Ten lectures on rough paths
(work in progress)

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Chapter 1
The Sewing Lemma

Given two continuous functions $X, Y: [0,T] \to \mathbb{R}$, the integral

$$\int_0^T Y_r \, \mathrm{d}X_r \tag{1.1}$$

can be defined as $\int_0^T Y_r \, \dot{X}_r \, \mathrm{d}r$ when X is differentiable or, more generally, as a Lebesgue integral when X is of bounded variation (so that $\mathrm{d}X$ is a signed measure). The key question we want to address is: *how to define the integral when X is neither differentiable nor of bounded variation?* This is an example of a more general problem: given a distribution $\dot{X}$ and a non-smooth function Y, how to define their product $Y\dot{X}$?

A motivation is given by $X = B$ with $(B_t)_{t \geq 0}$ a Brownian motion. In this special case, one can use probability theory to answer the question and define the integral in (1.1), but one sees that there are several possible definitions: for example Itô, Stratonovich, etc.

We are going to present the alternative answer provided by the theory of Rough Paths, originally introduced by Terry Lyons. This provides a robust construction of the integral in (1.1) and sheds a new "pathwise" light on stochastic integration.

The approach we follow is based on the *Sewing Lemma*, to which this chapter is devoted. In particular, we will show in Section 2.2 that the integral in (1.1) has a canonical definition (*Young integral*) when Y and X are Hölder continuous, under a constraint on their Hölder exponents. Going beyond this constraint requires Rough Paths, which will be studied in Chapter 5.

1.1. Local approximation

If X is of class C^1, we can define the integral function

$$I_t := \int_0^t Y_r \, \dot{X}_r \, \mathrm{d}r, \qquad t \in [0,T].$$

Then we have $I_0 = 0$ and for $0 \leqslant s \leqslant t \leqslant T$

$$I_t - I_s - Y_s (X_t - X_s) = \int_s^t (Y_r - Y_s) \, \dot{X}_r \, \mathrm{d}r = o(t-s)$$

as $t - s \to 0$, because $\dot{X}$ is bounded and $|Y_r - Y_s| = o(1)$ as $|r - s| \to 0$. Thus the integral function I_t satisfies

$$I_0 = 0, \qquad I_t - I_s = Y_s (X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T\,. \tag{1.2}$$

Remarkably, *this relation characterizes* $(I_t)_{t\in[0,T]}$. Indeed, if I^1 and I^2 satisfy (1.2) with the same functions X, Y, their difference $\Delta := I^1 - I^2$ satisfies

$$|\Delta_t - \Delta_s| = o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T,$$

which implies $\frac{\mathrm{d}}{\mathrm{d}t}\Delta_t \equiv 0$ and then $\Delta_t = \Delta_0 = 0$. This simple result deserves to be stated in a separate

LEMMA 1.1. *Given any functions* $X, Y: [0,T] \to \mathbb{R}$*, there can be at most one function* $I: [0,T] \to \mathbb{R}$ *satisfying (1.2).*

The formulation (1.2) is interesting also because the derivative $\dot{X}$ of X does not appear. Therefore, if we can find a function $I: [0,T] \to \mathbb{R}$ which satisfies (1.2), such a function is *unique* and we can take it as a *definition* of the integral (1.1).

We will see in Section 2.2 that this program can be accomplished when X and Y satisfy suitable Hölder regularity assumptions. In order to get there, in the next sections we will look at a more general problem.

Remark 1.2. Whenever we write $o(t-s)$ we always mean *uniformly for* $0 \leqslant s \leqslant t \leqslant T$, i.e.

$$\forall \epsilon > 0 \ \exists \delta > 0: \qquad 0 \leqslant s \leqslant t \leqslant T, \quad t - s \leq \delta \quad \text{implies} \quad |o(t-s)| \leq \epsilon (t-s)\,.$$

This will be implicitly assumed in the sequel.

1.2. A GENERAL PROBLEM

Let us generalise the problem (1.2). If we define for $n \geqslant 1$

$$[0,T]^n_{\leqslant} := \{(t_1, \ldots, t_n): 0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T\},$$

$$A: [0,T]^2_{\leqslant} \to \mathbb{R}, \qquad A_{st} := Y_s\,(X_t - X_s)\,, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.3}$$

we can decouple (1.2) in two relations

$$I_0 = 0, \qquad I_t - I_s = A_{st} + R_{st}\,, \tag{1.4}$$

$$R: [0,T]^2_{\leqslant} \to \mathbb{R}, \qquad R_{st} = o(t-s)\,. \tag{1.5}$$

The general problem is, given any continuous $A: [0,T]^2_{\leqslant} \to \mathbb{R}$, to find a pair of functions (I, R) satisfying (1.4)-(1.5). We call

- $A: [0,T]^2_{\leqslant} \to \mathbb{R}$ the *germ*,
- $I: [0,T] \to \mathbb{R}$ the *integral*,
- $R: [0,T]^2_{\leqslant} \to \mathbb{R}$ the *remainder*.

We are going to present conditions which allow to solve this problem.

Note that *we always have uniqueness*. Indeed, given (I^1, R^1) and (I^2, R^2) which solve (1.4)-(1.5) for the same A, by the same arguments which lead to Lemma 1.1 we have $\frac{\mathrm{d}}{\mathrm{d}t}(I_t^1 - I_t^2) \equiv 0$, hence $I^1 = I^2$ and then $R^1 = R^2$ by (1.4). We record this as

LEMMA 1.3. *Given any germ A, there can be at most one pair of functions (I, R) satisfying (1.4)-(1.5).*

We are going to work with continuous functions, so we define for $k \geq 1$

$$C_k := \{F: [0,T]_{\leqslant}^k \to \mathbb{R}: \quad F \text{ is continuous}\}.$$

We will actually only need the spaces C_1, C_2, C_3.

1.3. AN ALGEBRAIC LOOK

We first focus on relation (1.4) alone. For a fixed germ A, this equation has infinitely many solutions (I, R), because given *any* I we can simply *define* R so as to fulfill (1.4). Interestingly, all solutions admit an algebraic characterization in terms of R alone.

LEMMA 1.4. *Fix a function $A \in C_2$.*

1. *If a pair $(I, R) \in C_1 \times C_2$ satisfies (1.4), then R satisfies*

$$R_{st} - R_{su} - R_{ut} = -(A_{st} - A_{su} - A_{ut}), \qquad \forall 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{1.6}$$

2. *Viceversa, given any function $R \in C_2$ which satisfies (1.6), if we set $I_t := A_{0t} + R_{0t}$, the pair $(I, R) \in C_1 \times C_2$ satisfies (1.4).*

Proof. Relation (1.4) clearly implies (1.6), simply because

$$(I_t - I_s) - (I_s - I_u) - (I_t - I_u) = 0\,. \tag{1.7}$$

Viceversa, given R satisfying (1.6), we can define $L_{st} := A_{st} + R_{st}$ so that

$$L_{st} - L_{su} - L_{ut} = 0.$$

Applying this formula to $(s', u', t') = (0, s, t)$, we obtain that $I_t := L_{0t}$ satisfies

$$I_t - I_s = L_{0t} - L_{0s} = L_{st} = A_{st} + R_{st}$$

and the proof is complete because $I_0 := A_{00} + R_{00} = 0$, which follows by (1.6) for $s = u = 0$. □

Relations (1.4) and (1.6) contain operators which deserve an explicit definition:

$$\delta: C_1 \to C_2, \qquad \delta f_{st} := f_t - f_s\,, \tag{1.8}$$

$$\delta: C_2 \to C_3, \qquad \delta F_{sut} := F_{st} - F_{su} - F_{ut}\,. \tag{1.9}$$

Remark 1.5. We note that the maps

$$C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \tag{1.10}$$

satisfy $\delta \circ \delta = 0$, see (1.7). Moreover, for $F \in C_2$, the function $\delta F \in C_3$ measures how much F differs from being the increment δf of some $f \in C_1$. Indeed, *we have $\delta F \equiv 0$ if and only if $F = \delta f$ for some $f \in C_1$* (in other words, (1.10) defines an *exact cochain complex*). The proof of this fact, essentially contained in the proof of Lemma 1.4, is left as an exercise.

We can now rephrase Lemma 1.4 as follows.

PROPOSITION 1.6. *Fix $A \in C_2$. Finding a pair $(I, R) \in C_1 \times C_2$ satisfying (1.4) is equivalent to finding $R \in C_2$ such that*

$$\delta R_{sut} = -\delta A_{sut}, \qquad \forall\, 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{1.11}$$

In the special case $A_{st} = Y_s\, \delta X_{st}$ with $X, Y \in C_1$ as in (1.3), we have

$$\delta A_{sut} = -\delta Y_{su}\, \delta X_{ut}\,. \tag{1.12}$$

Proof. We only need to prove (1.12). When $A_{st} = Y_s\, \delta X_{st}$ we have

$$\begin{aligned} \delta A_{sut} &= Y_s\,(X_t - X_s) - Y_s\,(X_u - X_s) - Y_u\,(X_t - X_u) \\ &= Y_s\,(X_t - X_u) - Y_u\,(X_t - X_u) = -(Y_u - Y_s)\,(X_t - X_u), \end{aligned}$$

which completes the proof. □

1.4. ENTERS ANALYSIS: THE SEWING LEMMA

So far we have analyzed (1.4). We now let (1.5) enter the game, i.e. we look for a pair of functions $(I, R) \in C_1 \times C_2$ which fulfills (1.4)-(1.5), given a (general) germ $A \in C_2$.

We stress that condition (1.5) is essential to ensure *uniqueness*: without it, equation (1.4) admits infinitely many solutions, as discussed before Lemma 1.4. When we couple (1.4) with (1.5), uniqueness is guaranteed by Lemma 1.3, but *existence* is no longer obvious. This is what we now focus on.

We start with a simple necessary condition.

LEMMA 1.7. *For (1.4)-(1.5) to admit a solution, it is necessary that the germ A satisfies*

$$|\delta A_{sut}| = o(t-s), \qquad \textit{for } 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{1.13}$$

Proof. If (1.4) admits a solution, by Proposition 1.6 we have $|\delta A_{sut}| = |\delta R_{sut}|$. If furthermore R satisfies (1.5), we must have

$$|\delta R_{sut}| \leq |R_{st}| + |R_{su}| + |R_{ut}| = o(t-s) + o(u-s) + o(t-u)$$

and the conclusion follows since $|t-u|+|u-s|=|t-s|$. □

Remark 1.8. Choosing $u=s$ in (1.13) we obtain that $-A_{ss}=o(t-s)$, which means that $A_{ss}=0$. Therefore a necessary condition for (1.4)-(1.5) to admit a solution is that *A vanishes on the diagonal of Δ_T^2*.

Remarkably, the necessary condition in Lemma 1.7 is close to being sufficient: it is enough to upgrade $o(x)$ in $O(x^\eta)$ for some $\eta>1$. This is the content of the celebrated *Sewing Lemma*, which we next present. We first introduce some notation.

We denote by $\|\cdot\|_\infty$ the usual supremum norm:

$$\|F\|_\infty := \sup_{x\in[0,T]_\leqslant^k} |F_x| \qquad \text{for} \quad F\in C_k\,.$$

We recall the following notation: for $F,G\in C_k$ and $F,G\geq 0$ we write

$$F\lesssim G \quad \Longleftrightarrow \quad \exists C\in\mathbb{R}_+\colon \quad F_x\leqslant C\,G_x, \quad \forall\, x\in[0,T]_\leqslant^k.$$

Next, given $\eta\in(0,\infty)$, we define the following norms for $F\in C_2$ and $G\in C_3$:

$$\|F\|_\eta := \sup_{(s,t)\in[0,T]_\leqslant^2\colon\, s\neq t} \frac{|F_{st}|}{|t-s|^\eta}\,, \quad \|G\|_\eta := \sup_{(s,u,t)\in[0,T]_\leqslant^3\colon\, s\neq t} \frac{|G_{sut}|}{|t-s|^\eta}\,, \tag{1.14}$$

and we introduce the corresponding function spaces:

$$C_2^\eta := \{F\in C_2\colon \quad \|F\|_\eta<\infty\}\,, \qquad C_3^\eta := \{G\in C_3\colon \quad \|G\|_\eta<\infty\}\,.$$

It can be easily shown that C_2^η and C_3^η endowed with $|\cdot|_\eta$ are Banach spaces.

A finite sequence of ordered points $\mathcal{P}=\{a=t_0<t_1<\ldots<t_k=b\}$ is called *partition* of the interval $[a,b]$. The *cardinality* of a partition $\#\mathcal{P}=k$ is the number of intervals, while its *mesh* $|\mathcal{P}|:=\max_{i=1,\ldots,\#\mathcal{P}}|t_i-t_{i-1}|$ is the largest interval size.

We are now ready to state the Sewing Lemma (Gubinelli [3], Feyel-de La Pradelle [1]). This gives an explicit sufficient condition for the solvability of (1.4)-(1.5) in terms of a key property, that we call *coherence*.

Definition 1.9. (Coherence) *A germ $A\in C_2$ is called* coherent *if, for some $\eta>1$, it satisfies $\delta A\in C_3^\eta$, i.e. $\|\delta A\|_\eta<\infty$. More explicitly*

$$\exists\eta\in(1,\infty)\colon \qquad |\delta A_{sut}|\lesssim|t-s|^\eta\,, \qquad 0\leqslant s\leqslant u\leqslant t\leqslant T. \tag{1.15}$$

Theorem 1.10. (Sewing Lemma) *If a germ $A\in C_2$ is coherent, i.e. it satisfies (1.15) for some $\eta>1$, then there exists a unique pair $(I,R)\in C_1\times C_2$ such that*

$$I_0=0, \qquad I_t-I_s=A_{st}+R_{st}\,, \qquad R_{st}=o(t-s)\,.$$

Moreover:

- *The* integral $I \in C_1$ *is the limit of* Riemann sums *of the germ*

$$I_t := \lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{\#\mathcal{P}-1} A_{t_i t_{i+1}} \tag{1.16}$$

along arbitrary partitions $\mathcal{P}$ *of* $[0,t]$ *with vanishing mesh* $|\mathcal{P}| \to 0$.

- *The* remainder $R \in C_2$, *given by*

$$R_{st} := I_t - I_s - A_{st}\,, \tag{1.17}$$

satisfies $|R_{st}| \lesssim |t-s|^\eta$. *More precisely*

$$\|R\|_\eta \le K_\eta \|\delta A\|_\eta\,, \qquad \textit{where} \qquad K_\eta := (1-2^{1-\eta})^{-1}\,. \tag{1.18}$$

The Sewing Lemma is a cornerstone of the theory of *Rough Paths*, to be introduced in Chapter 5. We will already see in Chapter 2.2 an interesting application to *Young integrals*. The (instructive) proof of Theorem 1.10 is postponed to Section 1.6.

Remark 1.11. For a fixed partition $\mathcal{P}$ of $[0,t]$ we have by (1.17)

$$I_t = \sum_{i=0}^{\#\mathcal{P}-1} A_{t_i t_{i+1}} + \sum_{i=0}^{\#\mathcal{P}-1} R_{t_i t_{i+1}}.$$

Therefore, (1.16) is equivalent to

$$\lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{\#\mathcal{P}-1} R_{t_i t_{i+1}} = 0$$

which is the reason why one wants the remainder R to be small close to the diagonal. The information $R_{st} = o(|t-s|)$ is not enough in general to obtain the existence of (I,R), while a sufficient condition is the quantitative estimate $|R_{st}| \lesssim |t-s|^\eta$.

1.5. THE SEWING MAP

Given a coherent germ A, by Theorem 1.10 we can find an integral I and a remainder R which solve (1.4)-(1.5). We now look closer at the remainder R.

LEMMA 1.12. *In the setting of Theorem 1.10,* the remainder R is a function of δA*: given two coherent germs* A, A' *with* $\delta A = \delta A'$*, the corresponding remainders* R, R' *coincide. Moreover, the map* $\delta A \mapsto R$ *is linear.*

Proof. By Proposition 1.6 we have $\delta(R-R')=\delta(A-A')=0$, hence $R-R'=\delta f$ for some $f\in C_1$ (see Remark 1.5). Both $|R_{st}|$ and $|R'_{st}|$ are $o(|t-s|)$ by (1.5), hence $|f_t-f_s|=o(|t-s|)$. Then f must be constant and $R=R'$. Linearity of the map $\delta A\mapsto R$ is easy. □

Since R is a function of δA, we introduce a specific notation for this map:

$$R=-\Lambda(\delta A)$$

with minus sign for later convenience. Let us describe more precisely this map Λ. Throughout the following discussion, we fix arbitrarily $\eta\in(1,\infty)$.

- *Domain.* The map Λ is defined on δA for coherent germs A, see Definition 1.9. The domain of Λ is then $C_3^\eta\cap\delta C_2$, where we denote by $\delta C_2\subseteq C_3$ the image of the space C_2 under the operator δ in (1.9).
- *Codomain.* The map Λ sends δA to $-R$, and we have $|R_{st}|\lesssim|t-s|^\eta$, see (1.18). A natural choice of codomain for Λ is then C_2^η.
- *Characterization.* In view of Proposition 1.6 and Lemma 1.3, the function $-R=\Lambda(\delta A)$ is characterized by the properties

  $$\delta(-R)_{sut}=\delta A\,,\qquad |R_{st}|=o(|t-s|)\,.$$

  The second condition is already enforced by our choice C_2^η of codomain for Λ, which yields $|R_{st}|\lesssim|t-s|^\eta$. The first relation can be rewritten as $\delta(\Lambda(B))=B$ for all B in the domain of Λ, that is $\delta\circ\Lambda$ *is the identity map*.

In conclusion, we have proved the following result.

Theorem 1.13. (Sewing Map) *Let $\eta\in(1,\infty)$. There exists a unique map*

$$\Lambda\colon C_3^\eta\cap\delta C_2\longrightarrow C_2^\eta,$$

called the Sewing Map*, such that $\delta\circ\Lambda=\mathrm{id}$ is the identity on $C_3^\eta\cap\delta C_2$.*

- *The map Λ is linear and satisfies*

  $$\|\Lambda(B)\|_\eta\leqslant K_\eta\|B\|_\eta\qquad\forall B\in C_3^\eta\cap\delta C_2\,,\tag{1.19}$$

  where K_η is the same constant as in (1.18).
- *Given a coherent germ $A\in C_2$, i.e. $\delta A\in C_3^\eta$, the unique solution (I, R) of (1.4)-(1.5) is obtained by $R:=-\Lambda(\delta A)$ and $I_t:=A_{0t}+R_{0t}$.*

1.6. Proof of the Sewing Lemma

We prove Theorem 1.10.

Proof. We follow [2], page 6.

Step 1: construction of I. Let us fix $0 \leqslant s < t \leqslant T$. For a partition $\mathcal{P} = \{s = t_0 < t_1 < \ldots < t_n = t\}$ of $[s,t]$ with n points, we define

$$I_{\mathcal{P}} := \sum_{i=0}^{n-1} A_{t_i t_{i+1}} .$$

If there are $n \geq 2$ intervals in $\mathcal{P}$, there must exist $i \in \{1, \ldots, n-1\}$ such that $|t_{i+1} - t_{i-1}| \leq \frac{2}{n-1} |t-s|$. Indeed, if this is not the case, we must have

$$2\,|t-s| \geq \sum_{i=1}^{n-1} |t_{i+1} - t_{i-1}| > \sum_{i=1}^{n-1} \frac{2}{n-1} |t-s| = 2\,|t-s| .$$

Removing the point t_i from $\mathcal{P}$ yields a partition $\mathcal{P}'$ of $n-1$ intervals, for which

$$\begin{aligned} |I_{\mathcal{P}} - I_{\mathcal{P}'}| &= |A_{t_{i-1} t_i} + A_{t_i t_{i+1}} - A_{t_{i-1} t_{i+1}}| = |\delta A_{t_{i-1} t_i t_{i+1}}| \\ &\leqslant \|\delta A\|_\eta \frac{2\,|t-s|^\eta}{(n-1)^\eta} . \end{aligned} \tag{1.20}$$

Iterating this argument, until we arrive at the trivial partition $\{s,t\}$, we get

$$|I_{\mathcal{P}} - A_{st}| \leq C_\eta \, \|\delta A\|_\eta \, |t-s|^\eta , \qquad \text{with} \qquad C_\eta := \sum_{m \geq 1} \frac{2}{m^\eta} < \infty , \tag{1.21}$$

because $\eta > 1$. Similarly, if $\mathcal{Q} \supseteq \mathcal{P}$ is another partition of $[s,t]$,

$$|I_{\mathcal{Q}} - I_{\mathcal{P}}| \leq \sum_{i=0}^{\#\mathcal{P}-1} |I_{\mathcal{Q} \cap [t_i, t_{i+1}]} - A_{t_i t_{i+1}}| \leq C_\eta \, \|\delta A\|_\eta \sum_{i=0}^{\#\mathcal{P}-1} |t_{i+1} - t_i|^\eta$$

$$\leq C_\eta \, \|\delta A\|_\eta \, |\mathcal{P}|^{\eta-1} \sum_{i=0}^{\#\mathcal{P}-1} |t_{i+1} - t_i| \leq C_\eta \, \|\delta A\|_\eta \, T \, |\mathcal{P}|^{\eta-1} .$$

Finally, if $\mathcal{P}$ and $\mathcal{P}'$ are arbitrary partitions, setting $\mathcal{Q} := \mathcal{P} \cup \mathcal{P}'$ and applying the triangle inequality yields $|I_{\mathcal{P}'} - I_{\mathcal{P}}| \leq 2\, C_\eta \, \|\delta A\|_\eta \, T \, |\mathcal{P}|^{\eta-1}$. This means that the family $I_{\mathcal{P}}$ is Cauchy (for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $|\mathcal{P}|, |\mathcal{P}'| \leq \delta_\epsilon$ implies $|I_{\mathcal{P}'} - I_{\mathcal{P}}| \leq \epsilon$), hence it admits a limit as $|\mathcal{P}| \to 0$, that we call J_{st}. We note that J_{st} is only defined for $0 \leq s < t \leq T$.

We now *define* $I_t := J_{0t}$. We claim that

$$I_t - I_s = J_{st} \qquad \text{for all } 0 \leq s < t \leq T .$$

Indeed, if we consider partitions $\mathcal{P}'$ on $[0,s]$ and $\mathcal{P}$ of $[s,t]$, so that $\mathcal{P}'' := \mathcal{P} \cup \mathcal{P}'$ is a partition of $[0,t]$, then $I_{\mathcal{P}''} - I_{\mathcal{P}'} = I_{\mathcal{P}}$ and taking the limit of vanishing mesh we get the claim. Taking the limit of relation (1.21), since $I_{\mathcal{P}} \to I_t - I_s$, we obtain

$$|R_{st}| \leq C_\eta \, \|\delta A\|_\eta \, |t-s|^\eta , \qquad R_{st} := \delta I_{st} - A_{st} , \qquad 0 \leqslant s < t \leqslant T.$$

Therefore (1.18) holds, with K_η replaced by the worse constant C_η. This is because the estimate (1.21) holds for arbitrary partitions.

Step 2: Sewing bound with optimal constant. If we choose the sequence of dyadic partitions $\mathcal{P}_n := \{t_i^n := s + \frac{i}{2^n}(t-s) \colon 0 \le i \le 2^n\}$ of $[s,t]$ of order n, the arguments above give the sewing bound with the better constant $K_\eta = (1-2^{1-\eta})^{-1}$ instead of C_η. Indeed, let again $s<t$. If we remove all the "odd points" t_{2j+1}^n with $0 \le j \le 2^{n-1}-1$ from $\mathcal{P}_n$, we obtain $\mathcal{P}_{n-1}$. Then, in analogy with (1.20), we have for $n \ge 1$

$$\begin{aligned}|I_{\mathcal{P}_n} - I_{\mathcal{P}_{n-1}}| \le \sum_{j=0}^{2^{n-1}-1} |\delta A_{t_{2j}^n t_{2j+1}^n t_{2j+2}^n}| &\le 2^{n-1} \, \|\delta A\|_{\eta,[s,t]} \left(\frac{2\,|t-s|}{2^n}\right)^\eta \\ &= 2^{-(\eta-1)(n-1)} \|\delta A\|_{\eta,[s,t]} \, |t-s|^\eta ,\end{aligned}$$

where we set (also for future use)

$$\|\delta A\|_{\eta,[s,t]} := \sup_{a,b,c \in [s,t] \colon a \leqslant b \leqslant c, a<c} \frac{|\delta A_{abc}|}{|c-a|^\eta}. \tag{1.22}$$

Since $\mathcal{P}_0 = \{s,t\}$, we have $I_{\mathcal{P}_0} = A_{st}$ and we obtain for any $k \in \mathbb{N}$

$$\begin{aligned}|I_{\mathcal{P}_k} - A_{st}| &= |I_{\mathcal{P}_k} - I_{\mathcal{P}_0}| \\ &\le \sum_{n=1}^{k} |I_{\mathcal{P}_n} - I_{\mathcal{P}_{n-1}}| \\ &\leqslant \|\delta A\|_{\eta,[s,t]} \, |t-s|^\eta \sum_{n=1}^{k} 2^{-(\eta-1)(n-1)} \\ &\leqslant (1-2^{1-\eta})^{-1} \, \|\delta A\|_{\eta,[s,t]} \, |t-s|^\eta ,\end{aligned}$$

because $\sum_{n=1}^\infty 2^{-(\eta-1)(n-1)} = (1-2^{1-\eta})^{-1}$. For $k \to \infty$ we have $I_{\mathcal{P}_k} \to I_t - I_s$ and

$$|\delta I_{st} - A_{st}| \leqslant (1-2^{1-\eta})^{-1} \, \|\delta A\|_{\eta,[s,t]} \, |t-s|^\eta, \qquad s<t. \tag{1.23}$$

Since $\|\delta A\|_{\eta,[s,t]} \leqslant \|\delta A\|_\eta$, (1.18) is proven. □

1.7. Norms and distances

We collect here all the main definitions and properties of the different norms and distances we use in what follows. We fix $T>0$ and $k,d \in \mathbb{N}$ and we work on function spaces

$$C_k := \{F \colon [0,T]_\leqslant^k \to \mathbb{R}^d \colon F \text{ is continuous}\}.$$

For the convenience of the reader, we recall here some definitions already given in the previous sections. We consider for $F \in C_k$

$$\|F\|_\infty := \sup_{x \in [0,T]_\leqslant^k} |F_x|.$$

Next, given $\eta \in (0,\infty)$, we define for $G \in C_2$ and $H \in C_3$

$$\|G\|_\eta := \sup_{(s,t)\in[0,T]^2_\leqslant:\, s\neq t} \frac{|G_{st}|}{|t-s|^\eta}, \quad \|H\|_\eta := \sup_{(s,u,t)\in[0,T]^3_\leqslant:\, s\neq t} \frac{|H_{sut}|}{|t-s|^\eta},$$

and corresponding function spaces

$$C_2^\eta := \{G \in C_2 \colon \|G\|_\eta < \infty\}, \qquad C_3^\eta := \{H \in C_3 \colon \|H\|_\eta < \infty\}.$$

Now, we fix $\tau > 0$ and we introduce modified versions of the norms defined above: we set, for $F \in C_k$, $G \in C_2^\eta$ and $H \in C_3^\eta$,

$$\|F\|_{\infty,\tau} := \sup_{x=(x_1,\ldots,x_k)\in[0,T]^k_\leqslant} \exp\left(-\frac{x_k}{\tau}\right)|F_x|$$

$$\|G\|_{\eta,\tau} := \sup_{0\leqslant s\leqslant t\leqslant T} \mathbf{1}_{(0<|t-s|\leqslant\tau)} \exp\left(-\frac{t}{\tau}\right) \frac{|G_{st}|}{|t-s|^\eta},$$

$$\|H\|_{\eta,\tau} := \sup_{0\leqslant s\leqslant u\leqslant t\leqslant T} \mathbf{1}_{(0<|t-s|\leqslant\tau)} \exp\left(-\frac{t}{\tau}\right) \frac{|H_{sut}|}{|t-s|^\eta}.$$

Note in particular that

$$\lim_{\tau\to+\infty} \|F\|_{\infty,\tau} = \|F\|_\infty, \qquad \lim_{\tau\to+\infty} \|G\|_{\eta,\tau} = \|G\|_\eta, \qquad \lim_{\tau\to+\infty} \|H\|_{\eta,\tau} = \|H\|_\eta.$$

Also note that $\|\cdot\|_{\infty,\tau}$ is a norm on C_k, and we have the equivalence of norms

$$\|\cdot\|_{\infty,\tau} \leqslant \|\cdot\|_\infty \leqslant e^{\frac{T}{\tau}} \|\cdot\|_{\infty,\tau}. \tag{1.24}$$

On the other hand, $\|\cdot\|_{\eta,\tau}$ is only a semi-norm on C_k^η if $\tau < T$; we have at least

$$\|\cdot\|_{\eta,\tau} \leqslant \|\cdot\|_\eta \leqslant e^{\frac{T}{\tau}} \left(\|\cdot\|_{\eta,\tau} + \frac{1}{\tau^\eta} \|\cdot\|_{\infty,\tau} \right). \tag{1.25}$$

However, if $\tau \geq T$ we have again equivalence of norms

$$\|\cdot\|_{\eta,\tau} \leqslant \|\cdot\|_\eta \leqslant e^{\frac{T}{\tau}} \|\cdot\|_{\eta,\tau}, \qquad \tau \geq T. \tag{1.26}$$

Remark 1.14. The norms $\|\cdot\|_{\eta,\tau}$ are very useful to transform *local* results in *global* results: indeed, using the norms $\|\cdot\|_\eta$ requires sometimes the size $T > 0$ of the time interval $[0,T]$ to be *small*, which can be annoying. The norms $\|\cdot\|_{\eta,\tau}$ allow to keep $T > 0$ arbitrary by choosing a sufficiently small $\tau > 0$.

