\frac{t}{\tau}}|R_{st}|\leqslant C_\eta\,(t-s)^\eta\,\|\delta R\|_{\eta,\tau}^{\mathbb{T}},$$

and the proof is complete. □

We also have an analog of Lemma 1.16. We set for $f\colon\mathbb{T}\to\mathbb{R}$ and $\tau>0$

$$\|f\|_{\infty,\tau}^{\mathbb{T}}:=\sup_{t\in\mathbb{T}}\mathrm{e}^{-\frac{t}{\tau}}|f_t|\,.$$

LEMMA 1.19. (DISCRETE SUPREMUM-HÖLDER BOUND) *For* $\mathbb{T}:=\{0=t_1<\cdots<t_{\#\mathbb{T}}\}\subseteq\mathbb{R}_+$ *set*

$$M:=\max_{i=2,\ldots,\#\mathbb{T}}|t_i-t_{i-1}|.$$

Then for all $f\colon\mathbb{T}\to\mathbb{R}$, $\tau\geqslant 2M$ *and* $\eta>0$

$$\|f\|_{\infty,\tau}^{\mathbb{T}}\leqslant|f_0|+5\,\tau^\eta\,\|\delta f\|_{\eta,\tau}^{\mathbb{T}}\,.\tag{1.47}$$

Proof. We define $T_0:=0$ and for $i\geqslant 1$, as long as $\mathbb{T}\cap(T_{i-1},T_{i-1}+\tau]$ is not empty, we set

$$T_i:=\max\mathbb{T}\cap(T_{i-1},T_{i-1}+\tau],\qquad i=1,\ldots,N,$$

so that $T_N=\max\mathbb{T}$. We have by construction $T_i+M>T_{i-1}+\tau$ for all $i=1,\ldots,N-1$, and since $M\leqslant\frac{\tau}{2}$

$$T_i-T_{i-1}\geqslant\tau-M\geqslant\frac{\tau}{2}\,.$$

For $i=N$ we have only $T_N > T_{N-1}$. Therefore for $i=1,\ldots N$

$$\begin{aligned}
\mathrm{e}^{-\frac{T_i}{\tau}}|f_{T_i}| &\leqslant |f_0| + \sum_{k=1}^{i} (T_k - T_{k-1})^{\eta}\, \mathrm{e}^{-\frac{T_i - T_k}{\tau}} \left[\mathrm{e}^{-\frac{T_k}{\tau}} \frac{|f_{T_k} - f_{T_{k-1}}|}{(T_k - T_{k-1})^{\eta}} \right] \\
&\leqslant |f_0| + \tau^{\eta}\, \|\delta f\|_{\eta,\tau}^{\mathbb{T}} \sum_{k=1}^{i} \mathrm{e}^{-\frac{T_i - T_k}{\tau}} \\
&\leqslant |f_0| + \tau^{\eta}\, \|\delta f\|_{\eta,\tau}^{\mathbb{T}} \left(1 + \sum_{k=0}^{\infty} \mathrm{e}^{-\frac{k}{2}} \right) \\
&\leqslant |f_{t_0}| + 4\tau^{\eta}\, \|\delta f\|_{\eta,\tau}^{\mathbb{T}}.
\end{aligned}$$

Now for $t \in \mathbb{T} \setminus \{T_i\}_i$ we have $T_i < t < T_{i+1}$ for some i and then

$$\begin{aligned}
\mathrm{e}^{-\frac{t}{\tau}}|f_t| &\leqslant \mathrm{e}^{-\frac{t}{\tau}}|f_{T_i}| + (t - T_i)^{\eta}\, \mathrm{e}^{-\frac{t}{\tau}} \frac{|f_t - f_{T_i}|}{(t - T_i)^{\eta}} \leqslant \mathrm{e}^{-\frac{T_i}{\tau}}|f_{T_i}| + \tau^{\eta}\, \|\delta f\|_{\eta,\tau}^{\mathbb{T}} \\
&\leqslant |f_0| + 5\tau^{\eta}\, \|\delta f\|_{\eta,\tau}^{\mathbb{T}}.
\end{aligned}$$

The proof is complete. □

1.9. Extra (to be completed)

We also introduce the usual supremum norm, for $F \in C_2$ and $G \in C_3$:

$$\|F\|_{\infty} := \sup_{0 \leqslant s \leqslant t \leqslant T} |F_{st}|, \qquad \|G\|_{\infty} := \sup_{0 \leqslant s \leqslant u \leqslant t \leqslant T} |G_{sut}|,$$

and a corresponding weighted version, for $\tau \in (0, \infty)$:

$$\|F\|_{\infty,\tau} := \sup_{0 \leqslant s \leqslant t \leqslant T} \mathrm{e}^{-\frac{t}{\tau}} |F_{st}|, \qquad \|G\|_{\infty,\tau} := \sup_{0 \leqslant s \leqslant u \leqslant t \leqslant T} \mathrm{e}^{-\frac{t}{\tau}} |G_{sut}|. \tag{1.48}$$

Note that

$$\lim_{\tau \to +\infty} \|F\|_{\infty,\tau} = \|F\|_{\infty}, \qquad \lim_{\tau \to +\infty} \|G\|_{\eta,\tau} = \|G\|_{\eta}, \qquad \lim_{\tau \to +\infty} \|H\|_{\eta,\tau} = \|H\|_{\eta}.$$

We have

$$\|F\|_{\eta,\tau} \leqslant \|G\|_{\infty,\tau} \|H\|_{\eta}, \qquad (F_{sut} = G_{su} H_{ut}), \tag{1.49}$$

Note that $\|\cdot\|_{\eta,\tau}$ is only a semi-norm on C_n^{η} if $\tau < T$; we have at least

$$\|\cdot\|_{\eta,\tau} \leqslant \|\cdot\|_{\eta} \leqslant e^{\frac{T}{\tau}} \left(\|\cdot\|_{\eta,\tau} + \frac{1}{\tau^{\eta}} \|\cdot\|_{\infty,\tau} \right). \tag{1.50}$$

However, if $\tau \geq T$ we have again equivalence of norms

$$\|\cdot\|_{\eta,\tau} \leqslant \|\cdot\|_{\eta} \leqslant e^{\frac{T}{\tau}} \|\cdot\|_{\eta,\tau}, \qquad \tau \geq T. \tag{1.51}$$

Chapter 2

Difference equations: the Young case

Fix a time horizon $T>0$ and two dimensions $k,d\in\mathbb{N}$. We study the following *controlled difference equation* for an unknown path $Z\colon[0,T]\to\mathbb{R}^k$:

$$Z_t-Z_s=\sigma(Z_s)\,(X_t-X_s)+o(t-s)\,,\qquad 0\leqslant s\leqslant t\leqslant T,\tag{2.1}$$

where the "driving path" $X\colon[0,T]\to\mathbb{R}^d$ and the function $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ are given, and $o(t-s)$ is *uniform* for $0\leqslant s\leqslant t\leqslant T$ (see Remark 1.1).

The difference equation (2.1) is a natural generalized formulation of the *controlled differential equation*

$$\dot{Z}_t=\sigma(Z_t)\,\dot{X}_t\,,\qquad 0\leqslant t\leqslant T.\tag{2.2}$$

Indeed, as we showed in Chapter 1 (see Section 1.2), equations (2.1) and (2.2) are *equivalent* when X is continuously differentiable and σ is continuous, but (2.1) is meaningful also when X is non differentiable.

In this chapter we prove *well-posedness for the difference equation (2.1)* when the driving path $X\in\mathcal{C}^\alpha$ is Hölder continuous in the regime $\alpha\in\left]\frac{1}{2},1\right]$, called the *Young case*. The more challenging regime $\alpha\leqslant\frac{1}{2}$, called the *rough case*, is the object of the next Chapter 3, where new ideas will be introduced.

2.1. Summary

Using the increment notation $\delta f_{st}:=f_t-f_s$ from (1.11), we rewrite (2.1) as

$$\delta Z_{st}=\sigma(Z_s)\,\delta X_{st}+o(t-s),\qquad 0\leqslant s\leqslant t\leqslant T,\tag{2.3}$$

so that a solution of (2.3) is any path $Z\colon[0,T]\to\mathbb{R}^k$ such that the *"remainder"*

$$Z_{st}^{[2]}:=\delta Z_{st}-\sigma(Z_s)\,\delta X_{st}\qquad\text{satisfies}\qquad Z_{st}^{[2]}=o(t-s)\,.\tag{2.4}$$

We summarize the main results of this chapter stating *local and global existence, uniqueness of solutions and continuity of the solution map* for the difference equation (2.3) under natural assumptions on σ. We will actually prove more precise results, which yield quantitative estimates.

Theorem 2.1. (Well-posedness) *Let $X\colon[0,T]\to\mathbb{R}^d$ be of class $\mathcal{C}^\alpha$ with $\alpha\in\left]\frac{1}{2},1\right]$ and let $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$. Then we have:*

- ***local existence**: if σ is locally γ-Hölder with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$ (e.g. of class C^1), then for every $z_0\in\mathbb{R}^k$ there is a possibly shorter time horizon $T'=T'_{\alpha,X,\sigma}(z_0)\in\left]0,T\right]$ and a path $Z\colon[0,T']\to\mathbb{R}^k$ starting from $Z_0=z_0$ which solves (2.3) for $0\leqslant s\leqslant t\leqslant T'$;*

- ***global existence***: *if σ is globally γ-Hölder with $\gamma \in \left(\frac{1}{\alpha}-1, 1\right]$ (e.g. of class C^1 with $\|\nabla\sigma\|_\infty < \infty$), then we can take $T'_{\alpha,X,\sigma}(z_0) = T$ for any $z_0 \in \mathbb{R}^d$;*
- ***uniqueness***: *if σ is of class $\mathcal{C}^\gamma$ with $\gamma \in \left(\frac{1}{\alpha}, 2\right]$ (e.g. if σ is of class C^2), then there is exactly one solution Z of (2.3) with $Z_0 = z_0$;*
- ***continuity of the solution map***: *if σ is differentiable with bounded and globally $(\gamma-1)$-Hölder gradient with $\gamma \in \left(\frac{1}{\alpha}, 2\right]$ (i.e. $\|\nabla\sigma\|_\infty < \infty$, $[\nabla\sigma]_{\mathcal{C}^{\gamma-1}} < \infty$), then the solution Z of (2.3) is a continuous function of the starting point z_0 and driving path X: the map $(z_0, X) \mapsto Z$ is continuous from $\mathbb{R}^k \times \mathcal{C}^\alpha \to \mathcal{C}^\alpha$.*

In the first part of this chapter, we give for granted the existence of solutions and we focus on their properties: we prove *a priori estimates* in Section 2.3, *uniqueness of solutions* in Section 2.4 and *continuity of the solution map* in Section 2.5. A key role is played by the Sewing Bound from Chapter 1, see Theorems 1.9 and 1.17, and its discrete version, see Theorem 1.18.

The proof of local and global *existence of solutions of (2.3)* is given at the end of this chapter, see Section 2.6, exploiting a suitable Euler scheme.

2.2. SET-UP

We collect here some notions and tools that will be used extensively.

