

# Ten lectures on rough paths

(work in progress)

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# Part I

## Rough Equations







# CHAPTER 1

## THE SEWING BOUND

The problem of interest in this book is the study of differential equations driven by *irregular functions* (more specifically: continuous but not differentiable). This will be achieved through the powerful and elegant theory of *rough paths*. A key motivation comes from stochastic differential equations driven by Brownian motion, but the goal is to develop a general theory which does not rely on probability.

This first chapter is dedicated to an elementary but fundamental tool, the *Sewing Bound*, that will be applied extensively throughout the book. It is a general Hölder-type bound for functions of two real variables that can be understood by itself, see Theorem 1.9 below. To provide motivation, we present it as a natural a priori estimate for solutions of differential equations.

**Notation.** We fix a time horizon  $T > 0$  and two dimensions  $k, d \in \mathbb{N}$ . We use “path” as a synonymous of “function defined on  $[0, T]$ ” with values in  $\mathbb{R}^d$ . We denote by  $|\cdot|$  the Euclidean norm. The space of linear maps from  $\mathbb{R}^d$  to  $\mathbb{R}^k$ , identified by  $k \times d$  real matrices, is denoted by  $\mathbb{R}^k \otimes (\mathbb{R}^d)^* \simeq \mathbb{R}^{k \times d}$  and is equipped with the Hilbert-Schmidt norm  $|\cdot|$  (i.e. the Euclidean norm on  $\mathbb{R}^{k \times d}$ ). For  $A \in \mathbb{R}^k \otimes (\mathbb{R}^d)^*$  and  $v \in \mathbb{R}^d$  we have  $|Av| \leq |A| |v|$ .

### 1.1. CONTROLLED DIFFERENTIAL EQUATION

Consider the following *controlled ordinary differential equation (ODE)*: given a continuously differentiable path  $X: [0, T] \rightarrow \mathbb{R}^d$  and a continuous function  $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ , we look for a differentiable path  $Z: [0, T] \rightarrow \mathbb{R}^k$  such that

$$\dot{Z}_t = \sigma(Z_t) \dot{X}_t, \quad t \in [0, T]. \quad (1.1)$$

By the fundamental theorem of calculus, this is equivalent to

$$Z_t = Z_0 + \int_0^t \sigma(Z_s) \dot{X}_s ds, \quad t \in [0, T]. \quad (1.2)$$

In the special case  $k = d = 1$  and when  $\sigma(x) = \lambda x$  is linear (with  $\lambda \in \mathbb{R}$ ), we have the explicit solution  $Z_t = z_0 \exp(\lambda (X_t - X_0))$ , which has the interesting property of being well-defined also when  $X$  is non differentiable.

For any dimensions  $k, d \in \mathbb{N}$ , if we assume that  $\sigma(\cdot)$  is Lipschitz, classical results in the theory of ODEs guarantee that *equation (1.1)-(1.2) is well-posed for any continuously differentiable path  $X$* , namely for any  $Z_0 \in \mathbb{R}^k$  there is one and only one solution  $Z$  (with no explicit formula, in general).

Our aim is to extend such a well-posedness result to a setting where  $X$  is *continuous but not differentiable* (also in cases where  $\sigma(\cdot)$  may be non-linear). Of course, to this purpose it is first necessary to provide a generalized formulation of (1.1)-(1.2) where the derivative of  $X$  does not appear.

## 1.2. CONTROLLED DIFFERENCE EQUATION

Let us still suppose that  $X$  is continuously differentiable. We deduce by (1.1)-(1.2) that for  $0 \leq s \leq t \leq T$

$$Z_t - Z_s = \sigma(Z_s)(X_t - X_s) + \int_s^t (\sigma(Z_u) - \sigma(Z_s)) \dot{X}_u du, \quad (1.3)$$

which implies that  $Z$  satisfies the following *controlled difference equation*:

$$Z_t - Z_s = \sigma(Z_s)(X_t - X_s) + o(t - s), \quad 0 \leq s \leq t \leq T, \quad (1.4)$$

because  $u \mapsto \sigma(Z_u)$  is continuous and  $u \mapsto \dot{X}_u$  is (continuous, hence) bounded on  $[0, T]$ .

**Remark 1.1.** (UNIFORMITY) Whenever we write  $o(t - s)$ , as in (1.4), we always mean *uniformly for*  $0 \leq s \leq t \leq T$ , i.e.

$$\forall \varepsilon > 0 \exists \delta > 0: \quad 0 \leq s \leq t \leq T, \quad t - s \leq \delta \quad \text{implies} \quad |o(t - s)| \leq \varepsilon(t - s). \quad (1.5)$$

This will be implicitly assumed in the sequel.

Let us make two simple observations.

- If  $X$  is continuously differentiable we deduced (1.4) from (1.1), but we can easily deduce (1.1) from (1.4): in other terms, the two equations (1.1) and (1.4) are *equivalent*.
- If  $X$  is *not* continuously differentiable, equation (1.4) is still *meaningful*, unlike equation (1.1) which contains explicitly  $\dot{X}$ .

For these reasons, henceforth we focus on the difference equation (1.4), which provides a generalized formulation of the differential equation (1.1) when  $X$  is continuous but not necessarily differentiable.

The problem is now to prove *well-posedness* for the difference equation (1.4). We are going to show that this is possible assuming a suitable *Hölder regularity* on  $X$ , but non trivial ideas are required. In this chapter we illustrate some key ideas, showing how to prove uniqueness of solutions via *a priori estimates* (existence of solutions will be studied in the next chapters). We start from a basic result, which ensures the continuity of solutions; more precise result will be obtained later.

LEMMA 1.2. (CONTINUITY OF SOLUTIONS) *Let  $X$  and  $\sigma$  be continuous. Then any solution  $Z$  of (1.4) is a continuous path, more precisely it satisfies*

$$|Z_t - Z_s| \leq C |X_t - X_s| + o(t - s), \quad 0 \leq s \leq t \leq T, \quad (1.6)$$

for a suitable constant  $C < \infty$  which depends on  $Z$ .

**Proof.** Relation (1.6) follows by (1.4) with  $C := \|\sigma(Z)\|_\infty = \sup_{0 \leq t \leq T} |\sigma(Z_t)|$ , renaming  $|o(t-s)|$  as  $o(t-s)$ . We only have to prove that  $C < \infty$ . Since  $\sigma$  is continuous by assumption, it is enough to show that  $Z$  is *bounded*.