We now relate the different norms introduced so far.

Lemma 1.15. *For $G \in C_k$ and $k \in \{2,3\}$ we have*

$$\|G\|_{\eta,\tau} \leqslant (\tau \wedge T)^{\eta'-\eta} \|G\|_{\eta',\tau}, \qquad \eta' \geq \eta. \tag{1.27}$$

Moreover for $G \in C_1$ we have

$$\|G\|_\infty \leqslant |G_0| + T^\eta \|\delta G\|_\eta, \qquad \eta > 0, \tag{1.28}$$

$$\|G\|_{\infty,\tau} \leqslant |G_0| + 3(\tau \wedge T)^\eta \|\delta G\|_{\eta,\tau}, \qquad \eta > 0. \tag{1.29}$$

Proof. Let us first prove (1.27). For $G \in C_2^\eta$, we have

$$\begin{aligned}\|G\|_{\eta,\tau} &= \sup_{0 \leqslant s < t \leqslant T} \mathbf{1}_{(0 < |t-s| \leqslant \tau)} \exp\left(-\frac{t}{\tau}\right) |t-s|^{\eta'-\eta} \frac{|G_{st}|}{|t-s|^{\eta'}} \\ &\leqslant (T \wedge \tau)^{\eta'-\eta} \|G\|_{\eta',\tau}.\end{aligned}$$

The case $G \in C_3^\eta$ is analogous. Let us prove now (1.28): for any $G \in C_1$ and for $t \in \,]0, T]$ we have

$$|G_t| \leqslant |G_0| + |G_t - G_0| = |G_0| + t^\eta \frac{|G_t - G_0|}{t^\eta} \leqslant |G_0| + T^\eta \|\delta G\|_\eta.$$

The proof of (1.29) is slightly more complicated. If $t \in \,]0, \tau \wedge T]$, then

$$e^{-\frac{t}{\tau}} |G_t| \leqslant |G_0| + t^\eta e^{-\frac{t}{\tau}} \frac{|G_t - G_0|}{t^\eta} \leqslant |G_0| + (\tau \wedge T)^\eta \, \|\delta G\|_{\eta,\tau}.$$

Suppose now that $\tau < t \leqslant T$ and let $N := \min\{n \in \mathbb{N}: n\tau \geq t\} \geq 2$, so that $\frac{t}{N} \leqslant \tau$. We set $t_k = k \frac{t}{N}$, $k \geq 0$, so that $t_N = t$. Then

$$\begin{aligned}e^{-\frac{t}{\tau}} |G_t| &\leqslant |G_0| + \sum_{k=0}^{N-1} e^{-\frac{t - t_{k+1}}{\tau}} e^{-\frac{t_{k+1}}{\tau}} \frac{|G_{t_{k+1}} - G_{t_k}|}{(t_{k+1} - t_k)^\eta} (t_{k+1} - t_k)^\eta \\ &\leqslant |G_0| + (\tau \wedge T)^\eta \|\delta G\|_{\eta,\tau} \sum_{k=0}^{N-1} e^{-\frac{t - t_{k+1}}{\tau}}.\end{aligned}$$

By definition of N we have $(N-1)\tau < t$; since $\tau < t$ we obtain $N\tau < 2t$ and therefore $\frac{t}{N\tau} \geq \frac{1}{2}$. Since $t - t_{k+1} = (N - k - 1)\frac{t}{N}$, renaming $\ell := N - k - 1$ we obtain

$$\sum_{k=0}^{N-1} e^{-\frac{t - t_{k+1}}{\tau}} = \sum_{\ell=0}^{N-1} e^{-\ell \frac{t}{N\tau}} = \frac{1 - e^{-\frac{t}{\tau}}}{1 - e^{-\frac{t}{N\tau}}} \leqslant \frac{1}{1 - e^{-\frac{1}{2}}} \leqslant 3.$$

The proof is complete. □

LEMMA 1.16. *If $F_{st} = G_s H_{st}$ with $G \in C_1$ and $F, H \in C_2$,*

$$\|F\|_{\eta,\tau} \leqslant \|G\|_{\infty,\tau} \|H\|_\eta, \qquad (F_{st} = G_s H_{st}). \tag{1.30}$$

Analogously, if $F_{sut} = G_{su} H_{ut}$ with $G, H \in C_2$ and $F \in C_3$,

$$\|F\|_{\eta,\tau} \leqslant \|G\|_{\infty,\tau} \|H\|_\eta, \qquad (F_{sut} = G_{su} H_{ut}), \tag{1.31}$$

$$\|F\|_{\eta'+\eta,\tau} \leqslant \|G\|_{\eta',\tau} \|H\|_\eta, \qquad (F_{sut} = G_{su} H_{ut}). \tag{1.32}$$

Proof. For $F_{st} = G_s H_{st}$, we have

$$\|F\|_\eta = \sup_{0 \leqslant s < t \leqslant T} \mathbf{1}_{(0 < |t-s| \leqslant \tau)} \exp\left(-\frac{t}{\tau}\right) \frac{|G_s H_{st}|}{|t-s|^\eta}$$

$$\leqslant \left[\sup_{s \in [0,T]} \exp\left(-\frac{s}{\tau}\right) |G_s|\right] \left[\sup_{0 \leqslant s < t \leqslant T} \frac{|H_{st}|}{|t-s|^\eta}\right] = \|G\|_{\infty,\tau} \|H\|_\eta.$$

The other cases are analogous. □

We have a version of the estimate (1.19) for the Sewing map in our weighted spaces. We recall that $\|\cdot\|_{\eta,\tau}$ is a semi-norm rather than a norm.

LEMMA 1.17. *Let $\eta > 1$ and $\tau > 0$. For all $B \in C_3^\eta \cap \delta C_2$*

$$\|\Lambda B\|_{\eta,\tau} \leqslant K_\eta \|B\|_{\eta,\tau}. \tag{1.33}$$

Proof. Fix $s, t \in [0,T]$ with $s \leqslant t$. We know that $|\Lambda B_{st}| \leqslant K_\eta \|B\|_{\eta,[s,t]} |t-s|^\eta$ by (1.23). Let us suppose that $t - s \leqslant \tau$, then

$$e^{-\frac{t}{\tau}} \frac{|\Lambda B_{st}|}{|t-s|^\eta} \leqslant e^{-\frac{t}{\tau}} K_\eta \|B\|_{\eta,[s,t]} \leqslant K_\eta \|B\|_{\eta,\tau}\,.$$

The proof is complete. □

Chapter 2

The Young integral

We can finally come back to the problem we explained at the beginning of Chapter 1: given two continuous functions $X, Y: [0,T] \to \mathbb{R}$, we look for a function $I: [0,T] \to \mathbb{R}$ satisfying

$$I_0 = 0, \qquad I_t - I_s = Y_s (X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T\,. \tag{2.1}$$

This is equivalent to look for a pair of functions (I, R) satisfying

$$I_0 = 0, \qquad I_t - I_s = A_{st} + R_{st}\,, \tag{2.2}$$

$$R: [0,T]^2_{\leqslant} \to \mathbb{R}, \qquad R_{st} = o(t-s)\,. \tag{2.3}$$

with the germ $A_{st} = Y_s\, \delta X_{st}$. Recalling that $\delta A_{sut} = -\delta Y_{su}\, \delta X_{ut}$ by (1.12), we have for any $\alpha, \beta > 0$

$$|\delta A_{sut}| = |Y_u - Y_s||X_t - X_u| \quad \Longrightarrow \quad \|\delta A\|_{\alpha+\beta} \leqslant \|\delta X\|_{\alpha} \|\delta Y\|_{\beta}\,. \tag{2.4}$$

2.1. Hölder functions

In order to fulfill condition the condition (1.15) of the Sewing Lemma, it is natural to assume that the estimate (2.4) holds for $\alpha, \beta \in\,]0,1]$ such that $\alpha + \beta > 1$. We remark here that the space

$$\mathcal{C}^\alpha := \{f: [0,T] \to \mathbb{R}: \quad \|\delta f\|_\alpha < \infty\}, \qquad \alpha \in\,]0,1],$$

$$\text{where} \qquad \|\delta f\|_\alpha = \sup_{0 \leq s < t \leq T} \frac{|f_t - f_s|}{|t-s|^\alpha}\,,$$

is the classical space of Hölder functions with exponent α. For $\alpha = 1$ we have the usual space of Lipschitz functions. Moreover we recall that for $\alpha > 1$ the only functions $f \in C_1$ such that $\|\delta f\|_\alpha < +\infty$ are constant. Indeed we have the elementary

Lemma 2.1. *If $f: [0,T] \to \mathbb{R}$ and for some $c \geqslant 0$ the bound*

$$|f_t - f_s| \leq c|t-s|^\alpha, \qquad s, t \in [0,T],$$

holds for some $\alpha > 1$, then f is constant. More generally, if $|f_t - f_s| = o(|t-s|)$ uniformly as $|t-s| \to 0$, then f is constant.

Proof. By assumption, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|t - s| \leq \delta$ implies $|f(t) - f(s)| \leq \epsilon |t - s|$. Fix $[a, b] \subseteq [0, T]$. If $a = t_0 < t_1 < \ldots < t_n = b$ is a partition of $[a, b]$ with $(t_{i+1} - t_i) \leq \delta$, then $|f(t_{i+1}) - f(t_i)| \leq \epsilon |t_{i+1} - t_i|$ and we can write

$$|f(b) - f(a)| \leq \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| \leq \epsilon \sum_{i=0}^{n-1} |t_{i+1} - t_i| = \epsilon(b - a)$$

and sending $\epsilon \to 0$ we get $f(b) = f(a)$. Since $[a, b] \subseteq [0, T]$ was arbitrary, f is constant. □

The standard norm on $\mathcal{C}^\alpha$ is

$$\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha. \tag{2.5}$$

Let us denote by C^∞ the space of infinitely differentiable functions. Note that $C^\infty \subset C^\alpha$ for every $\alpha \in (0, 1)$, but C^∞ *is not dense in* C^α.

THEOREM 2.2. *The closure of C^∞ in C^α is the subset C_0^α defined by*

$$C_0^\alpha := \{f : |f(t) - f(s)| = o(|t - s|^\alpha) \text{ uniformly as } |t - s| \to 0\}.$$

Note that $f \in C_0^\alpha$ if and only if

$$\forall \epsilon > 0 \quad \exists \delta_\epsilon > 0: \qquad |f(t) - f(s)| \leq \epsilon |t - s|^\alpha \qquad \text{for } |t - s| \leq \delta_\epsilon. \tag{2.6}$$

Also note that the closure of C^1 in C^α is again C_0^α, simply because $C^\infty \subset C^1 \subset C_0^\alpha$.

A key tool for Theorem 2.2 is the following classical approximation result.

LEMMA 2.3. *For any continuous $f: [0, T] \to \mathbb{R}$ there is a sequence $f_n \in C^\infty$ such that $\|f_n - f\|_\infty \to 0$. One can take f_n with the same modulus of continuity as f, that is:*

$$\begin{array}{rll} \textit{if} & |f(t) - f(s)| \leq h(t - s) & \forall s, t \in [0, T], \\ \textit{then} & |f_n(t) - f_n(s)| \leq h(t - s) & \forall n \in \mathbb{N}, \forall s, t \in [0, T], \end{array} \tag{2.7}$$

where $h(\cdot)$ is arbitrary. It follows that $\|\delta f_n\|_\alpha \leq \|\delta f\|_\alpha$ for all $n \in \mathbb{N}$ and $\alpha \in (0, 1)$.

Remark 2.4. Lemma 2.3 holds with no change for functions $f: [0, T] \to R$, where R is an arbitrary Banach space. One only needs a notion of integral $\int_0^T f_s \, \mathrm{d}s$ when f is continuous, and for this one can take the Riemann integral, i.e. the limit of Riemann sums $\sum_i f(t_i)\,(t_{i+1} - t_i)$ along partitions (t_i) of $[0, T]$ with vanishing mesh $\max_i |t_{i+1} - t_i| \to 0$ (one can check that such Riemann sums form a Cauchy family). This integral satisfies the key usual properties: $f \mapsto \int_0^T f_s \, \mathrm{d}s$ is linear, $|\int_0^T f_s \, \mathrm{d}s| \leq \int_0^T |f_s| \, \mathrm{d}s$ and $\int_0^T f_s' \, \mathrm{d}s = f_T - f_0$.

We stress that C_0^α is strictly included in C^α, but what is left out is not so large.

Exercise 2.1. $C^{\alpha+\epsilon} \subset C_0^\alpha \subset C^\alpha$ for any $\epsilon > 0$ (inclusions are strict).

2.2. THE YOUNG INTEGRAL

As a corollary of Theorem 1.10, we obtain the existence of the *Young integral*, which provides a consistent definition of the integral (1.1) in this setting.

THEOREM 2.5. (YOUNG INTEGRAL) *Fix $\alpha, \beta \in\,]0,1]$ with $\alpha + \beta > 1$. For every $(X, Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ there is a (necessarily unique) function $I: [0, T] \to \mathbb{R}$ which satisfies (1.2), i.e.*

$$I_0 = 0, \qquad I_t - I_s = Y_s\,(X_t - X_s) + o(|t - s|)\,. \tag{2.8}$$

The functon I is called the Young integral *and is denoted $I_t = \int_0^t Y \mathrm{d}X$.*

The remainder $R_{st} := I_t - I_s - Y_s\,(X_t - X_s)$ *satisfies the bound*

$$\|R\|_{\alpha+\beta} \leq K_{\alpha+\beta}\, \|\delta X\|_\alpha\, \|\delta Y\|_\beta\,.$$

It follows that $I \in \mathcal{C}^\alpha$, more precisely

$$\|\delta I\|_\alpha \leq (\|Y\|_\infty + K_{\alpha+\beta} T^\beta \, \|\delta Y\|_\beta)\, \|\delta X\|_\alpha\,. \tag{2.9}$$

Proof. We recall that $\delta A_{sut} = -\delta Y_{su}\, \delta X_{ut}$ by (1.12). Therefore by (2.4), $\delta A \in C_3^\eta$ with $\eta = \alpha + \beta > 1$ and we can apply Theorem 1.10. In order to prove (2.9) we note that

$$\begin{aligned} \|\delta I\|_\alpha &\leqslant \|A\|_\alpha + \|R\|_\alpha \leqslant \|Y\|_\infty \|\delta X\|_\alpha + T^\beta \|R\|_{\alpha+\beta} \\ &\leqslant \|Y\|_\infty \|\delta X\|_\alpha + T^\beta K_{\alpha+\beta}\, \|\delta X\|_\alpha\, \|\delta Y\|_\beta\,. \end{aligned}$$

This concludes the proof. □

Remark 2.6. The setting of Theorem 2.5 provides a natural example of a germ $A_{st} := Y_s\, \delta X_{st}$ which is *not* in C_2^η for any $\eta > 1$ (excluding the trivial case when $Y \equiv 0$ on the intervals where X is not constant, hence $A \equiv 0$), but it satisfies $\delta A \in C_3^\eta$ with $\eta = \alpha + \beta > 1$.

Remark 2.7. It is natural to wonder what happens in Theorem 2.5 for $(X, Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ with $\alpha + \beta \leq 1$. In this case, *there might be no solution to (1.4)-(1.5)*, because the necessary condition (1.13) in Lemma 1.7 can fail. For instance, if we consider $X_t = t^\alpha$ and $Y_t = t^\beta$, $t \in [0, T]$, we note that for $s = 0$ and $u = \frac{t}{2}$ we have by (1.12)

$$|\delta A_{sut}| = |\delta A_{0\frac{t}{2}t}| = \left|\delta Y_{0\frac{t}{2}}\right| \left|\delta X_{\frac{t}{2}t}\right| = \left(\frac{t}{2}\right)^\beta \left(t^\alpha - \left(\frac{t}{2}\right)^\alpha\right) \gtrsim t^{\alpha+\beta}, \tag{2.10}$$

which is not $o(|t - s|) = o(t)$ for $\alpha + \beta \leq 1$.

In order to define a notion of integral (1.1) when $\alpha+\beta\leq 1$, we are going to relax condition (1.2) in Definition 5.1. This will lead to the notion of *Rough Paths*, described in Chapter 5.

2.3. PROPERTIES OF THE YOUNG INTEGRAL

The Young integral, defined in Theorem 2.5, enjoys several properties which are similar to those of the classical Riemann-Lebesgue integral. One of them is an *integration by parts formula*, which follows by the uniqueness of the solution for the problem (1.4)-(1.5), recall Lemma 1.3.

PROPOSITION 2.8. (INTEGRATION BY PARTS) *Fix $\alpha,\beta\in\,]0,1]$ with $\alpha+\beta>1$. For all $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$ the Young integral satisfies*

$$\int_0^t X\,\mathrm{d}Y+\int_0^t Y\,\mathrm{d}X = X_t\,Y_t - X_0\,Y_0\,. \tag{2.11}$$

Proof. Let us set d. By the property (2.8) we have

$$I'_t - I'_s = \underbrace{Y_s\,(X_t-X_s)+X_s\,(Y_t-Y_s)}_{A_{st}} + o(|t-s|)\,.$$

Next we set $I''_t := X_t\,Y_t - X_0\,Y_0$ and note that, by direct computation,

$$I''_t - I''_s = \underbrace{Y_s\,(X_t-X_s)+X_s\,(Y_t-Y_s)}_{A_{st}} + \underbrace{(X_t-X_s)\,(Y_t-Y_s)}_{R_{st}}\,,$$

where $|R_{st}|\leqslant\|\delta X\|_\alpha\,\|\delta Y\|_\beta\,|t-s|^{\alpha+\beta}=o(|t-s|)$. By Lemma 1.3, for any germ A, there is at most one function I which satisfies (1.4)-(1.5), hence $I'=I''$. □

The Young integral also satisfies another property of the classical Riemann-Lebesgue integral: the so-called chain rule.

PROPOSITION 2.9. (CHAIN RULE) *Fix $\alpha\in\,]1/2,1]$ and $\gamma\in\,]0,1]$ such that $\gamma>1/(1+\alpha)$. Let $X\in\mathcal{C}^\alpha$ and let $\varphi:\mathbb{R}\to\mathbb{R}$ be differentiable with $\varphi'\in\mathcal{C}^\gamma(\mathbb{R})$. Then $\varphi'\circ X\in\mathcal{C}^{\alpha\gamma}$ and*

$$\varphi(X_t)-\varphi(X_0)=\int_0^t\varphi'(X)\,\mathrm{d}X\,. \tag{2.12}$$

Proof. It is easy to see that $\varphi'\circ g\in\mathcal{C}^{\alpha\gamma}$. Then the right-hand side of (2.12) is well-defined as a Young integral. We see now that (2.12) is equivalent to

$$|\varphi(X_t)-\varphi(X_s)-\varphi'(X_s)\;(X_t-X_s)|\lesssim|t-s|^{\alpha+\beta}$$

By the classical Lagrange theorem, if say $X_t>X_s$, then

$$\varphi(X_t)-\varphi(X_s)-\varphi'(X_s)\;(X_t-X_s)=(\varphi'(\xi)-\varphi'(X_s))\,(X_t-X_s)$$

with $\xi \in]X_s, X_t[$. Therefore since φ' of class $\mathcal{C}^\gamma$

$$|\varphi(X_t) - \varphi(X_s) - \varphi'(X_s)\ (X_t - X_s)| \lesssim |X_t - X_s|^{\gamma+1} \lesssim |t-s|^{\alpha+\alpha\gamma}$$

and $\alpha + \alpha\,\gamma > 1$ by assumption, we can apply the Sewing Lemma and the proof is complete. □

More generally, we have

COROLLARY 2.10. *Let $X \in \mathcal{C}^\alpha$ with $\alpha \in]1/2, 1]$. Then for all $s \leqslant t$*

$$\varphi(X_t) - \varphi(X_s) = \varphi'(X_s)\,(X_t - X_s) + \int_s^t (\varphi'(X_r) - \varphi'(X_s))\,\mathrm{d}X_r\,. \qquad (2.13)$$

Proof. It is enough to note that

$$\begin{aligned}\varphi(X_t) - \varphi(X_s) &= \int_s^t \varphi'(X_r)\mathrm{d}X_r \\ &= \varphi'(X_s)(X_t - X_s) + \int_s^t (\varphi'(X_r) - \varphi'(X_s))\,\mathrm{d}X_r\,,\end{aligned}$$

where all integrals are in the Young sense. □

In particular, for $X \in \mathcal{C}^\alpha$ and $\alpha > 1/2$

$$\frac{X_t^2}{2} - \frac{X_s^2}{2} = X_s\,(X_t - X_s) + \int_s^t (X_r - X_s)\mathrm{d}X_r. \qquad (2.14)$$

Moreover we have obviously

$$\int_s^t (X_r - X_s)\,\mathrm{d}X_r = \frac{X_t^2}{2} - \frac{X_s^2}{2} - X_s\,(X_t - X_s) = \frac{(X_t - X_s)^2}{2}. \qquad (2.15)$$

Exercise 2.2. Show that Proposition 2.9 still holds if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable but the derivative F' is only locally Lipschitz (that is, for every compact interval $[-M, M]$ there exists $C_M < \infty$ such that $|F'(z) - F'(w)| \leq C\,|z - w|$ for all $z, w \in [-M, M]$).

Exercise 2.3. Show that Proposition 2.9 still holds if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable but the derivative F' is only (locally) Hölder of exponent η, provided $\eta > \frac{1-\alpha}{\alpha}$.

We conclude with a simple but important formula.

LEMMA 2.11. *Let $(X, Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ as in Theorem 2.5, with $\alpha, \beta \in (0, 1)$, $\alpha + \beta > 1$, and $I_t := \int_0^t Y_u\,\mathrm{d}X_u$ the Young integral. If we set $R \in C_2$*

$$R_{st} := I_t - I_s - Y_s\,(X_t - X_s), \qquad 0 \leqslant s \leqslant t \leqslant T,$$

then we have the explicit formula

$$R_{st} = \int_s^t (Y_u - Y_s)\,\mathrm{d}X_u, \qquad 0 \leqslant s \leqslant t \leqslant T, \qquad (2.16)$$

where the integral in the RHS is in the Young sense.

Proof. Note that the (constant) function $[s,t] \ni u \mapsto Y_s$ trivially belongs to C^β. By uniqueness of the Young integral, we obtain

$$\int_s^t Y_s \, \mathrm{d}X_u = Y_s \, (X_t - X_s).$$

Then by linearity of the Young integral we obtain

$$I_t - I_s - Y_s \, (X_t - X_s) = \int_s^t Y_u \, \mathrm{d}X_u - \int_s^t Y_s \, \mathrm{d}X_u = \int_s^t (Y_u - Y_s) \, \mathrm{d}X_u.$$

The proof is complete. □

2.4. UNIQUENESS OF THE YOUNG INTEGRAL

If X is continuous and $Y \in C^1$ is continuously differentiable, the classical integral $I_t^{\text{classical}} := \int_0^t X_s \, \dot{Y}_s \, \mathrm{d}s$ satisfies (2.8), as we already remarked. As a consequence, we can view the Young integral $(X, Y) \mapsto I_t^{\text{Young}}$ built in Theorem 2.5 as a *continuous extension to* $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ of the map $(X, Y) \mapsto I_t^{\text{classical}}$ defined on $\mathcal{C}^\alpha \times C^1$, for any fixed $\alpha, \beta \in (0,1)$ with $\alpha + \beta > 1$. It would be tempting to state that I_t^{Young} is the *unique* continuous extension of $I_t^{\text{classical}}$, but this is not true because $C^1 \subset \mathcal{C}^\beta$ *is not dense in* $\mathcal{C}^\beta$.

A way to circumvent this difficulty is to note that C^1 *is dense in* $\mathcal{C}^\beta$ *with respect to the topology of* $\mathcal{C}^{\beta'}$, *for any* $\beta' < \beta$ (observe that $\mathcal{C}^\beta \subset \mathcal{C}^{\beta'}$). Therefore, if we fix $\alpha \in (0,1)$, we can state that $(X, Y) \mapsto I_t^{\text{Young}}$ is the unique continuous extension to $\mathcal{C}^\alpha \times \bigcup_{\beta \in (1-\alpha, 1)} \mathcal{C}^\beta$ of the map $(X, Y) \mapsto I_t^{\text{classical}}$ defined on $\mathcal{C}^\alpha \times C^1$, provided we agree that convergence in $\bigcup_{\beta \in (1-\alpha, 1)} \mathcal{C}^\beta$ means convergence in some $\mathcal{C}^\beta$.

In order to make this precise, we introduce a weaker notion of convergence. The Young integral turns out to be continuous with respect to this notion of convergence.

DEFINITION 2.12. *Fix* $\alpha \in (0,1)$. *Given a sequence* $f_n, f \colon [0,T] \to \mathbb{R}$, *with* $n \in \mathbb{N}$, *we write*

$$f_n \rightsquigarrow_\alpha f \qquad \Longleftrightarrow \qquad \|f_n - f\|_\infty \to 0 \quad \textit{and} \quad \sup_{n \in \mathbb{N}} \|\delta f_n\|_\alpha < \infty \,. \tag{2.17}$$

In other terms, $f_n \rightsquigarrow_\alpha f$ *if and only if* $f_n \to f$ *in the sup-norm and* f_n *is bounded in* C^α.

Exercise 2.4. Fix $\alpha \in (0,1)$. Prove the following.

1. If $f_n \rightsquigarrow_\alpha f$, then $f \in C^\alpha$; more precisely $\|\delta f\|_\alpha \leq \sup_{n \in \mathbb{N}} \|\delta f_n\|_\alpha < \infty$.
2. If $f_n \rightsquigarrow_\alpha f$, then $f_n \to f$ in $C^{\alpha'}$ for any $\alpha' < \alpha$, but not necessarily $f_n \to f$ in C^α.
3. If $f_n \rightsquigarrow_\alpha f$ and $F \colon \mathbb{R} \to \mathbb{R}$ is Lipschitz, then $F(f_n) \rightsquigarrow_\alpha F(f)$.
4. In the definition (2.17) of $f_n \rightsquigarrow_\alpha f$, one can replace uniform convergence $\|f_n - f\|_\infty \to 0$ by pointwise convergence, i.e. $f_n(t) \to f(t)$ for every $t \in [0,T]$.

This notion of convergence provides the following characterization of the Young integral.

Theorem 2.13. *Fix $\alpha, \beta \in (0,1)$ with $\alpha + \beta > 1$. The Young integral $I(f, g) = \int_0^{\cdot} f\,\mathrm{d}g$ is the only operator $I: C^\alpha \times C^\beta \to C^\beta$ which, for $g \in C^1$, is defined by :*

$$I(f,g) = \int_a^{\cdot} f_s\, g'_s\, \mathrm{d}s \tag{2.18}$$

and such that

$$f_n \rightsquigarrow_\alpha f \text{ and } g_n \rightsquigarrow_\beta g \qquad \Longrightarrow \qquad I(f_n, g_n) \rightsquigarrow_\beta I(f,g)\,. \tag{2.19}$$

Proof. We start by proving the desired continuity property. Let $f_n \rightsquigarrow_\alpha f$ and $g_n \rightsquigarrow_\beta g$ and choose $\alpha' < \alpha$, $\beta' < \beta$ such that we still have $\alpha' + \beta' > 1$. By Exercise 2.4 we know that $f_n \to f$ in $C^{\alpha'}$ and $g_n \to g$ in $C^{\beta'}$.

The Young integral is a continuous bilinear operator $I: C^{\alpha'} \times C^{\beta'} \to C^{\beta'}$, thus one has $I(f_n, g_n) \to I(f,g)$ in $C^{\beta'}$ and, in particular, $\|I(f_n, g_n) - I(f, g)\|_\infty \to 0$. Moreover by (2.9)

$$\sup_n \|I(f_n, g_n)\|_\beta \leqslant \sup_n \, (\|f_n\|_\infty + c_{\alpha+\beta} T^\alpha \|\delta f_n\|_\alpha) \|g_n\|_\beta < \infty$$

As far as uniqueness is concerned assume that $J: C^\alpha \times C^\beta \to C^\beta$ coincides with I for $g \in C^1$ and verifies (2.19). Given $f \in C^\alpha$ and $g \in C^\beta$ we construct a sequence $(g_n) \subset C^1$ with $\|g_n - g\|_\infty \to 0$ and $\|g_n\|_\beta \leqslant \|g\|_\beta$. Then

$$I(f,g) = \lim_n I(f, g_n) = \lim_n J(f, g_n) = J(f,g)$$

where the limit has to be intended, for instance, in the $\|\cdot\|_\infty$ norm. □

2.5. Two proofs

It remains to prove Theorem 2.2 and Lemma 2.3.

