

Ten lectures on rough paths

(work in progress)

F. CARAVENNA M. GUBINELLI L. ZAMBOTTI

TABLE OF CONTENTS

I. Rough Equations	9
1. THE SEWING BOUND	13
1.1. Controlled differential equation	13
1.2. Controlled difference equation	14
1.3. Some useful function spaces	15
1.4. Local uniqueness of solutions	17
1.5. The Sewing bound	18
1.6. End of proof of uniqueness	20
1.7. Weighted norms	21
1.8. A discrete Sewing Bound	23
1.9. Extra (to be completed)	25
2. DIFFERENCE EQUATIONS: THE YOUNG CASE	27
2.1. Summary	27
2.2. Set-up	28
2.3. A priori estimates	29
2.4. Uniqueness	31
2.5. Continuity of the solution map	34
2.6. Euler scheme and local/global existence	37
First part: globally Hölder case.	37
Second part: locally Lipschitz case.	39
2.7. Error estimate in the Euler scheme	40
2.8. Extra: a value for $\hat{\tau}$	41
3. DIFFERENCE EQUATIONS: THE ROUGH CASE	43
3.1. Enhanced Taylor expansion	43
3.2. Rough paths	45
3.3. Rough difference equations	47
3.4. Set-up	48
3.5. A priori estimates	49
3.6. Uniqueness	51
3.7. Continuity of the solution map	54
3.8. Global existence and uniqueness	58
3.9. Milstein scheme and local existence	59
4. STOCHASTIC DIFFERENTIAL EQUATIONS	63
4.1. Local expansion of stochastic integrals	64
4.2. Brownian rough path and SDE	65
4.3. SDE with a drift	66

4.4. Itô versus Stratonovich	68
4.5. Wong-Zakai	69
4.6. A refined Kolmogorov criterion	73
4.7. Proof of Theorem 4.3	76
II. Rough Integration	83
5. THE SEWING LEMMA	87
5.1. Local approximation	87
5.2. A general problem	88
5.3. An algebraic look	89
5.4. Enters analysis: the Sewing Lemma	89
5.5. The Sewing Map	91
5.6. Proof of the Sewing Lemma	92
6. THE YOUNG INTEGRAL	95
6.1. Construction of the Young integral	95
6.2. Integral formulation of Young equations	97
6.3. Local existence via contraction	97
6.4. Properties of the Young integral	99
6.5. More on Hölder spaces	101
6.6. Uniqueness of the Young integral	102
6.7. Two technical proofs	104
7. ROUGH PATHS	107
7.1. Integral beyond Young	108
7.2. A negative result	109
7.3. A choice	110
7.4. One-dimensional rough paths	111
7.5. The vector case	112
7.6. Distance on rough paths	114
7.7. Canonical rough paths for $\alpha > \frac{1}{2}$	115
7.8. Lack of continuity	117
7.9. Proof of Proposition 7.5	118
8. GEOMETRIC ROUGH PATHS	123
8.1. Geometric rough paths	123
8.2. The Stratonovich rough path	125
8.3. Non-geometric rough paths	126
8.4. Pure area rough paths	127
8.5. Doss-Sussmann	128
8.6. Lack of continuity (again)	129
9. ROUGH INTEGRATION	131
9.1. Controlled paths	131

9.2. The rough integral	132
9.3. Continuity properties of the rough integral	133
9.4. Stochastic and rough integrals	136
9.5. Properties in the geometric case	136
10. ROUGH INTEGRAL EQUATIONS	139
10.1. Localization argument	140
10.2. Invariance	140
10.3. Local Lipschitz continuity	142
10.4. Contraction	144
11. ALGEBRA	147
11.1. A non-commutative group	147
11.2. Shuffle group	148
11.3. Algebra and generalised integral	148
11.4. Unordered times	150
11.5. An example: the Brownian case	151
12. NEW TOPICS	153
BIBLIOGRAPHY	155

Part I

Rough Equations

CHAPTER 1

THE SEWING BOUND

The problem of interest in this book is the study of differential equations driven by *irregular functions* (more specifically: continuous but not differentiable). This will be achieved through the powerful and elegant theory of *rough paths*. A key motivation comes from stochastic differential equations driven by Brownian motion, but the goal is to develop a general theory which does not rely on probability.

This first chapter is dedicated to an elementary but fundamental tool, the *Sewing Bound*, that will be applied extensively throughout the book. It is a general Hölder-type bound for functions of two real variables that can be understood by itself, see Theorem 1.9 below. To provide motivation, we present it as a natural a priori estimate for solutions of differential equations.

Notation. We fix a time horizon $T > 0$ and two dimensions $k, d \in \mathbb{N}$. We use “path” as a synonym of “function defined on $[0, T]$ ” with values in \mathbb{R}^d . We denote by $|\cdot|$ the Euclidean norm. The space of linear maps from \mathbb{R}^d to \mathbb{R}^k , identified by $k \times d$ real matrices, is denoted by $\mathbb{R}^k \otimes (\mathbb{R}^d)^* \simeq \mathbb{R}^{k \times d}$ and is equipped with the Hilbert-Schmidt norm $|\cdot|$ (i.e. the Euclidean norm on $\mathbb{R}^{k \times d}$). For $A \in \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ and $v \in \mathbb{R}^d$ we have $|Av| \leq |A| |v|$.

1.1. CONTROLLED DIFFERENTIAL EQUATION

Consider the following *controlled ordinary differential equation (ODE)*: given a continuously differentiable path $X: [0, T] \rightarrow \mathbb{R}^d$ and a continuous function $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$, we look for a differentiable path $Z: [0, T] \rightarrow \mathbb{R}^k$ such that

$$\dot{Z}_t = \sigma(Z_t) \dot{X}_t, \quad t \in [0, T]. \quad (1.1)$$

By the fundamental theorem of calculus, this is equivalent to

$$Z_t = Z_0 + \int_0^t \sigma(Z_s) \dot{X}_s ds, \quad t \in [0, T]. \quad (1.2)$$

In the special case $k = d = 1$ and when $\sigma(x) = \lambda x$ is linear (with $\lambda \in \mathbb{R}$), we have the explicit solution $Z_t = z_0 \exp(\lambda (X_t - X_0))$, which has the interesting property of being well-defined also when X is non differentiable.

For any dimensions $k, d \in \mathbb{N}$, if we assume that $\sigma(\cdot)$ is Lipschitz, classical results in the theory of ODEs guarantee that *equation (1.1)-(1.2) is well-posed for any continuously differentiable path X* , namely for any $Z_0 \in \mathbb{R}^k$ there is one and only one solution Z (with no explicit formula, in general).

Our aim is to extend such a well-posedness result to a setting where X is *continuous but not differentiable* (also in cases where $\sigma(\cdot)$ may be non-linear). Of course, to this purpose it is first necessary to provide a generalized formulation of (1.1)-(1.2) where the derivative of X does not appear.

1.2. CONTROLLED DIFFERENCE EQUATION

Let us still suppose that X is continuously differentiable. We deduce by (1.1)-(1.2) that for $0 \leq s \leq t \leq T$

$$Z_t - Z_s = \sigma(Z_s)(X_t - X_s) + \int_s^t (\sigma(Z_u) - \sigma(Z_s)) \dot{X}_u du, \quad (1.3)$$

which implies that Z satisfies the following *controlled difference equation*:

$$Z_t - Z_s = \sigma(Z_s)(X_t - X_s) + o(t - s), \quad 0 \leq s \leq t \leq T, \quad (1.4)$$

because $u \mapsto \sigma(Z_u)$ is continuous and $u \mapsto \dot{X}_u$ is (continuous, hence) bounded on $[0, T]$.

Remark 1.1. (UNIFORMITY) Whenever we write $o(t - s)$, as in (1.4), we always mean *uniformly for* $0 \leq s \leq t \leq T$, i.e.

$$\forall \varepsilon > 0 \exists \delta > 0: \quad 0 \leq s \leq t \leq T, \quad t - s \leq \delta \quad \text{implies} \quad |o(t - s)| \leq \varepsilon(t - s). \quad (1.5)$$

This will be implicitly assumed in the sequel.

Let us make two simple observations.

- If X is continuously differentiable we deduced (1.4) from (1.1), but we can easily deduce (1.1) from (1.4): in other terms, the two equations (1.1) and (1.4) are *equivalent*.
- If X is *not* continuously differentiable, equation (1.4) is still *meaningful*, unlike equation (1.1) which contains explicitly \dot{X} .

For these reasons, henceforth we focus on the difference equation (1.4), which provides a generalized formulation of the differential equation (1.1) when X is continuous but not necessarily differentiable.

The problem is now to prove *well-posedness* for the difference equation (1.4). We are going to show that this is possible assuming a suitable *Hölder regularity* on X , but non trivial ideas are required. In this chapter we illustrate some key ideas, showing how to prove uniqueness of solutions via *a priori estimates* (existence of solutions will be studied in the next chapters). We start from a basic result, which ensures the continuity of solutions; more precise result will be obtained later.

LEMMA 1.2. (CONTINUITY OF SOLUTIONS) *Let X and σ be continuous. Then any solution Z of (1.4) is a continuous path, more precisely it satisfies*

$$|Z_t - Z_s| \leq C |X_t - X_s| + o(t - s), \quad 0 \leq s \leq t \leq T, \quad (1.6)$$

for a suitable constant $C < \infty$ which depends on Z .

Proof. Relation (1.6) follows by (1.4) with $C := \|\sigma(Z)\|_\infty = \sup_{0 \leq t \leq T} |\sigma(Z_t)|$, renaming $|o(t-s)|$ as $o(t-s)$. We only have to prove that $C < \infty$. Since σ is continuous by assumption, it is enough to show that Z is *bounded*.

Since $o(t-s)$ is uniform, see (1.5), we can fix $\bar{\delta} > 0$ such that $|o(t-s)| \leq 1$ for all $0 \leq s \leq t \leq T$ with $|t-s| \leq \bar{\delta}$. It follows that Z is bounded in any interval $[\bar{s}, \bar{t}]$ with $|\bar{t} - \bar{s}| \leq \bar{\delta}$, because by (1.4) we can bound

$$\sup_{t \in [\bar{s}, \bar{t}]} |Z_t| \leq |Z_{\bar{s}}| + |\sigma(Z_{\bar{s}})| \sup_{t \in [\bar{s}, \bar{t}]} |X_t - X_{\bar{s}}| + 1 < \infty.$$

We conclude that Z is bounded in the whole interval $[0, T]$, because we can write $[0, T]$ as a finite union of intervals $[\bar{s}, \bar{t}]$ with $|\bar{t} - \bar{s}| \leq \bar{\delta}$. \square

Remark 1.3. (COUNTEREXAMPLES) The weaker requirement that (1.4) holds for any fixed $s \in [0, T]$ as $t \downarrow s$ is not enough for our purposes, since in this case Z needs not be continuous. An easy counterexample is the following: given any continuous path $X: [0, 2] \rightarrow \mathbb{R}$, we define $Z: [0, 2] \rightarrow \mathbb{R}$ by

$$Z_t := \begin{cases} X_t & \text{if } 0 \leq t < 1, \\ X_t + 1 & \text{if } 1 \leq t \leq 2. \end{cases}$$

Note that $Z_t - Z_s = X_t - X_s$ when either $0 \leq s \leq t < 1$ or $1 \leq s \leq t \leq 2$, hence Z satisfies the difference equation (1.4) with $\sigma(\cdot) \equiv 1$ for any fixed $s \in [0, 2)$ as $t \downarrow s$, but not uniformly for $0 \leq s \leq t \leq 2$, since Z is discontinuous at $t = 1$.