Since  $o(t-s)$  is uniform, see (1.5), we can fix  $\bar{\delta} > 0$  such that  $|o(t-s)| \leq 1$  for all  $0 \leq s \leq t \leq T$  with  $|t-s| \leq \bar{\delta}$ . It follows that  $Z$  is bounded in any interval  $[\bar{s}, \bar{t}]$  with  $|\bar{t} - \bar{s}| \leq \bar{\delta}$ , because by (1.4) we can bound

$$\sup_{t \in [\bar{s}, \bar{t}]} |Z_t| \leq |Z_{\bar{s}}| + |\sigma(Z_{\bar{s}})| \sup_{t \in [\bar{s}, \bar{t}]} |X_t - X_{\bar{s}}| + 1 < \infty.$$

We conclude that  $Z$  is bounded in the whole interval  $[0, T]$ , because we can write  $[0, T]$  as a finite union of intervals  $[\bar{s}, \bar{t}]$  with  $|\bar{t} - \bar{s}| \leq \bar{\delta}$ .  $\square$

**Remark 1.3.** (COUNTEREXAMPLES) The weaker requirement that (1.4) holds for any fixed  $s \in [0, T]$  as  $t \downarrow s$  is not enough for our purposes, since in this case  $Z$  needs not be continuous. An easy counterexample is the following: given any continuous path  $X: [0, 2] \rightarrow \mathbb{R}$ , we define  $Z: [0, 2] \rightarrow \mathbb{R}$  by

$$Z_t := \begin{cases} X_t & \text{if } 0 \leq t < 1, \\ X_t + 1 & \text{if } 1 \leq t \leq 2. \end{cases}$$

Note that  $Z_t - Z_s = X_t - X_s$  when either  $0 \leq s \leq t < 1$  or  $1 \leq s \leq t \leq 2$ , hence  $Z$  satisfies the difference equation (1.4) with  $\sigma(\cdot) \equiv 1$  for any fixed  $s \in [0, 2)$  as  $t \downarrow s$ , but not uniformly for  $0 \leq s \leq t \leq 2$ , since  $Z$  is discontinuous at  $t = 1$ .

For another counterexample, which is even unbounded, consider

$$Z_t := \begin{cases} \frac{1}{1-t} & \text{if } 0 \leq t < 1, \\ 0 & \text{if } 1 \leq t \leq 2, \end{cases}$$

which satisfies (1.4) as  $t \downarrow s$  for any fixed  $s \in [0, 2]$ , for  $X_t \equiv t$  and  $\sigma(z) = z^2$ .

### 1.3. SOME USEFUL FUNCTION SPACES

For  $n \geq 1$  we define the simplex

$$[0, T]_{\leq}^n := \{(t_1, \dots, t_n) : 0 \leq t_1 \leq \dots \leq t_n \leq T\} \quad (1.7)$$

(note that  $[0, T]_{\leq}^1 = [0, T]$ ). We then write  $C_n = C([0, T]_{\leq}^n, \mathbb{R}^k)$  as a shorthand for the space of continuous functions from  $[0, T]_{\leq}^n$  to  $\mathbb{R}^k$ :

$$C_n := C([0, T]_{\leq}^n, \mathbb{R}^k) := \{F: [0, T]_{\leq}^n \rightarrow \mathbb{R}^k : F \text{ is continuous}\}. \quad (1.8)$$

We are going to work with functions of one ( $f_s$ ), two ( $F_{st}$ ) or three ( $G_{sut}$ ) ordered variables in  $[0, T]$ , hence we focus on the spaces  $C_1, C_2, C_3$ .

- On the spaces  $C_2$  and  $C_3$  we introduce a Hölder-like structure: given any  $\eta \in (0, \infty)$ , we define for  $F \in C_2$  and  $G \in C_3$

$$\|F\|_\eta := \sup_{0 \leq s < t \leq T} \frac{|F_{st}|}{(t-s)^\eta}, \quad \|G\|_\eta := \sup_{\substack{0 \leq s \leq u \leq t \leq T \\ s < t}} \frac{|G_{sut}|}{(t-s)^\eta}, \quad (1.9)$$

and we denote by  $C_2^\eta$  and  $C_3^\eta$  the corresponding function spaces:

$$C_2^\eta := \{F \in C_2: \|F\|_\eta < \infty\}, \quad C_3^\eta := \{G \in C_3: \|G\|_\eta < \infty\}, \quad (1.10)$$

which are Banach spaces endowed with the norm  $\|\cdot\|_\eta$  (exercise).

- On the space  $C_1$  of continuous functions  $f: [0, T] \rightarrow \mathbb{R}^k$  we consider the usual Hölder structure. We first introduce the *increment*  $\delta f$  by

$$(\delta f)_{st} := f_t - f_s, \quad 0 \leq s \leq t \leq T, \quad (1.11)$$

and note that  $\delta f \in C_2$  for any  $f \in C_1$ . Then, for  $\alpha \in (0, 1]$ , we define the classical space  $\mathcal{C}^\alpha = \mathcal{C}^\alpha([0, T], \mathbb{R}^k)$  of  $\alpha$ -Hölder functions

$$\mathcal{C}^\alpha := \left\{ f: [0, T] \rightarrow \mathbb{R}^k: \|\delta f\|_\alpha = \sup_{0 \leq s < t \leq T} \frac{|f_t - f_s|}{(t-s)^\alpha} < \infty \right\} \quad (1.12)$$

(for  $\alpha = 1$  it is the space of Lipschitz functions). Note that  $\|\delta f\|_\alpha$  in (1.12) is consistent with (1.11) and (1.9).

**Remark 1.4.** (HÖLDER SEMI-NORM) We stress that  $f \mapsto \|\delta f\|_\alpha$  is a semi-norm on  $\mathcal{C}^\alpha$  (it vanishes on constant functions). The standard norm on  $\mathcal{C}^\alpha$  is

$$\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha, \quad (1.13)$$

where we define the standard sup norm

$$\|f\|_\infty := \sup_{t \in [0, T]} |f_t|. \quad (1.14)$$

For  $f: [0, T] \rightarrow \mathbb{R}^k$  we can bound  $\|f\|_\infty \leq |f(0)| + T^\alpha \|\delta f\|_\alpha$  (see (1.39) below), hence

$$\|f\|_{\mathcal{C}^\alpha} \leq |f(0)| + (1 + T^\alpha) \|\delta f\|_\alpha. \quad (1.15)$$

This explains why it is often enough to focus on the semi-norm  $\|\delta f\|_\alpha$ .