Proof of Lemma 2.3. We extend $f: [0,T] \to \mathbb{R}$ as a constant function in $(-\infty, 0]$ and $[T, \infty)$ (that is $f(x) := f(0)$ for $x \leq 0$ and $f(x) := f(T)$ for $x \geq T$). Let us fix an arbitrary probability density $\phi: \mathbb{R} \to [0, \infty)$ with compact support, say in $[-1,1]$. Then $\phi_n(x) := n\phi(nx)$ is again a density, for any $n \in \mathbb{N}$, and we set

$$f_n(x) := (f * \phi_n)(x) = \int_{\mathbb{R}} f(x - v)\, \phi_n(v)\, \mathrm{d}v = \int_{\mathbb{R}} f(x - \tfrac{w}{n})\, \phi(w)\, \mathrm{d}w\,.$$

Note that, for every $x \in \mathbb{R}$,

$$|f_n(x) - f(x)| \leq \int_{\mathbb{R}} |f(x - \tfrac{w}{n}) - f(x)|\, \phi(w)\, \mathrm{d}w \leq \int_{\mathbb{R}} \omega_f(|\tfrac{w}{n}|)\phi(w)\, \mathrm{d}w \leq \omega_f(\tfrac{1}{n})\,,$$

where $\omega_f(\delta) := \sup_{|h| \leq \delta} |f(t+h) - f(t)|$ is the modulus of continuity of f, and the last inequality holds because ϕ is a density supported in $[-1,1]$. Note that $\omega_f(\delta) \to 0$ as $\delta \to 0$ (because f is uniformly continuous), hence $\|f_n - f\|_\infty \to 0$.

It remains to prove (2.7), but this is easy:

$$\begin{aligned}|f_n(x)-f_n(y)| &\leqslant \int_{\mathbb{R}} |f(x-v)-f(y-v)|\,\phi_n(v)\,\mathrm{d}v\\ &\leqslant h(x-y)\int_{\mathbb{R}}\phi_n(v)\,\mathrm{d}v = h(x-y)\,.\end{aligned}$$

The proof is complete. □

Proof of Theorem 2.2. First we show that C_0^α is closed in C^α. Given f_n in C_0^α and $f\in C^\alpha$ such that $\|f_n-f\|_\alpha\to 0$, we have to show that $f\in C_0^\alpha$, that is (2.6) holds. For $s<t$ and for every $n\in\mathbb{N}$ we can write

$$\frac{|f(t)-f(s)|}{(t-s)^\alpha}\leq \|\delta f-\delta f_n\|_\alpha+\frac{|f_n(t)-f_n(s)|}{(t-s)^\alpha}\,. \tag{2.20}$$

Fix $n=\bar{n}_\epsilon$ such that $\|\delta f_{\bar{n}_\epsilon}-\delta f\|_\alpha<\frac{\epsilon}{2}$. Since $f_{\bar{n}_\epsilon}\in C_0^\alpha$, relation (2.6) holds for $f_{\bar{n}_\epsilon}$, so we can fix $\delta_\epsilon>0$ such that for $|t-s|\leq\delta$ the last term in (2.20) is $\leq\frac{\epsilon}{2}$ and we are done.

It remains to show that, for any $f\in C_0^\alpha$, there is a sequence $f_n\in C^\infty$ such that $\|f_n-f\|_\infty+\|\delta f_n-\delta f\|_\alpha\to 0$. We take the sequence $f_n\in C^\infty$ provided by Lemma 2.3, so we only need to show that $\|\delta f_n-\delta f\|_\alpha\to 0$.

Since $f\in C_0^\alpha$, the inequality (2.6) holds. The same inequality holds replacing f by f_n, uniformly for $n\in\mathbb{N}$, thanks to relation (2.7). This means that for any $\epsilon>0$, for all $0\leqslant s<t\leqslant T$ with $|t-s|\leqslant\delta_\epsilon$, and for any $n\in\mathbb{N}$, we can write

$$\frac{|(f_n-f)(t)-(f_n-f)(s)|}{(t-s)^\alpha}\leqslant\frac{|f_n(t)-f_n(s)|}{(t-s)^\alpha}+\frac{|f(t)-f(s)|}{(t-s)^\alpha}\leqslant 2\,\epsilon\,.$$

We now fix $\bar{n}_\epsilon>0$ such that $\|f_n-f\|_\infty\leqslant\epsilon\,(\delta_\epsilon)^\alpha$ for all $n\geq\bar{n}_\epsilon$. Then for $|t-s|>\delta_\epsilon$

$$\frac{|(f_n-f)(t)-(f_n-f)(s)|}{(t-s)^\alpha}\leqslant\frac{2\|f_n-f\|_\infty}{(\delta_\epsilon)^\alpha}\leqslant\epsilon\,.$$

Altogether, for $n\geq\bar{n}_\epsilon$ we have $\|\delta f_n-\delta f\|_\alpha\leqslant 2\epsilon$. This shows that $\|\delta f_n-\delta f\|_\alpha\to 0$. □

Chapter 3

Finite difference equations in the Young case

3.1. Differential versus integral equations

In the theory of ordinary differential equations (ODEs), one can give two equivalent formulations of such an equation:

$$\dot{x}_t = b(x_t), \qquad x_t = x_0 + \int_0^t b(x_s)\,\mathrm{d}s, \qquad t \geq 0, \tag{3.1}$$

the equivalence being of course a consequence of the fundamental theorem of calculus.

We are interested also in studying solutions $Y: [0,T] \to \mathbb{R}^k$ to an ordinary differential equation *controlled* by a smooth function $X: [0,T] \to \mathbb{R}^d$

$$\dot{Y}_t = \sigma(Y_t)\,\dot{X}_t, \tag{3.2}$$

which is equivalent to the *integral equation*

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\,\dot{X}_s\,\mathrm{d}s, \tag{3.3}$$

where $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ is sufficiently smooth. Because of the fundamental theorem of calculus, (3.2) and (3.3) are *the same equation*.

Let us rewrite (3.3), for $s < t$,

$$\begin{aligned} Y_t - Y_s &= \int_s^t \dot{Y}_r\,\mathrm{d}r = \int_s^t \sigma(Y_r)\,\dot{X}_r\,\mathrm{d}r = \\ &= \sigma(Y_s)(X_t - X_s) + \int_s^t (\sigma(Y_r) - \sigma(Y_s))\,\dot{X}_r\,\mathrm{d}r \\ &= \sigma(Y_s)(X_t - X_s) + R_{st}. \end{aligned} \tag{3.4}$$

If σ is at least continuous, then by uniform continuity of $r \mapsto \sigma(Y_r)$ we can see that

$$R_{st} = o(t-s)$$

in the uniform sense of Remark 1.2.

Suppose now that $X: [0,T] \to \mathbb{R}^d$ is of class C^α. We would like to give an analog of the controlled equation (3.2). For that, we define

$$\delta X_{st} := X_t - X_s, \qquad |\delta X_{st}| \lesssim |t-s|^\alpha, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Taking inspiration from (3.4) we look for $\mathrm{y}: [0,T] \to \mathbb{R}^d$ such that

$$\delta \mathrm{y}_{st} = \sigma(\mathrm{y}_s)\,\delta X_{st} + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.5}$$

recall Remark 1.2.

DEFINITION 3.1. *Let* $\alpha > 1/2$ *and* $X \in C^{\alpha}([0,T];\mathbb{R}^d)$*. A solution to (3.5) is a* $\mathrm{y} \in C^{\alpha}([0,T];\mathbb{R}^k)$ *such that for some* $\zeta > 1$

$$\mathrm{y}^2_{st} := \delta \mathrm{y}_{st} - \sigma(\mathrm{y}_s)\, \delta X_{st}, \qquad |\mathrm{y}^2_{st}| \lesssim |t-s|^{\zeta}, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.6}$$

namely $\mathrm{y}^2 \in C_2^{\zeta}$.

From the computation done in (3.4), we see that this definition extends the classical equation (3.3) which holds in the case of a differentiable driving path $X: [0,T] \to \mathbb{R}^d$, namely we have the following

PROPOSITION 3.2. *Let* $X: [0,T] \to \mathbb{R}^d$ *of class* C^1 *and* σ *locally of class* C^{δ} *with* $\delta \in (0,1)$*. If* $\mathrm{y}: [0,T] \to \mathbb{R}^d$ *satisfies (3.3), then* y *also satisfies (3.6) with* $\zeta = 1 + \delta$*.*

Proof. By the Taylor expansion in time (3.4) we have (3.6) with $\zeta = 1 + \delta$. □

By the Sewing Lemma, if y^2 satisfies (3.6) then it actually satisfies the same property with $\zeta = 2\alpha$.

LEMMA 3.3. *Let* y *be a solution to (3.5) as in Definition 3.1. Then* y^2 *defined by (3.6) also satisfies* $\mathrm{y}^2 \in C_2^{2\alpha}$*, namely*

$$|\mathrm{y}^2_{st}| \lesssim |t-s|^{2\alpha}, \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{3.7}$$

Proof. Since $\delta \circ \delta = 0$, by (3.6) and (1.12) we have

$$\delta \mathrm{y}^2_{sut} = \delta\sigma(\mathrm{y})_{su}\, \delta X_{ut} = (\sigma(\mathrm{y}_u) - \sigma(\mathrm{y}_s))\, \delta X_{ut}. \tag{3.8}$$

By (3.8) we obtain that $\delta \mathrm{y}^2 \in C_3^{2\alpha}$, so that, by the Sewing Lemma, $\Lambda(\delta \mathrm{y}^2) \in C_2^{2\alpha}$. Then $\mathrm{y}^2 - \Lambda(\delta \mathrm{y}^2) \in C_2^{\zeta \wedge (2\alpha)}$ and $\delta(\mathrm{y}^2 - \Lambda(\delta \mathrm{y}^2)) = 0$, which implies that $\mathrm{y}^2 - \Lambda(\delta \mathrm{y}^2) = 0$ by the uniqueness statement of Lemma 1.3. □

We first state a *local existence* result.

PROPOSITION 3.4. *Let* $y_0 \in \mathbb{R}^d$ *and* σ *is of class* C^1 *and globally Lipschitz, namely* $\|\nabla\sigma\|_{\infty} < +\infty$*. There exists* $T_{M,D,\alpha} > 0$ *such that, for all* $T \in (0, T_{M,D,\alpha})$ *and* $X \in C^{\alpha}([0,T];\mathbb{R}^d)$ *such that* $\|\delta X\|_{\alpha} \leqslant M$*, there exists a solution* y *to (3.6) on the interval* $[0,T]$ *such that* $\mathrm{y}_0 = y_0$ *and*

$$\|\mathrm{y}\|_{\alpha} \leqslant 12\, |\sigma(\mathrm{y}_0)|\, \|\delta X\|_{\alpha}. \tag{3.9}$$

The proof of this Proposition is not based on the Sewing Lemma but on a discretization argument. For the reader's convenience, it is postponed to section 3.6 below.

3.2. UNIQUENESS

Let us suppose that $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ is of class C^2, without any boundedness assumption. We want to show that this implies uniqueness of solutions to (3.6).

We first have an elementary but fundamental estimate, which is the main technical tool of this chapter, together with the Sewing Lemma. For any Ψ: $\mathbb{R}^k \to \mathbb{R}^m$ we introduce the notation

$$C_{\Psi,R} := \sup \{|\Psi(x)|: x \in \mathbb{R}^k, |x| \leqslant R\}. \tag{3.10}$$

LEMMA 3.5. *Let $\psi: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^\ell$ of class C^2. Then for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^k$ with norm less than R we have*

$$\begin{aligned} &|[\psi(x) - \psi(y)] - [\psi(\bar{x}) - \psi(\bar{y})]| \leqslant \\ &\leqslant C_{\nabla\psi,R}|(x-y) - (\bar{x}-\bar{y})| + C_{\nabla^2\psi,R}(|x-y| + |\bar{x}-\bar{y}|)|x-\bar{x}|. \end{aligned} \tag{3.11}$$

Proof. Note that for $x, \bar{x}, y, \bar{y} \in \mathbb{R}^k$

$$\psi(x) - \psi(\bar{x}) = \hat{\psi}(x, \bar{x})\,(x - \bar{x}), \qquad \hat{\psi}(x, \bar{x}) := \int_0^1 \nabla\psi(\bar{x} + u(x - \bar{x}))\;\mathrm{d}u.$$

Therefore

$$\begin{aligned} &[\psi(x) - \psi(y)] - [\psi(\bar{x}) - \psi(\bar{y})] = \hat{\psi}(x, \bar{x})\,(x - \bar{x}) - \hat{\psi}(y, \bar{y})\,(y - \bar{y}) = \\ &= \hat{\psi}(y, \bar{y})[(x - y) - (\bar{x} - \bar{y})] + (\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{y}))\,(x - \bar{x}). \end{aligned}$$

Now

$$\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{y}) = \hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{x}) + \hat{\psi}(y, \bar{x}) - \hat{\psi}(y, \bar{y}).$$

We can argue now as for $\psi(x) - \psi(\bar{x})$ and write

$$\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{x}) = \hat{\hat{\psi}}(x, y; \bar{x})\,(x - y),$$

and similarly for $\hat{\psi}(y, \bar{x}) - \hat{\psi}(y, \bar{y})$. Therefore (3.11) follows by using the notation (3.10). □

Now we can prove our uniqueness result.

THEOREM 3.6. (UNIQUENESS) *Let $\alpha > 1/2$ and $X: [0, T] \to \mathbb{R}^d$ of class C^α. If $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ is of class C^2, then for every $\mathrm{y}_0 \in \mathbb{R}^d$ there exists at most one solution y to (3.6).*

Proof. If y and $\bar{\mathrm{y}}$ are two solutions, set $\mathrm{z} := \mathrm{y} - \bar{\mathrm{y}}$. We want to show that, for $\tau \in \,]0, 1]$ small enough, $\|\mathrm{z}\|_{\infty,\tau} \leqslant 2|\mathrm{z}_0|$. First, we know by (1.29) that

$$\|\mathrm{z}\|_{\infty,\tau} \leqslant |\mathrm{z}_0| + 3\tau^\alpha \|\delta \mathrm{z}\|_{\alpha,\tau}. \tag{3.12}$$

We set as in (3.6)

$$\mathrm{z}^2_{st} := \mathrm{y}^2_{st} - \bar{\mathrm{y}}^2_{st} = \delta \mathrm{z}_{st} - (\sigma(\mathrm{y}_s) - \sigma(\bar{\mathrm{y}}_s))\, \delta X_{st}. \tag{3.13}$$

Using the notation in (3.10) we set $L := C_{\nabla\sigma, \|\mathrm{y}\|_\infty \vee \|\bar{\mathrm{y}}\|_\infty}$. Then

$$\|\sigma(\mathrm{y}) - \sigma(\bar{\mathrm{y}})\|_{\infty,\tau} \leqslant L\, \|\mathrm{z}\|_{\infty,\tau},$$

Therefore by (3.13) and (3.7) we obtain

$$\|\delta \mathrm{z}\|_{\alpha,\tau} \leqslant L\, \|\mathrm{z}\|_{\infty,\tau} + \tau^\alpha \|\mathrm{z}^2\|_{2\alpha,\tau}. \tag{3.14}$$

We estimate now $\|\mathrm{z}^2\|_{2\alpha,\tau}$. By (3.13) and (3.8) we have

$$\delta \mathrm{z}^2_{sut} = (\delta\sigma(\mathrm{y})_{su} - \delta\sigma(\bar{\mathrm{y}})_{su})\, \delta X_{ut}.$$

Therefore by (3.11) there is a constant $C_{\mathrm{y},\bar{\mathrm{y}},X} > 0$ such that

$$\|\delta \mathrm{z}^2\|_{2\alpha,\tau} \leqslant \|\delta X\|_\alpha \|\delta\sigma(\mathrm{y}) - \delta\sigma(\bar{\mathrm{y}})\|_{\alpha,\tau} \leqslant C_{\mathrm{y},\bar{\mathrm{y}},X}(\|\mathrm{z}\|_{\infty,\tau} + \|\delta \mathrm{z}\|_{\alpha,\tau}).$$

Therefore by the Sewing estimate in weighted spaces (1.33)

$$\|\mathrm{z}^2\|_{2\alpha,\tau} \leqslant K_{2\alpha}\, \|\delta \mathrm{z}^2\|_{2\alpha,\tau} \leqslant K_{2\alpha} C_{\mathrm{y},\bar{\mathrm{y}},X}(\|\mathrm{z}\|_{\infty,\tau} + \tau^\alpha \|\mathrm{z}^2\|_{2\alpha,\tau}).$$

By choosing $\tau > 0$ small enough, we obtain $\|\mathrm{z}^2\|_{2\alpha,\tau} \leqslant 2\, K_{2\alpha} C_{\mathrm{y},\bar{\mathrm{y}},X}\, \|\mathrm{z}\|_{\infty,\tau}$, so that by (3.14)

$$\|\delta \mathrm{z}\|_{\alpha,\tau} \leqslant (2\, K_{2\alpha} C_{\mathrm{y},\bar{\mathrm{y}},X} + L) \|\mathrm{z}\|_{\infty,\tau}.$$

Then by (3.12)

$$\|\mathrm{z}\|_{\infty,\tau} \leqslant |\mathrm{z}_0| + 3\tau^\alpha \|\delta \mathrm{z}\|_{\alpha,\tau} \leqslant |\mathrm{z}_0| + 3\tau^\alpha (2K_{2\alpha} C_{\mathrm{y},\bar{\mathrm{y}},X} + L) \|\mathrm{z}\|_{\infty,\tau}$$

and by choosing $\tau > 0$ even smaller if necessary, we obtain $\|\mathrm{z}\|_{\infty,\tau} \leqslant 2|\mathrm{z}_0|$. In particular, if $\mathrm{z}_0 = 0$, then $\mathrm{z} \equiv 0$. □

3.3. A PRIORI ESTIMATES

In this section we suppose that σ is of class C^1 and globally Lipschitz, namely $\|\nabla\sigma\|_\infty < +\infty$ (without boundedness assumptions on σ). We fix

$$D \geq \|\nabla\sigma\|_\infty.$$

LEMMA 3.7. *Let $M > 0$. There exists a constant $C_{M,D}$ such that for any X such that $\|\delta X\|_\alpha \leqslant M$, any solution to (3.6) satisfies*

$$\|\mathrm{y}^2\|_{2\alpha,\tau} \leqslant C_{M,D} \|\delta \mathrm{y}\|_{\alpha,\tau},$$

where y^2 is defined as in (3.6). Moreover there is $\varepsilon_{M,D} > 0$ such that, if $(\tau \wedge T)^\alpha \leqslant \varepsilon_{M,D}$, then

$$\|\delta \mathrm{y}\|_{\alpha,\tau} \leqslant 2\, \|\delta X\|_\alpha |\sigma(\mathrm{y}_0)|. \tag{3.15}$$

Proof. Since

$$\|\delta\sigma(\mathrm{y})\|_{\alpha,\tau} \leqslant \|\nabla\sigma\|_\infty \|\delta \mathrm{y}\|_{\alpha,\tau}$$

we obtain by (3.8)

$$\|\delta \mathrm{y}^2_{sut}\|_{2\alpha,\tau} \leqslant \|\nabla\sigma\|_\infty \|\delta X\|_\alpha \|\delta \mathrm{y}\|_{\alpha,\tau}.$$

By the Sewing estimate in weighted spaces (1.33)

$$\|\mathrm{y}^2\|_{2\alpha,\tau} \leqslant K_{2\alpha} \|\nabla\sigma\|_\infty \|\delta X\|_\alpha \|\delta \mathrm{y}\|_{\alpha,\tau},$$

which proves the first assertion with $C_{M,D} = K_{2\alpha} D M$.

Let us now prove the second assertion. We set $\varepsilon := (\tau \wedge T)^\alpha$. Since

$$\|\delta \mathrm{y}\|_{\alpha,\tau} \leqslant \|\delta X\|_\alpha \|\sigma(\mathrm{y})\|_{\infty,\tau} + \varepsilon \|\mathrm{y}^2\|_{2\alpha,\tau}$$

and by (1.29)

$$\|\sigma(\mathrm{y})\|_{\infty,\tau} \leqslant |\sigma(\mathrm{y}_0)| + 3D\varepsilon \, \|\delta \mathrm{y}\|_{\alpha,\tau},$$

we have for $\varepsilon \in \,]0,1]$

$$\|\delta \mathrm{y}\|_{\alpha,\tau} \leqslant \|\delta X\|_\alpha (|\sigma(\mathrm{y}_0)| + 3D\varepsilon \, \|\delta \mathrm{y}\|_{\alpha,\tau}) + \varepsilon K_{2\alpha} D M \|\delta \mathrm{y}\|_{\alpha,\tau}$$

and for $\varepsilon \leqslant \varepsilon_{M,D} := (2DM(K_{2\alpha}+3))^{-1}$ we obtain

$$\|\delta \mathrm{y}\|_{\alpha,\tau} \leqslant 2 \, \|\delta X\|_\alpha |\sigma(\mathrm{y}_0)|.$$

The proof is complete. □

3.4. Global existence and uniqueness

Let us suppose that $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ is of class C^2 with $\|\nabla\sigma\|_\infty < +\infty$.

THEOREM 3.8. *If $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ is of class C^2 with $\|\nabla\sigma\|_\infty < +\infty$ then for every $\mathrm{y}_0 \in \mathbb{R}^d$ and $T > 0$ there is a unique solution $(\mathrm{y}_t)_{t\in[0,T]}$ to (3.6).*

Proof. By Theorem 3.6 we have at most one solution. We now construct a solution on an arbitrary finite interval $[0,T]$. We define $\Lambda \subseteq [0,T]$ as the set of all s such that there is a solution $(\mathrm{y}_t)_{t\in[0,s]}$ to (3.6). By Proposition 3.4, Λ is an open subset of $[0,T]$ and contains 0. By the a priori estimates of Lemma 3.7, Λ is a closed subset of $[0,T]$. Therefore $\Lambda = [0,T]$. □

3.5. Continuity of the solution map

We consider now the map $\mathbb{R}^d \times C^\alpha \ni (\mathrm{y}_0, X) \mapsto \mathrm{y} = \Phi(\mathrm{y}_0, X) \in C^\alpha$, where y is the unique solution to (3.6) constructed in Theorem 3.8. We want to show that this map, called *the solution map*, is continuous. This property is highly non-trivial, since y solves (3.3) when X is of class C^1, and this equation is based on the derivative in time $\dot{X}$ of X. We shall see in the next chapters that this property can be proved also in more complex situations, where $\alpha \leqslant 1/2$ and which cover the case of X a Brownian motion and y the solution to a SDE.

We suppose in this section that σ is of class C^2, with $\|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty < +\infty$ (without boundedness assumptions on σ). We fix

$$D \geq \|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty.$$

We also introduce the seminorm

$$\|G\|_{\eta,\tau}^{\text{No-exp}} := \sup_{0\leqslant s<t\leqslant T} \mathbf{1}_{(0<|t-s|\leqslant\tau)} \frac{|G_{st}|}{|t-s|^\eta}, \qquad G \in C_2^\eta \tag{3.16}$$

to be compared with the definitions of $\|\cdot\|_{\eta,\tau}$ in Section 1.7. We have

$$\|\cdot\|_{\eta,\tau} \leqslant \|\cdot\|_{\eta,\tau}^{\text{No-exp}} \leqslant e^{\frac{T}{\tau}} \|\cdot\|_{\eta,\tau}. \tag{3.17}$$

Then by (3.11) we have for $f, \bar{f} \in \mathcal{C}^\alpha([0,T];\mathbb{R}^k)$

$$\|\delta\sigma(f) - \delta\sigma(\bar{f})\|_{\alpha,\tau} \leqslant c_{\sigma,f,\bar{f}} (\|f - \bar{f}\|_{\infty,\tau} + \|\delta f - \delta\bar{f}\|_{\alpha,\tau}), \tag{3.18}$$

where

$$c_{\sigma,f,\bar{f}} = D(\|\delta f\|_{\alpha,\tau}^{\text{No-exp}} + \|\delta\bar{f}\|_{\alpha,\tau}^{\text{No-exp}} + 1)\,.$$

PROPOSITION 3.9. *Let $M>0$ and $\max\{|\sigma(\mathrm{y}_0)|, \|\delta X\|_\alpha, |\sigma(\bar{\mathrm{y}}_0)|, \|\delta\bar{X}\|_\alpha\} \leqslant M$. Then for every $T>0$ there are $\hat{\tau}_{M,D,T}, C_{M,D,T}>0$ such that for $\tau \in]0, \hat{\tau}_{M,D,T}]$*

$$\|\mathrm{y} - \bar{\mathrm{y}}\|_{\infty,\tau} + \|\delta\mathrm{y} - \delta\bar{\mathrm{y}}\|_{\alpha,\tau} \leqslant C_{M,D,T}(|\mathrm{y}_0 - \bar{\mathrm{y}}_0| + \|\delta X - \delta\bar{X}\|_\alpha).$$

Proof. If y and $\bar{\mathrm{y}}$ are two solutions, set $\mathrm{z} := \mathrm{y} - \bar{\mathrm{y}}$. First, we know by (1.29) that

$$\|\mathrm{z}\|_{\infty,\tau} \leqslant |\mathrm{z}_0| + 3\tau^\alpha \|\delta\mathrm{z}\|_{\alpha,\tau}. \tag{3.19}$$

Let us set $\mathrm{z}^2 := \mathrm{y}^2 - \bar{\mathrm{y}}^2$, so that by (3.6)

$$\begin{aligned} \mathrm{z}_{st}^2 &= \delta\mathrm{z}_{st} - \sigma(\mathrm{y}_s)\delta X_{st} + \sigma(\bar{\mathrm{y}}_s)\,\delta\bar{X}_{st} \\ &= \delta\mathrm{z}_{st} - (\sigma(\mathrm{y}_s) - \sigma(\bar{\mathrm{y}}_s))\,\delta X_{st} - \sigma(\bar{\mathrm{y}}_s)(\delta X - \delta\bar{X})_{st}. \end{aligned}$$

By (3.8) we have

$$\delta\mathrm{z}_{sut}^2 = \delta(\sigma(\mathrm{y}) - \sigma(\bar{\mathrm{y}}))_{su}\,\delta X_{ut} + \delta\sigma(\bar{\mathrm{y}})_{su}\,(\delta X - \delta\bar{X})_{ut}. \tag{3.20}$$

By Lemma 3.7 we know that for $(\tau_{M,D})^\alpha = \varepsilon_{M,D}$ we have

$$\|\delta\mathrm{y}\|_{\alpha,\tau_{M,D}} + \|\delta\bar{\mathrm{y}}\|_{\alpha,\tau_{M,D}} \leqslant 4M^2,$$

and by (3.17) we obtain the bound

$$\|\delta\mathrm{y}\|_{\alpha,\tau_{M,D}}^{\text{No-exp}} + \|\delta\bar{\mathrm{y}}\|_{\alpha,\tau_{M,D}}^{\text{No-exp}} \leqslant e^{\frac{T}{\tau_{M,D}}} 4M^2.$$

By (3.18) we have for $\tau \leqslant \tau_{M,D}$

$$\begin{aligned}&\|\delta\sigma(\mathrm{y})-\delta\sigma(\bar{\mathrm{y}})\|_{\alpha,\tau}+\|\delta\sigma_2(\mathrm{y})-\delta\sigma_2(\bar{\mathrm{y}})\|_{\alpha,\tau}\leqslant\\ &\leqslant D(\|\delta\mathrm{y}\|^{\text{No-exp}}_{\alpha,\tau_{M,D}}+\|\delta\bar{\mathrm{y}}\|^{\text{No-exp}}_{\alpha,\tau_{M,D}}+1)(\|\mathrm{z}\|_{\infty,\tau}+\|\delta\mathrm{z}\|_{\alpha,\tau})\\ &\leqslant C_{D,M,T}(\|\mathrm{z}\|_{\infty,\tau}+\|\delta\mathrm{z}\|_{\alpha,\tau}).\end{aligned}$$

By (3.20) we obtain

$$\|\delta\mathrm{z}^2\|_{2\alpha,\tau}\leqslant C_{D,M,T}(\|\mathrm{z}\|_{\infty,\tau}+\|\delta\mathrm{z}\|_{\alpha,\tau})+D\|\delta\bar{\mathrm{y}}\|_{\alpha,\tau}\|\delta X-\delta\bar{X}\|_{\alpha}.$$

Therefore by the Sewing estimate in weighted spaces (1.33) and by (3.14)

$$\|\mathrm{z}^2\|_{2\alpha,\tau}\leqslant K_{2\alpha}\|\delta\mathrm{z}^2\|_{2\alpha,\tau}\lesssim_{M,D,T}\|\mathrm{z}\|_{\infty,\tau}+\tau^{\alpha}\|\mathrm{z}^2\|_{2\alpha,\tau}+\|\delta X-\delta\bar{X}\|_{\alpha}.$$

By choosing $\tau\in(0,\tau_{M,D})$ small enough, we obtain $\|\mathrm{z}^2\|_{2\alpha,\tau}\lesssim_{M,D,T}\|\mathrm{z}\|_{\infty,\tau}+\|\delta X-\delta\bar{X}\|_{\alpha}$, so that by (3.14)

$$\|\delta\mathrm{z}\|_{\alpha,\tau}\lesssim\|\mathrm{z}\|_{\infty,\tau}+\|\delta X-\delta\bar{X}\|_{\alpha}. \tag{3.21}$$

Then by (3.12)

$$\|\mathrm{z}\|_{\infty,\tau}\leqslant|\mathrm{z}_0|+3\tau^{\alpha}\|\delta\mathrm{z}\|_{\alpha,\tau}\lesssim_{M,D,T}|\mathrm{z}_0|+\tau^{\alpha}\|\mathrm{z}\|_{\infty,\tau}+\|\delta X-\delta\bar{X}\|_{\alpha}$$

and there exists $\hat{\tau}_{M,D,T}\leqslant\tau_{M,D}$ such that, for $\tau\in\,]0,\hat{\tau}_{M,D,T}]$, we obtain

$$\|\mathrm{z}\|_{\infty,\tau}\lesssim_{M,D,T}|\mathrm{z}_0|+\|\delta X-\delta\bar{X}\|_{\alpha}.$$

Finally by (3.21) we obtain

$$\|\delta\mathrm{z}\|_{\alpha,\tau}\lesssim_{M,D,T}|\mathrm{z}_0|+\|\delta X-\delta\bar{X}\|_{\alpha}.$$

The proof is complete. □

3.6. Euler scheme and local existence

In this section we prove the local existence result of Proposition 3.4, using a discretization in time argument. We stress that no results of this section rely on the material of the preceding sections, and it is only for the reader's convenience that we have postponed the proof of Proposition 3.4. In particular, this section and the next do not use the Sewing Lemma.