For another counterexample, which is even unbounded, consider

$$Z_t := \begin{cases} \frac{1}{1-t} & \text{if } 0 \leq t < 1, \\ 0 & \text{if } 1 \leq t \leq 2, \end{cases}$$

which satisfies (1.4) as $t \downarrow s$ for any fixed $s \in [0, 2]$, for $X_t \equiv t$ and $\sigma(z) = z^2$.

1.3. SOME USEFUL FUNCTION SPACES

For $n \geq 1$ we define the simplex

$$[0, T]_{\leq}^n := \{(t_1, \dots, t_n) : 0 \leq t_1 \leq \dots \leq t_n \leq T\} \quad (1.7)$$

(note that $[0, T]_{\leq}^1 = [0, T]$). We then write $C_n = C([0, T]_{\leq}^n, \mathbb{R}^k)$ as a shorthand for the space of continuous functions from $[0, T]_{\leq}^n$ to \mathbb{R}^k :

$$C_n := C([0, T]_{\leq}^n, \mathbb{R}^k) := \{F: [0, T]_{\leq}^n \rightarrow \mathbb{R}^k : F \text{ is continuous}\}. \quad (1.8)$$

We are going to work with functions of one (f_s), two (F_{st}) or three (G_{sut}) ordered variables in $[0, T]$, hence we focus on the spaces C_1, C_2, C_3 .

- On the spaces C_2 and C_3 we introduce a Hölder-like structure: given any $\eta \in (0, \infty)$, we define for $F \in C_2$ and $G \in C_3$

$$\|F\|_\eta := \sup_{0 \leq s < t \leq T} \frac{|F_{st}|}{(t-s)^\eta}, \quad \|G\|_\eta := \sup_{\substack{0 \leq s \leq u \leq t \leq T \\ s < t}} \frac{|G_{sut}|}{(t-s)^\eta}, \quad (1.9)$$

and we denote by C_2^η and C_3^η the corresponding function spaces:

$$C_2^\eta := \{F \in C_2: \|F\|_\eta < \infty\}, \quad C_3^\eta := \{G \in C_3: \|G\|_\eta < \infty\}, \quad (1.10)$$

which are Banach spaces endowed with the norm $\|\cdot\|_\eta$ (exercise).

- On the space C_1 of continuous functions $f: [0, T] \rightarrow \mathbb{R}^k$ we consider the usual Hölder structure. We first introduce the *increment* δf by

$$(\delta f)_{st} := f_t - f_s, \quad 0 \leq s \leq t \leq T, \quad (1.11)$$

and note that $\delta f \in C_2$ for any $f \in C_1$. Then, for $\alpha \in (0, 1]$, we define the classical space $\mathcal{C}^\alpha = \mathcal{C}^\alpha([0, T], \mathbb{R}^k)$ of α -Hölder functions

$$\mathcal{C}^\alpha := \left\{ f: [0, T] \rightarrow \mathbb{R}^k: \|\delta f\|_\alpha = \sup_{0 \leq s < t \leq T} \frac{|f_t - f_s|}{(t-s)^\alpha} < \infty \right\} \quad (1.12)$$

(for $\alpha = 1$ it is the space of Lipschitz functions). Note that $\|\delta f\|_\alpha$ in (1.12) is consistent with (1.11) and (1.9).

Remark 1.4. (HÖLDER SEMI-NORM) We stress that $f \mapsto \|\delta f\|_\alpha$ is a semi-norm on \mathcal{C}^α (it vanishes on constant functions). The standard norm on \mathcal{C}^α is

$$\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha, \quad (1.13)$$

where we define the standard sup norm

$$\|f\|_\infty := \sup_{t \in [0, T]} |f_t|. \quad (1.14)$$

For $f: [0, T] \rightarrow \mathbb{R}^k$ we can bound $\|f\|_\infty \leq |f(0)| + T^\alpha \|\delta f\|_\alpha$ (see (1.39) below), hence

$$\|f\|_{\mathcal{C}^\alpha} \leq |f(0)| + (1 + T^\alpha) \|\delta f\|_\alpha. \quad (1.15)$$

This explains why it is often enough to focus on the semi-norm $\|\delta f\|_\alpha$.

Remark 1.5. (HÖLDER EXPONENTS) We only consider the Hölder space \mathcal{C}^α for $\alpha \in (0, 1]$ because for $\alpha > 1$ the only functions in \mathcal{C}^α are constant functions (note that $\|\delta f\|_\alpha < \infty$ for $\alpha > 1$ implies $\dot{f}_t = 0$ for every $t \in [0, T]$).

On the other hand, the spaces C_2^η and C_3^η in (1.10) are interesting for any exponent $\eta \in (0, \infty)$. For instance, the condition $\|F\|_\eta < \infty$ for a function $F \in C_2$ means that $|F_{st}| \leq C(t-s)^\eta$, which does not imply $F \equiv 0$ when $\eta > 1$ (unless $F = \delta f$ is the increment of some function $f \in C_1$).

In our results below we will have to assume that the non-linearity $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ belongs to classes of Hölder functions, in the following sense.

DEFINITION 1.6. Let $\gamma > 0$. A function $F: \mathbb{R}^k \rightarrow \mathbb{R}^N$ is said to be globally γ -Hölder (or globally of class \mathcal{C}^γ) if

- for $\gamma \in (0, 1]$ we have

$$[F]_{\mathcal{C}^\gamma} := \sup_{x, y \in \mathbb{R}^k, x \neq y} \frac{|F(x) - F(y)|}{|x - y|^\gamma} < +\infty$$

- for $\gamma \in (n, n+1]$ and $n = \{1, 2, \dots\}$, F is n times continuously differentiable and

$$[D^{(n)}F]_{\mathcal{C}^\gamma} := \sup_{x, y \in \mathbb{R}^k, x \neq y} \frac{|D^{(n)}F(x) - D^{(n)}F(y)|}{|x - y|^{\gamma-n}} < +\infty$$

where $D^{(n)}$ is the n -fold differential of F .

Moreover $F: \mathbb{R}^k \rightarrow \mathbb{R}^N$ is said to be locally γ -Hölder (or locally of class \mathcal{C}^γ) if

- for $\gamma \in (0, 1]$ we have for all $R > 0$

$$\sup_{\substack{x, y \in \mathbb{R}^k, x \neq y \\ |x|, |y| \leq R}} \frac{|F(x) - F(y)|}{|x - y|^\gamma} < +\infty$$

- for $\gamma \in (n, n+1]$ and $n = \{1, 2, \dots\}$, F is n times continuously differentiable and

$$\sup_{\substack{x, y \in \mathbb{R}^k, x \neq y \\ |x|, |y| \leq R}} \frac{|D^{(n)}F(x) - D^{(n)}F(y)|}{|x - y|^{\gamma-n}} < +\infty.$$

We stress that in the previous definition we do not assume F or $D^{(n)}F$ to be bounded. The case $\gamma = 1$ corresponds to the classical *Lipschitz* condition.

1.4. LOCAL UNIQUENESS OF SOLUTIONS

We prove *uniqueness of solutions* for the controlled difference equation (1.4) when $X \in \mathcal{C}^\alpha$ is an Hölder path of exponent $\alpha > \frac{1}{2}$. For simplicity, we focus on the case when $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is a linear application: $\sigma \in (\mathbb{R}^k \otimes (\mathbb{R}^d)^*) \otimes (\mathbb{R}^k)^*$, and we write σZ instead of $\sigma(Z)$ (we discuss non linear $\sigma(\cdot)$ in Chapter 2).

THEOREM 1.7. (LOCAL UNIQUENESS OF SOLUTIONS, LINEAR CASE) *Fix a path $X: [0, T] \rightarrow \mathbb{R}^d$ in \mathcal{C}^α , with $\alpha \in]\frac{1}{2}, 1]$, and a linear map $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$. If $T > 0$ is small enough (depending on X, α, σ), then for any $z_0 \in \mathbb{R}^k$ there is at most one path $Z: [0, T] \rightarrow \mathbb{R}^k$ with $Z_0 = z_0$ which solves the linear controlled difference equation (1.4), that is (recalling (1.11))*

$$\delta Z_{st} - (\sigma Z_s) \delta X_{st} = o(t - s), \quad 0 \leq s \leq t \leq T. \quad (1.16)$$

Proof. Suppose that we have two paths $Z, \bar{Z}: [0, T] \rightarrow \mathbb{R}^k$ satisfying (1.16) with $Z_0 = \bar{Z}_0$ and define $Y := Z - \bar{Z}$. Our goal is to show that $Y = 0$.

Let us introduce the function $R \in C_2 = C([0, T]_{\leq}^2, \mathbb{R}^k)$ defined by

$$R_{st} := \delta Y_{st} - (\sigma Y_s) \delta X_{st}, \quad 0 \leq s \leq t \leq T, \quad (1.17)$$

and note that by (1.16) and linearity we have

$$R_{st} = o(t - s). \quad (1.18)$$

Recalling (1.9), we can estimate

$$\|\delta Y\|_\alpha \leq |\sigma| \|Y\|_\infty \|\delta X\|_\alpha + \|R\|_\alpha,$$

and since $R_{st} = o(t-s) = o((t-s)^\alpha)$, we have $\|R\|_\alpha < +\infty$ and therefore $\|\delta Y\|_\alpha < +\infty$. Since $Y_0 = 0$, we can bound

$$\|Y\|_\infty \leq |Y_0| + \sup_{0 \leq t \leq T} |Y_t - Y_0| \leq T^\alpha \|\delta Y\|_\alpha.$$

Since $1 \leq T^\alpha (t-s)^{-\alpha}$ for $0 \leq s < t \leq T$, we can also bound

$$\|R\|_\alpha \leq T^\alpha \|R\|_{2\alpha},$$

so that

$$\|\delta Y\|_\alpha \leq T^\alpha (|\sigma| \|\delta Y\|_\alpha \|\delta X\|_\alpha + \|R\|_{2\alpha}).$$

Suppose we can prove that, for some constant $C = C(X, \alpha, \sigma) < \infty$,

$$\|R\|_{2\alpha} \leq C \|\delta Y\|_\alpha. \quad (1.19)$$

Then we obtain

$$\|\delta Y\|_\alpha \leq T^\alpha (|\sigma| \|\delta X\|_\alpha + C) \|\delta Y\|_\alpha.$$

If we fix T small enough, so that $T^\alpha (|\sigma| \|\delta X\|_\alpha + C) < 1$, we get $\|\delta Y\|_\alpha = 0$, hence $\delta Y \equiv 0$. This means that $Y_t = Y_s$ for all $s, t \in [0, T]$, and since $Y_0 = 0$ we obtain $Y \equiv 0$, namely our goal $Z \equiv \bar{Z}$. This completes the proof *assuming the estimate (1.19)* (where the hypothesis $\alpha > \frac{1}{2}$ will play a key role). \square

To actually complete the proof of Theorem 1.7, it remains to show that the inequality (1.19) holds. This is performed in the next two sections:

- in Section 1.5 we present a fundamental estimate, the *Sewing Bound*, which applies to *any function* $R_{st} = o(t-s)$ (recall (1.18));
- in Section 1.6 we apply the Sewing Bound to R_{st} in (1.17) and we prove the desired estimate (1.19) for $\alpha > \frac{1}{2}$ (see the assumptions of Theorem 1.7).