**Remark 1.5.** (HÖLDER EXPONENTS) We only consider the Hölder space  $\mathcal{C}^\alpha$  for  $\alpha \in (0, 1]$  because for  $\alpha > 1$  the only functions in  $\mathcal{C}^\alpha$  are constant functions (note that  $\|\delta f\|_\alpha < \infty$  for  $\alpha > 1$  implies  $\dot{f}_t = 0$  for every  $t \in [0, T]$ ).

On the other hand, the spaces  $C_2^\eta$  and  $C_3^\eta$  in (1.10) are interesting for any exponent  $\eta \in (0, \infty)$ . For instance, the condition  $\|F\|_\eta < \infty$  for a function  $F \in C_2$  means that  $|F_{st}| \leq C(t-s)^\eta$ , which does not imply  $F \equiv 0$  when  $\eta > 1$  (unless  $F = \delta f$  is the increment of some function  $f \in C_1$ ).

In our results below we will have to assume that the non-linearity  $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$  belongs to classes of Hölder functions, in the following sense.

**DEFINITION 1.6.** Let  $\gamma > 0$ . A function  $F: \mathbb{R}^k \rightarrow \mathbb{R}^N$  is said to be globally  $\gamma$ -Hölder (or globally of class  $\mathcal{C}^\gamma$ ) if

- for  $\gamma \in (0, 1]$  we have

$$[F]_{\mathcal{C}^\gamma} := \sup_{x, y \in \mathbb{R}^k, x \neq y} \frac{|F(x) - F(y)|}{|x - y|^\gamma} < +\infty$$



- for  $\gamma \in (n, n+1]$  and  $n = \{1, 2, \dots\}$ ,  $F$  is  $n$  times continuously differentiable and

$$[D^{(n)}F]_{C^\gamma} := \sup_{x, y \in \mathbb{R}^k, x \neq y} \frac{|D^{(n)}F(x) - D^{(n)}F(y)|}{|x - y|^{\gamma-n}} < +\infty$$

where  $D^{(n)}$  is the  $n$ -fold differential of  $F$ .

Moreover  $F: \mathbb{R}^k \rightarrow \mathbb{R}^N$  is said to be locally  $\gamma$ -Hölder (or locally of class  $C^\gamma$ ) if

- for  $\gamma \in (0, 1]$  we have for all  $R > 0$

$$\sup_{\substack{x, y \in \mathbb{R}^k, x \neq y \\ |x|, |y| \leq R}} \frac{|F(x) - F(y)|}{|x - y|^\gamma} < +\infty$$

- for  $\gamma \in (n, n+1]$  and  $n = \{1, 2, \dots\}$ ,  $F$  is  $n$  times continuously differentiable and

$$\sup_{\substack{x, y \in \mathbb{R}^k, x \neq y \\ |x|, |y| \leq R}} \frac{|D^{(n)}F(x) - D^{(n)}F(y)|}{|x - y|^{\gamma-n}} < +\infty.$$

We stress that in the previous definition we do not assume  $F$  or  $D^{(n)}F$  to be bounded. The case  $\gamma = 1$  corresponds to the classical *Lipschitz* condition.

## 1.4. LOCAL UNIQUENESS OF SOLUTIONS

We prove *uniqueness of solutions* for the controlled difference equation (1.4) when  $X \in \mathcal{C}^\alpha$  is an Hölder path of exponent  $\alpha > \frac{1}{2}$ . For simplicity, we focus on the case when  $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$  is a linear application:  $\sigma \in (\mathbb{R}^k \otimes (\mathbb{R}^d)^*) \otimes (\mathbb{R}^k)^*$ , and we write  $\sigma Z$  instead of  $\sigma(Z)$  (we discuss non linear  $\sigma(\cdot)$  in Chapter 2).

**THEOREM 1.7.** (LOCAL UNIQUENESS OF SOLUTIONS, LINEAR CASE) *Fix a path  $X: [0, T] \rightarrow \mathbb{R}^d$  in  $\mathcal{C}^\alpha$ , with  $\alpha \in ]\frac{1}{2}, 1]$ , and a linear map  $\sigma: \mathbb{R}^k \rightarrow \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ . If  $T > 0$  is small enough (depending on  $X, \alpha, \sigma$ ), then for any  $z_0 \in \mathbb{R}^k$  there is at most one path  $Z: [0, T] \rightarrow \mathbb{R}^k$  with  $Z_0 = z_0$  which solves the linear controlled difference equation (1.4), that is (recalling (1.11))*

$$\delta Z_{st} - (\sigma Z_s) \delta X_{st} = o(t - s), \quad 0 \leq s \leq t \leq T. \quad (1.16)$$

**Proof.** Suppose that we have two paths  $Z, \bar{Z}: [0, T] \rightarrow \mathbb{R}^k$  satisfying (1.16) with  $Z_0 = \bar{Z}_0$  and define  $Y := Z - \bar{Z}$ . Our goal is to show that  $Y = 0$ .

Let us introduce the function  $R \in C_2 = C([0, T]_{\leq}^2, \mathbb{R}^k)$  defined by

$$R_{st} := \delta Y_{st} - (\sigma Y_s) \delta X_{st}, \quad 0 \leq s \leq t \leq T, \quad (1.17)$$

and note that by (1.16) and linearity we have

$$R_{st} = o(t - s). \quad (1.18)$$

Recalling (1.9), we can estimate

$$\|\delta Y\|_\alpha \leq |\sigma| \|Y\|_\infty \|\delta X\|_\alpha + \|R\|_\alpha,$$

and since  $R_{st} = o(t-s) = o((t-s)^\alpha)$ , we have  $\|R\|_\alpha < +\infty$  and therefore  $\|\delta Y\|_\alpha < +\infty$ . Since  $Y_0 = 0$ , we can bound

$$\|Y\|_\infty \leq |Y_0| + \sup_{0 \leq t \leq T} |Y_t - Y_0| \leq T^\alpha \|\delta Y\|_\alpha.$$

Since  $1 \leq T^\alpha (t-s)^{-\alpha}$  for  $0 \leq s < t \leq T$ , we can also bound

$$\|R\|_\alpha \leq T^\alpha \|R\|_{2\alpha},$$

so that

$$\|\delta Y\|_\alpha \leq T^\alpha (|\sigma| \|\delta Y\|_\alpha \|\delta X\|_\alpha + \|R\|_{2\alpha}).$$