We suppose now that σ is of class C^1 and globally Lipschitz, namely $\|\nabla\sigma\|_\infty<+\infty$ (without boundedness assumptions on σ). We recall that $\alpha\in\left]\frac{1}{2},1\right]$. To construct a solution to (4.13) in the sense of Def. 3.1, we fix $T>0$, $n\in\mathbb{N}$ and we set $t_i:=\frac{i}{n}$, $i\geqslant 0$. Then we set

$$y_{i+1}=y_i+\sigma(y_i)\,\delta X_{t_i t_{i+1}},\qquad i\geqslant 0. \tag{3.22}$$

We set $D:=\|\nabla\sigma\|_\infty$ and

$$\delta y_{ij}:=y_j-y_i,\qquad \|\delta y\|_\alpha:=n^{\alpha}\sup_{0<i<j\leqslant nT}\frac{|y_j-y_i|}{|j-i|^{\alpha}},\qquad A_{ij}=\sigma(y_i)\,\delta X_{t_i t_j}.$$

The main technical estimate is the following

LEMMA 3.10. *Let $M > 0$. There exists $T_{M,D,\alpha} > 0$ such that, for all $T \in (0, T_{M,D,\alpha})$ and $X \in C^\alpha([0,T];\mathbb{R}^d)$ such that $\|\delta X\|_\alpha \leqslant M$, we have*

$$\|\delta y\|_\alpha \leqslant 4|\sigma(y_0)|M, \tag{3.23}$$

$$|\delta y_{ik} - A_{ik}| \lesssim_{M,D,\alpha} |\sigma(y_0)|\left(\frac{|k-i|}{n}\right)^{2\alpha}, \qquad 0 \leqslant i \leqslant k \leqslant nT. \tag{3.24}$$

Proof. We want to obtain, setting

$$L(y) := \frac{2DM\,\|\delta y\|_\alpha}{1 - 2^{1-2\alpha}}, \tag{3.25}$$

that

$$|\delta y_{ik} - A_{ik}| \leqslant L(y)\left(\frac{|k-i|}{n}\right)^{2\alpha}, \qquad \forall 0 \leqslant i \leqslant k \leqslant nT. \tag{3.26}$$

Note that (3.26) holds if $k \in \{i, i+1\}$. Let $m \geqslant 1$ and suppose that (3.26) holds for all $i, k \leqslant nT$ such that $0 \leqslant k - i \leqslant m$. We want to show that (3.26) holds for all $i, k \leqslant nT$ such that $k - i = m + 1$; for such i, k, we set $j = i + \left\lfloor \frac{k-i}{2} \right\rfloor$, so that $0 \leqslant j - i \leqslant \frac{k-i}{2} \leqslant m$ and $0 \leqslant k - j - 1 \leqslant \frac{k-i}{2} \leqslant m$. Now, since $2\alpha > 1$, we have

$$|j-i|^{2\alpha} + |k-j-1|^{2\alpha} \leqslant 2^{1-2\alpha}\,|k-i|^{2\alpha}.$$

We set

$$\delta A_{ijk} := A_{ik} - A_{ij} - A_{jk}.$$

Since $A_{j(j+1)} = y_{j+1} - y_j$, we obtain

$$\begin{aligned}
|\delta y_{ik} - A_{ik}| &\leqslant |\delta A_{ijk}| + |\delta y_{ij} - A_{ij}| + |\delta y_{jk} - A_{jk}| \\
&\leqslant |\delta A_{ijk}| + |\delta y_{ij} - A_{ij}| + |\delta A_{j(j+1)k}| + |\delta y_{(j+1)k} - A_{j+1k}| \\
&\leqslant |\delta A_{ijk}| + |\delta A_{j(j+1)k}| + L(y)\,2^{1-2\alpha}\left(\frac{|k-i|}{n}\right)^{2\alpha},
\end{aligned}$$

where we have used the recurrence assumption in the third inequality. Now

$$\delta A_{ijk} = (\sigma(y_i) - \sigma(y_j))\,\delta X_{t_j t_k} \Rightarrow |\delta A_{ijk}| \leqslant \|\nabla\sigma\|_\infty \|\delta X\|_\alpha \|\delta y\|_\alpha \left(\frac{|k-i|}{n}\right)^{2\alpha}$$

and analogously for $\delta A_{j(j+1)k}$. Therefore

$$|\delta y_{ik} - A_{ik}|\left(\frac{|k-i|}{n}\right)^{-2\alpha} \leqslant 2DM\|\delta y\|_\alpha + L(y)\,2^{1-2\alpha} = L(y),$$

so that (3.26) is proved. Note now that for $i \leqslant nT$

$$|\sigma(y_i)| \leqslant |\sigma(y_0)| + |\sigma(y_i) - \sigma(y_0)| \leqslant |\sigma(y_0)| + \|\nabla\sigma\|_\infty\,\|\delta y\|_\alpha\,T^\alpha.$$

Now we obtain by (3.25) and (3.26)

$$\begin{aligned}
\|\delta y\|_\alpha &\leqslant n^\alpha \sup_{0 < i < j \leqslant nT} \frac{|\delta y_{ij} - A_{ij}| + |A_{ij}|}{|j-i|^\alpha} \leqslant \\
&\leqslant L(y)\,T^\alpha + (|\sigma(y_0)| + D\|\delta y\|_\alpha T^\alpha)M.
\end{aligned}$$

For $T^{\alpha} \leqslant (2DM)^{-1}$, we obtain

$$\|\delta y\|_{\alpha} \leqslant 2L(y)T^{\alpha} + 2|\sigma(y_0)|M \leqslant \frac{4DM\|\delta y\|_{\alpha}}{1-2^{1-2\alpha}}T^{\alpha} + 2|\sigma(y_0)|M$$

and, if $T^{\alpha} \leqslant (1-2^{1-2\alpha})(8DM)^{-1}$, we obtain finally

$$\|\delta y\|_{\alpha} \leqslant 4|\sigma(y_0)|M, \qquad L(y) \leqslant \frac{8DM^2|\sigma(y_0)|}{1-2^{1-2\alpha}} =: K$$

and by (3.26)

$$|\delta y_{ik} - \sigma(y_i)\,\delta X_{t_i t_k}| \leqslant K\left(\frac{|k-i|}{n}\right)^{2\alpha}, \qquad \forall 0 \leqslant i \leqslant k \leqslant nT.$$

□

Now we can prove Proposition 3.4 above.

Proof of Proposition 3.4. We call $\mathrm{y}^n: [0,T] \to \mathbb{R}^d$ the continuous function which is affine on each interval $\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]$ and such that $\mathrm{y}^n_{\frac{i}{2^n}} = y_i$, $0 \leqslant i \leqslant 2^n T$. Then we have by (3.23)

$$\|\mathrm{y}^n\|_{\alpha} \leqslant 3\|y\|_{\alpha} \leqslant 12|\sigma(y_0)|\,\|\delta X\|_{\alpha}.$$

In particular the sequence $(\mathrm{y}^n)_n$ is compact in $C([0,T])$ and we call y one of its limit points. By (3.24), for all $s,t \in \bigcup_n \left\{\frac{i}{2^n}: 0 \leqslant i \leqslant 2^n T\right\}$ we have

$$|\delta \mathrm{y}_{st} - \sigma(\mathrm{y}_s)\,\delta X_{st}| \lesssim |t-s|^{2\alpha}.$$

By the density of dyadic numbers, we obtain that y is indeed a solution to (3.6). □

3.7. Error estimate in the Euler scheme

We suppose in this section that σ is of class C^2 with $\|\nabla\sigma\|_{\infty} + \|\nabla^2\sigma\|_{\infty} < +\infty$.

Theorem 3.11. *The Euler scheme converges at speed $n^{2\alpha-1}$.*

Proof. Let us set $z_i := \partial y_i / \partial y_0$, where $(y_i)_{i \geqslant 0}$ is defined by (3.22). Then

$$z_{i+1} = z_i + \nabla\sigma(y_i)\,z_i\,\delta X_{t_i t_{i+1}}, \qquad i \geqslant 0.$$

This shows that the pair $(y_i, z_i)_{i \geqslant 0}$ satisfies a recurrence which is similar to (3.22) with a map Σ of class C^1 and therefore we can apply the above results to obtain that $|z_i| \leqslant$ const. In particular the map $y_0 \to y_k$ is Lipschitz-continuous, uniformly over $k \geqslant 0$.

Let us call, for $k \geqslant 0$, $\left(z_{\ell}^{(k)}\right)_{\ell \geqslant k}$ as the sequence which satisfies (3.22) but has initial value $z_k^{(k)} = y(t_k)$. Since $(y(t))_{t \geqslant 0}$ is a solution to (3.6), we have

$$\left|z_{k+1}^{(k)} - y(t_{k+1})\right| \lesssim n^{-2\alpha}.$$

Since the map $y_0 \to y_k$ is Lipschitz-continuous uniformly over $k \geqslant 0$, we have

$$\left|z_\ell^{(k)} - z_\ell^{(k+1)}\right| \lesssim \left|z_{k+1}^{(k)} - y(t_{k+1})\right| \lesssim n^{-2\alpha}, \qquad \ell \geqslant k+1.$$

Therefore

$$|y_\ell - y(t_\ell)| = \left|z_\ell^{(0)} - z_\ell^{(\ell)}\right| \leqslant \sum_{k=0}^{\ell-1} \left|z_\ell^{(k)} - z_\ell^{(k+1)}\right| \lesssim \frac{\ell}{n^{2\alpha}} = \frac{t_\ell}{n^{2\alpha-1}} \to 0$$

as t_ℓ is bounded and $n \to \infty$. □

3.8. INTEGRAL FORMULATION

In this section we explain why we call (3.6) a *Young* equation. In fact, we can interpret the finite difference equation (3.6) as an *integral equation*, using the Young integral of section 2.2.

PROPOSITION 3.12. *Let* $\mathrm{y} \in C^\alpha([0,T];\mathbb{R}^d)$ *with* $\alpha > \frac{1}{2}$. *Then* y *satisfies (3.6) if and only if*

$$\mathrm{y}_t = \mathrm{y}_0 + \int_0^t \sigma(\mathrm{y}_s)\,\mathrm{d}X_s, \qquad t \in [0,T], \tag{3.27}$$

where the integral is in the Young sense.

Proof. We consider the germ $A_{st} := \sigma(\mathrm{y}_s)\,\delta X_{st}$, $0 \leqslant s \leqslant t \leqslant T$. By (2.4)

$$|\delta A_{sut}| = |\sigma(\mathrm{y}_u) - \sigma(\mathrm{y}_s)||X_t - X_u| \quad \Longrightarrow \quad \|\delta A\|_{2\alpha} \leqslant \|\nabla\sigma\|_\infty \|\delta X\|_\alpha \|\delta \mathrm{y}\|_\alpha.$$

Therefore arguing as in Lemma 3.3 we obtain that (3.6) is equivalent to (2.8) above. □

3.9. LOCAL EXISTENCE VIA CONTRACTION

As an application of the estimates on the Young integral of Theorem 2.5, we want to give a local existence result for equation (3.6) which does not rely on compactness and which can be therefore used also in infinite dimension.

Let $\mathrm{y}_0 \in \mathbb{R}$ and $X \in \mathcal{C}^\alpha$ be given, $\sigma: \mathbb{R} \to \mathbb{R}$ smooth and the unknown $\mathrm{y}: [0, T] \to \mathbb{R}$ is such that $\sigma(\mathrm{y}) \in \mathcal{C}$ and $2\alpha > 1$, so that the right-hand side of (3.27) can be interpreted as a Young integral. We want now to show the following

THEOREM 3.13. (CONTRACTION FOR YOUNG DIFFERENTIAL EQUATIONS) *Let* $\sigma: \mathbb{R} \to \mathbb{R}$ *be of class* C^2 *with* $\nabla\sigma$ *and* $\nabla\sigma'$ *bounded. Let* $\alpha \in \,]1/2, 1]$ *and* $X \in \mathcal{C}^\alpha$ *fixed. It* $T > 0$ *is small enough, then for any* $\mathrm{y}_0 \in \mathbb{R}$ *there exists a unique* $\mathrm{y} \in \mathcal{C}^\alpha$ *which satisfies (3.27).*

Proof. For all $f \in \mathcal{C}^\alpha$ we have

$$|\sigma(f_t) - \sigma(f_s)| \leqslant \|\nabla\sigma\|_\infty\, |f_t - f_s|$$

so that

$$\|\delta\sigma(f)\|_\alpha \leqslant \|\nabla\sigma\|_\infty \|\delta f\|_\alpha.$$

By (2.9) with $\alpha = \beta$ we obtain for all $f \in \mathcal{C}^\alpha$ satisfying (3.27)

$$\|\delta f\|_\alpha \leqslant (|\sigma(f_0)| + (1 + K_{2\alpha})\, T^\alpha \|\nabla\sigma\|_\infty \|\delta f\|_\alpha)\, \|\delta X\|_\alpha$$

since

$$\|\sigma(f)\|_\infty \leqslant |\sigma(f_0)| + T^\alpha \|\delta\sigma(f)\|_\alpha.$$

Therefore, if T satisfies

$$T^\alpha \leqslant \frac{1}{2} \frac{1}{(1 + K_{2\alpha})\, \|\nabla\sigma\|_\infty \|\delta X\|_\alpha}$$

then we have the following a priori estimate on solutions to (3.27)

$$\|\delta\, \mathrm{y}\|_\alpha \leqslant 2|\sigma(\mathrm{y}_0)|\, \|\delta X\|_\alpha\,.$$

We fix such T and we set $\mathcal{C}^\alpha(\mathrm{y}_0) := \{f \in \mathcal{C}^\alpha\colon f_0 = \mathrm{y}_0, \|\delta f\|_\alpha \leqslant 2|\sigma(\mathrm{y}_0)|\, \|\delta X\|_\alpha\}$. Then we define $\Lambda\colon \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ given by

$$\Lambda(f) := h, \qquad h_t := \mathrm{y}_0 + \int_0^t \sigma(f_s)\, \mathrm{d}X_s, \qquad t \in [0, T].$$

It is easy to see, arguing as above, that Λ acts on $\mathcal{C}^\alpha(\mathrm{y}_0)$, namely $\Lambda\colon \mathcal{C}^\alpha(\mathrm{y}_0) \to \mathcal{C}^\alpha(\mathrm{y}_0)$. Note that the map $\mathcal{C}^\alpha(\mathrm{y}_0) \times \mathcal{C}^\alpha(\mathrm{y}_0) \ni (a, b) \mapsto \|\delta a - \delta b\|_\alpha$ defines a distance on $\mathcal{C}^\alpha(\mathrm{y}_0)$ which induces the same topology as $\|\cdot\|_{\mathcal{C}^\alpha}$. We want to show that Λ is a contraction for this distance if T is small enough. By (2.9) we have for $\alpha = \beta$

$$\begin{aligned}\|\delta\Lambda(a) - \delta\Lambda(b)\|_\alpha &\leqslant (\|\sigma(a) - \sigma(b)\|_\infty + K_{2\alpha} T^\alpha \|\delta\sigma(a) - \delta\sigma(b)\|_\alpha)\, \|\delta X\|_\alpha\,. \\ &\leqslant T^\alpha\, (1 + K_{2\alpha})\, \|\delta X\|_\alpha \|\delta\sigma(a) - \delta\sigma(b)\|_\alpha\,.\end{aligned}$$

We now need to estimate $\|\delta\sigma(a) - \delta\sigma(b)\|_\alpha$. By Lemma 3.5

$$\|\delta\sigma(a) - \delta\sigma(b)\|_\alpha \leqslant \|\nabla\sigma\|_\infty \|\delta a - \delta b\|_\alpha + \|\nabla^2\sigma\|_\infty (\|\delta a\|_\alpha + \|\delta b\|_\alpha)\, \|a - b\|_\infty.$$

Since, as usual, $\|a - b\|_\infty \leqslant T^\alpha \|\delta a - \delta b\|_\alpha$, we obtain

$$\|\delta\sigma(a) - \delta\sigma(b)\|_\alpha \leqslant (\|\nabla\sigma\|_\infty + T^\alpha \|\nabla^2\sigma\|_\infty (\|\delta a\|_\alpha + \|\delta b\|_\alpha)) \|\delta a - \delta b\|_\alpha. \qquad (3.28)$$

Therefore, for all $a, b \in \mathcal{C}^\alpha(\mathrm{y}_0)$

$$\|\delta\Lambda(a) - \delta\Lambda(b)\|_\alpha \leqslant C_T \|\delta a - \delta b\|_\alpha,$$

where $C_T := T^\alpha (1 + K_{2\alpha})\, \|\delta X\|_\alpha\, (\|\nabla\sigma\|_\infty + T^\alpha \|\nabla^2\sigma\|_\infty 4|\sigma(\mathrm{y}_0)|\, \|\delta X\|_\alpha)$. It is now enough to consider T small enough so that $C_T < 1$. □

Chapter 4

Finite difference equations in the rough case

The initial motivation for rough integration was to give a robust theory of *stochastic integration* and *stochastic differential equations* (SDE). A SDE is in fact written as an *integral equation* of the form

$$x_t = x_0 + \int_0^t \sigma(x_s)\,\mathrm{dB}_s \tag{4.1}$$

with $(B_t)_{t\geq 0}$ a d-dimensional standard Brownian motion, $x_0 \in \mathbb{R}^k$, $\sigma\colon \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$, and the integral is in the Itô sense (see ??? below). It is common in stochastic analysis to use the *differential* notation

$$\mathrm{d}x_t = \sigma(x_t)\,\mathrm{dB}_t \tag{4.2}$$

but this is just intended as another notation for the integral version (4.1). We note that the SDE (4.1) is an extension of the controlled ODE to a setting where the control X is replaced by the non smooth function B.

A *rough differential equation* is an equation which generalises and includes all equations above, however for a driving path X which is deterministic (unlike the Brownian motion B) and typically non-smooth. Neither classical nor Itô integration are available in this case, and are replaced by the *rough integral* of Chapter 8, namely by an extensive use of the Sewing Lemma 1.10.

4.1. Taylor expansion to second order

If $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$, then we have to modify the argument we used in chapter 3 for the smooth controlled equation

$$\dot{Y}_t = \sigma(Y_t)\,\dot{X}_t. \tag{4.3}$$

We suppose for the moment that $X \in C^1([0,T];\mathbb{R}^d)$. We rewrite, for $s < t$,

$$\begin{aligned}
Y_t - Y_s &= \int_s^t \dot{Y}_r\,\mathrm{d}r\\
&= \int_s^t \sigma(Y_r)\,\dot{X}_r\,\mathrm{d}r\\
&= \int_s^t \left(\sigma(Y_s) + \int_s^r \frac{\mathrm{d}}{\mathrm{d}v}(\sigma(Y_v))\,\mathrm{d}v\right)\dot{X}_r\,\mathrm{d}r\\
&= \sigma(Y_s)(X_t - X_s) + \int_s^t \left(\int_s^r \nabla\sigma(Y_v)\,\sigma(Y_v)\,\dot{X}_v\,\mathrm{d}v\right)\dot{X}_r\,\mathrm{d}r.
\end{aligned}$$

We next expand, for $s<r$,

$$\begin{aligned}
&\int_s^r \nabla\sigma(Y_v)\,\sigma(Y_v)\,\dot{X}_v\,\mathrm{d}v=\\
=&\int_s^r \left(\nabla\sigma(Y_s)\,\sigma(Y_s)+\int_s^v \frac{\mathrm{d}}{\mathrm{d}w}\,(\nabla\sigma(Y_w)\,\sigma(Y_w))\,\mathrm{d}w\right)\dot{X}_v\,\mathrm{d}v\\
=&\nabla\sigma(Y_s)\,\sigma(Y_s)\,(X_r-X_s)+\int_s^r O(|v-s|)\,\dot{X}_v\,\mathrm{d}v\\
=&\nabla\sigma(Y_s)\,\sigma(Y_s)\,(X_r-X_s)+O(|r-s|^2),
\end{aligned}$$

hence

$$\begin{aligned}
&Y_t-Y_s=\\
=&\sigma(Y_s)(X_t-X_s)+\int_s^t \nabla\sigma(Y_s)\,\sigma(Y_s)\,(X_r-X_s)\otimes\dot{X}_r\,\mathrm{d}r+\int_s^t O(|r-s|^2)\,\dot{X}_r\,\mathrm{d}r\\
=&\sigma(Y_s)(X_t-X_s)+\sigma_2(Y_s)\int_s^t (X_r-X_s)\otimes\dot{X}_r\,\mathrm{d}r+O(|t-s|^3), \qquad (4.4)
\end{aligned}$$

where, for $x,y\in\mathbb{R}^d$, we define $x\otimes y\in\mathbb{R}^{d\times d}$ by

$$x\otimes y:=(x_i\,y_j)_{1\leqslant i,j\leqslant d}\,,$$

and where we introduce the notation $\sigma_2\colon\mathbb{R}^k\to\mathbb{R}^k\otimes\mathbb{R}^d\otimes\mathbb{R}^d$

$$\sigma_2(y):=\nabla\sigma(y)\,\sigma(y), \qquad [\sigma_2(y)]^{ijm}:=\sum_{a=1}^k \nabla_a\,\sigma^{ij}(y)\,\sigma^{am}(y).$$

Here we introduce the notations $\mathbb{X}^1\colon[0,T]^2_\leqslant\to\mathbb{R}^d$, $\mathbb{X}^2\colon[0,T]^2_\leqslant\to\mathbb{R}^d\otimes\mathbb{R}^d$

$$\mathbb{X}^1_{st}:=X_t-X_s, \qquad \mathbb{X}^2_{st}:=\int_s^t (X_r-X_s)\otimes\dot{X}_r\,\mathrm{d}r, \qquad 0\leqslant s\leqslant t\leqslant T. \qquad (4.5)$$

We note now the following interesting formula

$$\mathbb{X}^2_{st}-\mathbb{X}^2_{su}-\mathbb{X}^2_{ut}=\mathbb{X}^1_{su}\otimes\mathbb{X}^1_{ut}, \qquad 0\leqslant s\leqslant u\leqslant t\leqslant T, \qquad (4.6)$$

which follows from

$$\mathbb{X}^2_{st}-\mathbb{X}^2_{su}-\mathbb{X}^2_{ut} \;=\; \int_u^t (X_u-X_s)\otimes\dot{X}_r\,\mathrm{d}r=(X_u-X_s)\otimes(X_t-X_u).$$

Moreover

$$|\mathbb{X}^1_{st}|\lesssim|t-s|, \qquad |\mathbb{X}^2_{st}|\lesssim|t-s|^2. \qquad (4.7)$$

The controlled equation (4.3) can be rewritten therefore

$$Y_t-Y_s\;=\sigma(Y_s)\,\mathbb{X}^1_{st}+\sigma_2(Y_s)\,\mathbb{X}^2_{st}+O(|t-s|^3), \qquad 0\leqslant s\leqslant t\leqslant T. \qquad (4.8)$$

Note that we have, in coordinates,

$$(\sigma_2(y)\,\mathbb{X}^2_{st})^i = \sum_{j,m=1}^{d} [\sigma_2(y)]^{ijm}\,(\mathbb{X}^2_{st})^{mj} = \sum_{j,m=1}^{d}\sum_{a=1}^{k} \nabla_a\,\sigma^{ij}(y)\,\sigma^{am}(y)\,(\mathbb{X}^2_{st})^{mj}.$$

4.2. Finite difference equations

Suppose now that $X\colon[0,T]\to\mathbb{R}^d$ is of class C^α with $\alpha\in\left(\frac{1}{3},\frac{1}{2}\right]$. We would like to give an analog of the controlled equation (4.3). In order to do so, one can use a generalisation of (4.4). For that, we define again

$$\mathbb{X}^1_{st} := X_t - X_s, \qquad |\mathbb{X}^1_{st}| \lesssim |t-s|^\alpha, \tag{4.9}$$

but the definition of $\mathbb{X}^2$ in (4.5) does not make sense anymore.

We are going to show in this chapter that, remarkably, it is always possible to construct a robust theory for the controlled equation (4.3) with X of class C^α with $\alpha\in\left(\frac{1}{3},\frac{1}{2}\right)$, provided we *choose* a function $\mathbb{X}^2\colon[0,T]^2_\leqslant\to\mathbb{R}^d\otimes\mathbb{R}^d$ satisfying for $0\leqslant s\leqslant u\leqslant t\leqslant T$

$$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su}\otimes\mathbb{X}^1_{ut}, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}, \tag{4.10}$$

recall (4.6)-(4.7). The existence of such a choice will be proved below.

Definition 4.1. *Let $\alpha\in(1/3,1/2]$, $d\in\mathbb{N}$ and $X\in C^\alpha([0,T];\mathbb{R}^d)$. A d-dimensional α-rough path over X is a pair $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ with $\mathbb{X}^1\colon[0,T]^2_\leqslant\to\mathbb{R}^d$, $\mathbb{X}^2\colon[0,T]^2_\leqslant\to\mathbb{R}^d\otimes\mathbb{R}^d$ satisfying for $0\leqslant s\leqslant u\leqslant t\leqslant T$*

$$\begin{aligned}&\mathbb{X}^1_{st} := X_t - X_s, \qquad \mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su}\otimes\mathbb{X}^1_{ut},\\ &|\mathbb{X}^1_{st}|\lesssim|t-s|^\alpha, \qquad |\mathbb{X}^2_{st}|\lesssim|t-s|^{2\alpha}.\end{aligned} \tag{4.11}$$

We call $\mathcal{R}_{\alpha,d}(X)$ the set of d-dimensional α-rough paths over X and $\mathcal{R}_{\alpha,d}$ the set of all d-dimensional α-rough paths.

In the Young case $\alpha>\frac{1}{2}$, the smooth controlled equation (3.2) was reformulated as the finite difference equation (3.5). In the case $\alpha>\frac{1}{3}$, taking inspiration from (4.8) we look for $\mathrm{y}\colon[0,T]\to\mathbb{R}^d$ such that

$$\delta\mathrm{y}_{st} = \sigma(\mathrm{y}_s)\,\mathbb{X}^1_{st} + \sigma_2(\mathrm{y}_s)\,\mathbb{X}^2_{st} + o(t-s), \qquad 0\leqslant s\leqslant t\leqslant T. \tag{4.12}$$

This equation expresses a *generalised Taylor expansion* of the solution y with respect to the rough path $\mathbb{X}$. More precisely, we give the following

Definition 4.2. *Let $\alpha>1/3$ and $\mathbb{X}\in\mathcal{R}_{\alpha,d}$ a rough path. A solution to (4.12) is a $\mathrm{y}\in\mathcal{C}^\alpha([0,T];\mathbb{R}^k)$ such that for some $\zeta>1$*

$$|\mathrm{y}^3_{st}|\lesssim|t-s|^\zeta, \qquad \mathrm{y}^3_{st} = \delta\mathrm{y}_{st} - \sigma(\mathrm{y}_s)\,\mathbb{X}^1_{st} - \sigma_2(\mathrm{y}_s)\,\mathbb{X}^2_{st}. \tag{4.13}$$

This definition extends the classical one in the case of differentiable driving path $X\colon [0,T] \to \mathbb{R}^d$. By the Sewing Lemma, if y^3 satisfies (4.13) then it actually satisfies the same property with $\zeta = 3\alpha$.

LEMMA 4.3. *Let* y *be a solution to (4.13) as in Definition 4.2. Then* y^3 *defined by (4.13) also satisfies*

$$|\mathrm{y}^3_{st}| \lesssim |t-s|^{3\alpha}, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Proof. Since $\delta \circ \delta = 0$, by (4.11) and (4.13) we have, analogously to (3.8),

$$\delta \mathrm{y}^3_{sut} = (\sigma(\mathrm{y}_u) - \sigma(\mathrm{y}_s) - \sigma_2(\mathrm{y}_s)\, \mathbb{X}^1_{su})\, \mathbb{X}^1_{ut} + (\sigma_2(\mathrm{y}_u) - \sigma_2(\mathrm{y}_s))\, \mathbb{X}^2_{ut}. \tag{4.14}$$

By (3.8) we obtain that $\delta \mathrm{y}^3 \in C_3^{3\alpha}$, so that, by the Sewing Lemma, $\Lambda(\delta \mathrm{y}^3) \in C_2^{3\alpha}$. Then $\mathrm{y}^3 - \Lambda(\delta \mathrm{y}^3) \in C_2^{\zeta \wedge (2\alpha)}$ and $\delta(\mathrm{y}^3 - \Lambda(\delta \mathrm{y}^3)) = 0$, which implies that $\mathrm{y}^3 - \Lambda(\delta \mathrm{y}^3) = 0$ by the uniqueness statement of Lemma 1.3. □

PROPOSITION 4.4. *Let* $X\colon [0,T] \to \mathbb{R}^d$ *of class* C^1 *and let us consider the canonical rough path* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *defined in (4.5). If* $Y\colon [0,T] \to \mathbb{R}^d$ *is a solution to (4.3), then* $y := Y$ *satisfies (4.13) for any* $\alpha < 1$.

Proof. By a Taylor expansion in time we have (4.13) for any $\alpha < 1$. □

As in Proposition 3.4 below for Young equations, we first state a *local existence* result.