1.5. THE SEWING BOUND

Let us fix an arbitrary function $R \in C_2 = C([0, T]_{\leq}^2, \mathbb{R}^k)$ with $R_{st} = o(t-s)$. Our goal is to bound $|R_{ab}|$ for any given $0 \leq a < b \leq T$.

We first show that we can express R_{ab} via ‘‘Riemann sums’’ along partitions $\mathcal{P} = \{a = t_0 < t_1 < \dots < t_m = b\}$ of $[a, b]$. These are defined by

$$I_{\mathcal{P}}(R) := \sum_{i=1}^{\#\mathcal{P}} R_{t_{i-1}t_i}, \quad (1.20)$$

where we denote by $\#\mathcal{P} := m$ the number of intervals of the partition \mathcal{P} . Let us denote by $|\mathcal{P}| := \max_{1 \leq i \leq m} (t_i - t_{i-1})$ the *mesh* of \mathcal{P} .

LEMMA 1.8. (RIEMANN SUMS) *Given any $R \in C_2$ with $R_{st} = o(t-s)$, for any $0 \leq a < b \leq T$ and for any sequence $(\mathcal{P}_n)_{n \geq 0}$ of partitions of $[a, b]$ with vanishing mesh $\lim_{n \rightarrow \infty} |\mathcal{P}_n| = 0$ we have*

$$\lim_{n \rightarrow \infty} I_{\mathcal{P}_n}(R) = 0.$$

If furthermore $\mathcal{P}_0 = \{a, b\}$ is the trivial partition, then we can write

$$R_{ab} = \sum_{n=0}^{\infty} (I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)), \quad 0 \leq a < b \leq T. \quad (1.21)$$

Proof. Writing $\mathcal{P}_n = \{a = t_0^n < t_1^n < \dots < t_{\#\mathcal{P}_n}^n = b\}$, we can estimate

$$|I_{\mathcal{P}_n}(R)| \leq \sum_{i=1}^{\#\mathcal{P}_n} |R_{t_{i-1}^n t_i^n}| \leq \left\{ \max_{j=1, \dots, \#\mathcal{P}_n} \frac{|R_{t_{j-1}^n t_j^n}|}{(t_j^n - t_{j-1}^n)} \right\} \sum_{j=1}^{\#\mathcal{P}_n} (t_j^n - t_{j-1}^n),$$

hence $|I_{\mathcal{P}_n}(R)| \rightarrow 0$ as $n \rightarrow \infty$, because the final sum equals $b - a$ and the bracket vanishes (since $R_{st} = o(t - s)$ and $|\mathcal{P}_n| = \max_{1 \leq j \leq \#\mathcal{P}_n} (t_j^n - t_{j-1}^n) \rightarrow 0$).

We deduce relation (1.21) by the telescopic sum

$$I_{\mathcal{P}_0}(R) - I_{\mathcal{P}_N}(R) = \sum_{n=0}^{N-1} (I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)),$$

because $\lim_{N \rightarrow \infty} I_{\mathcal{P}_N}(R) = 0$ while $I_{\mathcal{P}_0}(R) = R_{ab}$ for $\mathcal{P}_0 = \{a, b\}$. \square

If we remove a single point t_i from a partition $\mathcal{P} = \{t_0 < t_1 < \dots < t_m\}$, we obtain a new partition \mathcal{P}' for which, recalling (1.20), we can write

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = R_{t_{i-1} t_{i+1}} - R_{t_{i-1} t_i} - R_{t_i t_{i+1}}. \quad (1.22)$$

The expression in the RHS deserves a name: given any two-variables function $F \in C_2$, we define its increment $\delta F \in C_3$ as the three-variables function

$$\delta F_{sut} := F_{st} - F_{su} - F_{ut}, \quad 0 \leq s \leq u \leq t \leq T. \quad (1.23)$$

We can then rewrite (1.22) as

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = \delta R_{t_{i-1} t_i t_{i+1}}, \quad (1.24)$$

and recalling (1.9) we obtain the following estimate, for any $\eta > 0$:

$$|I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R)| \leq \|\delta R\|_{\eta} |t_{i+1} - t_{i-1}|^{\eta}. \quad (1.25)$$

We are now ready to state and prove the Sewing Bound.

THEOREM 1.9. (SEWING BOUND) *Given any $R \in C_2$ with $R_{st} = o(t - s)$, the following estimate holds for any $\eta \in (1, \infty)$ (recall (1.9)):*

$$\|R\|_{\eta} \leq K_{\eta} \|\delta R\|_{\eta} \quad \text{where} \quad K_{\eta} := (1 - 2^{1-\eta})^{-1}. \quad (1.26)$$

Proof. Fix $R \in C_2$ such that $\|\delta R\|_{\eta} < \infty$ for some $\eta > 1$ (otherwise there is nothing to prove). Also fix $0 \leq a < b \leq T$ and consider for $n \geq 0$ the dyadic partitions $\mathcal{P}_n := \{t_i^n := a + \frac{i}{2^n}(b - a) : 0 \leq i \leq 2^n\}$ of $[a, b]$. Since $\mathcal{P}_0 = \{a, b\}$ is the trivial partition, we can apply (1.21) to bound

$$|R_{ab}| \leq \sum_{n=0}^{\infty} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)|. \quad (1.27)$$

If we remove from \mathcal{P}_{n+1} all the “odd points” t_{2j+1}^{n+1} , with $0 \leq j \leq 2^n - 1$, we obtain \mathcal{P}_n . Then, iterating relations (1.24)-(1.25), we have

$$\begin{aligned} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)| &\leq \sum_{j=0}^{2^n-1} |\delta R_{t_{2j}^{n+1} t_{2j+1}^{n+1} t_{2j+2}^{n+1}}| \\ &\leq 2^n \|\delta R\|_\eta \left(\frac{2(b-a)}{2^{n+1}} \right)^\eta \\ &= 2^{-(\eta-1)n} \|\delta R\|_\eta (b-a)^\eta. \end{aligned} \quad (1.28)$$

Plugging this into (1.27), since $\sum_{n=0}^{\infty} 2^{-(\eta-1)n} = (1 - 2^{1-\eta})^{-1}$, we obtain

$$|R_{ab}| \leq (1 - 2^{1-\eta})^{-1} \|\delta R\|_\eta (b-a)^\eta, \quad 0 \leq a < b \leq T, \quad (1.29)$$

which proves (1.26). \square

Remark 1.10. Recalling (1.11) and (1.23), we have defined linear maps

$$C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \quad (1.30)$$

which satisfy $\delta \circ \delta = 0$. Indeed, for any $f \in C_1$ we have

$$\delta(\delta f)_{sut} = (f_t - f_s) - (f_u - f_s) - (f_t - f_u) = 0.$$

Intuitively, $\delta F \in C_3$ measures how much a function $F \in C_2$ differs from being the increment δf of some $f \in C_1$, because $\delta F \equiv 0$ if and only if $F = \delta f$ for some $f \in C_1$ (it suffices to define $f_t := F_{0t}$ and to check that $\delta f_{st} = \delta F_{0st} + F_{st} = F_{st}$).

Remark 1.11. The assumption $R_{st} = o(t-s)$ in Theorem 1.9 cannot be avoided: if $R := \delta f$ for a non constant $f \in C_1$, then $\delta R = 0$ while $\|R\|_\eta > 0$.

1.6. END OF PROOF OF UNIQUENESS

In this section, we apply the Sewing Bound (1.26) to the function R_{st} defined in (1.17), in order to prove the estimate (1.19) for $\alpha > \frac{1}{2}$.

We first determine the increment δR through a simple and instructive computation: by (1.17), since $\delta(\delta Z) = 0$ (see Remark 1.10), we have

$$\begin{aligned} \delta R_{sut} &:= R_{st} - R_{su} - R_{ut} \\ &= (Y_t - Y_s) - (Y_u - Y_s) - (Y_t - Y_u) \\ &\quad - (\sigma Y_s)(X_t - X_s) + (\sigma Y_s)(X_u - X_s) + (\sigma Y_u)(X_t - X_u) \\ &= [\sigma(Y_u - Y_s)](X_t - X_u). \end{aligned} \quad (1.31)$$

Recalling (1.9), this implies

$$\|\delta R\|_{2\alpha} \leq |\sigma| \|\delta Y\|_\alpha \|\delta X\|_\alpha.$$

We next note that if $\alpha > \frac{1}{2}$ (as it is assumed in Theorem 1.7) we can apply the Sewing Bound (1.26) for $\eta = 2\alpha > 1$ to obtain

$$\|R\|_{2\alpha} \leq K_{2\alpha} \|\delta R\|_{2\alpha} \leq K_{2\alpha} |\sigma| \|\delta Y\|_\alpha \|\delta X\|_\alpha.$$

This is precisely our goal (1.19) with $C = C(X, \alpha, \sigma) := K_{2\alpha} |\sigma| \|\delta X\|_\alpha$.

Summarizing: thanks to the Sewing bound (1.26), we have obtained the estimate (1.19) and completed the proof of Theorem 1.7, showing uniqueness of solutions to the difference equation (1.4) for any $X \in \mathcal{C}^\alpha$ with $\alpha \in]\frac{1}{2}, 1]$. In the next chapters we extend this approach to non-linear $\sigma(\cdot)$ and to situations where $X \in \mathcal{C}^\alpha$ with $\alpha \leq \frac{1}{2}$.