Suppose we can prove that, for some constant  $C = C(X, \alpha, \sigma) < \infty$ ,

$$\|R\|_{2\alpha} \leq C \|\delta Y\|_\alpha. \quad (1.19)$$

Then we obtain

$$\|\delta Y\|_\alpha \leq T^\alpha (|\sigma| \|\delta X\|_\alpha + C) \|\delta Y\|_\alpha.$$

If we fix  $T$  small enough, so that  $T^\alpha (|\sigma| \|\delta X\|_\alpha + C) < 1$ , we get  $\|\delta Y\|_\alpha = 0$ , hence  $\delta Y \equiv 0$ . This means that  $Y_t = Y_s$  for all  $s, t \in [0, T]$ , and since  $Y_0 = 0$  we obtain  $Y \equiv 0$ , namely our goal  $Z \equiv \bar{Z}$ . This completes the proof assuming the estimate (1.19) (where the hypothesis  $\alpha > \frac{1}{2}$  will play a key role).  $\square$

To actually complete the proof of Theorem 1.7, it remains to show that the inequality (1.19) holds. This is performed in the next two sections:

- in Section 1.5 we present a fundamental estimate, the *Sewing Bound*, which applies to any function  $R_{st} = o(t-s)$  (recall (1.18));
- in Section 1.6 we apply the Sewing Bound to  $R_{st}$  in (1.17) and we prove the desired estimate (1.19) for  $\alpha > \frac{1}{2}$  (see the assumptions of Theorem 1.7).

## 1.5. THE SEWING BOUND

Let us fix an arbitrary function  $R \in C_2 = C([0, T]_{\leq}^2, \mathbb{R}^k)$  with  $R_{st} = o(t-s)$ . Our goal is to bound  $|R_{ab}|$  for any given  $0 \leq a < b \leq T$ .

We first show that we can express  $R_{ab}$  via ‘‘Riemann sums’’ along partitions  $\mathcal{P} = \{a = t_0 < t_1 < \dots < t_m = b\}$  of  $[a, b]$ . These are defined by

$$I_{\mathcal{P}}(R) := \sum_{i=1}^{\#\mathcal{P}} R_{t_{i-1}t_i}, \quad (1.20)$$

where we denote by  $\#\mathcal{P} := m$  the number of intervals of the partition  $\mathcal{P}$ . Let us denote by  $|\mathcal{P}| := \max_{1 \leq i \leq m} (t_i - t_{i-1})$  the *mesh* of  $\mathcal{P}$ .

LEMMA 1.8. (RIEMANN SUMS) *Given any  $R \in C_2$  with  $R_{st} = o(t-s)$ , for any  $0 \leq a < b \leq T$  and for any sequence  $(\mathcal{P}_n)_{n \geq 0}$  of partitions of  $[a, b]$  with vanishing mesh  $\lim_{n \rightarrow \infty} |\mathcal{P}_n| = 0$  we have*

$$\lim_{n \rightarrow \infty} I_{\mathcal{P}_n}(R) = 0.$$

If furthermore  $\mathcal{P}_0 = \{a, b\}$  is the trivial partition, then we can write

$$R_{ab} = \sum_{n=0}^{\infty} (I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)), \quad 0 \leq a < b \leq T. \quad (1.21)$$

**Proof.** Writing  $\mathcal{P}_n = \{a = t_0^n < t_1^n < \dots < t_{\#\mathcal{P}_n}^n = b\}$ , we can estimate

$$|I_{\mathcal{P}_n}(R)| \leq \sum_{i=1}^{\#\mathcal{P}_n} |R_{t_{i-1}^n t_i^n}| \leq \left\{ \max_{j=1, \dots, \#\mathcal{P}_n} \frac{|R_{t_{j-1}^n t_j^n}|}{(t_j^n - t_{j-1}^n)} \right\} \sum_{j=1}^{\#\mathcal{P}_n} (t_j^n - t_{j-1}^n),$$

hence  $|I_{\mathcal{P}_n}(R)| \rightarrow 0$  as  $n \rightarrow \infty$ , because the final sum equals  $b - a$  and the bracket vanishes (since  $R_{st} = o(t - s)$  and  $|\mathcal{P}_n| = \max_{1 \leq j \leq \#\mathcal{P}_n} (t_j^n - t_{j-1}^n) \rightarrow 0$ ).

We deduce relation (1.21) by the telescopic sum

$$I_{\mathcal{P}_0}(R) - I_{\mathcal{P}_N}(R) = \sum_{n=0}^{N-1} (I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)),$$

because  $\lim_{N \rightarrow \infty} I_{\mathcal{P}_N}(R) = 0$  while  $I_{\mathcal{P}_0}(R) = R_{ab}$  for  $\mathcal{P}_0 = \{a, b\}$ .  $\square$

If we remove a single point  $t_i$  from a partition  $\mathcal{P} = \{t_0 < t_1 < \dots < t_m\}$ , we obtain a new partition  $\mathcal{P}'$  for which, recalling (1.20), we can write

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = R_{t_{i-1}t_{i+1}} - R_{t_{i-1}t_i} - R_{t_it_{i+1}}. \quad (1.22)$$

The expression in the RHS deserves a name: given any two-variables function  $F \in C_2$ , we define its increment  $\delta F \in C_3$  as the three-variables function

$$\delta F_{sut} := F_{st} - F_{su} - F_{ut}, \quad 0 \leq s \leq u \leq t \leq T. \quad (1.23)$$

We can then rewrite (1.22) as

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = \delta R_{t_{i-1}t_it_{i+1}}, \quad (1.24)$$

and recalling (1.9) we obtain the following estimate, for any  $\eta > 0$ :

$$|I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R)| \leq \|\delta R\|_{\eta} |t_{i+1} - t_{i-1}|^{\eta}. \quad (1.25)$$

We are now ready to state and prove the Sewing Bound.