We recall that C_1 denotes the space of continuous functions $f: [0,T] \to \mathbb{R}^k$. Similarly, C_2 and C_3 are the spaces of continuous functions of two and three ordered variables, i.e. defined on $[0,T]^2_\leqslant$ and $[0,T]^3_\leqslant$, see (1.7)-(1.8).

We are going to exploit the *weighted semi-norms* $\|\cdot\|_{\eta,\tau}$, see (1.33)-(1.34) (see also (1.9) for the original norm $\|\cdot\|_\eta$). These are useful to bound the *weighted supremum norm* $\|f\|_{\infty,\tau}$ of a function $f \in C_1$, see (1.37) and (1.40):

$$\|f\|_{\infty,\tau} \leqslant |f_0| + 3\,(\tau \wedge T)^\eta\, \|\delta f\|_{\eta,\tau}, \qquad \forall \eta, \tau > 0. \tag{2.5}$$

It follows directly from the definitions (1.33)-(1.34) that

$$\|\cdot\|_{\eta,\tau} \leqslant (\tau \wedge T)^{\eta'}\, \|\cdot\|_{\eta+\eta',\tau}\,, \qquad \forall \eta, \eta' > 0, \tag{2.6}$$

because $(t-s)^\eta \geqslant (t-s)^{\eta+\eta'}(\tau \wedge T)^{-\eta'}$ for $0 \leqslant s \leqslant t \leqslant T$ with $t - s \leqslant \tau$.

Remark 2.2. The factor $(\tau \wedge T)^{\eta'}$ in the RHS of (2.6) can be made small *by choosing τ small while keeping T fixed*. This is why we included the indicator function $\mathbb{1}_{\{0<t-s\leqslant\tau\}}$ in the definition (1.33)-(1.34) of the norms $\|\cdot\|_{\eta,\tau}$: without this indicator function, instead of $(\tau \wedge T)^{\eta'}$ we would have $T^{\eta'}$, which is small only when T is small.

We will often work with functions $F \in C_2$ or $F \in C_3$ that are *product of two factors*, like $F_{st} = g_s H_{st}$ or $F_{sut} = G_{su} H_{ut}$. We show in the next result that the semi-norm $\|F\|_{\eta,\tau}$ can be controlled by a product of suitable norms for each factor.

LEMMA 2.3. (WEIGHTED BOUNDS) *For any $\eta, \eta' \in (0, \infty)$ and $\tau > 0$, we have*

$$\text{if} \quad F_{st} = g_s H_{st} \quad \text{or} \quad F_{st} = g_t H_{st} \qquad \text{then} \qquad \|F\|_{\eta,\tau} \leqslant \|g\|_{\infty,\tau} \|H\|_{\eta}, \tag{2.7}$$

$$\text{if} \quad F_{sut} = G_{su} H_{ut} \qquad \text{then} \qquad \|F\|_{\eta+\eta',\tau} \leqslant \|G\|_{\eta,\tau} \|H\|_{\eta'}. \tag{2.8}$$

Proof. If $F_{st} = g_t H_{st}$, by (1.37) we can estimate $e^{-t/\tau} |g_t| \leqslant \|g\|_{\infty,\tau}$ to get (2.7). If $F_{st} = g_s H_{st}$, for $s \leqslant t$ we can bound $e^{-t/\tau} \leqslant e^{-s/\tau}$ in the definition (1.33)-(1.34) of $\|\cdot\|_{\eta,\tau}$, hence again by (1.37) we can estimate $e^{-s/\tau} |g_s| \leqslant \|g\|_{\infty,\tau}$ to get (2.7).

If $F_{sut} = G_{su} H_{ut}$, we can further bound $(t-s)^{\eta+\eta'} \geqslant (t-u)^{\eta} (u-s)^{\eta'}$ in (1.34) and then estimate $e^{-s/\tau} G_{su} / (u-s)^{\eta} \leqslant \|G\|_{\eta,\tau}$, which yields (2.8). □

We stress that in the RHS of (2.7) and (2.8) *only one factor gets the weighted norm or semi-norm*, while the other factor gets the non-weigthed norm $\|\cdot\|_{\eta}$. We will sometimes need an extra weight, which can be introduced as follows.

LEMMA 2.4. (EXTRA WEIGHT) *For any $\eta, \bar{\tau} \in (0, \infty)$ and $0 < \tau \leqslant \bar{\tau}$, we have*

$$\text{if} \quad F_{st} = g_s H_{st} \quad \text{or} \quad F_{st} = g_t H_{st} \qquad \text{then} \qquad \|F\|_{\eta,\tau} \leqslant \|g\|_{\infty,\tau} e^{\frac{T}{\bar{\tau}}} \|H\|_{\eta,\bar{\tau}}. \tag{2.9}$$

Proof. Recall the definition (1.33)-(1.34) of $\|\cdot\|_{\eta,\tau}$ and note that for $0 \leqslant s \leqslant t \leqslant T$ we have $e^{-t/\tau} |g_t| \leqslant \|g\|_{\infty,\tau}$ and $e^{-s/\tau} |g_s| \leqslant \|g\|_{\infty,\tau}$ (see the proof of Lemma 2.3). Finally, for $t - s \leqslant \tau \leqslant \bar{\tau}$ we can estimate $|H_{st}| \leqslant e^{T/\bar{\tau}} e^{-t/\bar{\tau}} |H_{st}| \leqslant e^{T/\bar{\tau}} \|H\|_{\eta,\bar{\tau}} (t-s)^{\eta}$. □

We recall that $\mathbb{R}^k \otimes (\mathbb{R}^d)^* \simeq \mathbb{R}^{k \times d}$ is the space of linear applications from $\mathbb{R}^d$ to $\mathbb{R}^k$ equipped with the Hilbert-Schmidt (Euclidean) norm $|\cdot|$. We say that a function is of class C^m if it is continuously differentiable m times. Given $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ of class C^2, that we represent by $\sigma^i_j(z)$ with $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, d\}$, we denote by $\nabla\sigma$: $\mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^k)^*$ its gradient and by $\nabla^2\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^k)^* \otimes (\mathbb{R}^k)^*$ its Hessian, represented for $i, a, b \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, d\}$ by

$$(\nabla\sigma(z))^i_{ja} = \frac{\partial \sigma^i_j}{\partial z_a}(z), \qquad (\nabla^2\sigma(z))^i_{jab} = \frac{\partial^2 \sigma^i_j}{\partial z_a \partial z_b}(z).$$

Remark 2.5. (NORM OF THE GRADIENT OF LIPSCHITZ FUNCTIONS) For a *locally Lipschitz function* $\psi: \mathbb{R}^k \to \mathbb{R}^{\ell}$ we can define the "norm of the gradient" at any point (even where ψ may not be differentiable):

$$|\nabla\psi(z)| := \limsup_{y \to z} \frac{|\psi(y) - \psi(z)|}{|y - z|} \in [0, \infty).$$

Similarly, $|\nabla^2\psi(z)|$ is well defined as soon as ψ is *differentiable with locally Lipschitz gradient* $\nabla\psi$ (which is slightly less than requiring $\psi \in C^2$).

2.3. A PRIORI ESTIMATES

In this section we prove *a priori estimates* for solutions of (2.3) assuming that σ is *globally Lipschitz*, that is $\|\nabla\sigma\|_{\infty} < \infty$ (recall Remark 2.5).

We first observe that if the driving path X is of class $\mathcal{C}^\alpha$, then any solution Z of (2.3) is also of class $\mathcal{C}^\alpha$, as soon as σ is continuous.

LEMMA 2.6. (HÖLDER REGULARITY) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha \in]0,1]$ and let σ be continuous. Then any solution Z of (2.3) is of class $\mathcal{C}^\alpha$.*

Proof. We know by Lemma 1.2 that Z is continuous, more precisely by (1.6) we have $|\delta Z_{st}| \leqslant C\,|\delta X_{st}| + o(t-s)$ with $C < \infty$. Since $|\delta X_{st}| \leqslant \|\delta X\|_\alpha (t-s)^\alpha$ and $o(t-s) = o((t-s)^\alpha)$ for any $\alpha \leqslant 1$, it follows that $Z \in \mathcal{C}^\alpha$. □

We next formulate the announced a priori estimates. It is convenient to use the weighted semi-norms $\|\cdot\|_{\eta,\tau}$ in (1.33)-(1.34) (note that the usual norms $\|\cdot\|_\eta$ in (1.9) can be recovered by letting $\tau \to \infty$).

THEOREM 2.7. (A PRIORI ESTIMATES) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2},1\right]$ and let σ be globally γ-Hölder with $\gamma \in \left(\frac{1}{\alpha}-1,1\right]$. Then, for any solution $Z\colon [0,T] \to \mathbb{R}^k$ of (2.3), the remainder $Z^{[2]}_{st} := \delta Z_{st} - \sigma(Z_s)\,\delta X_{st}$ satisfies $Z^{[2]} \in C_2^{(\gamma+1)\alpha}$, more precisely for any $\tau > 0$*

$$\|Z^{[2]}\|_{(\gamma+1)\alpha,\tau} \leqslant C_{\alpha,\gamma,X,\sigma}\,\|\delta Z\|^\gamma_{\alpha,\tau} \qquad \text{with } C_{\alpha,\gamma,X,\sigma} := K_{(\gamma+1)\alpha}\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}\,, \tag{2.10}$$

where $K_\eta = (1-2^{1-\eta})^{-1}$. Moreover, if either T or τ is small enough, we have

$$\|\delta Z\|_{\alpha,\tau} \leqslant 1 \vee (2\,\|\delta X\|_\alpha\,|\sigma(Z_0)|) \qquad \text{for } (\tau \wedge T)^{\alpha\gamma} \leqslant \varepsilon_{\alpha,\gamma,X,\sigma}, \tag{2.11}$$

where we define

$$\varepsilon_{\alpha,\gamma,X,\sigma} := \frac{1}{2\,(K_{(\gamma+1)\alpha}+3)\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}}\,. \tag{2.12}$$

If σ is globally Lipschitz, namely if we can take $\gamma = 1$, we can improve (2.11) to

$$\|\delta Z\|_{\alpha,\tau} \leqslant 2\,\|\delta X\|_\alpha\,|\sigma(Z_0)| \qquad \text{for } (\tau \wedge T)^\alpha \leqslant \varepsilon_{\alpha,1,X,\sigma}\,. \tag{2.13}$$

Proof. We first prove (2.10). Since $Z^{[2]}_{st} = o(t-s)$ by definition of solution, see (2.4), *we can estimate $Z^{[2]}$ in terms of $\delta Z^{[2]}$*, by the weighted Sewing Bound (1.41). Let us compute $\delta Z^{[2]}_{sut} = Z^{[2]}_{st} - Z^{[2]}_{su} - Z^{[2]}_{ut}$: recalling (2.4) and (1.32), since $\delta \circ \delta = 0$, we have

$$\delta Z^{[2]}_{sut} = \delta\sigma(Z)_{su}\,\delta X_{ut} = (\sigma(Z_u) - \sigma(Z_s))\,(X_t - X_u)\,. \tag{2.14}$$

Since $|\sigma(z) - \sigma(\bar{z})| \leqslant [\sigma]_{\mathcal{C}^\gamma}\,|z-\bar{z}|^\gamma$ for all $z, \bar{z} \in \mathbb{R}^d$, we can bound

$$\|\delta\sigma(Z)\|_{\gamma\alpha,\tau} \leqslant [\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|^\gamma_{\alpha,\tau}, \tag{2.15}$$

hence by (2.8) we obtain

$$\|\delta Z^{[2]}\|_{(\gamma+1)\alpha,\tau} \leqslant \|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|^\gamma_{\alpha,\tau}.$$

Applying the weighted Sewing Bound (1.41), for $(\gamma+1)\alpha > 1$ we then obtain

$$\|Z^{[2]}\|_{(\gamma+1)\alpha,\tau} \leqslant K_{(\gamma+1)\alpha}\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|^\gamma_{\alpha,\tau}\,, \tag{2.16}$$

which proves (2.10).