Remark 1.12. For later purpose, let us record the computation (1.31) without σ : given any (say, real) paths X and Y , if

$$A_{st} = Y_s \delta X_{st}, \quad \forall 0 \leq s \leq t \leq T,$$

then

$$\delta A_{sut} = -\delta Y_{su} \delta X_{ut}, \quad \forall 0 \leq s \leq u \leq t \leq T. \quad (1.32)$$

1.7. WEIGHTED NORMS

We conclude this chapter defining *weighted versions* $\|\cdot\|_{\eta, \tau}$ of the norms $\|\cdot\|_\eta$ introduced in (1.9): given $F \in C_2$ and $G \in C_3$, we set for $\eta, \tau \in (0, \infty)$

$$\|F\|_{\eta, \tau} := \sup_{0 \leq s \leq t \leq T} \mathbb{1}_{\{0 < t-s \leq \tau\}} e^{-\frac{t}{\tau}} \frac{|F_{st}|}{(t-s)^\eta}, \quad (1.33)$$

$$\|G\|_{\eta, \tau} := \sup_{0 \leq s \leq u \leq t \leq T} \mathbb{1}_{\{0 < t-s \leq \tau\}} e^{-\frac{t}{\tau}} \frac{|G_{sut}|}{(t-s)^\eta}, \quad (1.34)$$

where C_2 and C_3 are the spaces of continuous functions from $[0, T]_{\leq}^2$ and $[0, T]_{\leq}^3$ to \mathbb{R}^k , see (1.8). Note that as $\tau \rightarrow \infty$ we recover the usual norms:

$$\|\cdot\|_\eta = \lim_{\tau \rightarrow \infty} \|\cdot\|_{\eta, \tau}. \quad (1.35)$$

Remark 1.13. (NORMS VS. SEMI-NORMS) While $\|\cdot\|_\eta$ is a norm, $\|\cdot\|_{\eta, \tau}$ is a norm for $\tau \geq T$ but *it is only a semi-norm for $\tau < T$* (for instance, $\|F\|_{\eta, \tau} = 0$ for $F \in C_2$ implies $F_{st} = 0$ only for $t-s \leq \tau$: no constraint is imposed on F_{st} for $t-s > \tau$).

However, if $F = \delta f$, that is $F_{st} = f_t - f_s$ for some $f \in C_1$, we have the equivalence

$$\|\delta f\|_{\eta, \tau} \leq \|\delta f\|_\eta \leq \left(1 + \frac{T}{\tau}\right) e^{\frac{T}{\tau}} \|\delta f\|_{\eta, \tau}. \quad (1.36)$$

The first inequality is clear. For the second one, given $0 \leq s < t \leq T$, we can write $s = t_0 < t_1 < \dots < t_N = t$ with $t_i - t_{i-1} \leq \tau$ and $N \leq 1 + \frac{T}{\tau}$ (for instance, we can consider $t_i = s + i \frac{t-s}{N}$ where $N := \lceil \frac{t-s}{\tau} \rceil$); we then obtain $\delta f_{st} = \sum_{i=1}^N \delta f_{t_{i-1}t_i}$ and $|\delta f_{t_{i-1}t_i}| \leq \|\delta f\|_{\eta, \tau} e^{t_i/\tau} (t_i - t_{i-1})^\eta \leq \|\delta f\|_{\eta, \tau} e^{T/\tau} (t-s)^\eta$, which yields (1.36).

Remark 1.14. (FROM LOCAL TO GLOBAL) The weighted semi-norms $\|\cdot\|_{\eta, \tau}$ will be useful to transform *local* results in *global* results. Indeed, using the standard norms $\|\cdot\|_\eta$ often requires the size $T > 0$ of the time interval $[0, T]$ to be *small*, as in Theorem 1.7, which can be annoying. Using $\|\cdot\|_{\eta, \tau}$ will allow us to *keep* $T > 0$ *arbitrary*, by choosing a sufficiently small $\tau > 0$.

Recalling the supremum norm $\|f\|_\infty$ of a function $f \in C_1$, see (1.14), we define the corresponding weighted version

$$\|f\|_{\infty,\tau} := \sup_{0 \leq t \leq T} e^{-\frac{t}{\tau}} |f_t|. \quad (1.37)$$

We stress that $\|\cdot\|_{\infty,\tau}$ is a norm equivalent to $\|\cdot\|_\infty$ for any $\tau > 0$, since

$$\|\cdot\|_{\infty,\tau} \leq \|\cdot\|_\infty \leq e^{\frac{T}{\tau}} \|\cdot\|_{\infty,\tau}. \quad (1.38)$$

Remark 1.15. (EQUIVALENT HÖLDER NORM) It follows by (1.36) and (1.38) that $\|\cdot\|_{\infty,\tau} + \|\cdot\|_{\alpha,\tau}$ is a norm equivalent to $\|\cdot\|_{C^\alpha} := \|\cdot\|_\infty + \|\cdot\|_\alpha$ on the space C^α of Hölder functions, see Remark 1.4, for any $\tau > 0$.

We will often use the Hölder semi-norms $\|\delta f\|_\alpha$ and $\|\delta f\|_{\alpha,\tau}$ to bound the supremum norms $\|f\|_\infty$ and $\|f\|_{\infty,\tau}$, thanks to the following result.

LEMMA 1.16. (SUPREMUM-HÖLDER BOUND) For any $f \in C_1$ and $\eta \in (0, \infty)$

$$\|f\|_\infty \leq |f_0| + T^\eta \|\delta f\|_\eta, \quad (1.39)$$

$$\|f\|_{\infty,\tau} \leq |f_0| + 3(\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau}, \quad \forall \tau > 0. \quad (1.40)$$

Proof. Let us prove (1.39): for any $f \in C_1$ and for $t \in]0, T]$ we have

$$|f_t| \leq |f_0| + |f_t - f_0| = |f_0| + t^\eta \frac{|f_t - f_0|}{t^\eta} \leq |f_0| + T^\eta \|\delta f\|_\eta.$$

The proof of (1.40) is slightly more involved. If $t \in]0, \tau \wedge T]$, then

$$e^{-\frac{t}{\tau}} |f_t| \leq |f_0| + t^\eta e^{-\frac{t}{\tau}} \frac{|f_t - f_0|}{t^\eta} \leq |f_0| + (\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau},$$

which, in particular, implies (1.40) when $\tau \geq T$. When $\tau < T$, it remains to consider $\tau < t \leq T$: in this case, we define $N := \min \{n \in \mathbb{N} : n\tau \geq t\} \geq 2$ so that $\frac{t}{N} \leq \tau$. We set $t_k = k \frac{t}{N}$ for $k \geq 0$, so that $t_N = t$. Then

$$\begin{aligned} e^{-\frac{t}{\tau}} |f_t| &\leq |f_0| + \sum_{k=1}^N (t_k - t_{k-1})^\eta e^{-\frac{t-t_k}{\tau}} \left[e^{-\frac{t_k}{\tau}} \frac{|f_{t_k} - f_{t_{k-1}}|}{(t_k - t_{k-1})^\eta} \right] \\ &\leq |f_0| + (\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau} \sum_{k=1}^N e^{-\frac{t-t_k}{\tau}}. \end{aligned}$$

By definition of N we have $(N-1)\tau < t$; since $\tau < t$ we obtain $N\tau < 2t$ and therefore $\frac{t}{N\tau} \geq \frac{1}{2}$. Since $t - t_k = (N-k) \frac{t}{N}$, renaming $\ell := N - k$ we obtain

$$\sum_{k=1}^N e^{-\frac{t-t_k}{\tau}} = \sum_{\ell=0}^{N-1} e^{-\ell \frac{t}{N\tau}} = \frac{1 - e^{-\frac{t}{\tau}}}{1 - e^{-\frac{t}{N\tau}}} \leq \frac{1}{1 - e^{-\frac{1}{2}}} \leq 3.$$

The proof is complete. \square

We finally show that the Sewing Bound (1.26) still holds if we replace $\|\cdot\|_\eta$ by $\|\cdot\|_{\eta,\tau}$, for any $\tau > 0$.

THEOREM 1.17. (WEIGHTED SEWING BOUND) *Given any $R \in C_2$ with $R_{st} = o(t-s)$, the following estimate holds for any $\eta \in (1, \infty)$ and $\tau > 0$:*

$$\|R\|_{\eta,\tau} \leq K_\eta \|\delta R\|_{\eta,\tau} \quad \text{where} \quad K_\eta := (1 - 2^{1-\eta})^{-1}. \quad (1.41)$$

Proof. Given $0 \leq a \leq b \leq T$, let us define

$$\|\delta R\|_{\eta,[a,b]} := \sup_{\substack{s,u,t \in [a,b]: \\ s \leq u \leq t, s < t}} \frac{|\delta R_{sut}|}{(t-s)^\eta}. \quad (1.42)$$

Following the proof of Theorem 1.9, we can replace $\|\delta R\|_\eta$ by $\|\delta R\|_{\eta,[a,b]}$ in (1.28) and in (1.29), hence we obtain $|R_{ab}| \leq K_\eta \|\delta R\|_{\eta,[a,b]} (b-a)^\eta$. Then for $b-a \leq \tau$ we can estimate

$$e^{-\frac{b}{\tau}} \frac{|R_{ab}|}{(b-a)^\eta} \leq e^{-\frac{b}{\tau}} K_\eta \|\delta R\|_{\eta,[a,b]} \leq K_\eta \|\delta R\|_{\eta,\tau},$$

and (1.41) follows taking the supremum over $0 \leq a \leq b \leq T$ with $b-a \leq \tau$. \square

1.8. A DISCRETE SEWING BOUND

We can prove a version of the Sewing Bound for functions $R = (R_{st})_{s < t \in \mathbb{T}}$ defined on a *finite set of points* $\mathbb{T} := \{0 = t_1 < \dots < t_{\#\mathbb{T}}\} \subseteq \mathbb{R}_+$ (this will be useful to construct solutions to difference equations via Euler schemes, see Sections 2.6 and 3.9). The condition $R_{st} = o(t-s)$ from Theorem 1.9 is now replaced by the requirement that R *vanishes on consecutive points of \mathbb{T}* , i.e. $R_{t_i t_{i+1}} = 0$ for all $1 \leq i < \#\mathbb{T}$.

We define versions $\|\cdot\|_{\eta,\tau}^\mathbb{T}$ of the norms $\|\cdot\|_{\eta,\tau}$ restricted on \mathbb{T} for $\tau > 0$, recall (1.33)-(1.34):

$$\|A\|_{\eta,\tau}^\mathbb{T} := \sup_{\substack{0 \leq s < t \\ s,t \in \mathbb{T}}} \mathbb{1}_{\{0 < t-s \leq \tau\}} e^{-\frac{t}{\tau}} \frac{|A_{st}|}{|t-s|^\eta}, \quad (1.43)$$

$$\|B\|_{\eta,\tau}^\mathbb{T} := \sup_{\substack{0 \leq s \leq u \leq t \\ s,u,t \in \mathbb{T}, s < t}} \mathbb{1}_{\{0 < t-s \leq \tau\}} e^{-\frac{t}{\tau}} \frac{|B_{sut}|}{|t-s|^\eta} \quad (1.44)$$

for $A: \{(s,t) \in \mathbb{T}^2: 0 \leq s < t\} \rightarrow \mathbb{R}$ and $B: \{(s,u,t) \in \mathbb{T}^3: 0 \leq s \leq u \leq t, s < t\} \rightarrow \mathbb{R}$.