PROPOSITION 4.5. *Let* $y_0 \in \mathbb{R}^d$. *We suppose that* σ *and* σ_2 *are of class* C^1 *and globally Lipschitz, namely* $\|\nabla \sigma\|_\infty + \|\nabla \sigma_2\|_\infty < +\infty$. *Let* $D := \max\{1, \|\nabla \sigma\|_\infty, \|\nabla \sigma_2\|_\infty\}$ *and* $M > 0$.

There exists $T_{M,D,\alpha} > 0$ *such that, for all* $T \in (0, T_{M,D,\alpha})$ *and* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{\alpha,d}$ *such that* $\|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} \leqslant M$, *there exists a solution* y *to (4.13) on the interval* $[0,T]$ *such that* $\mathrm{y}_0 = y_0$ *and*

$$\|\mathrm{y}\|_\alpha \leqslant 15\, M(|\sigma(y_0)| + |\sigma_2(y_0)|). \tag{4.15}$$

The proof of this Proposition is not based on the Sewing Lemma but on a discretization argument. For the reader's convenience, it is postponed to section 4.8 below.

4.3. MAIN TECHNICAL TOOL

In Chapter 3 the main tool to study the Young equation (3.6), besides the Sewing Lemma, was the Lipschitz estimate of Lemma 3.5. In this chapter, these tools are still crucial, but an additional ingredient is needed. This is provided by the next elementary

LEMMA 4.6. *Let $y_1, y_2 \in \mathbb{R}^d$ and $x \in \mathbb{R}^k$. If σ is of class C^1, then denoting $\delta y := y_2 - y_1$, we have*

$$\begin{aligned}&\sigma(y_2) - \sigma(y_1) - \sigma_2(y_1)x = \\ &= \nabla\sigma(y_1)(\delta y - \sigma(y_1)x) + \int_0^1 [\nabla\sigma(y_1 + r\delta y) - \nabla\sigma(y_1)]\,\mathrm{d}r\,\delta y,\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}\sigma(y_2) - \sigma(y_1) - \sigma_2(y_1)\,x \;=\; & \int_0^1 (\sigma_2(y_1 + u\delta y) - \sigma_2(y_1))\,\mathrm{d}u\,x + \\ & + \int_0^1 \nabla\sigma(y_1 + u\delta y)\,\mathrm{d}u\,(\delta y - \sigma(y_1)\,x) + \\ & - \int_0^1 \nabla\sigma(y_1 + u\delta y)\int_0^u \nabla\sigma(y_1 + v\delta y)\,\mathrm{d}v\,\mathrm{d}u\,\delta y\,x.\end{aligned} \tag{4.17}$$

Proof. The first formula is based on elementary manipulations and on the fact that

$$\sigma(y_2) - \sigma(y_1) = \int_0^1 \nabla\sigma(y_1 + r\delta y)\,\mathrm{d}r\,\delta y.$$

For the second formula we start with the same remark, we write

$$\begin{aligned}&\sigma(y_2) - \sigma(y_1) = \int_0^1 \nabla\sigma(y_1 + r\delta y)\,\mathrm{d}r\,\delta y = \\ &= \int_0^1 \nabla\sigma(y_1 + u\delta y)\,\mathrm{d}u\,(\delta y - \sigma(y_1)x) + \underbrace{\int_0^1 \nabla\sigma(y_1 + u\delta y)\,\mathrm{d}u\,\sigma(y_1)x}_{A}\end{aligned}$$

and then

$$A \;=\; \int_0^1 \sigma_2(y_1 + u\delta y)\,\mathrm{d}u\,x - \underbrace{\int_0^1 \nabla\sigma(y_1 + u\delta y)\,(\sigma(y_1 + u\delta y) - \sigma(y_1))\,\mathrm{d}u\,x}_{B}.$$

Finally

$$B \;=\; \int_0^1 \nabla\sigma(y_1 + u\delta y)\int_0^u \nabla\sigma(y_1 + v\delta y)\,\mathrm{d}v\,\mathrm{d}u\,\delta y\,x$$

and (4.16) follows easily. □

We'll see below that (4.16) is very useful for the comparison between *two solutions*, as in the proofs of uniqueness (Theorem 4.7) and continuity of the solution map (Theorem 4.10), while (4.16) is well suited for a priori estimates on a single solution (Lemma 4.8) or on a discretization scheme (Lemma 4.11).

4.4. UNIQUENESS

Let us suppose that $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ is of class C^3, without any boundedness assumption. We show that this implies uniqueness of solutions to (4.13).

THEOREM 4.7. (UNIQUENESS) *Let $\alpha > 1/3$ and $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{\alpha,d}$ a rough path. If σ is of class C^3, then for every $\mathrm{y}_0 \in \mathbb{R}^d$ there exists at most one solution y to (4.13).*

Proof. If y and $\bar{\mathrm{y}}$ are two solutions, set $\mathrm{z} := \mathrm{y} - \bar{\mathrm{y}}$. We want to show that, for $\tau \in \,]0,1]$ small enough, $\|\mathrm{z}\|_{\infty,\tau} \leqslant 2|\mathrm{z}_0|$. We recall that by (4.14) for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\delta \mathrm{y}^3_{sut} = \underbrace{(\sigma(\mathrm{y}_u) - \sigma(\mathrm{y}_s) - \sigma_2(\mathrm{y}_s)\, \mathbb{X}^1_{su})}_{B_{su}}\, \mathbb{X}^1_{ut} + (\delta\sigma_2(\mathrm{y}))_{su}\, \mathbb{X}^2_{ut},$$

and analogously for $\delta\bar{\mathrm{y}}^3$. By (4.16) we have for $0 \leqslant s \leqslant t \leqslant T$

$$B_{st} \;=\; \underbrace{\nabla\sigma(\mathrm{y}_s)\, \mathrm{y}^2_{st}}_{E_{st}} + \underbrace{\int_0^1 [\nabla\sigma(\mathrm{y}_s + r\delta \mathrm{y}_{st}) - \nabla\sigma(\mathrm{y}_s)]\, \mathrm{d}r\, \delta \mathrm{y}_{st}}_{F_{st}}$$

with analogous notations for $\bar{B}_{st}$, etc. We set z^2 as in (3.13) and $\mathrm{z}^3_{st} := \mathrm{y}^3_{st} - \bar{\mathrm{y}}^3_{st}$ as in (4.13). By (4.14) we have

$$\delta \mathrm{z}^3_{sut} = (B_{su} - \bar{B}_{su})\, \mathbb{X}^1_{ut} + (\delta\sigma_2(\mathrm{y}) - \delta\sigma_2(\bar{\mathrm{y}}))_{su}\, \mathbb{X}^2_{ut}.$$

Using the notation in (3.10) we set

$$R := \|\mathrm{y}\|_\infty + \|\bar{\mathrm{y}}\|_\infty, \qquad L := C_{\nabla\sigma,R} + C_{\nabla^2\sigma,R} + C_{\nabla^3\sigma,R}.$$

We want to estimate $\|\delta \mathrm{z}^3\|_{3\alpha,\tau}$. We use a number of times the elementary estimate

$$|ab - cd| = |ab - ac + ac - cd| \leqslant |a|\,|b - c| + |c|\,|a - d|$$

for $a, b, c, d \in \mathbb{R}$. We start by estimating

$$\|E - \bar{E}\|_{2\alpha,\tau} \;\leqslant\; L(\|\mathrm{z}^2\|_{2\alpha,\tau} + \|\mathrm{y}^2\|_{2\alpha} \|\mathrm{z}\|_{\infty,\tau}).$$

Now, by (3.11)

$$\begin{aligned}&|(\nabla\sigma(\mathrm{y}_s + r\delta \mathrm{y}_{st}) - \nabla\sigma(\mathrm{y}_s)) - \nabla\sigma(\bar{\mathrm{y}}_s + r\delta\bar{\mathrm{y}}_{st}) - \nabla\sigma(\bar{\mathrm{y}}_s)|\\ &\leqslant L(|\delta \mathrm{z}_{st}| + (|\delta \mathrm{y}_{st}| + |\delta\bar{\mathrm{y}}_{st}|)|\mathrm{z}_s|),\end{aligned}$$

so that

$$\|F - \bar{F}\|_{2\alpha,\tau} \;\leqslant\; L\,(\|\delta \mathrm{y}\|_\alpha + \|\delta\bar{\mathrm{y}}\|_\alpha)(\|\delta \mathrm{y}\|_\alpha \|\mathrm{z}\|_{\infty,\tau} + \|\delta \mathrm{z}\|_{\alpha,\tau}).$$

Moreover by (3.11)

$$\|\delta\sigma_2(\mathrm{y}) - \delta\sigma_2(\bar{\mathrm{y}})\|_{\alpha,\tau} \;\leqslant\; L(\|\delta \mathrm{z}\|_{\alpha,\tau} + (\|\delta \mathrm{y}\|_\alpha + \|\delta\bar{\mathrm{y}}\|_\alpha)\|\mathrm{z}\|_{\infty,\tau}).$$

Therefore there is a constant $C_{\mathrm{y},\bar{\mathrm{y}}} > 0$ such that

$$\|\delta \mathrm{z}^3\|_{3\alpha,\tau} \leqslant MLC_{\mathrm{y},\bar{\mathrm{y}}}(\|\mathrm{z}\|_{\infty,\tau} + \|\delta \mathrm{z}\|_{\alpha,\tau} + \|\mathrm{z}^2\|_{2\alpha,\tau})$$

where $M := \|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha}$. By the Sewing estimate in weighted spaces (1.33)

$$\|\mathrm{z}^3\|_{3\alpha,\tau} \leqslant K_{3\alpha}\|\delta \mathrm{z}^3\|_{3\alpha,\tau} \;\leqslant\; K_{3\alpha}MLC_{\mathrm{y},\bar{\mathrm{y}}}(\|\mathrm{z}\|_{\infty,\tau} + \|\delta \mathrm{z}\|_{\alpha,\tau} + \|\mathrm{z}^2\|_{2\alpha,\tau}).$$

We estimate now $\|\mathrm{z}\|_{\infty,\tau} + \|\delta \mathrm{z}\|_{\alpha,\tau} + \|\mathrm{z}^2\|_{2\alpha,\tau}$. First, we know by (1.29) that

$$\|\mathrm{z}\|_{\infty,\tau} \leqslant |\mathrm{z}_0| + 3\tau^\alpha \|\delta \mathrm{z}\|_{\alpha,\tau}. \tag{4.18}$$

Now, note that

$$\|\sigma(\mathrm{y}) - \sigma(\bar{\mathrm{y}})\|_{\infty,\tau} + \|\sigma_2(\mathrm{y}) - \sigma_2(\bar{\mathrm{y}})\|_{\infty,\tau} \leqslant L\|\mathrm{z}\|_{\infty,\tau}.$$

By (4.13) this implies

$$\|\delta \mathrm{z}\|_{\alpha,\tau} \leqslant LM\|\mathrm{z}\|_{\infty,\tau} + \tau^{2\alpha}\|\mathrm{z}^3\|_{3\alpha,\tau}. \tag{4.19}$$

By the definitions of y^2 and y^3

$$|\mathrm{z}^2_{st}| = |\mathrm{y}^2_{st} - \bar{\mathrm{y}}^2_{st}| \;\leqslant\; |\mathrm{z}^3_{st}| + |(\sigma_2(\mathrm{y}_s) - \sigma_2(\bar{\mathrm{y}}_s))|\,|\mathbb{X}^2_{st}|,$$

so that

$$\|\mathrm{z}^2\|_{2\alpha,\tau} \;\leqslant\; LM\|\mathrm{z}\|_{\infty,\tau} + \tau^\alpha\|\mathrm{z}^3\|_{3\alpha,\tau}.$$

Therefore there exists a constant $C_{M,L,\mathrm{y},\bar{\mathrm{y}}}$ such that

$$\|\mathrm{z}^3\|_{3\alpha,\tau} \;\leqslant\; C_{M,L,\mathrm{y},\bar{\mathrm{y}}}\,(\|\mathrm{z}\|_{\infty,\tau} + (\tau^\alpha + \tau^{2\alpha})\|\mathrm{z}^3\|_{3\alpha,\tau}).$$

By choosing $\tau > 0$ small enough, we have $\|\mathrm{z}^3\|_{3\alpha,\tau} \leqslant 2C_{M,L,\mathrm{y},\bar{\mathrm{y}}}\,\|\mathrm{z}\|_{\infty,\tau}$. Using (4.18) and (4.19) we have now for another constant $C'_{M,L,\mathrm{y},\bar{\mathrm{y}}} > 0$

$$\|\mathrm{z}\|_{\infty,\tau} \leqslant |\mathrm{z}_0| + \tau^\alpha\, C'_{M,L,\mathrm{y},\bar{\mathrm{y}}}\,\|\mathrm{z}\|_{\infty,\tau}$$

and by choosing if necessary $\tau > 0$ even smaller, we have $\|\mathrm{z}\|_{\infty,\tau} \leqslant 2|\mathrm{z}_0|$. In particular, if $\mathrm{z}_0 = 0$, then $\mathrm{z} \equiv 0$. □

4.5. A PRIORI ESTIMATE

In this section we suppose that σ and σ_2 are of class C^1 and globally Lipschitz, namely $\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty < +\infty$ (without boundedness assumptions on σ and σ_2). We fix

$$D \geq \|\nabla\sigma\|_\infty + \|\nabla\sigma\|_\infty^2 + \|\nabla\sigma_2\|_\infty.$$

LEMMA 4.8. *Let $M > 0$ and $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{\alpha,d}$ such that $\|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} \leqslant M$. There there is $\varepsilon_{M,D} > 0$ such that, if $(\tau \wedge T)^\alpha \leqslant \varepsilon_{M,D}$, then any solution to (4.13) satisfies*

$$\|\delta \mathrm{y}\|_{\alpha,\tau} + \|\mathrm{y}^2\|_{2\alpha,\tau} \leqslant 4M(|\sigma(\mathrm{y}_0)| + |\sigma_2(\mathrm{y}_0)|) \tag{4.20}$$

and

$$\|\delta\sigma(\mathrm{y}) - \sigma_2(\mathrm{y})\,\mathbb{X}^2\|_{2\alpha,\tau} \leqslant 4M(M\vee 1)D(|\sigma(\mathrm{y}_0)| + |\sigma_2(\mathrm{y}_0)|). \tag{4.21}$$

Proof. We recall that by (4.14) for $0\leqslant s\leqslant u\leqslant t\leqslant T$

$$\delta\mathrm{y}^3_{sut} = \underbrace{(\sigma(\mathrm{y}_u) - \sigma(\mathrm{y}_s) - \sigma_2(\mathrm{y}_s)\,\mathbb{X}^1_{su})}_{B_{su}}\,\mathbb{X}^1_{ut} + (\delta\sigma_2(\mathrm{y}))_{su}\,\mathbb{X}^2_{ut}.$$

By (4.16), for $0\leqslant s\leqslant u\leqslant T$

$$\begin{aligned} B_{su} \;=\; & \int_0^1 (\sigma_2(\mathrm{y}_s + u\delta\mathrm{y}_{su}) - \sigma_2(\mathrm{y}_s))\,\mathrm{d}u\,\mathbb{X}^1_{su} + \int_0^1 \nabla\sigma(\mathrm{y}_s + u\delta\mathrm{y}_{su})\,\mathrm{d}u\,\mathrm{y}^2_{su} \\ & - \int_0^1 \nabla\sigma(\mathrm{y}_s + u\delta\mathrm{y}_{su}) \int_0^u \nabla\sigma(\mathrm{y}_s + v\delta\mathrm{y}_{su})\,\mathrm{d}v\,\mathrm{d}u\,\delta\mathrm{y}_{su}\,\mathbb{X}^1_{su}, \end{aligned}$$

so that

$$\|B\|_{2\alpha,\tau} \leqslant M(\|\nabla\sigma_2\|_\infty + \|\nabla\sigma\|_\infty^2)\|\delta\mathrm{y}\|_{\alpha,\tau} + \|\nabla\sigma\|_\infty \|\mathrm{y}^2\|_{2\alpha,\tau}.$$

By (4.14) we obtain

$$\|\delta\mathrm{y}^3\|_{3\alpha,\tau} \leqslant M(M+1)D\,(\|\delta\mathrm{y}\|_{\alpha,\tau} + \|\mathrm{y}^2\|_{2\alpha,\tau}), \tag{4.22}$$

and by the Sewing estimate in weighted spaces (1.33)

$$\|\mathrm{y}^3\|_{3\alpha,\tau} \leqslant C_{M,D}(\|\delta\mathrm{y}\|_{\alpha,\tau} + \|\mathrm{y}^2\|_{2\alpha,\tau}),$$

where $C_{M,D}>0$ is an explicit constant whose value can vary from a line to the next. Since $\mathrm{y}^2 = \mathrm{y}^3 + \sigma_2(\mathrm{y})\,\mathbb{X}^2$, we have denoting $\varepsilon := (\tau\wedge T)^\alpha$

$$\|\mathrm{y}^2\|_{2\alpha,\tau} \leqslant M\,\|\sigma_2(\mathrm{y})\|_{\infty,\tau} + \varepsilon\, C_{M,D}(\|\delta\mathrm{y}\|_{\alpha,\tau} + \|\mathrm{y}^2\|_{2\alpha,\tau}),$$

and since $\delta\mathrm{y} = \mathrm{y}^2 + \sigma(\mathrm{y})\,\mathbb{X}^1$

$$\|\delta\mathrm{y}\|_{\alpha,\tau} \leqslant M\,\|\sigma(\mathrm{y})\|_{\infty,\tau} + M\varepsilon\,\|\sigma_2(\mathrm{y})\|_{\infty,\tau} + \varepsilon^2\, C_{M,D}(\|\delta\mathrm{y}\|_{\alpha,\tau} + \|\mathrm{y}^2\|_{2\alpha,\tau}).$$

Since by (1.29)

$$\|\sigma(\mathrm{y})\|_{\infty,\tau} + \|\sigma_2(\mathrm{y})\|_{\infty,\tau} \leqslant |\sigma(\mathrm{y}_0)| + |\sigma_2(\mathrm{y}_0)| + 3\varepsilon D\|\delta\mathrm{y}\|_{\alpha,\tau},$$

we have

$$\begin{aligned} \|\delta\mathrm{y}\|_{\alpha,\tau} + \|\mathrm{y}^2\|_{2\alpha,\tau} \;\leqslant\; & (1+\varepsilon)M(|\sigma(\mathrm{y}_0)| + |\sigma_2(\mathrm{y}_0)|) + \\ & + C_{M,D}\,(\varepsilon + \varepsilon^2)\,(\|\delta\mathrm{y}\|_{\alpha,\tau} + \|\mathrm{y}^2\|_{2\alpha,\tau}). \end{aligned}$$

If $\varepsilon = \varepsilon_{M,D} \in (0,1)$ is such that $C_{M,D}(\varepsilon + \varepsilon^2) \leqslant \frac{1}{2}$, then if $(\tau\wedge T)^\alpha \in (0, \varepsilon_{M,D})$ we obtain

$$\|\delta\mathrm{y}\|_{\alpha,\tau} + \|\mathrm{y}^2\|_{2\alpha,\tau} \leqslant 4M(|\sigma(\mathrm{y}_0)| + |\sigma_2(\mathrm{y}_0)|).$$

Finally we obtain (4.21) since

$$\|B\|_{2\alpha,\tau} \leqslant 4M(M\vee 1)D(|\sigma(\mathrm{y}_0)| + |\sigma_2(\mathrm{y}_0)|).$$

The proof is complete. □

4.6. Global existence and uniqueness

Let us suppose that $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes\mathbb{R}^d$ is of class C^3 with $\|\nabla\sigma\|_\infty+\|\nabla\sigma_2\|_\infty<+\infty$.

Theorem 4.9. *If $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes\mathbb{R}^d$ is of class C^3 with $\|\nabla\sigma\|_\infty+\|\nabla\sigma_2\|_\infty<+\infty$ then for every $\mathrm{y}_0\in\mathbb{R}^d$ and $T>0$ there is a unique solution $(\mathrm{y}_t)_{t\in[0,T]}$ to (4.13).*

Proof. By Theorem 4.7 we have at most one solution. We now construct a solution on an arbitrary finite interval $[0,T]$. We define $\Lambda\subseteq[0,T]$ as the set of all s such that there is a solution $(\mathrm{y}_t)_{t\in[0,s]}$ to (4.13). By Proposition 4.5, Λ is an open subset of $[0,T]$ and contains 0. By the a priori estimates of Lemma 4.8, Λ is a closed subset of $[0,T]$. Therefore $\Lambda=[0,T]$. □

4.7. Continuity of the solution map

We consider now the map $\mathbb{R}^d\times\mathcal{R}_{\alpha,d}\ni(\mathrm{y}_0,\mathbb{X})\mapsto\mathrm{y}=\Phi(\mathrm{y}_0,\mathbb{X})\in C^\alpha$, where y is the unique solution to (3.6) constructed in Theorem 3.8 and $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$. We want to show that this map, called *the solution map*, is continuous. This property is highly non-trivial.

We suppose in this section that σ is of class C^3, with $\|\nabla\sigma\|_\infty+\|\nabla^2\sigma\|_\infty+\|\nabla^3\sigma\|_\infty+\|\nabla\sigma_2\|_\infty+\|\nabla^2\sigma_2\|_\infty<+\infty$ (without boundedness assumptions on σ and σ_2). We fix

$$D\geq\|\nabla\sigma\|_\infty+\|\nabla^2\sigma\|_\infty+\|\nabla^3\sigma\|_\infty+\|\nabla\sigma_2\|_\infty+\|\nabla^2\sigma_2\|_\infty.$$

Proposition 4.10. *Let $M>0$ and let us suppose that*

$$\max\{|\sigma(\mathrm{y}_0)|+|\sigma(\bar{\mathrm{y}}_0)|+|\sigma_2(\bar{\mathrm{y}}_0)|,\|\mathbb{X}^1\|_\alpha+\|\mathbb{X}^2\|_{2\alpha},\|\bar{\mathbb{X}}^1\|_\alpha+\|\bar{\mathbb{X}}^2\|_{2\alpha}\}\leqslant M.$$

Then for every $T>0$ there are $\hat{\tau}_{M,D,T},C_{M,D,T}>0$ such that for $\tau\in\,]0,\hat{\tau}_{M,D,T}]$

$$\begin{aligned}&\|\mathrm{y}-\bar{\mathrm{y}}\|_{\infty,\tau}+\|\delta\mathrm{y}-\delta\bar{\mathrm{y}}\|_{\alpha,\tau}+\|\mathrm{y}^2-\bar{\mathrm{y}}^2\|_{2\alpha,\tau}\leqslant\\&\quad\leqslant C_{M,D,T}\,(|\mathrm{y}_0-\bar{\mathrm{y}}_0|+\|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\alpha+\|\mathbb{X}^2-\bar{\mathbb{X}}^2\|_{2\alpha}).\end{aligned}$$

Proof. If y and $\bar{\mathrm{y}}$ are two solutions, set $\mathrm{z}:=\mathrm{y}-\bar{\mathrm{y}}$. We recall that by (4.14) for $0\leqslant s\leqslant u\leqslant t\leqslant T$

$$\delta\mathrm{y}^3_{sut}=\underbrace{(\sigma(\mathrm{y}_u)-\sigma(\mathrm{y}_s)-\sigma_2(\mathrm{y}_s)\,\mathbb{X}^1_{su})}_{B_{su}}\,\mathbb{X}^1_{ut}+(\delta\sigma_2(\mathrm{y}))_{su}\,\mathbb{X}^2_{ut},$$

and analogously for $\delta\bar{\mathrm{y}}^3$ and $\bar{B}_{su}$. In particular for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\begin{aligned}\delta \mathrm{z}^3_{sut} &= (B_{su} - \bar{B}_{su})\, \mathbb{X}^1_{ut} + \bar{B}_{su}\, (\mathbb{X}^1 - \bar{\mathbb{X}}^1)_{ut} \\ &\quad + (\delta\sigma_2(\mathrm{y}) - \delta\sigma_2(\bar{\mathrm{y}}))_{su}\, \mathbb{X}^2_{ut} + \delta\sigma_2(\bar{\mathrm{y}})_{su}\, (\mathbb{X}^2 - \bar{\mathbb{X}}^2)_{ut}. \end{aligned} \tag{4.23}$$

By (4.16) we have for $0 \leqslant s \leqslant t \leqslant T$

$$B_{st} = \underbrace{\nabla\sigma(\mathrm{y}_s)\, \mathrm{y}^2_{st}}_{E_{st}} + \underbrace{\int_0^1 [\nabla\sigma(\mathrm{y}_s + r\delta\mathrm{y}_{st}) - \nabla\sigma(\mathrm{y}_s)]\, \mathrm{d}r\, \delta\mathrm{y}_{st}}_{F_{st}}$$

with analogous notations for $\bar{E}_{st}$ and $\bar{F}_{st}$. We set z^2 as in (3.13) and $\mathrm{z}^3_{st} := \mathrm{y}^3_{st} - \bar{\mathrm{y}}^3_{st}$ as in (4.13).

We want to estimate $\|\delta \mathrm{z}^3\|_{3\alpha,\tau}$. We use a number of times the elementary estimate

$$|ab - cd| = |ab - ac + ac - cd| \leqslant |a|\, |b - c| + |c|\, |a - d|$$

for $a, b, c, d \in \mathbb{R}$. We start by estimating

$$\|E - \bar{E}\|_{2\alpha,\tau} \leqslant D(\|\mathrm{z}^2\|_{2\alpha,\tau} + \|\mathrm{y}^2\|^{\text{No-exp}}_{2\alpha,\tau} \|\mathrm{z}\|_{\infty,\tau}),$$

where the seminorms $\|\cdot\|^{\text{No-exp}}_{\eta,\tau}$ are defined in (3.16). Now, by (3.11)

$$\begin{aligned}&|(\nabla\sigma(\mathrm{y}_s + r\delta\mathrm{y}_{st}) - \nabla\sigma(\mathrm{y}_s)) - \nabla\sigma(\bar{\mathrm{y}}_s + r\delta\bar{\mathrm{y}}_{st}) - \nabla\sigma(\bar{\mathrm{y}}_s)| \\ &\quad \leqslant D(|\delta \mathrm{z}_{st}| + (|\delta\mathrm{y}_{st}| + |\delta\bar{\mathrm{y}}_{st}|)|\mathrm{z}_s|),\end{aligned}$$

so that

$$\|F - \bar{F}\|_{2\alpha,\tau} \leqslant D\, \|\delta\mathrm{y}\|^{\text{No-exp}}_{\alpha,\tau} c_{\mathrm{y},\bar{\mathrm{y}}}\, (\|\mathrm{z}\|_{\infty,\tau} + \|\delta\mathrm{z}\|_{\alpha,\tau}).$$

By (3.11) we have

$$\begin{aligned}&\|\delta\sigma(\mathrm{y}) - \delta\sigma(\bar{\mathrm{y}})\|_{\alpha,\tau} + \|\delta\sigma_2(\mathrm{y}) - \delta\sigma_2(\bar{\mathrm{y}})\|_{\alpha,\tau} + \|\delta\nabla\sigma(\mathrm{y}) - \delta\nabla\sigma(\bar{\mathrm{y}})\|_{\alpha,\tau} \leqslant \\ &\quad \leqslant D c_{\mathrm{y},\bar{\mathrm{y}}}\, (\|\mathrm{y} - \bar{\mathrm{y}}\,\|_{\infty,\tau} + \|\delta\mathrm{y} - \delta\bar{\mathrm{y}}\|_{\alpha,\tau}),\end{aligned} \tag{4.24}$$

where

$$c_{\mathrm{y},\bar{\mathrm{y}}} = (\|\delta\mathrm{y}\|^{\text{No-exp}}_{\alpha,\tau} + \|\delta\bar{\mathrm{y}}\,\|^{\text{No-exp}}_{\alpha,\tau} + 1)\,.$$

In particular

$$\|\delta\sigma_2(\mathrm{y}) - \delta\sigma_2(\bar{\mathrm{y}})\|_{\alpha,\tau} \leqslant D c_{\mathrm{y},\bar{\mathrm{y}}}\, (\|\mathrm{z}\|_{\infty,\tau} + \|\delta\mathrm{z}\|_{\alpha,\tau}).$$

By Lemma 4.8, if $\tau^{\alpha}_{M,D} = \varepsilon_{M,D}$, then

$$\begin{aligned}\|\delta\mathrm{y}\|^{\text{No-exp}}_{\alpha,\tau_{M,D}} + \|\mathrm{y}^2\|^{\text{No-exp}}_{\alpha,\tau_{M,D}} + \|\delta\bar{\mathrm{y}}\,\|^{\text{No-exp}}_{\alpha,\tau_{M,D}} &\leqslant e^{\frac{T}{\tau_{M,D}}}\, 8M^2 \\ \|\bar{B}\|_{2\alpha,\tau} &\leqslant 4D(M+1)^3.\end{aligned}$$

Therefore there is a constant $C_{T,M,D} > 0$ such that for all $\tau \in (0, \tau_{M,D})$

$$\begin{aligned}\|\delta\mathrm{z}^3\|_{3\alpha,\tau} &\leqslant C_{T,M,D}(\|\mathrm{z}\|_{\infty,\tau} + \|\delta\mathrm{z}\|_{\alpha,\tau} + \|\mathrm{z}^2\|_{2\alpha,\tau}) + \\ &\quad + C_{T,M,D}(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_{\alpha} + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}).\end{aligned}$$