We next prove (2.11). To simplify notation, let us set $\varepsilon := (\tau \wedge T)^{\alpha}$. Recalling (2.7) and (2.6), we obtain by (2.4)

$$\begin{aligned} \|\delta Z\|_{\alpha,\tau} &\leqslant \|\sigma(Z)\,\delta X\|_{\alpha,\tau} + \|Z^{[2]}\|_{\alpha,\tau} \\ &\leqslant \|\sigma(Z)\|_{\infty,\tau}\,\|\delta X\|_{\alpha} + \varepsilon^{\gamma}\,\|Z^{[2]}\|_{(\gamma+1)\alpha,\tau}\,. \end{aligned} \tag{2.17}$$

We can estimate $\|\sigma(Z)\|_{\infty,\tau}$ by (2.5) and (2.15):

$$\|\sigma(Z)\|_{\infty,\tau} \leqslant |\sigma(Z_0)| + 3\,\varepsilon^{\gamma}\,[\sigma]_{\mathcal{C}^{\gamma}}\,\|\delta Z\|_{\alpha,\tau}^{\gamma}\,.$$

Plugging this and (2.16) into (2.17), we get

$$\begin{aligned} \|\delta Z\|_{\alpha,\tau} &\leqslant (|\sigma(Z_0)| + 3\,\varepsilon^{\gamma}\,[\sigma]_{\mathcal{C}^{\gamma}}\,\|\delta Z\|_{\alpha,\tau}^{\gamma})\,\|\delta X\|_{\alpha} + \\ &\qquad + \varepsilon^{\gamma}\,K_{(\gamma+1)\alpha}\,\|\delta X\|_{\alpha}\,[\sigma]_{\mathcal{C}^{\gamma}}\,\|\delta Z\|_{\alpha,\tau}^{\gamma} \\ &= \|\delta X\|_{\alpha}\,|\sigma(Z_0)| + \frac{1}{2}\,\frac{\varepsilon^{\gamma}}{\varepsilon_{\alpha,\gamma,X,\sigma}}\,\|\delta Z\|_{\alpha,\tau}^{\gamma}\,, \end{aligned}$$

where $\varepsilon_{\alpha,\gamma,X,\sigma}$ is defined in (2.12). For $\varepsilon^{\gamma} \leqslant \varepsilon_{\alpha,\gamma,X,\sigma}$ the last term is bounded by $\frac{1}{2}\|\delta Z\|_{\alpha,\tau}^{\gamma}$ *which is finite* by Lemma 2.6. If $\|\delta Z\|_{\alpha,\tau} \leqslant 1$ then (2.11) holds trivially; if not, $\frac{1}{2}\|\delta Z\|_{\alpha,\tau}^{\gamma} \leqslant \frac{1}{2}\|\delta Z\|_{\alpha,\tau}$. Bringing this term in the LHS we obtain (2.11).

To prove (2.13), we argue as for (2.11) and since $\gamma = 1$ we obtain

$$\|\delta Z\|_{\alpha,\tau} \leqslant \|\delta X\|_{\alpha}\,|\sigma(Z_0)| + \frac{1}{2}\,\frac{\varepsilon}{\varepsilon_{\alpha,1,X,\sigma}}\,\|\delta Z\|_{\alpha,\tau}.$$

For $\varepsilon \leqslant \varepsilon_{\alpha,1,X,\sigma}$ the last term is bounded by $\frac{1}{2}\|\delta Z\|_{\alpha,\tau}$ *which is finite* by Lemma 2.6. Bringing this term in the LHS we obtain (2.13), and this completes the proof. □

2.4. UNIQUENESS

In this section we prove uniqueness of solutions to (2.3) assuming that σ is of class C^1 *with locally Hölder gradient* (we stress that we make no boundedness assumption on σ). This improves on Theorem 1.7, both because we allow for non-linear σ and because we do not require that the time horizon $T > 0$ is small.

We first need an elementary but fundamental estimate on the difference of increments of a function. Given $\Psi: \mathbb{R}^k \to \mathbb{R}^{\ell}$, we use the notation

$$C_{\Psi,R} := \sup\,\{|\Psi(x)|:\ \ x \in \mathbb{R}^k,\ |x| \leqslant R\}\,. \tag{2.18}$$

LEMMA 2.8. (DIFFERENCE OF INCREMENTS) *Let $\psi: \mathbb{R}^k \to \mathbb{R}^{\ell}$ be of class $\mathcal{C}^{1+\rho}_{\mathrm{loc}}$ for some $0 < \rho \leqslant 1$ (i.e. ψ is differentiable with $\nabla\psi$ of class $\mathcal{C}^{\rho}_{\mathrm{loc}}$). Then for any $R > 0$ and for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^k$ with $\max\{|x|, |y|, |\bar{x}|, |\bar{y}|\} \leqslant R$ we can estimate*

$$\begin{aligned} &|[\psi(x) - \psi(y)] - [\psi(\bar{x}) - \psi(\bar{y})]| \\ &\quad \leqslant C_R'\,|(x-y) - (\bar{x} - \bar{y})| + C_R''\,\{|x-y|^{\rho} + |\bar{x} - \bar{y}|^{\rho}\}\,|y - \bar{y}|\,, \end{aligned} \tag{2.19}$$

where $C_R' := \sup\,\{|\nabla\psi(x)|:\ \ |x| \leqslant R\}$ and $C_R'' := \sup\left\{\frac{|\nabla\psi(x) - \nabla\psi(y)|}{|x-y|^{\rho}}:\ \ |x|, |y| \leqslant R\right\}$.

Proof. For $z, w \in \mathbb{R}^k$ we can write

$$\psi(z) - \psi(w) = \hat{\psi}(z, w)\,(z - w),$$

where $\hat{\psi}(z, w) := \int_0^1 \nabla\psi(u\,z + (1-u)\,w)\,\mathrm{d}u \in \mathbb{R}^\ell \otimes (\mathbb{R}^k)^*$, therefore

$$\begin{aligned}[\psi(x) - \psi(y)] - [\psi(\bar{x}) - \psi(\bar{y})] &= [\psi(x) - \psi(\bar{x})] - [\psi(y) - \psi(\bar{y})] \\ &= \hat{\psi}(x, \bar{x})\,(x - \bar{x}) - \hat{\psi}(y, \bar{y})\,(y - \bar{y}) \\ &= \hat{\psi}(x, \bar{x})\,[(x - \bar{x}) - (y - \bar{y})] \\ &\quad + [\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{y})]\,(y - \bar{y})\,.\end{aligned}$$

By definition of C_R' and C_R'' we have $|\hat{\psi}(x, \bar{x})| \leqslant C_R'$ and

$$\begin{aligned}|\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{y})| &\leqslant |\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{x})| + |\hat{\psi}(y, \bar{x}) - \hat{\psi}(y, \bar{y})| \\ &\leqslant C_R''\{|x - y|^\rho + |\bar{x} - \bar{y}|^\rho\},\end{aligned}$$

hence (2.19) follows. □

We are now ready to state and prove the announced uniqueness result.

THEOREM 2.9. (UNIQUENESS) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$ and let σ be of class $\mathcal{C}^\gamma$ for some $\gamma > \frac{1}{\alpha}$ (for instance, we can take $\sigma \in \mathcal{C}^2$). Then for every $z_0 \in \mathbb{R}^k$ there exists at most one solution Z to (2.3) with $Z_0 = z_0$.*

Proof. Let Z and $\bar{Z}$ be two solutions of (2.3), i.e. they satisfy (2.4), and set

$$Y := Z - \bar{Z}\,.$$

We want to show that, for $\tau > 0$ small enough, we have

$$\|Y\|_{\infty,\tau} \leqslant 2\,|Y_0|,$$

where the weighted norm $\|\cdot\|_{\infty,\tau}$ was defined in (1.37). In particular, if we assume that $Z_0 = \bar{Z}_0$, we obtain $\|Y\|_{\infty,\tau} = 0$ and hence $Z = \bar{Z}$.

We know by (2.5) that for any $\tau > 0$

$$\|Y\|_{\infty,\tau} \leqslant |Y_0| + 3\tau^\alpha\,\|\delta Y\|_{\alpha,\tau}\,, \tag{2.20}$$

where we recall that the weighted semi-norm $\|\cdot\|_{\alpha,\tau}$ was defined in (1.33). We now define $Y^{[2]}$ as the difference between the remainders $Z^{[2]}$ and $\bar{Z}^{[2]}$ of the solutions Z and $\bar{Z}$ as defined in (2.4), that is

$$Y_{st}^{[2]} := Z_{st}^{[2]} - \bar{Z}_{st}^{[2]} = \delta Y_{st} - (\sigma(Z_s) - \sigma(\bar{Z}_s))\,\delta X_{st}\,. \tag{2.21}$$

(We are slightly abusing notation, since $Y^{[2]}$ is not the remainder of Y when σ is not linear.) By assumption $\sigma \in \mathcal{C}^\gamma$ for some $\gamma > \frac{1}{\alpha}$: renaming γ as $\gamma \wedge 2$, we may assume that $\gamma \in \left]\frac{1}{\alpha}, 2\right]$. We are going to prove the following inequalities: for any $\tau > 0$

$$\|\delta Y\|_{\alpha,\tau} \leqslant c_1\,\|Y\|_{\infty,\tau} + \tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}\,, \tag{2.22}$$

$$\|Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant c_2\,\|Y\|_{\infty,\tau} + c_2'\,\tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}\,, \tag{2.23}$$

for finite constants c_i, c_i' *that may depend on* $X, \sigma, Z, \bar{Z}$ *but not on* τ.

Let us complete the proof assuming (2.22) and (2.23). Note that $(\gamma-1)\,\alpha>0$ by assumption. If we fix $\tau>0$ small, so that $c_2'\,\tau^{(\gamma-1)\alpha}<\frac{1}{2}$, from (2.23) we get $\|Y^{[2]}\|_{\gamma\alpha,\tau}\leqslant 2\,c_2\,\|Y\|_{\infty,\tau}$ which plugged into (2.22) yields $\|\delta Y\|_{\alpha,\tau}\leqslant 2\,c_1\,\|Y\|_{\infty,\tau}$ for $\tau>0$ small (it suffices that $2\,c_2\,\tau^{(\gamma-1)\alpha}<c_1$). Finally, plugging this into (2.20) and possibly choosing $\tau>0$ even smaller, we obtain our goal $\|Y\|_{\infty,\tau}\leqslant 2\,|Y_0|$ which completes the proof.