THEOREM 1.18. (DISCRETE SEWING BOUND) *If a function $R = (R_{st})_{s < t \in \mathbb{T}}$ vanishes on consecutive points of \mathbb{T} (i.e. $R_{t_i t_{i+1}} = 0$), then for any $\eta > 1$ and $\tau > 0$ we have*

$$\|R\|_{\eta,\tau}^\mathbb{T} \leq C_\eta \|\delta R\|_{\eta,\tau}^\mathbb{T} \quad \text{with} \quad C_\eta := 2^\eta \sum_{n \geq 1} \frac{1}{n^\eta} = 2^\eta \zeta(\eta) < \infty. \quad (1.45)$$

Proof. We fix $s, t \in \mathbb{T}$ with $s < t$ and we start by proving that

$$|R_{st}| \leq C_\eta \|\delta R\|_{\eta,\tau}^\mathbb{T} (t-s)^\eta.$$

We have $s = t_k$ and $t = t_{k+m}$ and we may assume that $m \geq 2$ (otherwise there is nothing to prove, since for $m = 1$ we have $R_{t_i t_{i+1}} = 0$).

Consider the partition $\mathcal{P} = \{s = t_k < t_{k+1} < \dots < t_{k+m} = t\}$ with m intervals. Note that for some index $i \in \{k+1, \dots, k+m-1\}$ we must have $t_{i+1} - t_{i-1} \leq \frac{2(t-s)}{m-1}$, otherwise we would get the contradiction

$$2(t-s) \geq \sum_{i=k+1}^{k+m-1} (t_{i+1} - t_{i-1}) > \sum_{i=k+1}^{k+m-1} \frac{2(t-s)}{m-1} = 2(t-s).$$

Removing the point t_i from \mathcal{P} we obtain a partition \mathcal{P}' with $m-1$ intervals. If we define $I_{\mathcal{P}}(R) := \sum_{i=k}^{k+m-1} R_{t_i t_{i+1}}$ as in (1.20), as in (1.24) we have

$$|I_{\mathcal{P}}(R) - I_{\mathcal{P}'}(R)| = |\delta R_{t_{i-1} t_i t_{i+1}}| \leq \frac{2^\eta (t-s)^\eta}{(m-1)^\eta} \sup_{\substack{s \leq u < v < w \leq t \\ u, v, w \in \mathbb{T}}} \frac{|\delta R_{uvw}|}{|w-u|^\eta}.$$

Iterating this argument, until we arrive at the trivial partition $\{s, t\}$, we get

$$|I_{\mathcal{P}}(R) - R_{st}| \leq C_\eta (t-s)^\eta \sup_{\substack{s \leq u < v < w \leq t \\ u, v, w \in \mathbb{T}}} \frac{|\delta R_{uvw}|}{|w-u|^\eta}, \quad (1.46)$$

with $C_\eta := \sum_{n \geq 1} \frac{2^n}{n^\eta} < \infty$ because $\eta > 1$. We finally note that $I_{\mathcal{P}}(R) = 0$ by the assumption $R_{t_i t_{i+1}} = 0$. Finally if $t-s \leq \tau$ then $w-u \leq \tau$ in the supremum in (1.46) and since $e^{-\frac{t}{\tau}} \leq e^{-\frac{w}{\tau}}$ we obtain

$$e^{-\frac{t}{\tau}} |R_{st}| \leq C_\eta (t-s)^\eta \|\delta R\|_{\eta, \tau}^{\mathbb{T}},$$

and the proof is complete. \square

We also have an analog of Lemma 1.16. We set for $f: \mathbb{T} \rightarrow \mathbb{R}$ and $\tau > 0$

$$\|f\|_{\infty, \tau}^{\mathbb{T}} := \sup_{t \in \mathbb{T}} e^{-\frac{t}{\tau}} |f_t|.$$

LEMMA 1.19. (DISCRETE SUPREMUM-HÖLDER BOUND) For $\mathbb{T} := \{0 = t_1 < \dots < t_{\#\mathbb{T}}\} \subseteq \mathbb{R}_+$ set

$$M := \max_{i=2, \dots, \#\mathbb{T}} |t_i - t_{i-1}|.$$

Then for all $f: \mathbb{T} \rightarrow \mathbb{R}$, $\tau \geq 2M$ and $\eta > 0$

$$\|f\|_{\infty, \tau}^{\mathbb{T}} \leq |f_0| + 5\tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}}. \quad (1.47)$$

Proof. We define $T_0 := 0$ and for $i \geq 1$, as long as $\mathbb{T} \cap (T_{i-1}, T_{i-1} + \tau]$ is not empty, we set

$$T_i := \max \mathbb{T} \cap (T_{i-1}, T_{i-1} + \tau], \quad i = 1, \dots, N,$$

so that $T_N = \max \mathbb{T}$. We have by construction $T_i + M > T_{i-1} + \tau$ for all $i = 1, \dots, N-1$, and since $M \leq \frac{\tau}{2}$

$$T_i - T_{i-1} \geq \tau - M \geq \frac{\tau}{2}.$$

For $i = N$ we have only $T_N > T_{N-1}$. Therefore for $i = 1, \dots, N$

$$\begin{aligned}
e^{-\frac{T_i}{\tau}} |f_{T_i}| &\leq |f_0| + \sum_{k=1}^i (T_k - T_{k-1})^\eta e^{-\frac{T_i - T_k}{\tau}} \left[e^{-\frac{T_k}{\tau}} \frac{|f_{T_k} - f_{T_{k-1}}|}{(T_k - T_{k-1})^\eta} \right] \\
&\leq |f_0| + \tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}} \sum_{k=1}^i e^{-\frac{T_i - T_k}{\tau}} \\
&\leq |f_0| + \tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}} \left(1 + \sum_{k=0}^{\infty} e^{-\frac{k}{2}} \right) \\
&\leq |f_{t_0}| + 4\tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}}.
\end{aligned}$$

Now for $t \in \mathbb{T} \setminus \{T_i\}_i$ we have $T_i < t < T_{i+1}$ for some i and then

$$\begin{aligned}
e^{-\frac{t}{\tau}} |f_t| &\leq e^{-\frac{t}{\tau}} |f_{T_i}| + (t - T_i)^\eta e^{-\frac{t}{\tau}} \frac{|f_t - f_{T_i}|}{(t - T_i)^\eta} \leq e^{-\frac{T_i}{\tau}} |f_{T_i}| + \tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}} \\
&\leq |f_0| + 5\tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}}.
\end{aligned}$$

The proof is complete. \square

1.9. EXTRA (TO BE COMPLETED)

We also introduce the usual supremum norm, for $F \in C_2$ and $G \in C_3$:

$$\|F\|_\infty := \sup_{0 \leq s \leq t \leq T} |F_{st}|, \quad \|G\|_\infty := \sup_{0 \leq s \leq u \leq t \leq T} |G_{sut}|,$$

and a corresponding weighted version, for $\tau \in (0, \infty)$:

$$\|F\|_{\infty, \tau} := \sup_{0 \leq s \leq t \leq T} e^{-\frac{t}{\tau}} |F_{st}|, \quad \|G\|_{\infty, \tau} := \sup_{0 \leq s \leq u \leq t \leq T} e^{-\frac{t}{\tau}} |G_{sut}|. \quad (1.48)$$

Note that

$$\lim_{\tau \rightarrow +\infty} \|F\|_{\infty, \tau} = \|F\|_\infty, \quad \lim_{\tau \rightarrow +\infty} \|G\|_{\eta, \tau} = \|G\|_\eta, \quad \lim_{\tau \rightarrow +\infty} \|H\|_{\eta, \tau} = \|H\|_\eta.$$

We have

$$\|F\|_{\eta, \tau} \leq \|G\|_{\infty, \tau} \|H\|_\eta, \quad (F_{sut} = G_{su} H_{ut}), \quad (1.49)$$

Note that $\|\cdot\|_{\eta, \tau}$ is only a semi-norm on C_n^η if $\tau < T$; we have at least

$$\|\cdot\|_{\eta, \tau} \leq \|\cdot\|_\eta \leq e^{\frac{T}{\tau}} \left(\|\cdot\|_{\eta, \tau} + \frac{1}{\tau^\eta} \|\cdot\|_{\infty, \tau} \right). \quad (1.50)$$

However, if $\tau \geq T$ we have again equivalence of norms

$$\|\cdot\|_{\eta, \tau} \leq \|\cdot\|_\eta \leq e^{\frac{T}{\tau}} \|\cdot\|_{\eta, \tau}, \quad \tau \geq T. \quad (1.51)$$

CHAPTER 2

DIFFERENCE EQUATIONS: THE YOUNG CASE

Fix a time horizon $T > 0$ and two dimensions $k, d \in \mathbb{N}$. We study the following *controlled difference equation* for an unknown path $Z: [0, T] \rightarrow \mathbb{R}^k$:

$$Z_t - Z_s = \sigma(Z_s)(X_t - X_s) + o(t - s), \quad 0 \leq s \leq t \leq T, \quad (2.1)$$

where the “driving path” $X: [0, T] \rightarrow \mathbb{R}^d$ and the function $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ are given, and $o(t - s)$ is *uniform* for $0 \leq s \leq t \leq T$ (see Remark 1.1).

The difference equation (2.1) is a natural generalized formulation of the *controlled differential equation*

$$\dot{Z}_t = \sigma(Z_t) \dot{X}_t, \quad 0 \leq t \leq T. \quad (2.2)$$

Indeed, as we showed in Chapter 1 (see Section 1.2), equations (2.1) and (2.2) are *equivalent* when X is continuously differentiable and σ is continuous, but (2.1) is meaningful also when X is non differentiable.

In this chapter we prove *well-posedness for the difference equation (2.1)* when the driving path $X \in \mathcal{C}^\alpha$ is Hölder continuous in the regime $\alpha \in]\frac{1}{2}, 1]$, called the *Young case*. The more challenging regime $\alpha \leq \frac{1}{2}$, called the *rough case*, is the object of the next Chapter 3, where new ideas will be introduced.

2.1. SUMMARY

Using the increment notation $\delta f_{st} := f_t - f_s$ from (1.11), we rewrite (2.1) as

$$\delta Z_{st} = \sigma(Z_s) \delta X_{st} + o(t - s), \quad 0 \leq s \leq t \leq T, \quad (2.3)$$

so that a solution of (2.3) is any path $Z: [0, T] \rightarrow \mathbb{R}^k$ such that the “*remainder*”

$$Z_{st}^{[2]} := \delta Z_{st} - \sigma(Z_s) \delta X_{st} \quad \text{satisfies} \quad Z_{st}^{[2]} = o(t - s). \quad (2.4)$$

We summarize the main results of this chapter stating *local and global existence, uniqueness of solutions and continuity of the solution map* for the difference equation (2.3) under natural assumptions on σ . We will actually prove more precise results, which yield quantitative estimates.