By the definitions of y^2 and y^3

$$|\mathrm{z}^2_{st}| = |\mathrm{y}^2_{st} - \bar{\mathrm{y}}^2_{st}| \leqslant |\mathrm{z}^3_{st}| + |(\sigma_2(\mathrm{y}_s) - \sigma_2(\bar{\mathrm{y}}_s))|\,|\mathbb{X}^2_{st}| + |\sigma_2(\bar{\mathrm{y}}_s)|\,|\mathbb{X}^2_{st} - \bar{\mathbb{X}}^2_{st}|.$$

The last term can be bounded as follows

$$\||\sigma_2(\bar{\mathrm{y}})|\,|\mathbb{X}^2 - \bar{\mathbb{X}}^2|\|_{2\alpha,\tau} \leqslant (|\sigma_2(\mathrm{y}_0)| + \tau^\alpha D \|\delta\bar{\mathrm{y}}\|^{\text{No-exp}}_{\alpha,\tau})\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}$$

and therefore

$$\|\mathrm{z}^2\|_{2\alpha,\tau} \lesssim_{M,D} \|\mathrm{z}\|_{\infty,\tau} + \tau^\alpha \|\mathrm{z}^3\|_{3\alpha,\tau} + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}.$$

Therefore by the Sewing estimate in weighted spaces (1.33) and by (4.19)

$$\|\mathrm{z}^3\|_{3\alpha,\tau} \lesssim_{M,D,T} \|\mathrm{z}\|_{\infty,\tau} + (\tau^\alpha + \tau^{2\alpha})\|\mathrm{z}^3\|_{3\alpha,\tau} + \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}.$$

By choosing $\tau > 0$ small enough, we obtain

$$\|\mathrm{z}^3\|_{3\alpha,\tau} \lesssim_{M,D,T} \|\mathrm{z}\|_{\infty,\tau} + \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha},$$

so that by (4.18)

$$\|\mathrm{z}\|_{\infty,\tau} \lesssim_{M,D,T} |\mathrm{z}_0| + \tau^\alpha \|\mathrm{z}\|_{\infty,\tau} + \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}$$

and there exists $\hat{\tau}_{M,D,T} \leqslant \tau_{M,D}$ such that, for $\tau \in \,]0, \hat{\tau}_{M,D,T}]$, we obtain

$$\|\mathrm{z}\|_{\infty,\tau} + \|\delta \mathrm{z}\|_{\alpha,\tau} + \|\mathrm{z}^2\|_{2\alpha,\tau} \lesssim_{M,D,T} |\mathrm{z}_0| + \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}.$$

The proof is complete. □

4.8. Milstein scheme and local existence

In this section we prove the local existence result of Proposition 4.5, under the assumption that σ, σ_2 are of class C^1 and uniformly Lipschitz. To construct a solution to (4.8), we set $t_i := \frac{i}{n}$, $i \geqslant 0$, and

$$y_{i+1} = y_i + \sigma(y_i)\,\mathbb{X}^1_{t_i t_{i+1}} + \sigma_2(y_i)\,\mathbb{X}^2_{t_i t_{i+1}}, \qquad i \geqslant 0.$$

We set $D := \max\{1, \|\nabla\sigma\|_\infty, \|\nabla\sigma_2\|_\infty\}$ and

$$\begin{aligned} \delta y_{ij} &:= y_j - y_i, \\ \|\delta y\|_\alpha &:= n^\alpha \sup_{0<i<j\leqslant nT} \frac{|y_j - y_i|}{|j-i|^\alpha}, \\ A_{ij} &:= \sigma(y_i)\,\mathbb{X}^1_{t_i t_j} + \sigma_2(y_i)\,\mathbb{X}^2_{t_i t_j}. \end{aligned}$$

The main technical estimate is the following

LEMMA 4.11. *Let $M > 0$. There exists $T_{M,D,\alpha} > 0$ such that, for all $T \in (0, T_{M,D,\alpha})$ and $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{\alpha,d}$ such that $\|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} \leqslant M$, we have*

$$\begin{aligned} \|\delta y\|_\alpha &\leqslant 5M(|\sigma(y_0)| + |\sigma_2(y_0)|), \\ |\delta y_{ik} - A_{ik}| &\lesssim_{M,D,\alpha} (|\sigma(y_0)| + |\sigma_2(y_0)|)\left(\frac{|k-i|}{n}\right)^{3\alpha}, \qquad 0 \leqslant i \leqslant k \leqslant nT. \end{aligned}$$

Proof. We set

$$L(y) \ := \ \frac{2}{1-2^{1-3\alpha}} 2DM(M|\sigma_2(y_0)|+2(DM+1)\,\|\delta y\|_\alpha).$$

We want first to obtain the following estimate

$$|\delta y_{ik}-A_{ik}| \leqslant L(y)\left(\frac{|k-i|}{n}\right)^{3\alpha}, \qquad \forall 0\leqslant i\leqslant k\leqslant nT. \tag{4.25}$$

Note that (4.25) holds if $k\in\{i,i+1\}$. Let $m\geqslant 1$ and suppose that (4.25) holds for all $i,k\leqslant nT$ such that $0\leqslant k-i\leqslant m$. We want to show that (3.26) holds for all $i,k\leqslant nT$ such that $k-i=m+1$; for such i,k, we set $j=i+\left\lfloor\frac{k-i}{2}\right\rfloor$, so that $0\leqslant j-i\leqslant\frac{k-i}{2}\leqslant m$ and $0\leqslant k-j-1\leqslant\frac{k-i}{2}\leqslant m$.

Now, since $3\alpha>1$, we have

$$|j-i|^{3\alpha}+|k-j-1|^{3\alpha}\leqslant 2^{1-3\alpha}\,|k-i|^{3\alpha}.$$

We set

$$\delta A_{ijk}:=A_{ik}-A_{ij}-A_{jk}.$$

Then, since $A_{j\,(j+1)}-y_{j+1}+y_j=0$,

$$\begin{aligned}
|\delta y_{ik}-A_{ik}| &\leqslant |\delta A_{ijk}|+|\delta y_{ij}-A_{ij}|+|\delta y_{jk}-A_{jk}|\\
&\leqslant |\delta A_{ijk}|+|\delta y_{ij}-A_{ij}|+|\delta A_{j\,(j+1)\,k}|+|\delta y_{(j+1)k}-A_{j+1k}|\\
&\leqslant |\delta A_{ijk}|+|\delta A_{j\,(j+1)\,k}|+L(y)\,2^{1-3\alpha}\left(\frac{|k-i|}{n}\right)^{3\alpha},
\end{aligned} \tag{4.26}$$

where we have used the recurrence assumption in the third inequality. Now, analogously to (4.14)

$$\delta A_{ijk} \ = \ \underbrace{\left(\sigma(y_j)-\sigma(y_i)-\sigma_2(y_i)\,\mathbb{X}^1_{t_it_j}\right)}_{B_{ij}}\otimes\mathbb{X}^1_{t_jt_k}+\underbrace{(\sigma_2(y_i)-\sigma_2(y_j))}_{C_{ij}}\,\mathbb{X}^2_{t_jt_k}.$$

We want an estimate $|\delta A_{ijk}|\lesssim\left(\frac{|k-i|}{n}\right)^{3\alpha}$. For that, it is enough to obtain $|B_{ij}|\lesssim\left(\frac{|j-i|}{n}\right)^{2\alpha}$ and $|C_{ij}|\lesssim\left(\frac{|j-i|}{n}\right)^{\alpha}$. We set

$$D_{ij} \ := \ \delta y_{ij}-\sigma(y_i)\,\mathbb{X}^1_{t_it_j},$$

and by (4.16) we obtain

$$\begin{aligned}
&B_{ij}=\sigma(y_j)-\sigma(y_i)-\sigma_2(y_i)\,\mathbb{X}^1_{t_it_j}=\\
&=\underbrace{\int_0^1(\sigma_2(y_i+u\delta y_{ij})-\sigma_2(y_i))\,\mathbb{X}^1_{t_it_j}\mathrm{d}u}_{E_{ij}}+\underbrace{\int_0^1\nabla\sigma(y_i+u\delta y_{ij})\,\mathrm{d}u\,D_{ij}}_{F_{ij}}\\
&\underbrace{-\int_0^1\nabla\sigma(y_i+u\delta y_{ij})(\sigma(y_i+u\delta y_{ij})-\sigma(y_i))\mathbb{X}^1_{t_it_j}\mathrm{d}u}_{G_{ij}}.
\end{aligned}$$

First

$$|E_{ij}| \leqslant \|\nabla\sigma_2\|_\infty \|\delta y\|_\alpha \|\mathbb{X}^1\|_\alpha \left(\frac{|j-i|}{n}\right)^{2\alpha}$$
$$\leqslant DM\|\delta y\|_\alpha \left(\frac{|j-i|}{n}\right)^{2\alpha}.$$

Similarly

$$|G_{ij}| \leqslant \|\nabla\sigma\|_\infty^2 \|\delta y\|_\alpha \|\mathbb{X}^1\|_\alpha \left(\frac{|j-i|}{n}\right)^{2\alpha}$$
$$\leqslant D^2 M\|\delta y\|_\alpha \left(\frac{|j-i|}{n}\right)^{2\alpha}.$$

By the induction hypothesis and the definition of A_{ij}

$$|D_{ij}| \leqslant |\delta y_{ij} - A_{ij}| + |\sigma_2(y_i)\, \mathbb{X}^2_{t_i t_j}|$$
$$\leqslant (T^\alpha L(y) + (|\sigma_2(y_0)| + T^\alpha \|\nabla\sigma_2\|_\infty \|\delta y\|_\alpha)\|\mathbb{X}^2\|_{2\alpha}) \left(\frac{|j-i|}{n}\right)^{2\alpha}$$
$$\leqslant (T^\alpha L(y) + M|\sigma_2(y_0)| + T^\alpha DM\|\delta y\|_\alpha) \left(\frac{|j-i|}{n}\right)^{2\alpha}.$$

Therefore

$$|F_{ij}| \leqslant D|D_{ij}|$$
$$\leqslant D(T^\alpha L(y) + M|\sigma_2(y_0)| + T^\alpha DM\|\delta y\|_\alpha) \left(\frac{|j-i|}{n}\right)^{2\alpha}.$$

Finally

$$|B_{ij}| \leqslant |E_{ij}| + |F_{ij}| + |G_{ij}|$$
$$\leqslant D\,[M|\sigma_2(y_0)| + T^\alpha L(y) + DM(2+T^\alpha)\|\delta y\|_\alpha] \left(\frac{|j-i|}{n}\right)^{2\alpha}.$$

Analogously

$$|C_{ij}| = |\sigma_2(y_j) - \sigma_2(y_i)| \leqslant D\|\delta y\|_\alpha \left(\frac{|j-i|}{n}\right)^{\alpha}.$$

Therefore

$$|\delta A_{ijk}| \leqslant$$
$$\leqslant DM(M|\sigma_2(y_0)| + T^\alpha L(y) + (2DM + T^\alpha DM + 1)\|\delta y\|_\alpha) \left(\frac{|k-i|}{n}\right)^{3\alpha},$$

and we have the same bound for $\delta A_{j(j+1)k}$. Therefore by (4.25), if $T^\alpha DM \leqslant 1$,

$$|\delta y_{ik} - A_{ik}| \leqslant \left(\frac{|k-i|}{n}\right)^{3\alpha} [2DM(M|\sigma_2(y_0)| + 2(DM+1)\|\delta y\|_\alpha) + (2^{1-3\alpha} + 2T^\alpha DM)L(y)].$$

If furthermore $2T^{\alpha}DM \leqslant \frac{1-2^{1-3\alpha}}{2}$, then $2^{1-3\alpha} + 2T^{\alpha}DM \leqslant \frac{1+2^{1-3\alpha}}{2}$ and by the definition of $L(y)$ we obtain

$$\begin{aligned}|\delta y_{ik} - A_{ik}| &\leqslant \left(\frac{|k-i|}{n}\right)^{3\alpha}\left(\frac{1-2^{1-3\alpha}}{2}L(y) + \frac{1+2^{1-3\alpha}}{2}L(y)\right)\\ &= \left(\frac{|k-i|}{n}\right)^{3\alpha} L(y),\end{aligned}$$

and (4.25) is proven for all n. Now we obtain by (4.25)

$$\begin{aligned}\|\delta y\|_{\alpha} &\leqslant n^{\alpha} \sup_{0<i<j\leqslant nT} \frac{|\delta y_{ij} - A_{ij}| + |A_{ij}|}{|j-i|^{\alpha}}\\ &\leqslant T^{2\alpha}L(y) + (|\sigma(y_0)| + |\sigma_2(y_0)| + 2DMT^{\alpha}\|\delta y\|_{\alpha})M.\end{aligned}$$

Since we have already assumed that $2DMT^{\alpha} \leqslant \frac{1}{2}$, we obtain

$$\|\delta y\|_{\alpha} \leqslant 2T^{2\alpha}L(y) + 2M(|\sigma(y_0)| + |\sigma_2(y_0)|).$$

By the definition of $L(y)$, if furthermore $\frac{8DM^2(1+4D)}{1-2^{1-3\alpha}}T^{2\alpha} \leqslant \frac{1}{2}$, we obtain finally

$$\begin{aligned}\|\delta y\|_{\alpha} &\leqslant 5M(|\sigma(y_0)| + |\sigma_2(y_0)|),\\ L(y) &\leqslant \frac{4DM^2}{1-2^{1-3\alpha}}(|\sigma_2(y_0)| + 10(DM+1)(|\sigma(y_0)| + |\sigma_2(y_0)|)) =: K,\end{aligned}$$

and by (4.25)

$$|\delta y_{ik} - A_{ik}| \leqslant K\left(\frac{|k-i|}{n}\right)^{3\alpha}, \qquad \forall 0 \leqslant i \leqslant k \leqslant nT. \qquad \square$$

Proof of Proposition 4.5. Arguing as in Proposition 3.4 we obtain the result of local existence for equation (4.13) of Proposition 4.5. $\square$

4.9. INTEGRAL FORMULATION

In this section we interpret the finite difference equation (4.13) as an *integral equation*. In section 3.8 we did this for the Young equation (3.6), using the Young integral of section 2.2. In the setting $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, the Young integral is not adapted, since the germ $A_{st} := \sigma(\mathrm{y}_s)\,\delta X_{st}$ has the property $\delta A \in C_2^{2\alpha}$ and $2\alpha \leqslant 1$, so that the Sewing Lemma can not be applied. However the equation (4.13) suggests *another* germ:

$$A_{st} := \sigma(\mathrm{y}_s)\,\mathbb{X}^1_{st} + \sigma_2(\mathrm{y}_s)\,\mathbb{X}^2_{st}, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Note that $A = \delta\mathrm{y} - \mathrm{y}^3$, in the notation (4.13). Then by (4.22) we know that $\delta A \in C_3^{3\alpha}$. Therefore we can interpret the formula

$$\delta\mathrm{y} = A - \Lambda(\delta A)$$

as

$$\mathrm{y}_t = \mathrm{y}_0 + \int_0^t \sigma(\mathrm{y}_s)\,\mathrm{d}\mathbb{X}_s, \qquad 0 \leqslant t \leqslant T,$$

which for the moment is only a notation that will be made more precise in chapter 8.

Chapter 5
Rough paths

We have seen in Chapter 4 that it is possible to build a robust theory for a controlled equation of the form $\dot{Y}_t = \sigma(Y_t)\,\dot{X}_t$ with $X: [0,T] \to \mathbb{R}^d$ of class C^α for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, provided we *choose* a function $\mathbb{X}^2: [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\delta\mathbb{X}^2_{sut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha},$$

see (4.10), where we denote $\mathbb{X}^1_{st} := \delta X_{st}$, $0 \leqslant s \leqslant t \leqslant T$. In coordinates, the former identity means

$$(\delta\mathbb{X}^2)^{ij}_{sut} = \delta X^i_{su} \otimes \delta X^j_{ut}, \qquad |(\mathbb{X}^2_{st})^{ij}| \lesssim |t-s|^{2\alpha}, \qquad i, j \in \{1, \ldots, d\}. \tag{5.1}$$

In Section 4.2 we left the problem of the existence of such a function $\mathbb{X}^2$ open.

We recall that, for X of class C^1, we have a natural choice for $\mathbb{X}^2$ given by

$$(\mathbb{X}^2_{st})^{ij} := \int_s^t (X^i_r - X^i_s)\,\dot{X}^j_r\,\mathrm{d}r, \qquad 0 \leqslant s \leqslant t \leqslant T,$$

see (4.5). In Lemma 2.11 we saw that, for $\alpha > \frac{1}{2}$ and $X \in C^\alpha([0,T]; \mathbb{R}^d)$, the (uniquely defined) Young integral $I^{ij}_t := \int_0^t X^i\,\mathrm{d}X^j$ satisfies

$$R^{ij}_{st} := I^{ij}_t - I^{ij}_s - X^i_s\,(X^j_t - X^j_s) = \int_s^t (X^i_r - X^i_s)\,\mathrm{d}X^j_r, \qquad |R^{ij}_{st}| \lesssim |t-s|^{2\alpha},$$

where the integral in the right-hand side is again of the Young type and $2\alpha > 1$.

There is a clear resemblance between the two last expressions, and indeed for $\alpha > \frac{1}{2}$ we show in Lemma 5.16 below that setting $(\mathbb{X}^2_{st})^{ij} := R^{ij}_{st}$ we obtain (5.1) and this is the only possible choice.

If now $\alpha \leqslant \frac{1}{2}$, neither of these formulae is well-defined, because for $2\alpha \leqslant 1$ we are not in the setting of the Young integral. However, we have seen in Chapter 4 that the bound $|\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}$ is enough for the whole theory of existence, uniqueness and stability of the rough equation (4.13) to work, even if $2\alpha \leqslant 1$.

This suggests that, for every $i, j \in \{1, \ldots, d\}$, the function $(\mathbb{X}^2_{st})^{ij}$ can be interpreted as the remainder R^{ij} associated with an integral I^{ij} of (X^i, X^j), where we *weaken* our requirements with respect to the Young integral, namely we only require that

$$I^{ij}_t - I^{ij}_s - X^i_s\,(X^j_t - X^j_s) = (\mathbb{X}^2_{st})^{ij}, \qquad |(\mathbb{X}^2_{st})^{ij}| \lesssim |t-s|^{2\alpha},$$

and now $2\alpha \leqslant 1$. Therefore the choice of the rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ over X is equivalent to the choice of a *generalised integral* $I = \int_0^{\cdot} X \otimes \mathrm{d}X \in C^{\alpha}([0,T]; \mathbb{R}^d \otimes \mathbb{R}^d)$, and in this case $\mathbb{X}^2$ plays the role of a generalised remainder with respect to the germ $(s,t) \mapsto X_s \otimes (X_t - X_s)$.

In this chapter we explore these notions and explain them in greater detail.

5.1. INTEGRAL BEYOND YOUNG

Let us fix $(X, Y) \in \mathcal{C}^{\alpha} \times \mathcal{C}^{\beta}$. We saw in Theorem 2.7 that when $\alpha + \beta > 1$ we can define the integral $I_t = \int_0^t Y \mathrm{d}X$ as the unique function which solves

$$I_0 = 0\,, \qquad \delta I_{st} = Y_s\, \delta X_{st} + R_{st}, \qquad R_{st} = o(|t-s|)\,. \tag{5.2}$$

This was based on the observation that for the germ $A_{st} := Y_s\, \delta X_{st}$ we have

$$\delta A_{sut} = -\delta Y_{su}\, \delta X_{ut} \qquad \Longrightarrow \qquad \|\delta A\|_{\alpha+\beta} \leqslant \|\delta X\|_{\alpha} \|\delta Y\|_{\beta}\,.$$

Therefore if $\eta := \alpha + \beta > 1$ we have $\|\delta A\|_{\eta} < \infty$, i.e. the germ A is coherent, see Definition 1.9, and the Sewing Lemma can be applied, see Theorem 1.10.

We now focus on the regime $\alpha + \beta \leqslant 1$. As we have already seen in (2.10) above, there exist germs A which allow *no function I solving (5.2)*. Indeed, we recall that choosing $X_t = t^{\alpha}$ and $Y_t = t^{\beta}$, $t \in [0,T]$, then the germ $A_{st} := Y_s\, \delta X_{st}$ satisfies $|\delta A_{0\frac{t}{2}t}| \gtrsim t^{\alpha+\beta}$, see (2.10), and therefore the necessary condition (1.13) in Lemma 1.7 is not satisfied.

A solution is to relax the requirement $R_{st} = o(|t-s|)$ in (5.2), say to

$$\exists \eta \leqslant 1: \qquad |R_{st}| \lesssim |t-s|^{\eta}. \tag{5.3}$$

Arguing as in 1.7, this would imply $|\delta R_{sut}| \lesssim |u-s|^{\eta} + |t-u|^{\eta}$. On the other hand, by 1.6 we have $|\delta R_{sut}| = |\delta A_{sut}| \lesssim |u-s|^{\beta} |t-u|^{\alpha}$. Choosing $|u-s| = |t-u|$ shows that the best we can hope for in (5.3) is $\eta = \alpha + \beta$.

Summarizing, given $(X, Y) \in \mathcal{C}^{\alpha} \times \mathcal{C}^{\beta}$ with $\alpha + \beta \leqslant 1$, it is natural to wonder whether there exists a function I which satisfies the following weakening of (5.2)

$$I_0 = 0\,, \qquad \delta I_{st} = Y_s\, \delta X_{st} + R_{st}\,, \qquad |R_{st}| \lesssim |t-s|^{\alpha+\beta}\,. \tag{5.4}$$

This would provide a "generalised notion of integral" $\int_0^{\cdot} Y \mathrm{d}X$. This justifies the following

DEFINITION 5.1. *Fix $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$. Given $(f, g) \in C^{\alpha} \times C^{\beta}$, if there exists a function $I: [0,T] \to \mathbb{R}$ which satisfies*

$$I_t - I_s = f_s\, (g_t - g_s) + O(|t-s|^{\alpha+\beta}) \qquad \textit{uniformly as } |t-s| \to 0, \tag{5.5}$$

we say that I is an integral of (f, g).

We stress that this new definition of integral extends the previous one (5.6), for $(f,g) \in C^\alpha \times C^\beta$ with $\alpha + \beta > 1$, because the term $o(|t-s|)$ is actually $O(|t-s|^{\alpha+\beta})$ in this case, by the key estimate for the Young integral (or, equivalently, for the sewing map).

On the positive side, *there is always existence for (5.4)* if $\alpha + \beta < 1$. This is a non-trivial result, due (in a more general setting) to Lyons and Victoir. We state this as a separate result, which is a consequence of Proposition 5.8 below.

Lemma 5.2. *Let $(X,Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ with $\alpha + \beta < 1$. There exists $(I,R) \in \mathcal{C}^\alpha \times C_2^{\alpha+\beta}$ satisfying (5.4).*

However it is an easy observation that *uniqueness can not hold for (5.4)*. Indeed, given I which solves (5.4), any function of the form $I'_t := I_t + h_t - h_0$ with $h \in \mathcal{C}^{\alpha+\beta}$ still solves (5.4). As a matter of fact, *all solutions are of this form*, because given two solutions I, I' of (5.4), with corresponding R, R', their difference $h := I' - I$ must satisfy $|\delta h_{st}| = |R'_{st} - R_{st}| \lesssim |t-s|^{\alpha+\beta}$.

5.2. Two negative results

The Young integral $I_t = \int_0^t f \, \mathrm{d}g$, defined in Theorem 2.5 for $(f,g) \in C^\alpha \times C^\beta$ with $\alpha + \beta > 1$, is the unique function $I: [0,T] \to \mathbb{R}$ with $I_0 = 0$ which satisfies

$$I_t - I_s = f_s (g_t - g_s) + o(|t-s|), \qquad \text{uniformly as } |t-s| \to 0\,. \tag{5.6}$$

We now turn to the regime $\alpha + \beta \leqslant 1$. Let us recall that

$$A_{st} := f_s \, \delta g_{st} \qquad \Longrightarrow \qquad \delta A_{sut} = -\delta f_{su} \, \delta g_{ut}\,. \tag{5.7}$$

We first show that *one cannot hope to find a solution of (5.6)* for generic $(f,g) \in C^\alpha \times C^\beta$ with $\alpha + \beta < 1$.

Lemma 5.3. *Fix $\alpha, \beta \in (0,1)$ with $\alpha + \beta \leqslant 1$. Then there are $(f,g) \in C^\alpha \times C^\beta$ such that there is no function $I: [0,T] \to \mathbb{R}$ which satisfies (5.6).*

The proof is based on the following general result, of independent interest.

Lemma 5.4. *Given $I \in C_1$ and $A \in C_2$, define $R \in C_2$ by*

$$I_t - I_s = A_{st} + R_{st}\,. \tag{5.8}$$

If $R_{st} = o(|t-s|)$, uniformly as $|t-s| \to 0$, then also $\delta A_{sut} = o(|t-s|)$.

Proof. We show that, more generally, if $|R_{st}| \leqslant h(|t-s|)$ for some non-decreasing function $h: [0,\infty) \to \mathbb{R}$, then $|\delta A_{sut}| \leqslant 3h(|t-s|)$. To this purpose, note that $\delta(\delta I) = 0$, hence relation (5.8) implies $\delta A = -\delta R$ and then

$$|\delta R_{sut}| = |R_{st} - R_{su} - R_{ut}| \leqslant |R_{st}| + |R_{su}| + |R_{ut}| \leqslant 3h(|t-s|)\,,$$

which completes the proof. □

Proof of Lemma 5.3. By Lemma 5.4 and relation (5.7), *a necessary condition* for the existence of a function I which satisfies (5.6) is that for $s<u<t$

$$\delta f_{su}\,\delta g_{ut}=o(|t-s|) \quad \text{uniformly as } |t-s|\to 0\,. \tag{5.9}$$

Given $\alpha,\beta\in(0,1)$ with $\alpha+\beta\leqslant 1$, it is easy to find $(f,g)\in C^\alpha\times C^\beta$ for which (5.9) fails. For instance, fix $\bar{u}\in(a,b)$ and define $f(x):=|x-\bar{u}|^\alpha$ and $g(x):=|x-\bar{u}|^\beta$. Choosing $s=\bar{u}-\delta$ and $t=\bar{u}+\delta$, we have

$$\delta f(s,\bar{u})\,\delta g(\bar{u},t)=|\bar{u}-s|^\alpha\,|t-\bar{u}|^\beta=\delta^{\alpha+\beta}=\frac{|t-s|^{\alpha+\beta}}{2^{\alpha+\beta}}$$

which is clearly not $o(|t-s|)$ as $|t-s|\to 0$. □

Next we show that the usual integral $I(f,g)=\int_0^t f_s\,g_s'\,\mathrm{d}s$, when $g\in C^1$, cannot be extended to a continuous operator on $C^{\alpha'}\times C^{\beta'}$, when $\alpha'+\beta'<1$.

LEMMA 5.5. *Set* $[0,T]=[0,1]$ *and define, for* $\alpha,\beta\in(0,1)$,

$$f_n(t):=\frac{1}{n^\alpha}\cos(nt)\,,\qquad g_n(t):=\frac{1}{n^\beta}\sin(nt)\,.$$

Then $f_n\rightsquigarrow_\alpha 0$ *and* $g_n\rightsquigarrow_\beta 0$ *(recall Definition 2.12), more precisely:*

$$\|f_n\|_\infty\to 0\,,\qquad \|f_n\|_\alpha\leqslant 2\,;\qquad\qquad \|g_n\|_\infty\to 0\,,\qquad \|g_n\|_\beta\leqslant 2\,. \tag{5.10}$$

(In particular, $f_n\to 0$ *in* $C^{\alpha'}$ *and* $g_n\to 0$ *in* $C^{\beta'}$ *for any* $\alpha'<\alpha$ *and* $\beta'<\beta$*.)*

However, if we fix $\alpha+\beta\leqslant 1$*, we have* $I(f_n,g_n)\not\to 0$*, because*

$$\forall t\in[0,1]:\qquad \lim_{n\to\infty} I(f_n,g_n)_t=\begin{cases}+\infty & \text{if } \alpha+\beta<1\\ \frac{1}{2}t & \text{if } \alpha+\beta=1\\ 0 & \text{if } \alpha+\beta>1\end{cases}.$$

Proof. Note that $\|f_n\|_\infty=n^{-\alpha}$ and $\|f_n'\|_\infty=n^{1-\alpha}$, hence

$$|f_{n_t}-f_{n_s}|\leqslant\min\{\|f_n'\|_\infty|t-s|,\,2\,\|f_n\|_\infty\}\leqslant\min\{n^{1-\alpha}|t-s|,2\,n^{-\alpha}\}\,.$$

Since $\min\{x,y\}\leqslant x^\gamma y^{1-\gamma}$, for any $\gamma\in[0,1]$, choosing $\gamma=\alpha$ we obtain

$$|f_n(t)-f_n(s)|\leqslant 2^{1-\alpha}\,|t-s|^\alpha\,,$$

hence $\|f_n\|_\alpha\leqslant 2^{1-\alpha}\leqslant 2$. Similar arguments apply to g_n, proving (5.10).