It remains to prove (2.22) and (2.23). Using the notation from Lemma 2.8 we set

$$\begin{aligned} C_1' &:= \sup\{|\nabla\sigma(x)|:\ |x|\leqslant\|Z\|_\infty\vee\|\bar{Z}\|_\infty\}\,,\\ C_1'' &:= \sup\left\{\frac{|\nabla\sigma(x)-\nabla\sigma(y)|}{|x-y|^\rho}:\ |x|,|y|\leqslant\|Z\|_\infty\vee\|\bar{Z}\|_\infty\right\}. \end{aligned}$$

so that $|\sigma(Z_t)-\sigma(\bar{Z}_t)|\leqslant C_1'\,|Z_t-\bar{Z}_t|$ and, therefore,

$$\|\sigma(Z)-\sigma(\bar{Z})\|_{\infty,\tau}\leqslant C_1'\,\|Y\|_{\infty,\tau}\,. \tag{2.24}$$

We now exploit (2.21) to estimate $\|\delta Y\|_{\alpha,\tau}$: applying (2.7) we obtain

$$\begin{aligned} \|\delta Y\|_{\alpha,\tau} &\leqslant \|\sigma(Z)-\sigma(\bar{Z})\|_{\infty,\tau}\,\|\delta X\|_\alpha+\|Y^{[2]}\|_{\alpha,\tau}\\ &\leqslant C_1'\,\|Y\|_{\infty,\tau}\,\|\delta X\|_\alpha+\tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}\,, \end{aligned}\tag{2.25}$$

where we note that $\|Y^{[2]}\|_{\alpha,\tau}\leqslant\tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}$ by (2.6). We have shown that (2.22) holds with $c_1=C_1'\,\|\delta X\|_\alpha$.

We finally prove (2.23). Since $Y_{st}^{[2]}=o(t-s)$, see (2.21) and (2.4), we bound $Z^{[2]}$ by its increment $\delta Z^{[2]}$ through the weighted Sewing Bound (1.41):

$$\|Y^{[2]}\|_{\gamma\alpha,\tau}\ \leqslant K_{\gamma\alpha}\,\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}\,, \tag{2.26}$$

hence we focus on $\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}$. By (2.21) and (1.32), since $\delta\circ\delta=0$, we have

$$\delta Y_{sut}^{[2]}=(\delta\sigma(Z)_{su}-\delta\sigma(\bar{Z})_{su})\,\delta X_{ut}\,. \tag{2.27}$$

Applying the estimate (2.19) for $x=Z_u$, $y=Z_s$, $\bar{x}=\bar{Z}_u$, $\bar{y}=\bar{Z}_s$, we can write

$$\begin{aligned} |\delta\sigma(Z)_{su}-\delta\sigma(\bar{Z})_{su}| &\leqslant C_1'\,|\delta Z_{su}-\delta\bar{Z}_{su}|+C_1''\,\{|\delta Z_{su}|^{\gamma-1}+|\delta\bar{Z}_{su}|^{\gamma-1}\}|Z_s-\bar{Z}_s|\\ &= C_1'\,|\delta Y_{su}|+C_1''\,\{|\delta Z_{su}|^{\gamma-1}+|\delta\bar{Z}_{su}|^{\gamma-1}\}\,|Y_s|. \end{aligned}\tag{2.28}$$

hence by (2.7) we get

$$\begin{aligned} \|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau} &\leqslant C_1'\,\|\delta Y\|_{(\gamma-1)\alpha,\tau}+\\ &+\ C_1''\,\{\|\delta Z\|_\alpha^{\gamma-1}+\|\delta\bar{Z}\|_\alpha^{\gamma-1}\}\,\|Y\|_{\infty,\tau}. \end{aligned}\tag{2.29}$$

If we take $\tau\leqslant 1$ we can bound $\|\delta Y\|_{(\gamma-1)\alpha,\tau}\leqslant\|\delta Y\|_{\alpha,\tau}$ by (2.6) (recall that we are assuming $\gamma\leqslant 2$). Then by (2.27) we obtain, recalling (2.8),

$$\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}\leqslant\|\delta X\|_\alpha\,\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\leqslant\tilde{c}_1\,(\|\delta Y\|_{\alpha,\tau}+\|Y\|_{\infty,\tau})\,,$$

for a suitable (explicit) constant $\tilde{c}_1=\tilde{c}_1(\sigma,Z,\bar{Z},X)$. Applying (2.22), we obtain

$$\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}\leqslant(c_1+1)\,\tilde{c}_1\,\|Y\|_{\infty,\tau}+\tilde{c}_1\,\tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau},$$

which plugged into (2.26) shows that (2.23) holds. The proof is complete. □

We conclude with an example of (2.19).

Example 2.10. If $\sigma\colon \mathbb{R}\to\mathbb{R}$ is $\sigma(x)=x^2$, then we have

$$\begin{aligned}(\sigma(x)-\sigma(y))-(\sigma(\bar{x})-\sigma(\bar{y}))\\ &=(x^2-y^2)-(\bar{x}^2-\bar{y}^2)=(x^2-\bar{x}^2)-(y^2-\bar{y}^2)\\ &=(x-\bar{x})\,(x+\bar{x})-(y-\bar{y})\,(y+\bar{y})\\ &=[(x-\bar{x})-(y-\bar{y})]\,(y+\bar{y})+(x-\bar{x})\,[(x+\bar{x})-(y+\bar{y})]\\ &=[(x-\bar{x})-(y-\bar{y})]\,(y+\bar{y})+(x-\bar{x})\,[(x-y)+(\bar{x}-\bar{y})],\end{aligned}$$

where in the second last equality we have summed and subtracted $(y-\bar{y})\,(x+\bar{x})$. If we use this formula for $x=Z_t$, $y=Z_s$ and $\bar{x}=\bar{Z}_t$, $\bar{y}=\bar{Z}_s$, then we obtain

$$\delta(Z^2-\bar{Z}^2)_{st}=\delta(Z-\bar{Z})_{st}\,(Z_s+\bar{Z}_s)+(Z_t-\bar{Z}_t)\,[\delta Z_{st}+\delta Z_{st}],$$

which is in the spirit of (2.19) with $\rho=1$. It follows that

$$\|\delta(Z^2-\bar{Z}^2)\|_\alpha\leqslant 2\,\|\bar{Z}\|_\infty\,\|\delta(Z-\bar{Z})\|_\alpha+\|Z-\bar{Z}\|_\infty\,[\|\delta Z\|_\alpha+\|\delta\bar{Z}\|_\alpha],$$

which is the form that (2.29) takes in this particular case.

2.5. CONTINUITY OF THE SOLUTION MAP

In this section we assume that σ is *globally Lipschitz* and of class C^1 with a *globally γ-Hölder gradient*, i.e. $\|\nabla\sigma\|_\infty<\infty$ and $[\nabla\sigma]_{\mathcal{C}^\gamma}<\infty$, with $\gamma>\frac{1}{\alpha}$. Under these assumptions, we have *global existence and uniqueness* of solutions $Z\colon[0,T]\to\mathbb{R}^k$ to (2.3) for any time horizon $T>0$, for any starting point $Z_0\in\mathbb{R}^k$ and for any driving path X of class $\mathcal{C}^\alpha$ with $\frac{1}{2}<\alpha\leqslant 1$ (as we will prove in Section 2.6).

We can thus consider the *solution map*:

$$\begin{aligned}\Phi\colon\ \mathbb{R}^k\times\mathcal{C}^\alpha &\longrightarrow \mathcal{C}^\alpha\\ (Z_0,X) &\longmapsto Z:=\begin{cases}\text{unique solution of (2.3) for } t\in[0,T]\\ \text{starting from } Z_0\end{cases}.\end{aligned}\qquad(2.30)$$

We prove in this section that this map is *continuous*, in fact *locally Lipschitz*.

Remark 2.11. The continuity of the solution map is a highly non-trivial property. Indeed, when X is of class C^1, note that Z solves the equation

$$Z_t=Z_0+\int_0^t\sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s\,,\qquad(2.31)$$

which is based on the derivative $\dot{X}$ of X. We instead consider driving paths $X\in\mathcal{C}^\alpha$ with $\alpha\in\left]\frac{1}{2},1\right]$ which are continuous but may be non-differentiable.

We shall see in the next chapters that the continuity of the solution map holds also in more complex situations such as $X\in\mathcal{C}^\alpha$ with $\alpha\leqslant\frac{1}{2}$, which cover the case when X is a Brownian motion and Z is the solution to a SDE.

Before stating the continuity of the solution map, we recall that the space $\mathcal{C}^\alpha$ is equipped with the norm $\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha$, see Remark 1.4, but *an equivalent norm is* $\|f\|_{\infty,\tau} + \|\delta f\|_{\alpha,\tau}$ *for any choice of the weight* $\tau > 0$, see Remark 1.15.

THEOREM 2.12. (CONTINUITY OF THE SOLUTION MAP) *Let σ be globally Lipschitz with a globally $(\gamma-1)$-Hölder gradient: $\|\nabla\sigma\|_\infty < \infty$ and $[\nabla\sigma]_{\mathcal{C}^{\gamma-1}} < \infty$, with $\gamma \in \left(\frac{1}{\alpha}, 2\right]$. Then, for any $T>0$ and $\alpha \in \left]\frac{1}{2}, 1\right]$, the solution map $(Z_0, X) \mapsto Z$ in (2.30) is locally Lipschitz.*

More explicitly, given $M_0, M, D < \infty$, if we assume that

$$\max\{\|\nabla\sigma\|_\infty, [\nabla\sigma]_{\mathcal{C}^{\gamma-1}}\} \leqslant D,$$

and we consider starting points $Z_0, \bar{Z}_0 \in \mathbb{R}^d$ and driving paths X, $\bar{X} \in \mathcal{C}^\alpha$ with

$$\max\{|\sigma(Z_0)|, |\sigma(\bar{Z}_0)|\} \leqslant M_0, \qquad \max\{\|\delta X\|_\alpha, \|\delta\bar{X}\|_\alpha\} \leqslant M, \tag{2.32}$$

then the corresponding solutions $Z = (Z_s)_{s\in[0,T]}$, $\bar{Z} = (\bar{Z}_s)_{s\in[0,T]}$ of (2.3) satisfy

$$\|Z - \bar{Z}\|_{\infty,\tau} + \|\delta Z - \delta\bar{Z}\|_{\alpha,\tau} \leqslant \mathfrak{C}_M\, |Z_0 - \bar{Z}_0| + 6\, M_0\, \|\delta X - \delta\bar{X}\|_\alpha, \tag{2.33}$$

provided $0 < \tau \wedge T \leqslant \hat{\tau}$ for a suitable $\hat{\tau} = \hat{\tau}_{\alpha,\gamma,T,D,M_0,M} > 0$, where we set

$$\mathfrak{C}_M := 2\,(\|\nabla\sigma\|_\infty M + 1) \leqslant 2\,(D\,M + 1)\,.$$

Proof. Let us define the constant

$$\mathfrak{c}_M := \|\nabla\sigma\|_\infty M \leqslant D\,M\,. \tag{2.34}$$

We fix two solutions Z and $\bar{Z}$ of (2.3) with respective driving paths X and $\bar{X}$. If we define $Y := Z - \bar{Z}$, we can rewrite our goal (2.33) as

$$\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau} \leqslant 6\, M_0\, \|\delta X - \delta\bar{X}\|_\alpha + 2\,(\mathfrak{c}_M + 1)\,|Y_0|\,. \tag{2.35}$$

Let us introduce the shorthand

$$\varepsilon := (\tau \wedge T)^\alpha$$

and let us agree that, whenever we write *for ε small enough* we mean *for $0 < \varepsilon \leqslant \varepsilon_0$ for a suitable $\varepsilon_0 > 0$ which depends on α, T, M_0, M, D*. By (2.5), *for ε small enough*,

$$\|Y\|_{\infty,\tau} \leqslant |Y_0| + \varepsilon\, \|\delta Y\|_{\alpha,\tau} \leqslant |Y_0| + \frac{1}{5}\, \|\delta Y\|_{\alpha,\tau}, \tag{2.36}$$

hence to prove (2.35) we can focus on $\|\delta Y\|_{\alpha,\tau}$.