THEOREM 2.1. (WELL-POSEDNESS) *Let $X: [0, T] \rightarrow \mathbb{R}^d$ be of class \mathcal{C}^α with $\alpha \in]\frac{1}{2}, 1]$ and let $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$. Then we have:*

- **local existence:** *if σ is locally γ -Hölder with $\gamma \in (\frac{1}{\alpha} - 1, 1]$ (e.g. of class C^1), then for every $z_0 \in \mathbb{R}^k$ there is a possibly shorter time horizon $T' = T'_{\alpha, X, \sigma}(z_0) \in]0, T]$ and a path $Z: [0, T'] \rightarrow \mathbb{R}^k$ starting from $Z_0 = z_0$ which solves (2.3) for $0 \leq s \leq t \leq T'$;*

- **global existence:** if σ is globally γ -Hölder with $\gamma \in (\frac{1}{\alpha} - 1, 1]$ (e.g. of class C^1 with $\|\nabla\sigma\|_\infty < \infty$), then we can take $T'_{\alpha,X,\sigma}(z_0) = T$ for any $z_0 \in \mathbb{R}^d$;
- **uniqueness:** if σ is of class C^γ with $\gamma \in (\frac{1}{\alpha}, 2]$ (e.g. if σ is of class C^2), then there is exactly one solution Z of (2.3) with $Z_0 = z_0$;
- **continuity of the solution map:** if σ is differentiable with bounded and globally $(\gamma - 1)$ -Hölder gradient with $\gamma \in (\frac{1}{\alpha}, 2]$ (i.e. $\|\nabla\sigma\|_\infty < \infty$, $[\nabla\sigma]_{C^{\gamma-1}} < \infty$), then the solution Z of (2.3) is a continuous function of the starting point z_0 and driving path X : the map $(z_0, X) \mapsto Z$ is continuous from $\mathbb{R}^k \times C^\alpha \rightarrow C^\alpha$.

In the first part of this chapter, we give for granted the existence of solutions and we focus on their properties: we prove *a priori estimates* in Section 2.3, *uniqueness of solutions* in Section 2.4 and *continuity of the solution map* in Section 2.5. A key role is played by the Sewing Bound from Chapter 1, see Theorems 1.9 and 1.17, and its discrete version, see Theorem 1.18.

The proof of local and global *existence of solutions of (2.3)* is given at the end of this chapter, see Section 2.6, exploiting a suitable Euler scheme.

2.2. SET-UP

We collect here some notions and tools that will be used extensively.

We recall that C_1 denotes the space of continuous functions $f: [0, T] \rightarrow \mathbb{R}^k$. Similarly, C_2 and C_3 are the spaces of continuous functions of two and three ordered variables, i.e. defined on $[0, T]_{\leq}^2$ and $[0, T]_{\leq}^3$, see (1.7)-(1.8).

We are going to exploit the *weighted semi-norms* $\|\cdot\|_{\eta,\tau}$, see (1.33)-(1.34) (see also (1.9) for the original norm $\|\cdot\|_\eta$). These are useful to bound the *weighted supremum norm* $\|f\|_{\infty,\tau}$ of a function $f \in C_1$, see (1.37) and (1.40):

$$\|f\|_{\infty,\tau} \leq |f_0| + 3(\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau}, \quad \forall \eta, \tau > 0. \quad (2.5)$$

It follows directly from the definitions (1.33)-(1.34) that

$$\|\cdot\|_{\eta,\tau} \leq (\tau \wedge T)^{\eta'} \|\cdot\|_{\eta+\eta',\tau}, \quad \forall \eta, \eta' > 0, \quad (2.6)$$

because $(t-s)^\eta \geq (t-s)^{\eta+\eta'} (\tau \wedge T)^{-\eta'}$ for $0 \leq s \leq t \leq T$ with $t-s \leq \tau$.

Remark 2.2. The factor $(\tau \wedge T)^{\eta'}$ in the RHS of (2.6) can be made small *by choosing τ small while keeping T fixed*. This is why we included the indicator function $\mathbb{1}_{\{0 < t-s \leq \tau\}}$ in the definition (1.33)-(1.34) of the norms $\|\cdot\|_{\eta,\tau}$: without this indicator function, instead of $(\tau \wedge T)^{\eta'}$ we would have $T^{\eta'}$, which is small only when T is small.

We will often work with functions $F \in C_2$ or $F \in C_3$ that are *product of two factors*, like $F_{st} = g_s H_{st}$ or $F_{sut} = G_{su} H_{ut}$. We show in the next result that the semi-norm $\|F\|_{\eta,\tau}$ can be controlled by a product of suitable norms for each factor.

LEMMA 2.3. (WEIGHTED BOUNDS) *For any $\eta, \eta' \in (0, \infty)$ and $\tau > 0$, we have*

$$\text{if } F_{st} = g_s H_{st} \text{ or } F_{st} = g_t H_{st} \quad \text{then} \quad \|F\|_{\eta, \tau} \leq \|g\|_{\infty, \tau} \|H\|_{\eta}, \quad (2.7)$$

$$\text{if } F_{sut} = G_{su} H_{ut} \quad \text{then} \quad \|F\|_{\eta + \eta', \tau} \leq \|G\|_{\eta, \tau} \|H\|_{\eta'}. \quad (2.8)$$

Proof. If $F_{st} = g_t H_{st}$, by (1.37) we can estimate $e^{-t/\tau} |g_t| \leq \|g\|_{\infty, \tau}$ to get (2.7). If $F_{st} = g_s H_{st}$, for $s \leq t$ we can bound $e^{-t/\tau} \leq e^{-s/\tau}$ in the definition (1.33)-(1.34) of $\|\cdot\|_{\eta, \tau}$, hence again by (1.37) we can estimate $e^{-s/\tau} |g_s| \leq \|g\|_{\infty, \tau}$ to get (2.7).

If $F_{sut} = G_{su} H_{ut}$, we can further bound $(t-s)^{\eta+\eta'} \geq (t-u)^\eta (u-s)^{\eta'}$ in (1.34) and then estimate $e^{-s/\tau} G_{su} / (u-s)^\eta \leq \|G\|_{\eta, \tau}$, which yields (2.8). \square

We stress that in the RHS of (2.7) and (2.8) *only one factor gets the weighted norm or semi-norm*, while the other factor gets the non-weighted norm $\|\cdot\|_{\eta}$. We will sometimes need an extra weight, which can be introduced as follows.

LEMMA 2.4. (EXTRA WEIGHT) *For any $\eta, \bar{\tau} \in (0, \infty)$ and $0 < \tau \leq \bar{\tau}$, we have*

$$\text{if } F_{st} = g_s H_{st} \text{ or } F_{st} = g_t H_{st} \quad \text{then} \quad \|F\|_{\eta, \tau} \leq \|g\|_{\infty, \tau} e^{\frac{\tau}{\bar{\tau}}} \|H\|_{\eta, \bar{\tau}}. \quad (2.9)$$

Proof. Recall the definition (1.33)-(1.34) of $\|\cdot\|_{\eta, \tau}$ and note that for $0 \leq s \leq t \leq T$ we have $e^{-t/\tau} |g_t| \leq \|g\|_{\infty, \tau}$ and $e^{-s/\tau} |g_s| \leq \|g\|_{\infty, \tau}$ (see the proof of Lemma 2.3). Finally, for $t-s \leq \tau \leq \bar{\tau}$ we can estimate $|H_{st}| \leq e^{T/\bar{\tau}} e^{-t/\bar{\tau}} |H_{st}| \leq e^{T/\bar{\tau}} \|H\|_{\eta, \bar{\tau}} (t-s)^\eta$. \square

We recall that $\mathbb{R}^k \otimes (\mathbb{R}^d)^* \simeq \mathbb{R}^{k \times d}$ is the space of linear applications from \mathbb{R}^d to \mathbb{R}^k equipped with the Hilbert-Schmidt (Euclidean) norm $|\cdot|$. We say that a function is of class C^m if it is continuously differentiable m times. Given $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ of class C^2 , that we represent by $\sigma_j^i(z)$ with $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, d\}$, we denote by $\nabla \sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^k)^*$ its gradient and by $\nabla^2 \sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^k)^* \otimes (\mathbb{R}^k)^*$ its Hessian, represented for $i, a, b \in \{1, \dots, k\}$ and $j \in \{1, \dots, d\}$ by

$$(\nabla \sigma(z))_{ja}^i = \frac{\partial \sigma_j^i}{\partial z_a}(z), \quad (\nabla^2 \sigma(z))_{jab}^i = \frac{\partial^2 \sigma_j^i}{\partial z_a \partial z_b}(z).$$

Remark 2.5. (NORM OF THE GRADIENT OF LIPSCHITZ FUNCTIONS) *For a locally Lipschitz function $\psi: \mathbb{R}^k \rightarrow \mathbb{R}^\ell$ we can define the “norm of the gradient” at any point (even where ψ may not be differentiable):*

$$|\nabla \psi(z)| := \limsup_{y \rightarrow z} \frac{|\psi(y) - \psi(z)|}{|y - z|} \in [0, \infty).$$

Similarly, $|\nabla^2 \psi(z)|$ is well defined as soon as ψ is *differentiable with locally Lipschitz gradient* $\nabla \psi$ (which is slightly less than requiring $\psi \in C^2$).

2.3. A PRIORI ESTIMATES

In this section we prove *a priori estimates* for solutions of (2.3) assuming that σ is *globally Lipschitz*, that is $\|\nabla \sigma\|_\infty < \infty$ (recall Remark 2.5).

We first observe that if the driving path X is of class \mathcal{C}^α , then any solution Z of (2.3) is also of class \mathcal{C}^α , as soon as σ is continuous.

LEMMA 2.6. (HÖLDER REGULARITY) *Let X be of class \mathcal{C}^α with $\alpha \in]0, 1]$ and let σ be continuous. Then any solution Z of (2.3) is of class \mathcal{C}^α .*

Proof. We know by Lemma 1.2 that Z is continuous, more precisely by (1.6) we have $|\delta Z_{st}| \leq C |\delta X_{st}| + o(t-s)$ with $C < \infty$. Since $|\delta X_{st}| \leq \|\delta X\|_\alpha (t-s)^\alpha$ and $o(t-s) = o((t-s)^\alpha)$ for any $\alpha \leq 1$, it follows that $Z \in \mathcal{C}^\alpha$. \square

We next formulate the announced a priori estimates. It is convenient to use the weighted semi-norms $\|\cdot\|_{\eta, \tau}$ in (1.33)-(1.34) (note that the usual norms $\|\cdot\|_\eta$ in (1.9) can be recovered by letting $\tau \rightarrow \infty$).