**THEOREM 1.9. (SEWING BOUND)** *Given any  $R \in C_2$  with  $R_{st} = o(t - s)$ , the following estimate holds for any  $\eta \in (1, \infty)$  (recall (1.9)):*

$$\|R\|_{\eta} \leq K_{\eta} \|\delta R\|_{\eta} \quad \text{where} \quad K_{\eta} := (1 - 2^{1-\eta})^{-1}. \quad (1.26)$$

**Proof.** Fix  $R \in C_2$  such that  $\|\delta R\|_{\eta} < \infty$  for some  $\eta > 1$  (otherwise there is nothing to prove). Also fix  $0 \leq a < b \leq T$  and consider for  $n \geq 0$  the dyadic partitions  $\mathcal{P}_n := \{t_i^n := a + \frac{i}{2^n}(b-a) : 0 \leq i \leq 2^n\}$  of  $[a, b]$ . Since  $\mathcal{P}_0 = \{a, b\}$  is the trivial partition, we can apply (1.21) to bound

$$|R_{ab}| \leq \sum_{n=0}^{\infty} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)|. \quad (1.27)$$

If we remove from  $\mathcal{P}_{n+1}$  all the “odd points”  $t_{2j+1}^{n+1}$ , with  $0 \leq j \leq 2^n - 1$ , we obtain  $\mathcal{P}_n$ . Then, iterating relations (1.24)-(1.25), we have

$$\begin{aligned} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)| &\leq \sum_{j=0}^{2^n-1} |\delta R_{t_{2j}^{n+1} t_{2j+1}^{n+1} t_{2j+2}^{n+1}}| \\ &\leq 2^n \|\delta R\|_\eta \left( \frac{2(b-a)}{2^{n+1}} \right)^\eta \\ &= 2^{-(\eta-1)n} \|\delta R\|_\eta (b-a)^\eta. \end{aligned} \quad (1.28)$$

Plugging this into (1.27), since  $\sum_{n=0}^{\infty} 2^{-(\eta-1)n} = (1 - 2^{1-\eta})^{-1}$ , we obtain

$$|R_{ab}| \leq (1 - 2^{1-\eta})^{-1} \|\delta R\|_\eta (b-a)^\eta, \quad 0 \leq a < b \leq T, \quad (1.29)$$

which proves (1.26).  $\square$

**Remark 1.10.** Recalling (1.11) and (1.23), we have defined linear maps

$$C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \quad (1.30)$$

which satisfy  $\delta \circ \delta = 0$ . Indeed, for any  $f \in C_1$  we have

$$\delta(\delta f)_{sut} = (f_t - f_s) - (f_u - f_s) - (f_t - f_u) = 0.$$

Intuitively,  $\delta F \in C_3$  measures how much a function  $F \in C_2$  differs from being the increment  $\delta f$  of some  $f \in C_1$ , because  $\delta F \equiv 0$  if and only if  $F = \delta f$  for some  $f \in C_1$  (it suffices to define  $f_t := F_{0t}$  and to check that  $\delta f_{st} = \delta F_{0st} + F_{st} = F_{st}$ ).

**Remark 1.11.** The assumption  $R_{st} = o(t-s)$  in Theorem 1.9 cannot be avoided: if  $R := \delta f$  for a non constant  $f \in C_1$ , then  $\delta R = 0$  while  $\|R\|_\eta > 0$ .

## 1.6. END OF PROOF OF UNIQUENESS

In this section, we apply the Sewing Bound (1.26) to the function  $R_{st}$  defined in (1.17), in order to prove the estimate (1.19) for  $\alpha > \frac{1}{2}$ .

We first determine the increment  $\delta R$  through a simple and instructive computation: by (1.17), since  $\delta(\delta Z) = 0$  (see Remark 1.10), we have

$$\begin{aligned} \delta R_{sut} &:= R_{st} - R_{su} - R_{ut} \\ &= (Y_t - Y_s) - (Y_u - Y_s) - (Y_t - Y_u) \\ &\quad - (\sigma Y_s)(X_t - X_s) + (\sigma Y_s)(X_u - X_s) + (\sigma Y_u)(X_t - X_u) \\ &= [\sigma(Y_u - Y_s)](X_t - X_u). \end{aligned} \quad (1.31)$$

Recalling (1.9), this implies

$$\|\delta R\|_{2\alpha} \leq |\sigma| \|\delta Y\|_\alpha \|\delta X\|_\alpha.$$

We next note that if  $\alpha > \frac{1}{2}$  (as it is assumed in Theorem 1.7) we can apply the Sewing Bound (1.26) for  $\eta = 2\alpha > 1$  to obtain

$$\|R\|_{2\alpha} \leq K_{2\alpha} \|\delta R\|_{2\alpha} \leq K_{2\alpha} |\sigma| \|\delta Y\|_\alpha \|\delta X\|_\alpha.$$

This is precisely our goal (1.19) with  $C = C(X, \alpha, \sigma) := K_{2\alpha} |\sigma| \|\delta X\|_\alpha$ .

Summarizing: thanks to the Sewing bound (1.26), we have obtained the estimate (1.19) and completed the proof of Theorem 1.7, showing uniqueness of solutions to the difference equation (1.4) for any  $X \in \mathcal{C}^\alpha$  with  $\alpha \in ]\frac{1}{2}, 1]$ . In the next chapters we extend this approach to non-linear  $\sigma(\cdot)$  and to situations where  $X \in \mathcal{C}^\alpha$  with  $\alpha \leq \frac{1}{2}$ .

**Remark 1.12.** For later purpose, let us record the computation (1.31) without  $\sigma$ : given any (say, real) paths  $X$  and  $Y$ , if

$$A_{st} = Y_s \delta X_{st}, \quad \forall 0 \leq s \leq t \leq T,$$

then

$$\delta A_{sut} = -\delta Y_{su} \delta X_{ut}, \quad \forall 0 \leq s \leq u \leq t \leq T. \quad (1.32)$$

## 1.7. WEIGHTED NORMS

We conclude this chapter defining *weighted versions*  $\|\cdot\|_{\eta, \tau}$  of the norms  $\|\cdot\|_\eta$  introduced in (1.9): given  $F \in C_2$  and  $G \in C_3$ , we set for  $\eta, \tau \in (0, \infty)$

$$\|F\|_{\eta, \tau} := \sup_{0 \leq s \leq t \leq T} \mathbb{1}_{\{0 < t-s \leq \tau\}} e^{-\frac{t}{\tau}} \frac{|F_{st}|}{(t-s)^\eta}, \quad (1.33)$$

$$\|G\|_{\eta, \tau} := \sup_{0 \leq s \leq u \leq t \leq T} \mathbb{1}_{\{0 < t-s \leq \tau\}} e^{-\frac{t}{\tau}} \frac{|G_{sut}|}{(t-s)^\eta}, \quad (1.34)$$

where  $C_2$  and  $C_3$  are the spaces of continuous functions from  $[0, T]_{\leq}^2$  and  $[0, T]_{\leq}^3$  to  $\mathbb{R}^k$ , see (1.8). Note that as  $\tau \rightarrow \infty$  we recover the usual norms:

$$\|\cdot\|_\eta = \lim_{\tau \rightarrow \infty} \|\cdot\|_{\eta, \tau}. \quad (1.35)$$

**Remark 1.13.** (NORMS VS. SEMI-NORMS) While  $\|\cdot\|_\eta$  is a norm,  $\|\cdot\|_{\eta, \tau}$  is a norm for  $\tau \geq T$  but *it is only a semi-norm for  $\tau < T$*  (for instance,  $\|F\|_{\eta, \tau} = 0$  for  $F \in C_2$  implies  $F_{st} = 0$  only for  $t-s \leq \tau$ : no constraint is imposed on  $F_{st}$  for  $t-s > \tau$ ).