Recalling (2.4), let us define $Y^{[2]} := Z^{[2]} - \bar{Z}^{[2]}$. We are going to establish the following two relations, *for ε small enough*:

$$\frac{4}{5}\, \|\delta Y\|_{\alpha,\tau} \leqslant 2\, M_0\, \|\delta X - \delta\bar{X}\|_\alpha + \mathfrak{c}_M\, |Y_0| + \|Y^{[2]}\|_{\alpha,\tau}\,, \tag{2.37}$$

$$\|Y^{[2]}\|_{\alpha,\tau} \leqslant M_0\, \|\delta X - \delta\bar{X}\|_\alpha + \frac{1}{2}|Y_0| + \frac{1}{5}\, \|\delta Y\|_{\alpha,\tau}\,. \tag{2.38}$$

Plugging (2.38) into (2.37) and applying (2.36), we obtain (2.35).

It remains to prove (2.37) and (2.38). We record some useful bounds. Let us set

$$\bar{\varepsilon} = \bar{\varepsilon}_{\alpha,D,M} := \frac{1}{2\,(K_{2\alpha}+3)\,D\,M}. \tag{2.39}$$

We exploit the a priori estimate (2.13) from Theorem 2.7: by (2.32), we have

$$\text{for } \varepsilon = (\tau \wedge T)^{\alpha} \leqslant \bar{\varepsilon}: \qquad \max\{\|\delta Z\|_{\alpha,\tau}, \|\delta \bar{Z}\|_{\alpha,\tau}\} \leqslant 2\,M_0\,M, \tag{2.40}$$

therefore

$$\|\delta\sigma(Z)\|_{\alpha,\tau} \leqslant \|\nabla\sigma\|_{\infty}\,\|\delta Z\|_{\alpha,\tau} \leqslant 2\,\|\nabla\sigma\|_{\infty}\,M_0\,M = 2\,M_0\,\mathfrak{c}_M, \tag{2.41}$$

and applying (2.5) and (2.32) we get, *for ε small enough*,

$$\|\sigma(Z)\|_{\infty,\tau} \leqslant |\sigma(Z_0)| + 3\varepsilon\,\|\delta\sigma(Z)\|_{\alpha,\tau} \leqslant M_0\,(1+6\,\mathfrak{c}_M\,\varepsilon) \leqslant 2\,M_0 \ . \tag{2.42}$$

We can now prove (2.37). Defining $Y^{[2]} := Z^{[2]} - \bar{Z}^{[2]}$, we obtain from (2.4)

$$\begin{aligned}\delta Y_{st} \;&=\; \delta Z_{st} - \delta \bar{Z}_{st} \;=\; \sigma(Z_s)\,\delta X_{st} - \sigma(\bar{Z}_s)\,\delta\bar{X}_{st} + Y^{[2]}_{st}\\ &\qquad\qquad\qquad\;\;=\; \sigma(Z_s)\,(\delta X - \delta\bar{X})_{st} + (\sigma(Z_s) - \sigma(\bar{Z}_s))\,\delta\bar{X}_{st} + Y^{[2]}_{st},\end{aligned}$$

hence by (2.7) we can bound

$$\begin{aligned}\|\delta Y\|_{\alpha,\tau} \;\leqslant\; &\|\sigma(Z)\|_{\infty,\tau}\,\|\delta X - \delta\bar{X}\|_{\alpha}\\ &+ \|\delta\bar{X}\|_{\alpha}\,\|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau} + \|Y^{[2]}\|_{\alpha,\tau}\,.\end{aligned} \tag{2.43}$$

Let us look at the second term in the RHS of (2.43): by (2.5)

$$\begin{aligned}\|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau} \;&\leqslant\; \|\nabla\sigma\|_{\infty}\,\|Z - \bar{Z}\|_{\infty,\tau}\\ &\leqslant\; \|\nabla\sigma\|_{\infty}\,(|Y_0| + 3\varepsilon\,\|\delta Y\|_{\alpha,\tau}).\end{aligned} \tag{2.44}$$

Hence by (2.32) and (2.34) we get, *for ε small enough*,

$$\|\delta\bar{X}\|_{\alpha}\,\|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau} \leqslant \mathfrak{c}_M\,|Y_0| + \frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}\,. \tag{2.45}$$

Plugging this into (2.43) we then obtain, by (2.42),

$$\frac{4}{5}\,\|\delta Y\|_{\alpha,\tau} \leqslant 2\,M_0\,\|\delta X - \delta\bar{X}\|_{\alpha} + \mathfrak{c}_M\,|Y_0| + \|Y^{[2]}\|_{\alpha,\tau}, \tag{2.46}$$

which proves (2.37).

We finally prove (2.38). Since $Y^{[2]}_{st} = Z^{[2]}_{st} - \bar{Z}^{[2]}_{st} = o(t-s)$, see (2.4), the weighted Sewing Bound (1.41) and (2.6) give

$$\|Y^{[2]}\|_{\alpha,\tau} \leqslant \varepsilon^{\gamma-1}\,\|Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta Y^{[2]}\|_{\gamma\alpha,\tau} \ . \tag{2.47}$$

To estimate $\delta Y^{[2]} = \delta Z^{[2]} - \delta\,\bar{Z}^{[2]}$, note that by (2.4) and (1.32) we can write

$$\delta Y^{[2]}_{sut} = \delta\sigma(Z)_{su}\,(\delta X - \delta\bar{X})_{ut} + (\delta\sigma(Z) - \delta\sigma(\bar{Z}))_{su}\,\delta\bar{X}_{ut}\,, \tag{2.48}$$

hence by (2.8)

$$\|\delta Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant \|\delta\sigma(Z)\|_{(\gamma-1)\alpha,\tau}\,\|\delta X - \delta\bar{X}\|_{\alpha} + \|\delta\bar{X}\|_{\alpha}\,\|\delta\sigma(Z) - \delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\,. \tag{2.49}$$

The first term is easy to control: by (2.41), *for ε small enough*,

$$K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta\sigma(Z)\|_{(\gamma-1)\alpha,\tau}\,\|\delta X-\delta\bar{X}\|_{\alpha}\leqslant M_0\,\|\delta X-\delta\bar{X}\|_{\alpha}\,. \tag{2.50}$$

Let us now focus on the second term. By (2.19) we have, see also (2.28),

$$|\delta\sigma(Z)_{su}-\delta\sigma(\bar{Z})_{su}|\leqslant\|\nabla\sigma\|_{\infty}\,|\delta Y_{su}|+[\nabla\sigma]_{\mathcal{C}^{\gamma-1}}\{|\delta Z_{su}|^{\gamma-1}+|\delta\bar{Z}_{su}|^{\gamma-1}\}\,|Y_s|\,.$$

We apply (2.9) for $H=\delta Z$, $g=Y$ and $\bar{\tau}=(\bar{\varepsilon})^{1/\alpha}$ from (2.39):

$$\begin{aligned}\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau} &\leqslant \|\nabla\sigma\|_{\infty}\,\|\delta Y\|_{(\gamma-1)\alpha,\tau}+\\ &\quad +[\nabla\sigma]_{\mathcal{C}^{\gamma-1}}\,\mathrm{e}^{\frac{T}{\bar{\tau}}}\,(\|\delta Z\|_{\alpha,\bar{\tau}}^{\gamma-1}+\|\delta\bar{Z}\|_{\alpha,\bar{\tau}}^{\gamma-1})\|Y\|_{\infty,\tau}\\ &\leqslant D\,\|\delta Y\|_{\alpha,\tau}+2\,(2M_0\,M)^{\gamma-1}\,\mathrm{e}^{\frac{T}{\bar{\tau}}}D\,\|Y\|_{\infty,\tau},\end{aligned} \tag{2.51}$$

where we applied (2.40). Hence by (2.51), recalling (2.32), *for ε small enough* we obtain

$$K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta\bar{X}\|_{\alpha}\,\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\leqslant\frac{1}{10}\,\|\delta Y\|_{\alpha,\tau}+\frac{1}{2}\|Y\|_{\infty,\tau}, \tag{2.52}$$

and since $\|Y\|_{\infty,\tau}\leqslant|Y_0|+\frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}$, see (2.36), we obtain

$$K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta\bar{X}\|_{\alpha}\,\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\leqslant\frac{1}{2}|Y_0|+\frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}.$$

Finally, plugging this bound and (2.50) into (2.49) and (2.47), we obtain

$$\|Y^{[2]}\|_{\alpha,\tau}\leqslant M_0\,\|\delta X-\delta\bar{X}\|_{\alpha}+\frac{1}{2}|Y_0|+\frac{1}{5}\|\delta Y\|_{\alpha,\tau},$$

which proves (2.38) and completes the proof. □

Remark 2.13. An explicit choice for $\hat{\tau}$ in Theorem 2.12 is

$$\hat{\tau}^{\alpha}:=\frac{\mathrm{e}^{-\frac{T}{\bar{\tau}}}}{10\,(K_{2\alpha}+3)\,(1+M_0)\,(1+D\,(M+M^2))}, \tag{2.53}$$

with $\bar{\tau}=\bar{\tau}_{\alpha,D,M}$ defined in (2.39). This is obtained by tracking all the points in the proof of Theorem 2.12 where $\varepsilon=(\tau\wedge T)^{\alpha}$ was assumed to be *small enough*: see Section 2.8 for the details.

2.6. Euler scheme and local/global existence

In this section we discuss *global existence of solutions*, under the assumption that σ is globally γ-Hölder with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$, i.e. $[\sigma]_{\mathcal{C}^{\gamma}}<\infty$ (again with no boundedness assumption on σ). We also state a result of *local existence of solutions* for equation (2.3), where we only assume that σ is *locally γ-Hölder* with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$ (with no boundedness assumption on σ).

We fix $X:[0,T]\to\mathbb{R}^d$ of class $\mathcal{C}^{\alpha}$ with $\alpha\in\left]\frac{1}{2},1\right]$ and a starting point $z_0\in\mathbb{R}^k$. We split the proof in two parts: we first assume that $\sigma:\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ is globally γ-Hölder, then we consider the case when σ is locally γ-Hölder.

First part: globally Hölder case.