THEOREM 2.7. (A PRIORI ESTIMATES) *Let X be of class \mathcal{C}^α with $\alpha \in]\frac{1}{2}, 1]$ and let σ be globally γ -Hölder with $\gamma \in (\frac{1}{\alpha} - 1, 1]$. Then, for any solution $Z: [0, T] \rightarrow \mathbb{R}^k$ of (2.3), the remainder $Z_{st}^{[2]} := \delta Z_{st} - \sigma(Z_s) \delta X_{st}$ satisfies $Z^{[2]} \in C_2^{(\gamma+1)\alpha}$, more precisely for any $\tau > 0$*

$$\|Z^{[2]}\|_{(\gamma+1)\alpha, \tau} \leq C_{\alpha, \gamma, X, \sigma} \|\delta Z\|_{\alpha, \tau}^\gamma \quad \text{with } C_{\alpha, \gamma, X, \sigma} := K_{(\gamma+1)\alpha} \|\delta X\|_\alpha [\sigma]_{\mathcal{C}^\gamma}, \quad (2.10)$$

where $K_\eta = (1 - 2^{1-\eta})^{-1}$. Moreover, if either T or τ is small enough, we have

$$\|\delta Z\|_{\alpha, \tau} \leq 1 \vee (2 \|\delta X\|_\alpha |\sigma(Z_0)|) \quad \text{for } (\tau \wedge T)^{\alpha\gamma} \leq \varepsilon_{\alpha, \gamma, X, \sigma}, \quad (2.11)$$

where we define

$$\varepsilon_{\alpha, \gamma, X, \sigma} := \frac{1}{2(K_{(\gamma+1)\alpha} + 3) \|\delta X\|_\alpha [\sigma]_{\mathcal{C}^\gamma}}. \quad (2.12)$$

If σ is globally Lipschitz, namely if we can take $\gamma = 1$, we can improve (2.11) to

$$\|\delta Z\|_{\alpha, \tau} \leq 2 \|\delta X\|_\alpha |\sigma(Z_0)| \quad \text{for } (\tau \wedge T)^\alpha \leq \varepsilon_{\alpha, 1, X, \sigma}. \quad (2.13)$$

Proof. We first prove (2.10). Since $Z_{st}^{[2]} = o(t-s)$ by definition of solution, see (2.4), we can estimate $Z^{[2]}$ in terms of $\delta Z^{[2]}$, by the weighted Sewing Bound (1.41). Let us compute $\delta Z_{sut}^{[2]} = Z_{st}^{[2]} - Z_{su}^{[2]} - Z_{ut}^{[2]}$: recalling (2.4) and (1.32), since $\delta \circ \delta = 0$, we have

$$\delta Z_{sut}^{[2]} = \delta \sigma(Z)_{su} \delta X_{ut} = (\sigma(Z_u) - \sigma(Z_s)) (X_t - X_u). \quad (2.14)$$

Since $|\sigma(z) - \sigma(\bar{z})| \leq [\sigma]_{\mathcal{C}^\gamma} |z - \bar{z}|^\gamma$ for all $z, \bar{z} \in \mathbb{R}^d$, we can bound

$$\|\delta \sigma(Z)\|_{\gamma\alpha, \tau} \leq [\sigma]_{\mathcal{C}^\gamma} \|\delta Z\|_{\alpha, \tau}^\gamma, \quad (2.15)$$

hence by (2.8) we obtain

$$\|\delta Z^{[2]}\|_{(\gamma+1)\alpha, \tau} \leq \|\delta X\|_\alpha [\sigma]_{\mathcal{C}^\gamma} \|\delta Z\|_{\alpha, \tau}^\gamma.$$

Applying the weighted Sewing Bound (1.41), for $(\gamma+1)\alpha > 1$ we then obtain

$$\|Z^{[2]}\|_{(\gamma+1)\alpha, \tau} \leq K_{(\gamma+1)\alpha} \|\delta X\|_\alpha [\sigma]_{\mathcal{C}^\gamma} \|\delta Z\|_{\alpha, \tau}^\gamma, \quad (2.16)$$

which proves (2.10).

We next prove (2.11). To simplify notation, let us set $\varepsilon := (\tau \wedge T)^\alpha$. Recalling (2.7) and (2.6), we obtain by (2.4)

$$\begin{aligned} \|\delta Z\|_{\alpha,\tau} &\leq \|\sigma(Z) \delta X\|_{\alpha,\tau} + \|Z^{[2]}\|_{\alpha,\tau} \\ &\leq \|\sigma(Z)\|_{\infty,\tau} \|\delta X\|_\alpha + \varepsilon^\gamma \|Z^{[2]}\|_{(\gamma+1)\alpha,\tau}. \end{aligned} \quad (2.17)$$

We can estimate $\|\sigma(Z)\|_{\infty,\tau}$ by (2.5) and (2.15):

$$\|\sigma(Z)\|_{\infty,\tau} \leq |\sigma(Z_0)| + 3\varepsilon^\gamma [\sigma]_{C^\gamma} \|\delta Z\|_{\alpha,\tau}^\gamma.$$

Plugging this and (2.16) into (2.17), we get

$$\begin{aligned} \|\delta Z\|_{\alpha,\tau} &\leq (|\sigma(Z_0)| + 3\varepsilon^\gamma [\sigma]_{C^\gamma} \|\delta Z\|_{\alpha,\tau}^\gamma) \|\delta X\|_\alpha + \\ &\quad + \varepsilon^\gamma K_{(\gamma+1)\alpha} \|\delta X\|_\alpha [\sigma]_{C^\gamma} \|\delta Z\|_{\alpha,\tau}^\gamma \\ &= \|\delta X\|_\alpha |\sigma(Z_0)| + \frac{1}{2} \frac{\varepsilon^\gamma}{\varepsilon_{\alpha,\gamma,X,\sigma}} \|\delta Z\|_{\alpha,\tau}^\gamma, \end{aligned}$$

where $\varepsilon_{\alpha,\gamma,X,\sigma}$ is defined in (2.12). For $\varepsilon^\gamma \leq \varepsilon_{\alpha,\gamma,X,\sigma}$ the last term is bounded by $\frac{1}{2} \|\delta Z\|_{\alpha,\tau}^\gamma$ which is finite by Lemma 2.6. If $\|\delta Z\|_{\alpha,\tau} \leq 1$ then (2.11) holds trivially; if not, $\frac{1}{2} \|\delta Z\|_{\alpha,\tau}^\gamma \leq \frac{1}{2} \|\delta Z\|_{\alpha,\tau}$. Bringing this term in the LHS we obtain (2.11).

To prove (2.13), we argue as for (2.11) and since $\gamma = 1$ we obtain

$$\|\delta Z\|_{\alpha,\tau} \leq \|\delta X\|_\alpha |\sigma(Z_0)| + \frac{1}{2} \frac{\varepsilon}{\varepsilon_{\alpha,1,X,\sigma}} \|\delta Z\|_{\alpha,\tau}.$$

For $\varepsilon \leq \varepsilon_{\alpha,1,X,\sigma}$ the last term is bounded by $\frac{1}{2} \|\delta Z\|_{\alpha,\tau}$ which is finite by Lemma 2.6. Bringing this term in the LHS we obtain (2.13), and this completes the proof. \square

2.4. UNIQUENESS

In this section we prove uniqueness of solutions to (2.3) assuming that σ is of class C^1 with locally Hölder gradient (we stress that we make no boundedness assumption on σ). This improves on Theorem 1.7, both because we allow for non-linear σ and because we do not require that the time horizon $T > 0$ is small.

We first need an elementary but fundamental estimate on the difference of increments of a function. Given $\Psi: \mathbb{R}^k \rightarrow \mathbb{R}^\ell$, we use the notation

$$C_{\Psi,R} := \sup \{ |\Psi(x)| : x \in \mathbb{R}^k, |x| \leq R \}. \quad (2.18)$$

LEMMA 2.8. (DIFFERENCE OF INCREMENTS) *Let $\psi: \mathbb{R}^k \rightarrow \mathbb{R}^\ell$ be of class $C_{\text{loc}}^{1+\rho}$ for some $0 < \rho \leq 1$ (i.e. ψ is differentiable with $\nabla \psi$ of class C_{loc}^ρ). Then for any $R > 0$ and for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^k$ with $\max \{|x|, |y|, |\bar{x}|, |\bar{y}|\} \leq R$ we can estimate*

$$\begin{aligned} &|[\psi(x) - \psi(y)] - [\psi(\bar{x}) - \psi(\bar{y})]| \\ &\leq C'_R |(x - y) - (\bar{x} - \bar{y})| + C''_R \{|x - y|^\rho + |\bar{x} - \bar{y}|^\rho\} |y - \bar{y}|, \end{aligned} \quad (2.19)$$

where $C'_R := \sup \{ |\nabla \psi(x)| : |x| \leq R \}$ and $C''_R := \sup \left\{ \frac{|\nabla \psi(x) - \nabla \psi(y)|}{|x - y|^\rho} : |x|, |y| \leq R \right\}$.

Proof. For $z, w \in \mathbb{R}^k$ we can write

$$\psi(z) - \psi(w) = \hat{\psi}(z, w)(z - w),$$

where $\hat{\psi}(z, w) := \int_0^1 \nabla \psi(uz + (1-u)w) du \in \mathbb{R}^\ell \otimes (\mathbb{R}^k)^*$, therefore

$$\begin{aligned} [\psi(x) - \psi(y)] - [\psi(\bar{x}) - \psi(\bar{y})] &= [\psi(x) - \psi(\bar{x})] - [\psi(y) - \psi(\bar{y})] \\ &= \hat{\psi}(x, \bar{x})(x - \bar{x}) - \hat{\psi}(y, \bar{y})(y - \bar{y}) \\ &= \hat{\psi}(x, \bar{x})[(x - \bar{x}) - (y - \bar{y})] \\ &\quad + [\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{y})](y - \bar{y}). \end{aligned}$$

By definition of C'_R and C''_R we have $|\hat{\psi}(x, \bar{x})| \leq C'_R$ and

$$\begin{aligned} |\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{y})| &\leq |\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{x})| + |\hat{\psi}(y, \bar{x}) - \hat{\psi}(y, \bar{y})| \\ &\leq C''_R \{|x - y|^\rho + |\bar{x} - \bar{y}|^\rho\}, \end{aligned}$$

hence (2.19) follows. \square

We are now ready to state and prove the announced uniqueness result.

THEOREM 2.9. (UNIQUENESS) *Let X be of class \mathcal{C}^α with $\alpha \in]\frac{1}{2}, 1]$ and let σ be of class \mathcal{C}^γ for some $\gamma > \frac{1}{\alpha}$ (for instance, we can take $\sigma \in \mathcal{C}^2$). Then for every $z_0 \in \mathbb{R}^k$ there exists at most one solution Z to (2.3) with $Z_0 = z_0$.*

Proof. Let Z and \bar{Z} be two solutions of (2.3), i.e. they satisfy (2.4), and set

$$Y := Z - \bar{Z}.$$

We want to show that, for $\tau > 0$ small enough, we have

$$\|Y\|_{\infty, \tau} \leq 2|Y_0|,$$

where the weighted norm $\|\cdot\|_{\infty, \tau}$ was defined in (1.37). In particular, if we assume that $Z_0 = \bar{Z}_0$, we obtain $\|Y\|_{\infty, \tau} = 0$ and hence $Z = \bar{Z}$.