However, if  $F = \delta f$ , that is  $F_{st} = f_t - f_s$  for some  $f \in C_1$ , we have the equivalence

$$\|\delta f\|_{\eta, \tau} \leq \|\delta f\|_\eta \leq \left(1 + \frac{T}{\tau}\right) e^{\frac{T}{\tau}} \|\delta f\|_{\eta, \tau}. \quad (1.36)$$

The first inequality is clear. For the second one, given  $0 \leq s < t \leq T$ , we can write  $s = t_0 < t_1 < \dots < t_N = t$  with  $t_i - t_{i-1} \leq \tau$  and  $N \leq 1 + \frac{T}{\tau}$  (for instance, we can consider  $t_i = s + i \frac{t-s}{N}$  where  $N := \lceil \frac{t-s}{\tau} \rceil$ ); we then obtain  $\delta f_{st} = \sum_{i=1}^N \delta f_{t_{i-1}t_i}$  and  $|\delta f_{t_{i-1}t_i}| \leq \|\delta f\|_{\eta, \tau} e^{t_i/\tau} (t_i - t_{i-1})^\eta \leq \|\delta f\|_{\eta, \tau} e^{T/\tau} (t-s)^\eta$ , which yields (1.36).

**Remark 1.14.** (FROM LOCAL TO GLOBAL) The weighted semi-norms  $\|\cdot\|_{\eta, \tau}$  will be useful to transform *local* results in *global* results. Indeed, using the standard norms  $\|\cdot\|_\eta$  often requires the size  $T > 0$  of the time interval  $[0, T]$  to be *small*, as in Theorem 1.7, which can be annoying. Using  $\|\cdot\|_{\eta, \tau}$  will allow us to *keep*  $T > 0$  *arbitrary*, by choosing a sufficiently small  $\tau > 0$ .

Recalling the supremum norm  $\|f\|_\infty$  of a function  $f \in C_1$ , see (1.14), we define the corresponding weighted version

$$\|f\|_{\infty,\tau} := \sup_{0 \leq t \leq T} e^{-\frac{t}{\tau}} |f_t|. \quad (1.37)$$

We stress that  $\|\cdot\|_{\infty,\tau}$  is a norm equivalent to  $\|\cdot\|_\infty$  for any  $\tau > 0$ , since

$$\|\cdot\|_{\infty,\tau} \leq \|\cdot\|_\infty \leq e^{\frac{T}{\tau}} \|\cdot\|_{\infty,\tau}. \quad (1.38)$$

**Remark 1.15.** (EQUIVALENT HÖLDER NORM) It follows by (1.36) and (1.38) that  $\|\cdot\|_{\infty,\tau} + \|\cdot\|_{\alpha,\tau}$  is a norm equivalent to  $\|\cdot\|_{C^\alpha} := \|\cdot\|_\infty + \|\cdot\|_\alpha$  on the space  $C^\alpha$  of Hölder functions, see Remark 1.4, for any  $\tau > 0$ .

We will often use the Hölder semi-norms  $\|\delta f\|_\alpha$  and  $\|\delta f\|_{\alpha,\tau}$  to bound the supremum norms  $\|f\|_\infty$  and  $\|f\|_{\infty,\tau}$ , thanks to the following result.

LEMMA 1.16. (SUPREMUM-HÖLDER BOUND) For any  $f \in C_1$  and  $\eta \in (0, \infty)$

$$\|f\|_\infty \leq |f_0| + T^\eta \|\delta f\|_\eta, \quad (1.39)$$

$$\|f\|_{\infty,\tau} \leq |f_0| + 3(\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau}, \quad \forall \tau > 0. \quad (1.40)$$

**Proof.** Let us prove (1.39): for any  $f \in C_1$  and for  $t \in ]0, T]$  we have

$$|f_t| \leq |f_0| + |f_t - f_0| = |f_0| + t^\eta \frac{|f_t - f_0|}{t^\eta} \leq |f_0| + T^\eta \|\delta f\|_\eta.$$

The proof of (1.40) is slightly more involved. If  $t \in ]0, \tau \wedge T]$ , then

$$e^{-\frac{t}{\tau}} |f_t| \leq |f_0| + t^\eta e^{-\frac{t}{\tau}} \frac{|f_t - f_0|}{t^\eta} \leq |f_0| + (\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau},$$

which, in particular, implies (1.40) when  $\tau \geq T$ . When  $\tau < T$ , it remains to consider  $\tau < t \leq T$ : in this case, we define  $N := \min \{n \in \mathbb{N} : n\tau \geq t\} \geq 2$  so that  $\frac{t}{N} \leq \tau$ . We set  $t_k = k \frac{t}{N}$  for  $k \geq 0$ , so that  $t_N = t$ . Then

$$\begin{aligned} e^{-\frac{t}{\tau}} |f_t| &\leq |f_0| + \sum_{k=1}^N (t_k - t_{k-1})^\eta e^{-\frac{t-t_k}{\tau}} \left[ e^{-\frac{t_k}{\tau}} \frac{|f_{t_k} - f_{t_{k-1}}|}{(t_k - t_{k-1})^\eta} \right] \\ &\leq |f_0| + (\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau} \sum_{k=1}^N e^{-\frac{t-t_k}{\tau}}. \end{aligned}$$

By definition of  $N$  we have  $(N-1)\tau < t$ ; since  $\tau < t$  we obtain  $N\tau < 2t$  and therefore  $\frac{t}{N\tau} \geq \frac{1}{2}$ . Since  $t - t_k = (N-k) \frac{t}{N}$ , renaming  $\ell := N - k$  we obtain

$$\sum_{k=1}^N e^{-\frac{t-t_k}{\tau}} = \sum_{\ell=0}^{N-1} e^{-\ell \frac{t}{N\tau}} = \frac{1 - e^{-\frac{t}{\tau}}}{1 - e^{-\frac{t}{N\tau}}} \leq \frac{1}{1 - e^{-\frac{1}{2}}} \leq 3.$$

The proof is complete.  $\square$

We finally show that the Sewing Bound (1.26) still holds if we replace  $\|\cdot\|_\eta$  by  $\|\cdot\|_{\eta,\tau}$ , for any  $\tau > 0$ .