We consider a finite set $\mathbb{T}=\{0=t_1<\cdots<t_{\#\mathbb{T}}\}\subset\mathbb{R}_+$ and we define an approximate solution $Z=Z^{\mathbb{T}}=(Z_t)_{t\in\mathbb{T}}$ through the *Euler scheme*

$$Z_0:=z_0, \qquad Z_{t_{i+1}}:=Z_{t_i}+\sigma(Z_{t_i})\,\delta X_{t_i,t_{i+1}} \qquad \text{for } 1\leqslant i\leqslant\#\mathbb{T}-1. \tag{2.54}$$

Let us define the "remainder"

$$R_{st}:=\delta Z_{st}-\sigma(Z_s)\,\delta X_{st} \qquad \text{for } s<t\in\mathbb{T}. \tag{2.55}$$

We assume that σ is *globally γ-Hölder,* namely $[\sigma]_{\mathcal{C}^\gamma}<\infty$, with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$. We set

$$\hat{\varepsilon}_{\alpha,\gamma,X,\sigma}:=\frac{1}{2\,(C_{(\gamma+1)\alpha}+5)\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}}, \tag{2.56}$$

where the constant C_η is defined in (1.45). We prove the following *a priori estimates* on the Euler scheme (2.54), which are analogous to those in Theorem 2.7.

LEMMA 2.14. *If σ is globally γ-Hölder, namely $[\sigma]_{\mathcal{C}^\gamma}<\infty$, with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$, then*

$$\|R\|_{(\gamma+1)\alpha}^{\mathbb{T}}\leqslant C_{(\gamma+1)\alpha}\,[\sigma]_{\mathcal{C}^\gamma}\,(\|\delta Z\|_\alpha^{\mathbb{T}})^\gamma\,\|\delta X\|_\alpha, \tag{2.57}$$

$$\text{and for } \tau^{\gamma\alpha}\leqslant\hat{\varepsilon}_{\alpha,\gamma,X,\sigma}\text{:} \qquad \|\delta Z\|_\alpha^{\mathbb{T}}\leqslant 1\vee(2\,|\sigma(z_0)|\,\|\delta X\|_\alpha)\,. \tag{2.58}$$

Proof. Since $\delta R_{sut}=(\sigma(Z_s)-\sigma(Z_u))\,\delta X_{ut}$, recall (1.32), and since $R_{t_it_{i+1}}=0$ by (2.54), we can apply the discrete Sewing Bound (1.45) with $\eta=(\gamma+1)\alpha>1$ to get

$$\|R\|_{(\gamma+1)\alpha,\tau}^{\mathbb{T}}\leqslant C_{(\gamma+1)\alpha}\,\|\delta R\|_{(\gamma+1)\alpha,\tau}^{\mathbb{T}}\leqslant C_{(\gamma+1)\alpha}\,[\sigma]_{\mathcal{C}^\gamma}\,(\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma\,\|\delta X\|_\alpha. \tag{2.59}$$

We have proved (2.57).

We next prove (2.58). Recalling (2.55) we can bound, by (2.6) for $\|\cdot\|_{\gamma\alpha,\mathbb{T}_n}$,

$$\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}}\leqslant\|\sigma(Z)\|_{\infty,\tau}^{\mathbb{T}}\,\|\delta X\|_\alpha+\tau^{\gamma\alpha}\,\|R\|_{(\gamma+1)\alpha,\tau}^{\mathbb{T}}\,.$$

By (1.47)

$$\|\sigma(Z)\|_{\infty,\tau}^{\mathbb{T}}\leqslant|\sigma(z_0)|+5\tau^{\gamma\alpha}\|\delta\sigma(Z)\|_{\gamma\alpha,\tau}^{\mathbb{T}}\leqslant|\sigma(z_0)|+5\tau^{\gamma\alpha}\,[\sigma]_{\mathcal{C}^\gamma}\,(\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma\,.$$

We thus obtain, combining the previous bounds,

$$\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}}\leqslant|\sigma(z_0)|\,\|\delta X\|_\alpha+\{\tau^{\gamma\alpha}\,(C_{\gamma\alpha}+5)\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta X\|_\alpha\}\,(\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma\,.$$

Now if $\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}}\leqslant 1$ then (2.58) is proved, otherwise $(\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma\leqslant\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}}$ and then for τ as in (2.56) the term in brackets is less than $\frac{1}{2}$ and we obtain (2.58). □

We can now prove the following

THEOREM 2.15. (GLOBAL EXISTENCE) *Let X be of class $\mathcal{C}^\alpha$, with $\alpha\in\left]\frac{1}{2},1\right]$, and let σ be globally γ-Hölder with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$, i.e. $[\sigma]_{\mathcal{C}^\gamma}<\infty$. For every $z_0\in\mathbb{R}^k$, with no restriction on $T>0$, there exists a solution $(Z_t)_{t\in[0,T]}$ of (2.3) with $Z_0=z_0$.*

Proof. Given $n \in \mathbb{N}$, we construct an approximate solution $Z^n = (Z_t^n)_{t \in \mathbb{T}_n}$ of (2.3) defined in the discrete set of times $\mathbb{T}_n := (\{i 2^{-n}:\ i = 0, 1, \ldots\} \cap [0, T]) \cup \{T\}$ through the *Euler scheme* (2.54).

$$Z_0^n := z_0, \qquad Z_{t_{i+1}}^n := Z_{t_i}^n + \sigma(Z_{t_i}^n)\, \delta X_{t_i, t_{i+1}} \qquad \text{for } t_i, t_{i+1} \in \mathbb{T}_n\,. \tag{2.60}$$

Let us define the "remainder"

$$R_{st}^n := \delta Z_{st}^n - \sigma(Z_s^n)\, \delta X_{st} \qquad \text{for } s < t \in \mathbb{T}_n\,. \tag{2.61}$$

We fix $T > 0$ such that

We extend Z^n by linear interpolation to a continuous function defined on $[0, T]$, still denoted by Z^n. Given two points $t_i \leqslant s < t \leqslant t_{i+1}$ inside the same interval $[t_i, t_{i+1}]$ of the partition $\mathbb{T}_n$, since $\delta Z_{st}^n = \frac{t-s}{t_{i+1} - t_i}\, \delta Z_{t_i t_{i+1}}^n$, we can bound for $\alpha \in (0, 1]$

$$\frac{|\delta Z_{st}^n|}{(t-s)^\alpha} = \left(\frac{t-s}{t_{i+1} - t_i}\right)^{1-\alpha} \frac{|\delta Z_{t_i t_{i+1}}^n|}{(t_{i+1} - t_i)^\alpha} \leqslant \frac{|\delta Z_{t_i t_{i+1}}^n|}{(t_{i+1} - t_i)^\alpha}\,.$$

Given two points $s < t$ in different intervals, say $t_i \leqslant s \leqslant t_{i+1} \leqslant t_j \leqslant t \leqslant t_{j+1}$ for some $i < j$, by the triangle inequality we can bound $|\delta Z_{st}^n| \leqslant |\delta Z_{s t_{i+1}}^n| + |\delta Z_{t_{i+1} t_j}^n| + |\delta Z_{t_j t}^n|$. Recalling (1.9) and (1.43), we then obtain $\|\cdot\|_\alpha \leqslant 3\, \|\cdot\|_\alpha^{\mathbb{T}_n}$, hence by (2.58) we get

$$\|\delta Z^n\|_{\alpha, \tau} \leqslant 3 \vee (6\, |\sigma(z_0)|\, \|\delta X\|_\alpha)\,. \tag{2.62}$$

The family $(Z^n)_{n \in \mathbb{N}}$ is *equi-continuous* by (2.62) and *equi-bounded*, since $Z_0^n = z_0$ for all $n \in \mathbb{N}$, hence by the Arzelà-Ascoli Theorem it is *compact* in the space $C([0, T], \mathbb{R}^k)$. Let us denote by $Z: [0, T] \to \mathbb{R}^k$ any limit point. Plugging (2.58) into (2.57), by (2.61) we can write

$$\text{if } T^\alpha \leqslant \hat{\varepsilon}_{\alpha, X, \sigma}: \qquad |\delta Z_{st}^n - \sigma(Z_s^n)\, \delta X_{st}| \leqslant c(z_0)\, (t-s)^{2\alpha} \qquad \forall s < t \in \mathbb{T}_n\,, \tag{2.63}$$

where $c(z_0) := C_{(\gamma+1)\alpha}\, [\sigma]_{\mathcal{C}^\gamma}\, (3 \vee (6\, |\sigma(z_0)|\, \|\delta X\|_\alpha))^\gamma\, \|\delta X\|_\alpha$. Letting $n \to \infty$ and observing that $\mathbb{T}_n \subseteq \mathbb{T}_{n+1}$, we see that (2.63) still holds with Z^n replaced by Z and $\mathbb{T}_n$ replaced by the set $\mathbb{T} := \bigcup_{\ell \in \mathbb{N}} \mathbb{T}_{2^\ell} = \left(\left\{\frac{i}{2^n}:\ i, n \in \mathbb{N}\right\} \cap [0, T]\right) \cup \{T\}$ of dyadic rationals:

$$\text{if } T^\alpha \leqslant \hat{\varepsilon}_{\alpha, X, \sigma}: \qquad |\delta Z_{st} - \sigma(Z_s)\, \delta X_{st}| \leqslant c(z_0)\, (t-s)^{2\alpha} \qquad \forall s < t \in \mathbb{T}.$$

Since $\mathbb{T}$ is dense in $[0, T]$ and Z is continuous, this bound extends to all $0 \leqslant s < t \leqslant T$, which shows that Z is a solution of (2.3). This completes the proof. □

Second part: locally Lipschitz case.

We now assume that σ is *locally γ-Hölder* and we fix $z_0 \in \mathbb{R}^k$. We also fix $T > 0$ such that $T \leqslant \tilde{\varepsilon}_{\alpha, X, \sigma}(z_0)$, see (2.64), and we prove that there exists a solution $Z: [0, T] \to \mathbb{R}^k$ of (2.3) with $Z_0 = z_0$.

Theorem 2.16. (Local existence) *Let X be of class $\mathcal{C}^\alpha$, with $\alpha \in \left]\frac{1}{2}, 1\right]$, and let σ be locally Lipschitz (e.g. of class C^1). For any $z_0 \in \mathbb{R}^k$ and for $T > 0$ small enough, i.e.*

$$T^\alpha \leqslant \tilde{\varepsilon}_{\alpha, X, \sigma}(z_0) := \frac{1}{2}\, \frac{1}{(C_{2\alpha} + 3)\, \|\delta X\|_\alpha\, \{1 + \sup_{|z - z_0| \leqslant |\sigma(z_0)|} |\nabla \sigma(z)|\}}\,, \tag{2.64}$$

there exists a solution $(Z_t)_{t\in[0,T]}$ *of (2.3) with* $Z_0 = z_0$.