We know by (2.5) that for any $\tau > 0$

$$\|Y\|_{\infty, \tau} \leq |Y_0| + 3\tau^\alpha \|\delta Y\|_{\alpha, \tau}, \quad (2.20)$$

where we recall that the weighted semi-norm $\|\cdot\|_{\alpha, \tau}$ was defined in (1.33). We now define $Y^{[2]}$ as the difference between the remainders $Z^{[2]}$ and $\bar{Z}^{[2]}$ of the solutions Z and \bar{Z} as defined in (2.4), that is

$$Y_{st}^{[2]} := Z_{st}^{[2]} - \bar{Z}_{st}^{[2]} = \delta Y_{st} - (\sigma(Z_s) - \sigma(\bar{Z}_s)) \delta X_{st}. \quad (2.21)$$

(We are slightly abusing notation, since $Y^{[2]}$ is not the remainder of Y when σ is not linear.) By assumption $\sigma \in \mathcal{C}^\gamma$ for some $\gamma > \frac{1}{\alpha}$: renaming γ as $\gamma \wedge 2$, we may assume that $\gamma \in]\frac{1}{\alpha}, 2]$. We are going to prove the following inequalities: for any $\tau > 0$

$$\|\delta Y\|_{\alpha, \tau} \leq c_1 \|Y\|_{\infty, \tau} + \tau^{(\gamma-1)\alpha} \|Y^{[2]}\|_{\gamma\alpha, \tau}, \quad (2.22)$$

$$\|Y^{[2]}\|_{\gamma\alpha, \tau} \leq c_2 \|Y\|_{\infty, \tau} + c'_2 \tau^{(\gamma-1)\alpha} \|Y^{[2]}\|_{\gamma\alpha, \tau}, \quad (2.23)$$

for finite constants c_i, c'_i that may depend on X, σ, Z, \bar{Z} but not on τ .

Let us complete the proof assuming (2.22) and (2.23). Note that $(\gamma - 1)\alpha > 0$ by assumption. If we fix $\tau > 0$ small, so that $c'_2 \tau^{(\gamma-1)\alpha} < \frac{1}{2}$, from (2.23) we get $\|Y^{[2]}\|_{\gamma\alpha, \tau} \leq 2c_2 \|Y\|_{\infty, \tau}$ which plugged into (2.22) yields $\|\delta Y\|_{\alpha, \tau} \leq 2c_1 \|Y\|_{\infty, \tau}$ for $\tau > 0$ small (it suffices that $2c_2 \tau^{(\gamma-1)\alpha} < c_1$). Finally, plugging this into (2.20) and possibly choosing $\tau > 0$ even smaller, we obtain our goal $\|Y\|_{\infty, \tau} \leq 2|Y_0|$ which completes the proof.

It remains to prove (2.22) and (2.23). Using the notation from Lemma 2.8 we set

$$\begin{aligned} C'_1 &:= \sup \{|\nabla\sigma(x)|: |x| \leq \|Z\|_{\infty} \vee \|\bar{Z}\|_{\infty}\}, \\ C''_1 &:= \sup \left\{ \frac{|\nabla\sigma(x) - \nabla\sigma(y)|}{|x - y|^\rho}: |x|, |y| \leq \|Z\|_{\infty} \vee \|\bar{Z}\|_{\infty} \right\}. \end{aligned}$$

so that $|\sigma(Z_t) - \sigma(\bar{Z}_t)| \leq C'_1 |Z_t - \bar{Z}_t|$ and, therefore,

$$\|\sigma(Z) - \sigma(\bar{Z})\|_{\infty, \tau} \leq C'_1 \|Y\|_{\infty, \tau}. \quad (2.24)$$

We now exploit (2.21) to estimate $\|\delta Y\|_{\alpha, \tau}$: applying (2.7) we obtain

$$\begin{aligned} \|\delta Y\|_{\alpha, \tau} &\leq \|\sigma(Z) - \sigma(\bar{Z})\|_{\infty, \tau} \|\delta X\|_{\alpha} + \|Y^{[2]}\|_{\alpha, \tau} \\ &\leq C'_1 \|Y\|_{\infty, \tau} \|\delta X\|_{\alpha} + \tau^{(\gamma-1)\alpha} \|Y^{[2]}\|_{\gamma\alpha, \tau}, \end{aligned} \quad (2.25)$$

where we note that $\|Y^{[2]}\|_{\alpha, \tau} \leq \tau^{(\gamma-1)\alpha} \|Y^{[2]}\|_{\gamma\alpha, \tau}$ by (2.6). We have shown that (2.22) holds with $c_1 = C'_1 \|\delta X\|_{\alpha}$.

We finally prove (2.23). Since $Y_{st}^{[2]} = o(t - s)$, see (2.21) and (2.4), we bound $Z^{[2]}$ by its increment $\delta Z^{[2]}$ through the weighted Sewing Bound (1.41):

$$\|Y^{[2]}\|_{\gamma\alpha, \tau} \leq K_{\gamma\alpha} \|\delta Y^{[2]}\|_{\gamma\alpha, \tau}, \quad (2.26)$$

hence we focus on $\|\delta Y^{[2]}\|_{\gamma\alpha, \tau}$. By (2.21) and (1.32), since $\delta \circ \delta = 0$, we have

$$\delta Y_{sut}^{[2]} = (\delta\sigma(Z)_{su} - \delta\sigma(\bar{Z})_{su}) \delta X_{ut}. \quad (2.27)$$

Applying the estimate (2.19) for $x = Z_u, y = Z_s, \bar{x} = \bar{Z}_u, \bar{y} = \bar{Z}_s$, we can write

$$\begin{aligned} |\delta\sigma(Z)_{su} - \delta\sigma(\bar{Z})_{su}| &\leq C'_1 |\delta Z_{su} - \delta \bar{Z}_{su}| + C''_1 \{|\delta Z_{su}|^{\gamma-1} + |\delta \bar{Z}_{su}|^{\gamma-1}\} |Z_s - \bar{Z}_s| \\ &= C'_1 |\delta Y_{su}| + C''_1 \{|\delta Z_{su}|^{\gamma-1} + |\delta \bar{Z}_{su}|^{\gamma-1}\} |Y_s|. \end{aligned} \quad (2.28)$$

hence by (2.7) we get

$$\begin{aligned} \|\delta\sigma(Z) - \delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha, \tau} &\leq C'_1 \|\delta Y\|_{(\gamma-1)\alpha, \tau} + \\ &\quad + C''_1 \{ \|\delta Z\|_{\alpha}^{\gamma-1} + \|\delta \bar{Z}\|_{\alpha}^{\gamma-1} \} \|Y\|_{\infty, \tau}. \end{aligned} \quad (2.29)$$

If we take $\tau \leq 1$ we can bound $\|\delta Y\|_{(\gamma-1)\alpha, \tau} \leq \|\delta Y\|_{\alpha, \tau}$ by (2.6) (recall that we are assuming $\gamma \leq 2$). Then by (2.27) we obtain, recalling (2.8),

$$\|\delta Y^{[2]}\|_{\gamma\alpha, \tau} \leq \|\delta X\|_{\alpha} \|\delta\sigma(Z) - \delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha, \tau} \leq \tilde{c}_1 (\|\delta Y\|_{\alpha, \tau} + \|Y\|_{\infty, \tau}),$$

for a suitable (explicit) constant $\tilde{c}_1 = \tilde{c}_1(\sigma, Z, \bar{Z}, X)$. Applying (2.22), we obtain

$$\|\delta Y^{[2]}\|_{\gamma\alpha, \tau} \leq (c_1 + 1) \tilde{c}_1 \|Y\|_{\infty, \tau} + \tilde{c}_1 \tau^{(\gamma-1)\alpha} \|Y^{[2]}\|_{\gamma\alpha, \tau},$$

which plugged into (2.26) shows that (2.23) holds. The proof is complete. \square

We conclude with an example of (2.19).

Example 2.10. If $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ is $\sigma(x) = x^2$, then we have

$$\begin{aligned} & (\sigma(x) - \sigma(y)) - (\sigma(\bar{x}) - \sigma(\bar{y})) \\ &= (x^2 - y^2) - (\bar{x}^2 - \bar{y}^2) = (x^2 - \bar{x}^2) - (y^2 - \bar{y}^2) \\ &= (x - \bar{x})(x + \bar{x}) - (y - \bar{y})(y + \bar{y}) \\ &= [(x - \bar{x}) - (y - \bar{y})](y + \bar{y}) + (x - \bar{x})[(x + \bar{x}) - (y + \bar{y})] \\ &= [(x - \bar{x}) - (y - \bar{y})](y + \bar{y}) + (x - \bar{x})[(x - y) + (\bar{x} - \bar{y})], \end{aligned}$$

where in the second last equality we have summed and subtracted $(y - \bar{y})(x + \bar{x})$. If we use this formula for $x = Z_t, y = Z_s$ and $\bar{x} = \bar{Z}_t, \bar{y} = \bar{Z}_s$, then we obtain

$$\delta(Z^2 - \bar{Z}^2)_{st} = \delta(Z - \bar{Z})_{st}(Z_s + \bar{Z}_s) + (Z_t - \bar{Z}_t)[\delta Z_{st} + \delta \bar{Z}_{st}],$$

which is in the spirit of (2.19) with $\rho = 1$. It follows that

$$\|\delta(Z^2 - \bar{Z}^2)\|_\alpha \leq 2\|\bar{Z}\|_\infty \|\delta(Z - \bar{Z})\|_\alpha + \|Z - \bar{Z}\|_\infty [\|\delta Z\|_\alpha + \|\delta \bar{Z}\|_\alpha],$$

which is the form that (2.29) takes in this particular case.

2.5. CONTINUITY OF THE SOLUTION MAP

In this section we assume that σ is *globally Lipschitz* and of class C^1 with a *globally γ -Hölder gradient*, i.e. $\|\nabla\sigma\|_\infty < \infty$ and $[\nabla\sigma]_{C^\gamma} < \infty$, with $\gamma > \frac{1}{\alpha}$. Under these assumptions, we have *global existence and uniqueness* of solutions $Z: [0, T] \rightarrow \mathbb{R}^k$ to (2.3) for any time horizon $T > 0$, for any starting point $Z_0 \in \mathbb{R}^k$ and for any driving path X of class C^α with $\frac{1}{2} < \alpha \leq 1$ (as we will prove in Section 2.6).

We can thus consider the *solution map*:

$$\begin{aligned} \Phi: \mathbb{R}^k \times C^\alpha &\longrightarrow C^\alpha \\ (Z_0, X) &\longmapsto Z := \begin{cases} \text{unique solution of (2.3) for } t \in [0, T] \\ \text{starting from } Z_0 \end{cases} \end{aligned} \quad (2.30)$$

We prove in this section that this map is *continuous*, in fact *locally Lipschitz*.

Remark 2.11. The continuity of the solution map is a highly non-trivial property. Indeed, when X is of class C^1 , note that Z solves the equation

$$Z_t = Z_0 + \int_0^t \sigma(Z_s) \dot{X}_s \, ds, \quad (2.31)$$

which is based on the derivative \dot{X} of X . We instead consider driving paths $X \in C^\alpha$ with $\alpha \in]\frac{1}{2}, 1]$ which are continuous but may be non-differentiable.

We shall see in the next chapters that the continuity of the solution map holds also in more complex situations such as $X \in C^\alpha$ with $\alpha \leq \frac{1}{2}$, which cover the case when X is a Brownian motion and Z is the solution to a SDE.