Next we observe that $\frac{1}{2\pi}\int_0^{2\pi}\cos^2(x)\,\mathrm{d}x=\frac{1}{2\pi}\int_0^{2\pi}\sin^2(x)\,\mathrm{d}x=\frac{1}{2}$. Then, for fixed $t>0$, as $n\to\infty$

$$\int_0^{nt}\cos^2(x)\,\mathrm{d}x=\int_0^{2\pi\lfloor\frac{nt}{2\pi}\rfloor}\cos^2(x)\,\mathrm{d}x+O(1)=\frac{1}{2}2\pi\left\lfloor\frac{nt}{2\pi}\right\rfloor+O(1)=\frac{t}{2}n+O(1)\,.$$

It follows that

$$I(f_n,g_n)_t=\frac{n}{n^{\alpha+\beta}}\int_0^t\cos^2(ns)\,\mathrm{d}s=\frac{1}{n^{\alpha+\beta}}\int_0^{nt}\cos^2(x)\,\mathrm{d}x\sim\frac{t}{2}\,n^{1-(\alpha+\beta)}\,.$$ □

In view of Lemma 5.3, in order to define a generalised integral $\int f\,\mathrm{d}g$ when $(f, g) \in C^\alpha \times C^\beta$ with $\alpha + \beta < 1$, we have to relax relation (5.6). We do this replacing the term $o(|t-s|)$ by $O(|t-s|^{\alpha+\beta})$. This is, in a sense, the best we can hope for, because the example built in Lemma 5.3, together with Lemma 5.4, shows that we cannot have $O(|t-s|^\gamma)$ with $\gamma > 1$.

Remark 5.6. Finding an integral I of (f, g) is equivalent to finding a function R_{st} with

$$\delta R_{sut} = \delta f_{su}\, \delta g_{ut}\,, \tag{5.11}$$

$$R_{st} = O(|t-s|^{\alpha+\beta}) \quad \text{uniformly as } |t-s| \to 0\,. \tag{5.12}$$

Indeed, if we define A as in (5.7), relation (5.11) implies that $\delta(A+R) = 0$, hence there exists $I\colon [0,T] \to \mathbb{R}$ which satisfies $\delta I = A + R$, which is exactly relation (5.5).

Remark 5.7. An integral I as in Definition 5.1 is necessarily of class C^β by (5.5).

We state now a result which implies Lemma 5.2 above.

PROPOSITION 5.8. (PARAINTEGRAL) *Fix $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$. There exists a (non unique) bilinear and continuous map $J_\prec\colon C^\alpha \times C^\beta \to C_2^{\alpha+\beta}$ such that*

$$\|J_\prec(f, g)\|_{\alpha+\beta} \leqslant \mathsf{C}\, \|\delta f\|_\alpha\, \|\delta g\|_\beta\,, \tag{5.13}$$

for a suitable $\mathsf{C} = \mathsf{C}(\alpha, \beta, T)$, with the property that, for all $s < u < t$,

$$\delta J_\prec(f, g)_{sut} = \delta f_{su}\, \delta g_{ut}\,. \tag{5.14}$$

It follows that any $(f, g) \in C^\alpha \times C^\beta$ admits an integral I as in Definition 5.1.

The proof of Proposition 5.8 is postponed to Section 5.9 below.

Remark 5.9. By Proposition 5.8, for all $(f, g) \in C^\alpha \times C^\beta$ with $\alpha + \beta < 1$ there exists an integral as in Definition 5.1. As a consequence, *there are infinitely many integrals and all of them differ by a function in $C^{\alpha+\beta}$.* Indeed, if I satisfies (5.5), given an arbitrary $h \in C^{\alpha+\beta}$ also $I + h$ satisfies (5.5). Viceversa, if I and I' satisfy (5.5), $h := I - I'$ satisfies $\delta h = O(|t-s|^{\alpha+\beta})$, that is $h \in C^{\alpha+\beta}$.

5.3. A CHOICE

We have seen in (2.16) above that, given $(X, Y) \in C^\alpha \times C^\beta$ with $\alpha + \beta > 1$, we have an *explicit formula* for the remainder $R_{st} = I_t - I_s - Y_s(X_t - X_s)$, given by

$$R_{st} = \int_s^t (Y_u - Y_s)\, \mathrm{d}X_u, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{5.15}$$

where $I_t = \int_0^t Y_u \, \mathrm{d}X_u$ is the *unique* function given by the Young integral of Theorem 2.5. Moreover $R_{st} = \int_s^t (Y_u - Y_s) \, \mathrm{d}X_u$ is the unique function in C_2 which satisfies

$$R \in C_2^{\alpha+\beta}, \qquad \delta R_{sut} = \delta Y_{su} \, \delta X_{ut}, \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{5.16}$$

In the regime $\alpha + \beta < 1$, the Young integral is not available anymore. However by Proposition 5.8 we know that we can find an integral $I \in C^\beta$ in the sense of Definition 5.1 by setting

$$\delta I_{st} := Y_s \, (X_t - X_s) - J_{\prec}(X, Y)_{st},$$

where $J_{\prec}$ is the paraintegral of Proposition 5.8, see also Remark 5.6. This shows that, in this setting, the remainder $R_{st} = I_t - I_s - Y_s \, (X_t - X_s)$ is not given by an explicit formula like (5.15) (which is now ill-defined), rather we have

$$R = -J_{\prec}(X, Y).$$

However formula (5.15) suggests that we can *define*

$$\int_s^t (Y_u - Y_s) \, \mathrm{d}X_u := R_{st} = -J_{\prec}(X, Y)_{st}, \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{5.17}$$

In other words, the left hand side of (5.17) is *chosen* to be equal to the remainder R associated with the integral I as in (5.4). We recall that $R = -J_{\prec}(X, Y)$ satisfies

$$R \in C_2^{\alpha+\beta}, \qquad \delta R_{sut} = \delta Y_{su} \, \delta X_{ut}, \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{5.18}$$

The difference between formula (5.18) and formula (5.16), is that in the former $\alpha + \beta < 1$ while in the latter $\alpha + \beta > 1$. Accordingly, in (5.18) the function R is *not* uniquely determined, while in (5.16) it is.

The comparison between formula (5.18) and formula (5.16), and the explicit expression (5.15) in the case $\alpha + \beta > 1$ show that (5.17) is a reasonable *definition* of the function $(s, t) \mapsto \int_s^t (Y_u - Y_s) \, \mathrm{d}X_u$ in the setting $\alpha + \beta \leqslant 1$.

We also stress that R in (5.18) *can not be uniquely determined.* Indeed, for any $h \in C^{\alpha+\beta}$, the function $R' := R + \delta h$ satisfies the same equality; the integral associated with R' as in (5.4) is $I' = I + h - h_0$. In fact, *all possible solutions are of this form*, because given two integrals I, I' with corresponding remainders R, R' as in (5.4), , their difference $h := I' - I$ must satisfy $|\delta h_{st}| = |R'_{st} - R_{st}| \lesssim |t - s|^{\alpha+\beta}$. In other words we have infinitely many possible choices given by

$$(I', R') = (I + h, R + \delta h), \qquad h \in \mathcal{C}^{\alpha+\beta}, h_0 = 0. \tag{5.19}$$

Remark 5.10. In the special case $X = Y$ and $\alpha = \beta \leqslant \frac{1}{2}$, (5.2) becomes

$$I_0 = 0\,, \qquad \delta I_{st} = X_s\, \delta X_{st} + R_{st}\,, \qquad |R_{st}| \lesssim |t-s|^{2\alpha}\,. \tag{5.20}$$

Now the germ is $A_{st} = X_s(X_t - X_s)$ and we have a simple canonical solution which does not rely on the paraintegral and is given by

$$I_t := \frac{1}{2}(X_t^2 - X_0^2), \qquad R_{st} := \frac{1}{2}(X_t - X_s)^2,$$

since

$$\underbrace{\frac{1}{2}\,(X_t^2 - X_s^2)}_{I_t - I_s} = \underbrace{X_s(X_t - X_s)}_{A_{st}} + \underbrace{\frac{1}{2}(X_t - X_s)^2}_{R_{st}}.$$

As we have seen in (2.14)-(2.15), if $\alpha > 1/2$ then (I, R) is the *only* solution of (5.20) and moreover

$$R_{st} = \int_s^t (X_r - X_s)\, \mathrm{dX}_r$$

where the integral is in the Young sense. If $\alpha \leqslant \frac{1}{2}$, then we have infinitely many possible solutions (I', R').

5.4. One-dimensional rough paths

We have seen at the beginning of this chapter that for every $i, j \in \{1, \ldots, d\}$, the function $(\mathbb{X}^2_{st})^{ij}$ plays the role of the remainder R^{ij} associated with a generalised integral I^{ij} of (X^i, X^j) in the sense of Definition 5.1 with $\alpha = \beta < \frac{1}{2}$: in other words the choice of $\mathbb{X}^2$ is *equivalent* to the choice of integrals (in the sense of Definition 5.1) $I^{ij} \in \mathcal{C}^\alpha$ for all $i, j \in \{1, \ldots, d\}$, such that

$$I^{ij}_0 = 0\,, \qquad \delta I^{ij}_{st} = X^i_s\, \delta X^j_{st} + (\mathbb{X}^2_{st})^{ij}\,, \qquad |(\mathbb{X}^2_{st})^{ij}| \lesssim |t-s|^{2\alpha}\,,$$

or, in more compact notations,

$$I_0 = 0\,, \qquad \delta I_{st} = X_s \otimes \mathbb{X}^1_{st} + \mathbb{X}^2_{st}\,, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}\,. \tag{5.21}$$

Existence of $\mathbb{X}^2$ satisfying (5.21) with $\alpha < \frac{1}{2}$ is therefore granted by Lemma 5.2, e.g. via the paraintegral of Theorem 5.8. We also know that in the regime $\alpha < \frac{1}{2}$ we have infinitely many possible choices for $(I, \mathbb{X}^2)$, all of the form (5.19) above.

Suppose first that we are in the setting $d = 1$. Then Definition 4.1 becomes

Definition 5.11. *Let $\alpha \in \,]1/3, 1/2]$ and $X \colon [0, T] \to \mathbb{R}$ of class $\mathcal{C}^\alpha$. A α-*Rough Path *over X is a pair $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2) \in C_2^\alpha \times C_2^{2\alpha}$ such that*

$$\mathbb{X}^1_{st} = X_t - X_s, \qquad \delta \mathbb{X}^2_{sut} = \mathbb{X}^1_{su}\, \mathbb{X}^1_{ut}. \tag{5.22}$$

The reason of the restriction $\alpha > 1/3$ will become clear in ???. We recall that the conditions $X \in \mathcal{C}^\alpha$ and $\mathbb{X}^1 = \delta X \in C_2^\alpha$ are equivalent, and that $(\mathbb{X}^1, \mathbb{X}^2) \in C_2^\alpha \times C_2^{2\alpha}$ is equivalent to

$$|\mathbb{X}_{st}^1| \lesssim |t-s|^\alpha, \qquad |\mathbb{X}_{st}^2| \lesssim |t-s|^{2\alpha}.$$

We have seen in Chapter 4 that it is possible to build an integration theory for every choice of the α-rough path $\mathbb{X}$ over X. In this theory we can recover existence *and* uniqueness of the integral function $\int_0^\cdot Y\,\mathrm{d}X$ for a large class of choices of Y. For this we have to give very different roles to the integrator X and to the integrand Y, whereas in the case of the Young integral the two functions play a symmetric role: X will be a component of a rough path and Y a component of a *controlled path*, see Chapter 8.

We note that the algebraic condition $\delta\mathbb{X}_{sut}^2 = \mathbb{X}_{su}^1\,\mathbb{X}_{ut}^1$ is *non-linear*, which implies that α-rough paths do not form a vector subspace of $C_2^\alpha \times C_2^{2\alpha}$.

For all $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, given any *real-valued* path $X \in C^\alpha([0,T];\mathbb{R})$, there is always a rough path lying above X. Indeed, $I_t := \frac{1}{2} X_t^2$ is an integral of (X, X) in the sense of Definition 5.1, because

$$\delta I_{st} = \frac{1}{2}(X_t^2 - X_s^2) = X_s\,\delta X_{st} + \frac{1}{2}(\delta X_{st})^2 = X_s\,\delta X_{st} + O(|t-s|^{2\alpha})\,.$$

Then, by Remark 5.10, we can define a rough path $\mathbb{X}$ by setting

$$\mathbb{X}_{st}^2 = \frac{1}{2}(\delta X_{st})^2\,. \tag{5.23}$$

More directly, note that (5.23) satisfies the Chen relation (5.25), and clearly $\mathbb{X}^2 \in C_2^{2\alpha}$.

5.5. The vector case

Let us consider now a *vector valued* path $X: [0,T] \to \mathbb{R}^d$, with $X_t = (X_t^1, \ldots, X_t^d)$. We suppose that X is of class $\mathcal{C}^\alpha$, namely that $X^i \in \mathcal{C}^\alpha$ for all $i = 1, \ldots, d$, with $\alpha > 1/3$.

We can now generalise Definition 5.11 to the vector case. The multi-dimensional case $d \geq 2$ is sensibly richer, because off-diagonal terms $\int X^i\,\mathrm{d}X^j$ with $i \neq j$ are integral of a function with respect to a *different* function.

Definition 5.12. *Let* $\alpha\,]1/3, 1/2]$, $d \geq 1$ *and* $X: [0,T] \to \mathbb{R}^d$ *of class* C^α. *A* α-Rough Path *on* $\mathbb{R}^d$ *over* X *is a pair* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$, *with*

- $\mathbb{X}^1 = (\delta X^i)_{i=1,\ldots,d} \in C_2^\alpha([0,T];\mathbb{R}^d)$
- $\mathbb{X}^2 = (R^{ij})_{i,j=1,\ldots,d} \in C_2^{2\alpha}([0,T]_\leqslant^2;\mathbb{R}^d \otimes \mathbb{R}^d)$

such that

$$(\delta\mathbb{X}_{sut}^2)^{ij} = (\mathbb{X}_{su}^1)^i\,(\mathbb{X}_{ut}^1)^j, \tag{5.24}$$

or equivalently

$$\mathbb{X}_{st}^2 - \mathbb{X}_{su}^2 - \mathbb{X}_{ut}^2 = \mathbb{X}_{su}^1 \otimes \mathbb{X}_{ut}^1. \tag{5.25}$$

We denote by $\mathcal{R}_{\alpha,d}$ the space of α-rough paths on $\mathbb{R}^d$ and by $\mathcal{R}_{\alpha,d}(X)$ the set of α-rough paths over X.

The condition (5.24)-(5.25) is the first instance of the celebrated *Chen relation*. As in the one-dimensional case, existence of $\mathbb{X}^2$ satisfying (5.24)-(5.25) with $\alpha < \frac{1}{2}$ is therefore granted by Lemma 5.2, e.g. via the paraintegral of Theorem 5.8. We also know that in the regime $\alpha < \frac{1}{2}$ we have infinitely many possible choices for $(I, \mathbb{X}^2)$, all of the form (5.19) above.

We are going to see in Chapter 8 that it is possible to build an integration theory for every choice of an α-rough path $\mathbb{X}$. Again, we note that the condition (5.24)-(5.25) is *non-linear*, which implies that α-rough paths do not form a vector space.

The following exercise is a simple summary of the discussion at the beginning of this chapter.

Exercise 5.1. Given a α-rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ over X in $\mathbb{R}^d$, a process $I \in C^\alpha([0, T]; \mathbb{R}^d \otimes \mathbb{R}^d)$ satisfying (5.21) is an integral of (X, X) in the sense of Definition 5.1.

Viceversa, given $X \in C^\alpha([0, T]; \mathbb{R}^d)$ and an integral $I \in C^\alpha([0, T]; \mathbb{R}^d \otimes \mathbb{R}^d)$ of (X, X), in the sense of Definition 5.1, defining $\mathbb{X}^2$ by (5.21) we obtain a α-rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ over X in $\mathbb{R}^d$.

In the multi-dimensional case $X \in C^\alpha([0, T]; \mathbb{R}^d)$ with $d \geq 2$, building a rough path over X is non-trivial, because one has to define off-diagonal integrals $\int X^i \, \mathrm{d}X^j$ for $i \neq j$. However, by the results we have proved on the existence of the paraintegral in Proposition 5.8, we can easily deduce the following.

Proposition 5.13. *For any $d \in \mathbb{N}$, $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and $X \in C^\alpha([0, T]; \mathbb{R}^d)$, there is a α-rough path $\mathbb{X}$ which lies above X (hence, by Lemma 5.17, there are infinitely many of them).*

Proof. For any fixed $i, j \in \{1, \ldots, d\}$, let I^{ij} be an integral of (X_i, X_j) in the sense of Definition 5.1, whose existence is guaranteed by the paraintegral of Proposition 5.8. Then, by Exercise 5.1, defining $\mathbb{X}^2$ by (5.21) we obtain a rough path $\mathbb{X}$ which lies above X. □

Let us "justify" the term rough *path*, even though $\mathbb{X}$ is a function of two variables.

Exercise 5.2. Let $\mathbb{X}$ be a rough path above X. Then $\mathbb{X}$ is determined by the paths $(X_t - X_0, \mathbb{X}^2_{0t})_{t \in [0,T]}$. [Hint: use the Chen relation.]

We conclude with an elementary observation, that will be useful later. By Exercise 5.1, any α-rough path $\mathbb{X}$ over $X \in C^\alpha([0, T]; \mathbb{R}^d)$ determines an integral I of (X, X), given by (5.21). Applying the latter relation in a telescopic fashion, we can write

$$I_t = \sum_{[t_i, t_{i+1}] \in \mathcal{P}} (X_{t_i} \delta X_{t_i t_{i+1}} + \mathbb{X}^2_{t_i t_{i+1}}), \tag{5.26}$$

where $\mathcal{P}=\{0=t_0<t_1<\ldots<t_k=t\}$ is an arbitrary partition of $[0,t]$. We will see later ??? that a generalization of (5.26), when we also take the limit of vanishing mesh $|\mathcal{P}|\to 0$, is the correct recipe for building "Riemann-sums", in order to define an integral of (h,X) for a wide class of functions h.

5.6. Distance on rough paths

We denote by $\mathcal{R}_{\alpha,d}$ the set of all α-rough paths in $\mathbb{R}^d$. For $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)\in\mathcal{R}_{\alpha,d}$ we set

$$\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}}:=\|\mathbb{X}^1\|_\alpha+\|\mathbb{X}^2\|_{2\alpha}=\sup_{0\le s<t\le T}\frac{|\mathbb{X}^1_{st}|}{|t-s|^\alpha}+\sup_{0\le s<t\le T}\frac{|\mathbb{X}^2_{st}|}{|t-s|^{2\alpha}}. \qquad (5.27)$$

We stress that $\mathcal{R}_{\alpha,d}$ *is not a vector space*, because the Chen relation (5.25) is not linear. However, it is meaningful to define for $\mathbb{X},\bar{\mathbb{X}}\in\mathcal{R}_{\alpha,d}$

$$d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}}):=\|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\alpha+\|\mathbb{X}^2-\bar{\mathbb{X}}^2\|_{2\alpha}. \qquad (5.28)$$

Exercise 5.3. $d_{\mathcal{R}_{\alpha,d}}$ is a distance on $\mathcal{R}_{\alpha,d}$.

When we talk of convergence in $\mathcal{R}_{\alpha,d}$, we mean with respect to the distance $d_{\mathcal{R}_{\alpha,d}}$. Note that $d_{\mathcal{R}_{\alpha,d}}$ is equal on $\mathcal{R}_{\alpha,d}$ to the distance induced by the natural norm $\|F\|_\alpha+\|G\|_{2\alpha}$ for $(F,G)\in C_2^\alpha\times C_2^{2\alpha}$. In particular $\mathbb{X}_n=(\mathbb{X}_n^1,\mathbb{X}_n^2)\to\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ in $\mathcal{R}_{\alpha,d}$ if and only if $\mathbb{X}_n^1\to\mathbb{X}^1$ in C_2^α and $\mathbb{X}_n^2\to\mathbb{X}^2$ in $C_2^{2\alpha}$.

Lemma 5.14. *The metric space $(\mathcal{R}_{\alpha,d},d_{\mathcal{R}_{\alpha,d}})$ is complete.*

Proof. Let $(\mathbb{X}_n)_{n\in\mathbb{N}}\subset\mathcal{R}_{\alpha,d}$ be a Cauchy sequence. Then, by definition of $d_{\mathcal{R}_{\alpha,d}}$, for every $\epsilon>0$ there is $\bar{n}_\epsilon<\infty$ such that for all $n,m\ge\bar{n}_\epsilon$ and $0\le s<t\le T$

$$|\mathbb{X}^1_n(s,t)-\mathbb{X}^1_m(s,t)|\le\epsilon|t-s|^\alpha,\quad |\mathbb{X}^2_n(s,t)-\mathbb{X}^2_m(s,t)|\le\epsilon|t-s|^{2\alpha}. \qquad (5.29)$$

Note that

$$d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})\ge\frac{\|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\infty}{T^\alpha}+\frac{\|\mathbb{X}^2-\bar{\mathbb{X}}^2\|_\infty}{T^{2\alpha}}.$$

It follows that the sequences of continuous functions $(\mathbb{X}^1_n)_{n\in\mathbb{N}}$ and $(\mathbb{X}^2_n)_{n\in\mathbb{N}}$ are Cauchy in the sup-norm, hence there are continuous functions $\mathbb{X}^1$ and $\mathbb{X}^2$ such that $\|\mathbb{X}^1_n-\mathbb{X}^1\|_\infty\to 0$ and $\|\mathbb{X}^2_n-\mathbb{X}^2\|_\infty\to 0$. In particular, we have pointwise convergence $\mathbb{X}^1_m(t)\to\mathbb{X}^1(t)$ and $\mathbb{X}^2_m(s,t)\to\mathbb{X}^2(s,t)$ as $m\to\infty$. Taking this limit in (5.29) shows that $d_{\mathcal{R}_{\alpha,d}}(\mathbb{X}_n,\mathbb{X})\le\epsilon$ for all $n\ge\bar{n}_\epsilon$. □

This allows to rephrase the continuity result of section 4.7. We fix

$$D\ge\|\nabla\sigma\|_\infty+\|\nabla^2\sigma\|_\infty+\|\nabla^3\sigma\|_\infty+\|\nabla\sigma_2\|_\infty+\|\nabla^2\sigma_2\|_\infty.$$

We obtain from Proposition 4.10

PROPOSITION 5.15. *We suppose in this section that* $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ *and* $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ *is of class* C^3*, with* $\|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty + \|\nabla\sigma_2\|_\infty + \|\nabla^2\sigma_2\|_\infty < +\infty$ *(without boundedness assumptions on* σ *and* σ_2*). For* $\mathbb{X} \in \mathcal{R}_{\alpha,d}$ *and* $\mathrm{y}_0 \in \mathbb{R}^k$ *we denote by* $\mathrm{y}: [0,T] \to \mathbb{R}^k$ *the unique solution to equation (4.13)*

$$|\mathrm{y}^3_{st}| \lesssim |t-s|^\zeta, \qquad \mathrm{y}^3_{st} = \delta \mathrm{y}_{st} - \sigma(\mathrm{y}_s)\, \mathbb{X}^1_{st} - \sigma_2(\mathrm{y}_s)\, \mathbb{X}^2_{st},$$

for some $\zeta > 1$*. Then the map* $\mathbb{R}^k \times \mathcal{R}_{\alpha,d} \ni (\mathrm{y}_0, \mathbb{X}) \mapsto \mathrm{y} \in C^\alpha$ *is locally Lipschitz continuous.*

5.7. CANONICAL ROUGH PATHS FOR $\alpha > \frac{1}{2}$

Let $\frac{1}{3} < \alpha' \leqslant \frac{1}{2} < \alpha < 1$. Then it is well known that $C^\alpha \subset C^{\alpha'}$. Therefore, if $X \in C^\alpha([0,T]; \mathbb{R}^d)$ we have in particular $X \in C^{\alpha'}([0,T]; \mathbb{R}^d)$ and therefore there is a α'-rough path $\mathbb{X}$ over X. However, is there a α-rough path over X? Note that we have restricted Definition 5.12 to the range $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, while here we are discussing the existence of $\mathbb{X}^2: [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying the Chen relation (5.25) and

$$|\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}$$

where now $\alpha > \frac{1}{2}$.

LEMMA 5.16. *Let* $\alpha \in \left(\frac{1}{2}, 1\right)$*. For every* $X \in C^\alpha([0,T]; \mathbb{R}^d)$*, there is a unique* $\mathbb{X}^2: [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ *satisfying the Chen relation (5.25) and such that* $\mathbb{X}^2 \in C_2^{2\alpha}$*. We have the explicit formula*

$$\mathbb{X}^2_{st} = \int_s^t \mathbb{X}^1_{su} \otimes \mathrm{d}X_u, \qquad \mathbb{X}^1_{st} = \delta X_{st}, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{5.30}$$

where the integral is in the Young sense. Moreover the map $C^\alpha \ni X \mapsto \mathbb{X}^2 \in C_2^{2\alpha}$ *is continuous (in particular, locally Lipschitz-continuous).*

Proof. It is easy to check that $\mathbb{X}^2$ in (5.30) satisfies the Chen relation (5.22), thanks to the bi-linearity of the Young integral. Indeed, we can rewrite (5.30) as

$$\mathbb{X}^2_{st} = \int_s^t X_u \otimes \mathrm{d}X_u - X_s \otimes (X_t - X_s)\,, \tag{5.31}$$

hence for $s \leqslant u \leqslant t$ we have that

$$\begin{aligned}(\delta\mathbb{X}^2)_{sut} &= -X_s \otimes (X_t - X_s) + X_s \otimes (X_u - X_s) + X_u \otimes (X_t - X_u)\\ &= -X_s \otimes (X_t - X_u) + X_u \otimes (X_t - X_u)\\ &= \delta X_{su} \otimes \delta X_{ut}\,.\end{aligned}$$

We show now that $\mathbb{X}^2 \in C_2^{2\alpha}$. We recall that the Young integral satisfies the following key estimate, for $f \in C^{\alpha}$ and $g \in C^{\beta}$ with $\alpha + \beta > 1$:

$$\left| \int_s^t f \, \mathrm{d}g - f_s \, (g_t - g_s) \right| \leqslant c_{\alpha+\beta} |t-s|^{\alpha+\beta}.$$

Choosing $f = X^i$ and $g = X^j$ shows that $\mathbb{X}^2$, given by (5.31), is $O(|t-s|^{2\alpha})$. Finally, we prove the continuity of $C^{\alpha} \ni X \mapsto \mathbb{X}^2 \in C_2^{2\alpha}$. Given $X, \bar{X} \in C^{\alpha}$and the respective $\mathbb{X}^2, \bar{\mathbb{X}}^2 \in C_2^{2\alpha}$, we have

$$\mathbb{X}_{st}^2 - \bar{\mathbb{X}}_{st}^2 = \int_s^t (\mathbb{X}_{su}^1 - \bar{\mathbb{X}}_{su}^1) \otimes \mathrm{d}X_u + \int_s^t \bar{\mathbb{X}}_{su}^1 \otimes \mathrm{d}(X - \bar{X})_u,$$

with all integrals in the Young sense. Then by the Sewing Lemma

$$\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} \leqslant K_{2\alpha}(\|\delta X\|_{\alpha} + \|\delta \bar{X}\|_{\alpha}) \|\delta X - \delta \bar{X}\|_{\alpha}.$$

The proof is complete. □

Therefore, we could extend Definition 5.12 to α-rough paths for $\alpha \in \left(\frac{1}{3}, 1\right]$. For $\alpha \in \left(\frac{1}{2}, 1\right]$ and $X \in C^{\alpha}([0,T]; R^d)$ there is a unique α-rough path over X, which we call the *canonical rough path* over X.

While for $\alpha > \frac{1}{2}$ there is a unique rough path lying above a given path $X \in C^{\alpha}$, for $\alpha < \frac{1}{2}$ there are infinitely many of them, that can be characterized explicitly.

LEMMA 5.17. *Let $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ be a α-rough path in $\mathbb{R}^d$, with $\alpha < \frac{1}{2}$. Then $\bar{\mathbb{X}} = (\mathbb{X}^1, \bar{\mathbb{X}}^2)$ is a α-rough path if and only if for some $f \in C^{2\alpha}([0,T]; \mathbb{R}^d \otimes \mathbb{R}^d)$ one has $\bar{\mathbb{X}}^2 = \mathbb{X}^2 + \delta f$, that is*

$$\bar{\mathbb{X}}_{st}^2 = \mathbb{X}_{st}^2 + f_t - f_s, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Proof. By assumption $\mathbb{X}^2$ and $\bar{\mathbb{X}}^2$ satisfy the Chen relation (5.25). If $\bar{\mathbb{X}}^2 = \mathbb{X}^2 + \delta f$ then $\mathbb{X}^2 \in C_2^{2\alpha}$ if and only if $\bar{\mathbb{X}}^2 \in C_2^{2\alpha}$ and $\delta \mathbb{X}^2 = \delta \bar{\mathbb{X}}^2$. Therefore, if $\mathbb{X}$ is a α-rough path then so is $\bar{\mathbb{X}}$.

Viceversa, if $\bar{\mathbb{X}}$ is a α-rough path, then $\delta \mathbb{X}^2 = \delta \bar{\mathbb{X}}^2$ because both $\mathbb{X}$ and $\bar{\mathbb{X}}$ satisfy the Chen relation (5.25) with the same $\mathbb{X}^1$, hence $\bar{\mathbb{X}}^2 = \mathbb{X}^2 + \delta f$ for some f. Since both $\mathbb{X}^2, \bar{\mathbb{X}}^2$ belong to $C_2^{2\alpha}$, then also $\delta f \in C_2^{2\alpha}$, which is the same as $f \in C^{2\alpha}$. □

Remark 5.18. We mainly work with α-Hölder rough pats for $\alpha \in (\frac{1}{3}, \frac{1}{2})$, excluding the boundary case $\alpha = \frac{1}{2}$ for technical reasons. Let us stress that, by doing so, *we are not throwing away any rough paths, but only giving up a tiny amount of regularity*, because any rough path of exponent $\frac{1}{2}$ is a rough path of exponent α, for any $\alpha < \frac{1}{2}$.