Let $\tilde{\sigma}$ be a *globally* γ*-Hölder* function (depending on z_0) such that

$$\tilde{\sigma}(z) = \sigma(z) \quad \forall |z - z_0| \leqslant \sigma(z_0) \qquad \text{and} \qquad [\tilde{\sigma}]_{\mathcal{C}^\gamma} = \sup_{|z - z_0| \leqslant \sigma(z_0)} |\nabla \sigma(z)| \,. \tag{2.65}$$

Since $T \leqslant \tilde{\varepsilon}_{\alpha,X,\sigma}(z_0) \leqslant \hat{\varepsilon}_{\alpha,X,\sigma}$, see (2.64) and (2.56), by the first part of the proof *there exists a solution Z of (2.3) with* $\tilde{\sigma}$ *in place of* σ and $Z_0 = z_0$. We will prove that

$$|Z_t - z_0| \leqslant \sigma(z_0) \ \text{ for all } \ t \in [0,T] \,, \tag{2.66}$$

therefore $\tilde{\sigma}(Z_t) = \sigma(Z_t)$ for all $t \in [0,T]$, see (2.65). This means that Z is a solution of the original (2.3) with σ, which completes the proof of Theorem 2.16.

To prove (2.66), we apply the a priori estimate (2.13) with $\tau = \infty$: we note that $T \leqslant \tilde{\varepsilon}_{\alpha,X,\sigma}(z_0) \leqslant \varepsilon_{\alpha,X,\sigma}$ (see (2.64) and (2.12), and note that $C_{2\alpha} \geqslant K_{2\alpha}$), therefore

$$\|\delta Z\|_\alpha \leqslant 2 \, \|\delta X\|_\alpha \, |\sigma(z_0)|,$$

because $\tilde{\sigma}(z_0) = \sigma(z_0)$. Then for every $t \in [0,T]$ we can bound

$$|Z_t - z_0| \leqslant T^\alpha \, \|\delta Z\|_\alpha \leqslant 2 \, T^\alpha \, \|\delta X\|_\alpha \, |\sigma(z_0)| \leqslant |\sigma(z_0)|,$$

where the last inequality holds because $T^\alpha \leqslant \tilde{\varepsilon}_{\alpha,X,\sigma}(z_0) \leqslant (2 \, \|\delta X\|_\alpha)^{-1}$, see (2.64). This completes the proof of (2.66).

2.7. Error estimate in the Euler scheme

We suppose in this section that σ is of class C^2 with $\|\nabla \sigma\|_\infty + \|\nabla^2 \sigma\|_\infty < +\infty$.

THEOREM 2.17. *The Euler scheme converges at speed* $n^{2\alpha-1}$.

Proof. Let us set $z_i := \partial y_i / \partial y_0$, where $(y_i)_{i \geqslant 0}$ is defined by (2.60). Then

$$z_{i+1} = z_i + \nabla \sigma(y_i) \, z_i \, \delta X_{t_i t_{i+1}}, \qquad i \geqslant 0.$$

This shows that the pair $(y_i, z_i)_{i \geqslant 0}$ satisfies a recurrence which is similar to (2.60) with a map Σ of class C^1 and therefore we can apply the above results to obtain that $|z_i| \leqslant \text{const}$. In particular the map $y_0 \to y_k$ is Lipschitz-continuous, uniformly over $k \geqslant 0$.

Let us call, for $k \geqslant 0$, $(z_\ell^{(k)})_{\ell \geqslant k}$ as the sequence which satisfies (2.60) but has initial value $z_k^{(k)} = y(t_k)$. Since $(y(t))_{t \geqslant 0}$ is a solution to (2.4), we have

$$|z_{k+1}^{(k)} - y(t_{k+1})| \lesssim n^{-2\alpha}.$$

Since the map $y_0 \to y_k$ is Lipschitz-continuous uniformly over $k \geqslant 0$, we have

$$|z_\ell^{(k)} - z_\ell^{(k+1)}| \lesssim |z_{k+1}^{(k)} - y(t_{k+1})| \lesssim n^{-2\alpha}, \qquad \ell \geqslant k+1.$$

Therefore

$$|y_\ell - y(t_\ell)| = |z_\ell^{(0)} - z_\ell^{(\ell)}| \leqslant \sum_{k=0}^{\ell-1} |z_\ell^{(k)} - z_\ell^{(k+1)}| \lesssim \frac{\ell}{n^{2\alpha}} = \frac{t_\ell}{n^{2\alpha-1}} \to 0$$

as t_ℓ is bounded and $n \to \infty$. □

2.8. EXTRA: A VALUE FOR $\hat{\tau}$

We can give an explicit expression for $\hat{\tau} = \hat{\tau}_{M_0,M,T}$ in Theorem 2.12, by tracking all the points in the proof where τ *is small enough*, namely:

- for (2.36) we need $\tau^\alpha \leqslant \frac{1}{15}$;
- for (2.40) we need $\tau^\alpha \leqslant (\hat{\rho}_M)^\alpha := (2\,(K_{2\alpha} + 3)\,\mathfrak{c}_M)^{-1}$;
- for (2.42) we need $\tau^\alpha \leqslant (6\,\mathfrak{c}_M)^{-1}$, for (2.45) we need $\tau^\alpha \leqslant (15\,\mathfrak{c}_M)^{-1}$;
- for (2.50) we need $\tau^{(\gamma-1)\alpha} \leqslant (2\,K_{\gamma\alpha}\,\mathfrak{c}_M)^{-1}$;
- for (2.52) we need $\tau^{(\gamma-1)\alpha} \leqslant (10\,K_{\gamma\alpha}\,\mathfrak{c}_M)^{-1}$ (first term in the RHS) and also $\tau^{(\gamma-1)\alpha} \leqslant \left(K_{\gamma\alpha}\,\mathrm{e}^{\frac{T}{\hat{\rho}_M}}\,M_0\,M^2\,\|\nabla^2\sigma\|_\infty\right)^{-1}$ (second term in the RHS).

Since $\mathfrak{c}_M = M\,\|\nabla\sigma\|_\infty$, see (2.34), it is easy to check that all these constraints are satisfied for $0 < \tau \leqslant \hat{\tau}$ given by formula (2.53) in Remark 2.13.

Chapter 3

Difference equations: the rough case

We have so far considered the difference equation (2.3), that is

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.1}$$

where X is given, Z is the unknown and $\sigma(\cdot)$ is sufficiently regular. This is a generalization of the differential equation $\dot{Z}_t = \sigma(Z_t)\,\dot{X}_t$ which is meaningful for non smooth X, as we showed in Chapter 2, where we proved *well-posedness* in the so-called *Young case*, i.e. assuming that $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$.

However, the restriction $\alpha > \frac{1}{2}$ is a substantial limitation: in particular, we cannot take $X = B$ as a typical path of Brownian motion, which is in $\mathcal{C}^\alpha$ only for $\alpha < \frac{1}{2}$. For this reason, we show in this chapter how to *enrich* the difference equation (3.1) and prove *well-posedness when $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$*, called the *rough case*. This will be applied to Brownian motion in the next Chapter 4, in order to obtain a robust formulation of classical *stochastic differential equations*.

Remark 3.1. (Young vs. Rough case) The restriction $\alpha > \frac{1}{2}$ for the study of the difference equation (3.1) has a substantial reason, namely *there is no solution to (3.1) for general $X \in \mathcal{C}^\alpha$ with $\alpha \leqslant \frac{1}{2}$*. Indeed, taking the "increment" δ of both sides of (3.1) and recalling (1.23) and (1.32), we obtain

$$(\sigma(Z_u) - \sigma(Z_s))\,(X_t - X_u) = o(t-s) \qquad \text{for } 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{3.2}$$

If $X \in \mathcal{C}^\alpha$, for any $\alpha \in (0,1]$, then we know from Lemma 2.6 that $Z \in \mathcal{C}^\alpha$, but not better in general (e.g. when $\sigma(\cdot) \equiv c$ is constant we have $Z = c\,X$), hence the LHS of (3.2) is $\lesssim (u-s)^\alpha\,(t-u)^\alpha \lesssim (t-s)^{2\alpha}$, but not better in general. This shows that the restriction $\alpha > \frac{1}{2}$ is generally necessary for (3.1) to have solutions.

3.1. Enhanced Taylor expansion

We fix $d, k \in \mathbb{N}$, a time horizon $T > 0$ and a sufficiently regular function $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$. Our goal is to give a meaning to the integral equation

$$Z_t = Z_0 + \int_0^t \sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s, \qquad 0 \leqslant t \leqslant T, \tag{3.3}$$

where $Z: [0,T] \to \mathbb{R}^k$ is the unknown and $X: [0,T] \to \mathbb{R}^d$ is a non smooth path, more precisely $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$.

The difference equation (3.1) is no longer enough, for the crucial reason that typically *it admits no solutions for* $\alpha \leqslant \frac{1}{2}$, see Remark 3.1. We are going to solve this problem by *enriching the RHS of (3.1)* in a suitable, but non canonical way: this leads to the key notion of *rough path* which is central in this book.

To provide motivation, suppose for the moment that X is continuously differentiable, so that (3.3) is meaningful. As we saw in (1.3), an integration yields for $s<t$

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + \int_s^t (\sigma(Z_u) - \sigma(Z_s))\,\dot{X}_u\,\mathrm{d}u\,. \tag{3.4}$$

In Chapter 1 we observed that the integral is $o(t-s)$, which leads to the difference equation (3.1). More precisely, the integral is $O((t-s)^2)$ if $X \in C^1$ and σ is locally Lipschitz (note that $Z \in C^1$). The idea is now to go further, expanding the integral to get a more accurate local description, with a better remainder $O((t-s)^3)$.

To this purpose, we assume that σ is differentiable and we introduce the key function $\sigma_2\colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*$ by

$$\sigma_2(z) := \nabla\sigma(z)\,\sigma(z), \qquad \text{i.e.} \qquad [\sigma_2(z)]^i_{j\ell} := \sum_{a=1}^k \frac{\partial \sigma^i_j}{\partial z_a}(z)\,\sigma^a_\ell(z)\,. \tag{3.5}$$

Since $\frac{\mathrm{d}}{\mathrm{d}r}\sigma(Z_r) = \nabla\sigma(Z_r)\,\dot{Z}_r = \sigma_2(Z_r)\,\dot{X}_r$ by (3.3), we can write for $s<u$

$$\begin{aligned} \sigma(Z_u) - \sigma(Z_s) &= \int_s^u \sigma_2(Z_r)\,\dot{X}_r\,\mathrm{d}r \\ &= \sigma_2(Z_s)\,(X_u - X_s) + \int_s^u (\sigma_2(Z_r) - \sigma_2(Z_s))\,\dot{X}_r\,\mathrm{d}r, \end{aligned} \tag{3.6}$$

where for $z \in \mathbb{R}^d$ and $a \in \mathbb{R}^d$ we define $\sigma_2(z)\,a \in \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ by

$$[\sigma_2(z)\,a]^i_j = \sum_{\ell=1}^d [\sigma_2(z)]^i_{j\ell}\,a^\ell.$$

If we assume that σ_2 is locally Lipschitz, then the last integral in (3.6) is $O((u-s)^2)$ (recall that $X \in C^1$). Plugging this into (3.4), we then obtain

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + \sigma_2(Z_s) \int_s^t (X_u - X_s) \otimes \dot{X}_u\,\mathrm{d}u + O((t-s)^3), \tag{3.7}$$

where now for $z \in \mathbb{R}^d$ and $B \in \mathbb{R}^d \otimes \mathbb{R}^d$ we define $\sigma_2(z)\,B \in \mathbb{R}^k$ by

$$[\sigma_2(z)\,B]^i = \sum_{\ell,m=1}^d [\sigma_2(z)]^i_{\ell m}\,B^{m\ell}. \tag{3.8}$$

Let us rewrite the integral in the right-hand side of (3.7) more conveniently. To this purpose we introduce the shorthands

$$\mathbb{X}^1_{st} := X_t - X_s\,, \qquad \mathbb{X}^2_{st} := \int_s^t (X_r - X_s) \otimes \dot{X}_r\,\mathrm{d}r, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.9}$$

so that $\mathbb{X}^1$: $[0,T]^2_{\leqslant} \to \mathbb{R}^d$ and $\mathbb{X}^2$: $[0,T]^2_{\leqslant} \to \mathbb{R}^d \otimes \mathbb{R}^d$, see (1.7). More explicitly:

$$(\mathbb{X}^2_{st})^{ij} := \int_s^t (X^i_r - X^i_s)\, \dot{X}^j_r \, \mathrm{d}r, \qquad i,j \in \{1,\ldots,d\}.$$

We can thus rewrite (3.7), replacing $O((t-s)^3)$ by $o(t-s)$, in the compact form

$$Z_t - Z_s = \sigma(Z_s)\, \mathbb{X}^1_{st} + \sigma_2(Z_s)\, \mathbb{X}^2_{st} + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.10}$$

where for the product $\sigma_2(Z_s)\,\mathbb{X}^2_{st}$ we use the contraction rule (3.8).