**THEOREM 1.17. (WEIGHTED SEWING BOUND)** *Given any  $R \in C_2$  with  $R_{st} = o(t-s)$ , the following estimate holds for any  $\eta \in (1, \infty)$  and  $\tau > 0$ :*

$$\|R\|_{\eta,\tau} \leq K_\eta \|\delta R\|_{\eta,\tau} \quad \text{where} \quad K_\eta := (1 - 2^{1-\eta})^{-1}. \quad (1.41)$$

**Proof.** Given  $0 \leq a \leq b \leq T$ , let us define

$$\|\delta R\|_{\eta,[a,b]} := \sup_{\substack{s,u,t \in [a,b]: \\ s \leq u \leq t, s < t}} \frac{|\delta R_{sut}|}{(t-s)^\eta}. \quad (1.42)$$

Following the proof of Theorem 1.9, we can replace  $\|\delta R\|_\eta$  by  $\|\delta R\|_{\eta,[a,b]}$  in (1.28) and in (1.29), hence we obtain  $|R_{ab}| \leq K_\eta \|\delta R\|_{\eta,[a,b]} (b-a)^\eta$ . Then for  $b-a \leq \tau$  we can estimate

$$e^{-\frac{b}{\tau}} \frac{|R_{ab}|}{(b-a)^\eta} \leq e^{-\frac{b}{\tau}} K_\eta \|\delta R\|_{\eta,[a,b]} \leq K_\eta \|\delta R\|_{\eta,\tau},$$

and (1.41) follows taking the supremum over  $0 \leq a \leq b \leq T$  with  $b-a \leq \tau$ .  $\square$

## 1.8. A DISCRETE SEWING BOUND

We can prove a version of the Sewing Bound for functions  $R = (R_{st})_{s < t \in \mathbb{T}}$  defined on a *finite set of points*  $\mathbb{T} := \{0 = t_1 < \dots < t_{\#\mathbb{T}}\} \subseteq \mathbb{R}_+$  (this will be useful to construct solutions to difference equations via Euler schemes, see Sections 2.6 and 3.9). The condition  $R_{st} = o(t-s)$  from Theorem 1.9 is now replaced by the requirement that  $R$  *vanishes on consecutive points of  $\mathbb{T}$* , i.e.  $R_{t_i t_{i+1}} = 0$  for all  $1 \leq i < \#\mathbb{T}$ .

We define versions  $\|\cdot\|_{\eta,\tau}^\mathbb{T}$  of the norms  $\|\cdot\|_{\eta,\tau}$  restricted on  $\mathbb{T}$  for  $\tau > 0$ , recall (1.33)-(1.34):

$$\|A\|_{\eta,\tau}^\mathbb{T} := \sup_{\substack{0 \leq s < t \\ s,t \in \mathbb{T}}} \mathbb{1}_{\{0 < t-s \leq \tau\}} e^{-\frac{t}{\tau}} \frac{|A_{st}|}{|t-s|^\eta}, \quad (1.43)$$

$$\|B\|_{\eta,\tau}^\mathbb{T} := \sup_{\substack{0 \leq s \leq u \leq t \\ s,u,t \in \mathbb{T}, s < t}} \mathbb{1}_{\{0 < t-s \leq \tau\}} e^{-\frac{t}{\tau}} \frac{|B_{sut}|}{|t-s|^\eta} \quad (1.44)$$

for  $A: \{(s,t) \in \mathbb{T}^2: 0 \leq s < t\} \rightarrow \mathbb{R}$  and  $B: \{(s,u,t) \in \mathbb{T}^3: 0 \leq s \leq u \leq t, s < t\} \rightarrow \mathbb{R}$ .

**THEOREM 1.18. (DISCRETE SEWING BOUND)** *If a function  $R = (R_{st})_{s < t \in \mathbb{T}}$  vanishes on consecutive points of  $\mathbb{T}$  (i.e.  $R_{t_i t_{i+1}} = 0$ ), then for any  $\eta > 1$  and  $\tau > 0$  we have*

$$\|R\|_{\eta,\tau}^\mathbb{T} \leq C_\eta \|\delta R\|_{\eta,\tau}^\mathbb{T} \quad \text{with} \quad C_\eta := 2^\eta \sum_{n \geq 1} \frac{1}{n^\eta} = 2^\eta \zeta(\eta) < \infty. \quad (1.45)$$

**Proof.** We fix  $s, t \in \mathbb{T}$  with  $s < t$  and we start by proving that

$$|R_{st}| \leq C_\eta \|\delta R\|_{\eta,\tau}^\mathbb{T} (t-s)^\eta.$$

We have  $s = t_k$  and  $t = t_{k+m}$  and we may assume that  $m \geq 2$  (otherwise there is nothing to prove, since for  $m = 1$  we have  $R_{t_i t_{i+1}} = 0$ ).

Consider the partition  $\mathcal{P} = \{s = t_k < t_{k+1} < \dots < t_{k+m} = t\}$  with  $m$  intervals. Note that for some index  $i \in \{k+1, \dots, k+m-1\}$  we must have  $t_{i+1} - t_{i-1} \leq \frac{2(t-s)}{m-1}$ , otherwise we would get the contradiction

$$2(t-s) \geq \sum_{i=k+1}^{k+m-1} (t_{i+1} - t_{i-1}) > \sum_{i=k+1}^{k+m-1} \frac{2(t-s)}{m-1} = 2(t-s).$$

Removing the point  $t_i$  from  $\mathcal{P}$  we obtain a partition  $\mathcal{P}'$  with  $m-1$  intervals. If we define  $I_{\mathcal{P}}(R) := \sum_{i=k}^{k+m-1} R_{t_i t_{i+1}}$  as in (1.20), as in (1.24) we have

$$|I_{\mathcal{P}}(R) - I_{\mathcal{P}'}(R)| = |\delta R_{t_{i-1} t_i t_{i+1}}| \leq \frac{2^\eta (t-s)^\eta}{(m-1)^\eta} \sup_{\substack{s \leq u < v < w \leq t \\ u, v, w \in \mathbb{T}}} \frac{|\delta R_{uvw}|}{|w-u|^\eta}.$$