We have obtained an *enhanced Taylor expansion*: comparing with (3.1), we added a "second order term" containing $\mathbb{X}^2_{st}$. The idea is to take this new difference equation, that we call *rough difference equation*, as a generalized formulation of the differential equation (3.3), just as we did in Chapter 1 (see Section 1.2). However, there is a problem: the term $\mathbb{X}^2_{st}$ *depends on the derivative* $\dot{X}$, see (3.9), so it is not clearly defined for a non-differentiable X.

To overcome this problem, we will *assign* a suitable function $\mathbb{X}^2 = (\mathbb{X}^2_{st})_{0 \leqslant s \leqslant t \leqslant T}$ playing the role of the integral (3.9) when X is not differentiable: this leads to the notion of *rough paths*, defined in the next section and studied in depth in Chapter 7. We will show in this chapter that rough paths are the key to a robust solution theory of rough difference equations when X of class $\mathcal{C}^\alpha$ with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$.

3.2. Rough paths

Let us fix a path X: $[0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$ with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. Motivated by the previous section, we are going to reformulate the ill-posed integral equation (3.3) as the difference equation (3.10), which contains $\mathbb{X}^1$ and $\mathbb{X}^2$.

We can certainly define $\mathbb{X}^1_{st} := X_t - X_s$ as in (3.9), but there is no canonical definition of $\mathbb{X}^2_{st} = \int_s^t (X_r - X_s) \otimes \dot{X}_r\, \mathrm{d}r$, since X may not be differentiable. We therefore *assign* a function $\mathbb{X}^2_{st}$ which satisfies *suitable properties*. Note that when X is continuously differentiable the function $\mathbb{X}^2$ in (3.9) satisfies:

- an algebraic identity known as *Chen's relation*: for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut} = (X_u - X_s) \otimes (X_t - X_u), \tag{3.11}$$

  which follows from (3.9) noting that

$$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \int_u^t (X_r - X_s) \otimes \dot{X}_r\, \mathrm{d}r = (X_u - X_s) \otimes (X_t - X_u);$$

- the analytic bounds

$$|\mathbb{X}^1_{st}| \lesssim |t-s|, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^2, \tag{3.12}$$

  which follow from the fact that $\dot{X}$ is bounded.

The algebraic relation (3.11) is still meaningful for non-differentiable X, while the analytic bounds (3.12) can naturally be adapted to the case of Hölder paths $X \in \mathcal{C}^\alpha$ by changing the exponents $1, 2$ to $\alpha, 2\alpha$. This leads to the following key definition.

DEFINITION 3.2. (ROUGH PATHS) *Fix* $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$, $d \in \mathbb{N}$ *and a path* $X: [0,T] \to \mathbb{R}^d$ *of class* $\mathcal{C}^\alpha$. *An* ***α-rough path over X*** *is a pair* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *where the functions* $\mathbb{X}^1: [0,T]^2_\leqslant \to \mathbb{R}^d$ *and* $\mathbb{X}^2: [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ *satisfy, for* $0 \leqslant s \leqslant u \leqslant t \leqslant T$:

- *the algebraic relations*

$$\mathbb{X}^1_{st} = X_t - X_s, \qquad \delta\mathbb{X}^2_{sut} := \mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}, \tag{3.13}$$

  where the second identity is called Chen's relation;

- *the analytic bounds*

$$|\mathbb{X}^1_{st}| \lesssim |t-s|^\alpha, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}. \tag{3.14}$$

We call $\mathcal{R}_{\alpha,d}(X)$ *the set of d-dimensional* α*-rough paths* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *over* X *and* $\mathcal{R}_{\alpha,d} = \bigcup_{X \in \mathcal{C}^\alpha} \mathcal{R}_{\alpha,d}(X)$ *the set of all d-dimensional* α*-rough paths.*

When X is of class C^1, the choice (3.9) yields by (3.11)-(3.12) a α-rough path for any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ which we call the *canonical rough path*, see Section 7.7 below.

When $X = B$ is Brownian motion, the theory of stochastic integration provides a natural candidate for $\mathbb{X}^2$, in fact *multiple candidates* (think of Ito vs. Stratonovich integration), as we discuss in Chapter 4 below. Incidentally, this makes it clear that the construction of $\mathbb{X}^2$ is in general *non canonical*, i.e. there are multiple choices of $\mathbb{X}^2$ for a given path X. *This is a strength of the theory of rough paths*, since it allows to treat different non equivalent forms of integration.

Remark 3.3. The existence of rough paths over any given path X (i.e. the fact that $\mathcal{R}_{\alpha,d}(X) \neq \emptyset$) is a non trivial fact, which will be proved in Chapter 7.

Remark 3.4. ($\mathbb{X}^2$ AS A "PATH") The two-parameters function $\mathbb{X}^2_{st}$ is determined by the one-parameter function

$$\mathbb{I}_t := \mathbb{X}^2_{0t} + X_0 \otimes (X_t - X_0), \tag{3.15}$$

which intuitively describes the integral $\int_0^t X_r \otimes \dot{X}_r \, \mathrm{d}r$. Indeed, we can write

$$\mathbb{X}^2_{st} = \mathbb{I}_t - \mathbb{I}_s - X_s \otimes (X_t - X_s), \tag{3.16}$$

since $\mathbb{X}^2_{st} = \mathbb{X}^2_{0t} - \mathbb{X}^2_{0s} - (X_s - X_0) \otimes (X_t - X_s)$ by Chen's relation (3.13).

Vice versa, given a function $\mathbb{I}: [0,T] \to \mathbb{R}^d$, if we *define* $\mathbb{X}^2$ by (3.16), then Chen's relation (3.13) is automatically satisfied (recall (1.32)). In order to satisfy the analytic bound in (3.14), we must require that

$$|\mathbb{I}_t - \mathbb{I}_s - X_s \otimes (X_t - X_s)| \lesssim (t-s)^{2\alpha}, \tag{3.17}$$

which is a natural estimate if $\mathbb{I}_t - \mathbb{I}_s$ should describe "$= \int_s^t X_r \otimes \dot{X}_r \, \mathrm{d}r$".

Summarizing: given any path $X: [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$, *it is equivalent to assign* $\mathbb{X}^2: [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ *satisfying (3.13)-(3.14) or to assign* $\mathbb{I}: [0,T] \to \mathbb{R}^d$ *satisfying (3.17)*, the correspondence being given by (3.15)-(3.16).

3.3. Rough difference equations

Given a time horizon $T>0$ and two dimensions $d,k\in\mathbb{N}$, let us fix:

- a path $X\colon[0,T]\to\mathbb{R}^d$ of class $\mathcal{C}^\alpha$ with $\alpha\in\left]\frac{1}{3},\frac{1}{2}\right]$;
- an α-rough path $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ over X, see Definition 3.2;
- a *differentiable* function $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$, which lets us define the function

$$\sigma_2\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*\otimes(\mathbb{R}^d)^* \qquad (\text{see } (3.5))\,.$$

Motivated by the previous discussions, see in particular (3.10), we study in this chapter the following *rough difference equation* for an unknown path $Z\colon[0,T]\to\mathbb{R}^k$:

$$\delta Z_{st}=\sigma(Z_s)\,\mathbb{X}^1_{st}+\sigma_2(Z_s)\,\mathbb{X}^2_{st}+o(t-s), \qquad 0\leqslant s\leqslant t\leqslant T, \tag{3.18}$$

where we recall the increment notation $\delta Z_{st}:=Z_t-Z_s$ and the contraction rule (3.8), and we stress that $o(t-s)$ is *uniform* for $0\leqslant s\leqslant t\leqslant T$, see Remark 1.1. In analogy with (2.3)-(2.4), a solution of (3.18) is a path $Z\colon[0,T]\to\mathbb{R}^k$ such that

$$Z^{[3]}_{st}:=\delta Z_{st}-\sigma(Z_s)\,\mathbb{X}^1_{st}-\sigma_2(Z_s)\,\mathbb{X}^2_{st}=o(t-s)\,. \tag{3.19}$$

We stress that the rough difference equation (3.18) is a generalization of the integral equation (3.3), as we show in the next result.

Proposition 3.5. *If X and σ are of class C^1 and σ_2 is locally Lipschitz (e.g. if σ is of class C^2), then any solution Z to the integral equation (3.3) satisfies the difference equation (3.18) for the canonical rough path $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ in (3.9).*

Proof. If $X\in C^1$, then $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ defined in (3.9) is an α-rough path over X for any $\alpha\in\left]\frac{1}{3},\frac{1}{2}\right]$, as we showed in (3.11)-(3.12). Given a solution Z of (3.3), if σ_2 is locally Lipschitz we derived the Taylor expansion (3.10), hence (3.18) holds. ☐

We now state *local and global existence, uniqueness of solutions and continuity of the solution map* for the rough difference equation (3.18) under natural assumptions on σ and σ_2, summarizing the main results of this chapter. We refer to the next sections for more precise and quantitative results.

To be completed.

Proposition 3.6. *Let $z_0\in\mathbb{R}^d$. We suppose that σ and σ_2 are of class C^1 and globally Lipschitz, namely $\|\nabla\sigma\|_\infty+\|\nabla\sigma_2\|_\infty<+\infty$. Let $D:=\max\{1,\|\nabla\sigma\|_\infty,\|\nabla\sigma_2\|_\infty\}$ and $M>0$.*

There exists $T_{M,D,\alpha}>0$ such that, for all $T\in(0,T_{M,D,\alpha})$ and $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)\in\mathcal{R}_{\alpha,d}$ such that $\|\mathbb{X}^1\|_\alpha+\|\mathbb{X}^2\|_{2\alpha}\leqslant M$, there exists a solution Z to (3.19) on the interval $[0,T]$ such that $Z_0=z_0$ and

$$\|Z\|_\alpha\leqslant 15\,M(|\sigma(z_0)|+|\sigma_2(z_0)|). \tag{3.20}$$

The proof of this Proposition, based on a discretization argument, is postponed to section 3.9 below.