Iterating this argument, until we arrive at the trivial partition  $\{s, t\}$ , we get

$$|I_{\mathcal{P}}(R) - R_{st}| \leq C_\eta (t-s)^\eta \sup_{\substack{s \leq u < v < w \leq t \\ u, v, w \in \mathbb{T}}} \frac{|\delta R_{uvw}|}{|w-u|^\eta}, \quad (1.46)$$

with  $C_\eta := \sum_{n \geq 1} \frac{2^\eta}{n^\eta} < \infty$  because  $\eta > 1$ . We finally note that  $I_{\mathcal{P}}(R) = 0$  by the assumption  $R_{t_i t_{i+1}} = 0$ . Finally if  $t-s \leq \tau$  then  $w-u \leq \tau$  in the supremum in (1.46) and since  $e^{-\frac{t}{\tau}} \leq e^{-\frac{w}{\tau}}$  we obtain

$$e^{-\frac{t}{\tau}} |R_{st}| \leq C_\eta (t-s)^\eta \|\delta R\|_{\eta, \tau}^{\mathbb{T}},$$

and the proof is complete.  $\square$

We also have an analog of Lemma 1.16. We set for  $f: \mathbb{T} \rightarrow \mathbb{R}$  and  $\tau > 0$

$$\|f\|_{\infty, \tau}^{\mathbb{T}} := \sup_{t \in \mathbb{T}} e^{-\frac{t}{\tau}} |f_t|.$$

LEMMA 1.19. (DISCRETE SUPREMUM-HÖLDER BOUND) For  $\mathbb{T} := \{0 = t_1 < \dots < t_{\#\mathbb{T}}\} \subseteq \mathbb{R}_+$  set

$$M := \max_{i=2, \dots, \#\mathbb{T}} |t_i - t_{i-1}|.$$

Then for all  $f: \mathbb{T} \rightarrow \mathbb{R}$ ,  $\tau \geq 2M$  and  $\eta > 0$

$$\|f\|_{\infty, \tau}^{\mathbb{T}} \leq |f_0| + 5\tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}}. \quad (1.47)$$

**Proof.** We define  $T_0 := 0$  and for  $i \geq 1$ , as long as  $\mathbb{T} \cap (T_{i-1}, T_{i-1} + \tau]$  is not empty, we set

$$T_i := \max \mathbb{T} \cap (T_{i-1}, T_{i-1} + \tau], \quad i = 1, \dots, N,$$

so that  $T_N = \max \mathbb{T}$ . We have by construction  $T_i + M > T_{i-1} + \tau$  for all  $i = 1, \dots, N-1$ , and since  $M \leq \frac{\tau}{2}$

$$T_i - T_{i-1} \geq \tau - M \geq \frac{\tau}{2}.$$



For  $i = N$  we have only  $T_N > T_{N-1}$ . Therefore for  $i = 1, \dots, N$

$$\begin{aligned}
e^{-\frac{T_i}{\tau}} |f_{T_i}| &\leq |f_0| + \sum_{k=1}^i (T_k - T_{k-1})^\eta e^{-\frac{T_i - T_k}{\tau}} \left[ e^{-\frac{T_k}{\tau}} \frac{|f_{T_k} - f_{T_{k-1}}|}{(T_k - T_{k-1})^\eta} \right] \\
&\leq |f_0| + \tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}} \sum_{k=1}^i e^{-\frac{T_i - T_k}{\tau}} \\
&\leq |f_0| + \tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}} \left( 1 + \sum_{k=0}^{\infty} e^{-\frac{k}{2}} \right) \\
&\leq |f_{t_0}| + 4\tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}}.
\end{aligned}$$

Now for  $t \in \mathbb{T} \setminus \{T_i\}_i$  we have  $T_i < t < T_{i+1}$  for some  $i$  and then

$$\begin{aligned}
e^{-\frac{t}{\tau}} |f_t| &\leq e^{-\frac{t}{\tau}} |f_{T_i}| + (t - T_i)^\eta e^{-\frac{t}{\tau}} \frac{|f_t - f_{T_i}|}{(t - T_i)^\eta} \leq e^{-\frac{T_i}{\tau}} |f_{T_i}| + \tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}} \\
&\leq |f_0| + 5\tau^\eta \|\delta f\|_{\eta, \tau}^{\mathbb{T}}.
\end{aligned}$$

The proof is complete.  $\square$

## 1.9. EXTRA (TO BE COMPLETED)

We also introduce the usual supremum norm, for  $F \in C_2$  and  $G \in C_3$ :

$$\|F\|_\infty := \sup_{0 \leq s \leq t \leq T} |F_{st}|, \quad \|G\|_\infty := \sup_{0 \leq s \leq u \leq t \leq T} |G_{sut}|,$$

and a corresponding weighted version, for  $\tau \in (0, \infty)$ :

$$\|F\|_{\infty, \tau} := \sup_{0 \leq s \leq t \leq T} e^{-\frac{t}{\tau}} |F_{st}|, \quad \|G\|_{\infty, \tau} := \sup_{0 \leq s \leq u \leq t \leq T} e^{-\frac{t}{\tau}} |G_{sut}|. \quad (1.48)$$

Note that

$$\lim_{\tau \rightarrow +\infty} \|F\|_{\infty, \tau} = \|F\|_\infty, \quad \lim_{\tau \rightarrow +\infty} \|G\|_{\eta, \tau} = \|G\|_\eta, \quad \lim_{\tau \rightarrow +\infty} \|H\|_{\eta, \tau} = \|H\|_\eta.$$

We have

$$\|F\|_{\eta, \tau} \leq \|G\|_{\infty, \tau} \|H\|_\eta, \quad (F_{sut} = G_{su} H_{ut}), \quad (1.49)$$

Note that  $\|\cdot\|_{\eta, \tau}$  is only a semi-norm on  $C_n^\eta$  if  $\tau < T$ ; we have at least

$$\|\cdot\|_{\eta, \tau} \leq \|\cdot\|_\eta \leq e^{\frac{T}{\tau}} \left( \|\cdot\|_{\eta, \tau} + \frac{1}{\tau^\eta} \|\cdot\|_{\infty, \tau} \right). \quad (1.50)$$

However, if  $\tau \geq T$  we have again equivalence of norms

$$\|\cdot\|_{\eta, \tau} \leq \|\cdot\|_\eta \leq e^{\frac{T}{\tau}} \|\cdot\|_{\eta, \tau}, \quad \tau \geq T. \quad (1.51)$$