To summarize, the situation is the following:

1. For $\alpha \in \left(\frac{1}{2}, 1\right]$ and $X \in C^\alpha([0,T]; R^d)$ there is a unique α-rough path over X
2. For $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and $X \in C^\alpha([0,T]; R^d)$, there are infinitely many α-rough paths over X
3. For $\alpha = \frac{1}{2}$, either there is no α-rough path over X, or there are infinitely many of them.

In the range $\alpha \in \left(\frac{1}{2}, 1\right]$, the unique α-rough path $\mathbb{X}$ above X can be called the *canonical rough path* over X. We let $\mathcal{R}_{1,d}$ be the set of all canonical rough paths over paths $X \in C^1$ (see Lemma 5.16).

5.8. LACK OF CONTINUITY

We have seen in Lemma 5.16 that, for $\alpha > \frac{1}{2}$, the map $C^\alpha \ni X \mapsto \mathbb{X}^2 \in C_2^{2\alpha}$ is continuous. It is a crucial fact that this continuity property can *not* be extended to $\alpha \leqslant \frac{1}{2}$, as shown by the next example.

For $n \in \mathbb{N}$ consider the smooth paths $X_n^1, X_n^2 \colon [0,1] \to \mathbb{R}$

$$X_n^1(t) := \frac{1}{\sqrt{n}} \cos(nt), \qquad X_n^2(t) := \frac{1}{\sqrt{n}} \sin(nt).$$

We have already shown in Lemma 5.5 that $X_n^1 \to 0$ and $X_n^2 \to 0$ in C^α, for all $\alpha \in (0, \frac{1}{2})$. More precisely, we have shown that $X_n^1 \rightsquigarrow_{\frac{1}{2}} 0$ and $X_n^2 \rightsquigarrow_{\frac{1}{2}} 0$, by showing that $\|\delta X_n^1\|_{\frac{1}{2}} \leq 2$, $\|\delta X_n^2\|_{\frac{1}{2}} \leq 2$ for all $n \in \mathbb{N}$ and, obviously, $\|X_n^1\|_\infty \to 0$, $\|X_n^2\|_\infty \to 0$. Next we set

$$I_n^{ij}(t) := \int_0^t X_n^i(u)\, \mathrm{d}X_n^j(u), \qquad \text{for } i, j \in \{1, 2\},$$

and correspondingly

$$(\mathbb{X}_n^2)_{st}^{ij} = \tag{5.32}$$
$$= \int_s^t (X_n^i(u) - X_n^i(s))\, \mathrm{d}X_n^j(u) = I_n^{ij}(t) - I_n^{ij}(s) - X_n^i(s)(X_n^j(t) - X_n^j(s)).$$

It is not difficult to show that $(\mathbb{X}_n^2)^{ij} \to (\mathbb{X}^2)^{ij}$ in C_2^θ, for any $\theta \in (0,1)$, where we define

$$(\mathbb{X}^2)_{st}^{ij} = \begin{pmatrix} 0 & \dfrac{t-s}{2} \\ -\dfrac{t-s}{2} & 0 \end{pmatrix} = \begin{cases} \frac{t-s}{2} & \text{if } i=1, j=2 \\ -\frac{t-s}{2} & \text{if } i=2, j=1 \\ 0 & \text{if } i=j \end{cases}. \tag{5.33}$$

As a consequence, for any $\alpha \in (\frac{1}{3}, \frac{1}{2})$, we have $\mathbb{X}_n^1 \to 0$ in C^α and $\mathbb{X}_n^2 \to \mathbb{X}^2$ in $C_2^{2\alpha}$, that is *the canonical ls* $(\mathbb{X}_n^1, \mathbb{X}_n^2)$ *converge in* $\mathcal{R}_{\alpha,d}$ *to the rough path* $(0, \mathbb{X}^2)$.

Let us prove that $(\mathbb{X}_n^2)^{ij} \to (\mathbb{X}^2)^{ij}$ in C_2^θ, for any $\theta \in (0,1)$. We have already shown the pointwise (actually uniform) convergence $I_n^{12}(t) \to \frac{1}{2}\, t$. With similar arguments, one shows the uniform convergence $I_n^{ij} \to I^{ij}$ defined by

$$I^{ij}(t) = \begin{pmatrix} 0 & \frac{t}{2} \\ -\frac{t}{2} & 0 \end{pmatrix} = \begin{cases} \frac{t}{2} & \text{if } i=1, j=2 \\ -\frac{t}{2} & \text{if } i=2, j=1 \\ 0 & \text{if } i=j \end{cases} .$$

It follows by (5.32) that we have the uniform convergence $(\mathbb{X}_n^2)_{st}^{ij} \to I^{ij}(t) - I^{ij}(s) = (\mathbb{X}^2)_{st}^{ij}$. To prove convergence in C_2^θ, it suffices to show a uniform "Lipschitz-like" bound $|(\mathbb{X}_n^2)_{st}^{ij}| \leq 2\,|t-s|$, which is easy:

$$\begin{aligned} |(\mathbb{X}_n^2)_{st}^{ij}| &\leq \int_s^t |X_n^i(u) - X_n^i(s)|\,|(X_n^j)'(u)|\,\mathrm{d}u \\ &\leq 2\,\|X_n^i\|_\infty\,\|(X_n^j)'\|_\infty |t-s| \\ &= 2\frac{1}{\sqrt{n}}\frac{n}{\sqrt{n}}|t-s| \\ &= 2\,|t-s|\,. \end{aligned}$$

5.9. Proof of Proposition 5.8

Given continuous functions $f, g\colon [0,T] \to \mathbb{R}$, let us define $R^1, R^2 \in C_2$

$$R^1(f,g)_{st} := -f_s\,\delta g_{st}\,, \qquad R^2(f,g)_{st} := g_t\,\delta f_{st}\,, \qquad 0 \leqslant s \leqslant t \leqslant T, \qquad (5.34)$$

and note that

$$R_{st}^2 = R_{st}^1 + f_t\,g_t - f_s\,g_s\,.$$

It is easy to check that both R^1 and R^2 satisfy (5.11), that is

$$\delta R^1(f,g)_{sut} = \delta R^2(f,g)_{sut} = \delta f_{su}\,\delta g_{ut}\,. \qquad (5.35)$$

Note that R^1 coincides with $-A$, defined in (5.7), while $R^2 = R^1 + \delta(fg)$, hence $\delta R^2 = \delta R^1$.

However, neither R^1 nor R^2 satisfy (5.12), because we can only estimate

$$\|R^1\|_\beta \leqslant \|f\|_\infty\,\|\delta g\|_\beta\,, \qquad \|R^2\|_\alpha \leqslant \|g\|_\infty\,\|\delta f\|_\alpha\,. \qquad (5.36)$$

We are going to show that, by combining R^1 and R^2 in a suitable way, one can build R which satisfies both (5.11) and (5.12). This yields the existence of an integral.

We start with a technical approximation lemma.

Lemma 5.19. *Given $f \in C^\alpha$, there is a sequence $\tilde{f}_n \in C^\infty$ such that*

$$f(x) = f(0) + \sum_{n \geq 0} \tilde{f}_n(x)\,, \qquad \forall x \in [0,T]\,. \qquad (5.37)$$

One can choose $\tilde{f}_n$ so that for every $n \geq 0$

$$\|\tilde{f}_n\|_\infty \leqslant C\, \|\delta f\|_\alpha\, 2^{-n\alpha}, \qquad \|\tilde{f}_n'\|_\infty \leqslant C\, \|\delta f\|_\alpha\, 2^{n(1-\alpha)}, \tag{5.38}$$

where $C \in (0, \infty)$ depends only on T (e.g. one can take $C = 2(T^\alpha + 1)$).

Proof. We may assume without loss of generality that $f(x) = 0$ (it suffices to redefine $f(x)$ as $f(x) - f(0)$, which does not change $\|\delta f\|_\alpha$.)

We extend $f: \mathbb{R} \to \mathbb{R}$ (e.g. with $f(x) := f(0)$ for $x \leqslant 0$ and $f(x) := f(T)$ for $x \geq T$) so that $\|f\|_\alpha$ is not changed. Then we fix a probability density $\phi: [-1, 1] \to \mathbb{R}$ with $\phi \in C^1$ and for $n \geq 0$ we define the rescaled density

$$\phi_n(x) := 2^n \phi(2^n x).$$

Next, for $n \geq 0$, we set $f_n(x) := (f * \phi_n)(x)$, that is

$$\begin{aligned} f_n(x) &:= \int_{\mathbb{R}} f(z)\, \phi_n(x-z)\, \mathrm{d}z = \int_{\mathbb{R}} f(x-z)\, \phi_n(z)\, \mathrm{d}z \\ &= \int_{\mathbb{R}} f(x - \tfrac{z}{2^n}) \phi(z)\, \mathrm{d}z. \end{aligned} \tag{5.39}$$

It is easy to check that $\|f_n - f\|_\infty \to 0$. Next we define

$$\tilde{f}_0(x) := f_0(x), \qquad \text{for } k \geq 1\text{:} \quad \tilde{f}_k(x) := f_k(x) - f_{k-1}(x).$$

Note that $\sum_{k=0}^n \tilde{f}_k = f_n$, hence relation (5.37) is proved (we recall that $f(0) = 0$).

We now prove the first relation in (5.38). Since $f(0) = 0$, for all $x \in [0, T]$ we can write

$$\begin{aligned} |\tilde{f}_0(x)| = |f_0(x)| &\leqslant \int_{\mathbb{R}} |f(x-z)|\, \phi(z)\, \mathrm{d}z = \int_{\mathbb{R}} |f(x-z) - f(0)| \phi(z)\, \mathrm{d}z \\ &\leqslant \|\delta f\|_\alpha \int_{\mathbb{R}} |x - z|^\alpha\, \phi(z)\, \mathrm{d}z \leqslant (T^\alpha + 1)\, \|\delta f\|_\alpha, \end{aligned}$$

where for the last inequality we have used $(x+y)^\alpha \leqslant x^\alpha + y^\alpha$ (for $\alpha < 1$ and $x, y \geq 0$), $x \leqslant T$ and $\int_{\mathbb{R}} |z|^\alpha\, \phi(z)\, \mathrm{d}z \leqslant \int_{[-1,1]} \phi(z)\, \mathrm{d}z = 1$, because ϕ is a density supported on $[-1, 1]$. For $k \geq 1$ we estimate

$$\begin{aligned} |\tilde{f}_k(x)| &= |f_k(x) - f_{k-1}(x)| \\ &\leqslant \int_{\mathbb{R}} |f(x - \tfrac{z}{2^k}) - f(x - \tfrac{z}{2^{k-1}})|\, \phi(z)\, \mathrm{d}z \\ &\leqslant 2^{-k\alpha}\, \|\delta f\|_\alpha \end{aligned}$$

again because $\int_{\mathbb{R}} |z|^\alpha \phi(z)\, \mathrm{d}z \leqslant 1$. We have proved the first relation in (5.38).

We finally prove the second relation in (5.38). Note that

$$f_n'(x) = \int_{\mathbb{R}} f(z)\, \phi_n'(x-z)\, \mathrm{d}z = 2^n \int_{\mathbb{R}} f(x - \tfrac{z}{2^n})\, \phi'(z)\, \mathrm{d}z,$$

which has the same form as $f_n(x)$, see the last integral in (5.39), just with an extra multiplicative factor 2^n and with ϕ replaced by ϕ'. Arguing as before, we obtain

$$|\tilde{f}_0'(x)| = |f_0'(x)| \leqslant (T^\alpha + 1) \left(\int_{[-1,1]} |\phi'(z)| \, \mathrm{d}z \right) \|\delta f\|_\alpha ,$$

$$|\tilde{f}_k'(x)| = |f_k'(x) - f_{k-1}'(x)| \leqslant 2^{k(1-\alpha)} \left(\int_{[-1,1]} |\phi'(z)| \, \mathrm{d}z \right) \|\delta f\|_\alpha,$$

for $k \geq 1$. We can choose ϕ to be symmetric, decreasing on $[0, 1]$, with $\phi(0) = 1$ and $\phi(1) = 0$, so that

$$\int_{[-1,1]} |\phi'(z)| \, \mathrm{d}z = 2 \int_0^1 (-\phi'(z)) \, \mathrm{d}z = 2 \, (\phi(0) - \phi(1)) = 2 ,$$

and this completes the proof. □

Proof of Proposition 5.8. The existence of an integral is an immediate consequence of Remark 5.6, because if we define $R_{st} := J_\prec(f, g)_{st}$, then both relations (5.11) and (5.12) are satisfied.

It remains to build $J_\prec$. Let us write, applying Lemma 5.19,

$$f(x) = f(0) + \sum_{m \geq 0} \tilde{f}_n(x) , \qquad g(x) = g(0) + \sum_{n \geq 0} \tilde{g}_m(x) .$$

Recalling (5.34), we define

$$J_\prec(f, g) := \sum_{0 \leqslant m \leqslant n} R^1(\tilde{f}_n, \tilde{g}_m) + \sum_{0 \leqslant n < m} R^2(\tilde{f}_n, \tilde{g}_m) . \tag{5.40}$$

We show below that the series converge uniformly. Note that $\sum_{n \geq 0} \tilde{f}_n(x) = f(x) - f(0)$, hence $\sum_{n \geq 0} \delta \tilde{f}_n = \delta(f - f(0)) = \delta f$, and similarly for g. Applying (5.35), we get

$$\begin{aligned} \delta J_\prec(f, g)_{sut} &= \sum_{0 \leqslant m \leqslant n} (\delta \tilde{f}_n)_{su} \, (\delta \tilde{g}_m)_{ut} + \sum_{0 \leqslant n < m} (\delta \tilde{f}_n)_{su} \, (\delta \tilde{g}_m)_{ut} \\ &= \left(\sum_{n \geq 0} (\delta \tilde{f}_n)_{su} \right) \left(\sum_{m \geq 0} (\delta \tilde{g}_m)_{ut} \right) = \delta f_{su} \, \delta g_{ut} , \end{aligned}$$

which proves (5.14). We now prove (5.13). Note that, by (5.38),

$$|(\delta \tilde{f}_n)_{st}| \leqslant \|\tilde{f}_n'\|_\infty \, |t - s| \leqslant C \, \|\delta f\|_\alpha \, 2^{-\alpha n} (2^n \, |t - s|) ,$$

but at the same time, always by (5.38),

$$|(\delta \tilde{f}_n)_{st}| \leqslant |\tilde{f}_n(s)| + |\tilde{f}_n(t)| \leqslant 2 \, \|\tilde{f}_n\|_\infty \leqslant 2C \, \|\delta f\|_\alpha \, 2^{-\alpha n} .$$

Altogether, using the notation $x \wedge y := \min \{x, y\}$,

$$|(\delta \tilde{f}_n)_{st}| \leqslant 2C \, \|\delta f\|_\alpha \, 2^{-\alpha n} \, (2^n |t - s| \wedge 1) .$$

Similarly

$$|(\delta\tilde{g}_m)_{st}| \leqslant 2C\,\|\delta g\|_\beta\, 2^{-\beta m}\,(2^m|t-s|\wedge 1)\,.$$

Recalling (5.34) and applying again (5.38), we get

$$\begin{aligned} |R^1(\tilde{f}_n,\tilde{g}_m)_{st}| &\leqslant \|\tilde{f}_n\|_\infty\,|(\delta\tilde{g}_m)_{st}| \\ &\leqslant 2C^2\,\|\delta f\|_\alpha\,\|\delta g\|_\beta\,2^{-\alpha n}\,2^{-\beta m}\,(2^m|t-s|\wedge 1) \end{aligned}$$

and similarly

$$\begin{aligned} |R^2(\tilde{f}_n,\tilde{g}_m)_{st}| &\leqslant \|\tilde{g}_m\|_\infty\,|(\delta\tilde{f}_n)_{st}| \\ &\leqslant 2C^2\,\|\delta f\|_\alpha\,\|\delta g\|_\beta\,2^{-\alpha n}\,2^{-\beta m}(2^n|t-s|\wedge 1)\,. \end{aligned}$$

These relations show that the series in (5.40) converge indeed uniformly. We now plug these estimates into (5.40), getting

$$\begin{aligned} |J_\prec(f,g)_{st}| \;\leqslant\; 2C^2\,\|\delta f\|_\alpha\,\|\delta g\|_\beta\Bigg(&\sum_{0\leqslant m\leqslant n} 2^{-\alpha n}\,2^{-\beta m}\,(2^m|t-s|\wedge 1) \\ &+\sum_{0\leqslant n<m} 2^{-\alpha n}\,2^{-\beta m}\,(2^n|t-s|\wedge 1)\Bigg). \end{aligned} \tag{5.41}$$

Let us set for convenience

$$\bar{k}=\bar{k}_{st}:=\log_2\frac{1}{|t-s|}\,,$$

so that $2^m|t-s|\leqslant 2$ if and only if $m\leqslant\bar{k}$. Since $\sum_{n=m}^\infty 2^{-\alpha n}\leqslant\frac{1}{1-2^{-\alpha}}\,2^{-\alpha m}$, the first sum in (5.41) can be bounded as follows (neglecting the prefactor $(1-2^{-\alpha})^{-1}$):

$$\begin{aligned} \sum_{m\geq 0} 2^{-(\alpha+\beta)m}\,(2^m|t-s|\wedge 1) &\leqslant |t-s|\sum_{0\leqslant m<\bar{k}} 2^{(1-\alpha-\beta)m}+\sum_{m\geq\bar{k}} 2^{-(\alpha+\beta)m} \\ &\leqslant |t-s|\frac{2^{(1-\alpha-\beta)\bar{k}}}{2^{1-\alpha-\beta}-1}+\frac{2^{-(\alpha+\beta)\bar{k}}}{1-2^{-(\alpha+\beta)}} \\ &\leqslant \left\{\frac{1}{2^{1-\alpha-\beta}-1}+\frac{1}{1-2^{-(\alpha+\beta)}}\right\}|t-s|^{\alpha+\beta}\,. \end{aligned}$$

The same estimates apply to the second sum in (5.41), hence (5.13) is proved. □

Chapter 6

Brownian rough paths

Let us fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ endowed with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,1]}$. Let $B = (B_t)_{t\in[0,T]} = (B_t^1, \ldots, B_t^d)_{t\in[0,T]}$ be a d-dimensional Brownian motion. We set $\mathbb{B}^1: [0,T]_{\leqslant}^2 \to \mathbb{R}^d$ and $\mathbb{B}^2: [0,T]_{\leqslant}^2 \to \mathbb{R}^d \otimes \mathbb{R}^d$,

$$\mathbb{B}_{st}^1 := B_t - B_s, \qquad \mathbb{B}_{st}^2 = \int_s^t (B_r - B_s) \otimes \mathrm{d}B_r, \quad 0 \leqslant s \leqslant t \leqslant T, \tag{6.1}$$

i.e.

$$(\mathbb{B}_{st}^2)^{ij} = \int_s^t (B_r^i - B_s^i)\, \mathrm{d}B_r^j,$$

where the integral is in the Itô sense. The main aim of this chapter is to show the following

Theorem 6.1. *For all $\alpha \in \left(0, \frac{1}{2}\right)$, almost surely $\mathbb{B} := (\mathbb{B}^1, \mathbb{B}^2)$ is a α-rough path, namely $\mathbb{B} \in \mathcal{R}_{\alpha,d}$.*

It is easy to see that $\mathbb{B}$ a.s. satisfies for all $\alpha \in \left(0, \frac{1}{2}\right)$

$$|\mathbb{B}_{st}^1| \lesssim |t-s|^{\alpha}$$

and the Chen relation (5.25)

$$\delta \mathbb{B}_{sut}^2 = \mathbb{B}_{su}^1 \otimes \mathbb{B}_{ut}^1, \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T.$$

Indeed, the former formula follows by the well-known Hölder continuity of Brownian motion, and the latter from

$$\begin{aligned} \delta(\mathbb{B}^2)_{sut}^{ij} &= \int_s^t (B_r^i - B_s^i)\, \mathrm{d}B_r^j - \int_s^u (B_r^i - B_s^i)\, \mathrm{d}B_r^j - \int_u^t (B_r^i - B_u^i)\, \mathrm{d}B_r^j \\ &= \int_u^t (B_u^i - B_s^i)\, \mathrm{d}B_r^j = (B_u^i - B_s^i)(B_t^j - B_u^j), \end{aligned}$$

which is a legitimate computation by the properties of the Itô integral and the fact that the times $s \leqslant u \leqslant t$ are ordered.

The non-trivial missing information is the analytic estimate for $\mathbb{B}_{st}^2$:

$$|\mathbb{B}_{st}^2| \lesssim |t-s|^{2\alpha}.$$

In this chapter we prove this formula with a refinement of the classical Kolmogorov continuity criterion.

In fact, we are going to prove a more general result. Given $(f, g) \in C^\alpha \times C^\beta$ with $\alpha + \beta < 1$, we recall that an integral of (f, g) in the sense of Definition 5.1 is a function $I: [0, T] \to \mathbb{R}$ satisfying

$$I_t - I_s = f_s(g_t - g_s) + O(|t-s|^{\alpha+\beta}) \qquad \text{uniformly as } |t-s| \to 0. \tag{6.2}$$

We want to show that Itô stochastic integrals with respect to Brownian motion are almost surely integrals in this sense.

6.1. Main result

Let $B = (B_t)_{t\in[0,1]}$ be a d-dimensional Brownian motion and let $h = (h_t)_{t\in[0,1]}$ be a $\mathbb{R}^d$-valued adapted process with continuous paths. In particular $\int_0^1 |h_s|^2 \, \mathrm{d}s < \infty$, hence the Itô integral

$$I_t := \int_0^t h_s \, \mathrm{d}B_s \tag{6.3}$$

is well-defined as a local martingale. It is well-known that the process $I = (I_t)_{t\in[0,1]}$ admits a version with continuous paths, which we always fix.

For $0 \leqslant s \leqslant t \leqslant 1$ we define the (random) continuous function

$$R_{st} := I_t - I_s - h_s (B_t - B_s). \tag{6.4}$$

We recall that a.s. $b \in C^\beta$ for every $\beta < \frac{1}{2}$. This is our main result.

Theorem 6.2. *Assume that a.s. $h \in C^\alpha$, for some $\alpha \in (0,1)$. Then, for any $\beta < \frac{1}{2}$, there is an a.s. finite random constant C such that*

$$|R_{st}| \leqslant C |t-s|^{\alpha+\beta}, \qquad \forall 0 \leqslant s \leqslant t \leqslant 1. \tag{6.5}$$

In particular, a.s. the Itô integral in (6.3) is an integral of (h, B) in the sense of (6.2).

Proof. First observation: if the claim holds under the stronger assumption $\|\delta h\|_\alpha \leqslant c$, for some deterministic $c < \infty$, then we can deduce the general result by localization. Indeed, if we only assume that $\|\delta h\|_\alpha < \infty$ a.s., we can define for $n \in \mathbb{N}$ the stopping times

$$\tau_n := \inf \{ t \in [0,1] : \|\delta h\|_{\alpha,[0,t]} > n \},$$

where $\|\delta h\|_{\alpha,[0,t]}$ is the Hölder semi-norm of h restricted to $[0, t]$ (equivalently, the Hölder semi-norm of $s \mapsto h_{s\wedge t}$ on the whole interval $s \in [0,1]$). Let us define

$$h_s^{(n)} := h_{s\wedge\tau_n}, \qquad I_t^{(n)} := \int_0^t h_s^{(n)} \mathrm{d}B_s, \qquad R_{st}^{(n)} := I_t^{(n)} - I_s^{(n)} - h_s^{(n)}(B_t - B_s).$$

Note that $\|\delta h^{(n)}\|_\alpha \leqslant n$, by definition of τ_n. (Indeed, $\|\delta h\|_{\alpha,[0,t]} \leqslant n$ for all $t < \tau_n$, which means that $|h(r) - h(s)| \leqslant n|r-s|^\alpha$ for all $r, s \in [0, \tau_n)$; then, by continuity, $|h(r) - h(s)| \leqslant n|r-s|^\alpha$ for all $r, s \in [0, \tau_n]$, which means that $\|\delta h\|_{\alpha,[0,\tau_n]} = \|\delta h^{(n)}\|_\alpha \leqslant n$.) Then

$$|R_{st}^{(n)}| \leqslant C^{(n)}|t-s|^{\alpha+\beta}, \qquad \forall 0 \leqslant s < t \leqslant 1, \tag{6.6}$$

for a suitable a.s. finite random constant $C^{(n)}$. Let us define the events

$$A_n := \{\tau_n = \infty\} = \{\|\delta h\|_\alpha \leqslant n\}$$

and note that $h = h^{(n)}$ on A_n. By the locality property of the stochastic integral, $I = I^{(n)}$ a.s. on A_n,[6.1] hence also $R = R^{(n)}$ a.s. on A_n. Redefining $C^{(n)} = \infty$ on the exceptional set $\{R = R^{(n)}\}^c$, we get by (6.6)

$$\text{on the event } A_n: \qquad |R_{st}| \leqslant C^{(n)}|t-s|^{\alpha+\beta}, \qquad \forall 0 \leqslant s < t \leqslant 1.$$

Note that $A := \bigcup_{n\in\mathbb{N}} A_n = \{\|\delta h\|_\alpha < \infty\}$, hence $\mathbb{P}(A) = 1$. If we define $C := C^{(n)}$ on $A_n \setminus A_{n-1}$ (with $A_0 := \emptyset$) and $C := \infty$ on A^c, we have $C < \infty$ a.s. and relation (6.5) holds.

Second observation: if relation (6.5) holds for all s, t in a (deterministic) dense subset $\mathbb{D} \subseteq [0,1]$, then it holds for all $s, t \in [0,1]$, because R_{st} is a continuous function of $s$nt.

In conclusion, the proof is reduced to showing (6.5) only for $s, t \in \mathbb{D}$, under the assumption that $\|\delta h\|_\alpha \leqslant c < \infty$. This technical result is formulated as a separate lemma. □

Lemma 6.3. *Assume that $\mathbb{E}[\|\delta h\|_\alpha^p] < \infty$, for some $\alpha \in (0,1)$ and for all $p > 0$. Then, for any $\beta < \frac{1}{2}$, there is an a.s. finite random constant C such that*

$$|R_{st}| \leqslant C|t-s|^{\alpha+\beta}, \qquad \forall s, t \in \mathbb{D} \quad \textit{with } s \leqslant t. \tag{6.7}$$

Equivalently, a.s. $R \in C_2^{\alpha+\beta}$.

Next, we suppose that h is as in the statement of Theorem 6.2 and moreover there exists another adapted process $h^1 = (h_t^1)_{t\in[0,1]}$ with values in $\mathbb{R}^d \otimes \mathbb{R}^d$ such that a.s.

$$|\delta h_{st} - h_s^1 \, \mathbb{B}_{st}^1| \lesssim |t-s|^{2\alpha}.$$

Then we define

$$R_{st}^2 := R_{st} - h_s^1 \, \mathbb{B}_{st}^2 = \delta I_{st} - h_s \, \mathbb{B}_{st}^1 - h_s^1 \, \mathbb{B}_{st}^2,$$

where $\mathbb{B}^2$ is defined in (6.1). Then we have

Theorem 6.4. *Assume that a.s. h and h^1 are of class C^α, for some $\alpha \in (0,1)$. Then, for any $\beta < \frac{1}{2}$, there is an a.s. finite random constant C such that*

$$|R_{st}^2| \leqslant C|t-s|^{2\alpha+\beta}, \qquad \forall 0 \leqslant s \leqslant t \leqslant 1. \tag{6.8}$$

6.1. We mean that $I^{(n)}$ and I are indistinguishable on A_n: for a.e. $\omega \in A_n$ one has $I_t^{(n)}(\omega) = I_t(\omega)$ for all $t \in [0,1]$ (we recall that we always fix continuous versions of the stochastic integrals).

Arguing as in the proof of Theorem 6.2 we see that Theorem 6.4 follows from the following

LEMMA 6.5. *Assume that* $\mathbb{E}[\|\delta h\|_\alpha^p + \|\delta h - h^1\mathbb{B}^1\|_{2\alpha}^p] < \infty$*, for some* $\alpha \in (0, 1)$ *and for all* $p > 0$*. Then, for any* $\beta < \frac{1}{2}$*, there is an a.s. finite random constant* C *such that*

$$|R_{st}^2| \leqslant C|t-s|^{2\alpha+\beta}, \qquad \forall s,t \in \mathbb{D} \quad \text{with } s \leqslant t. \tag{6.9}$$

Equivalently, a.s. $R^2 \in C_2^{2\alpha+\beta}$.

Lemmas 6.3 and 6.5 will be proved in Section 6.4 below. First we show in Section 6.2 that Theorems 6.2 and 6.4 allow to connect Stochastic Differential Equations (SDEs) and rough finite difference equations.

6.2. APPLICATIONS TO SDES

Let us consider now a k-dimensional SDE

$$y_t = y_0 + \int_0^t \sigma(y_s)\,\mathrm{d}B_s, \qquad t \geqslant 0, \tag{6.10}$$

where $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ and $(B_t)_{t\geqslant 0}$ is a d-dimensional Brownian motion. We suppose that σ is of class C^3 and satisfies $\|\nabla\sigma\|_\infty < +\infty$. We want to show that

THEOREM 6.6. *The unique solution to the SDE (6.10) is a.s. equal to the unique solution to the rough finite difference equation (4.13) associated with the Itô rough path*

$$\mathbb{B}_{st}^1 := \delta B_{st}, \qquad \mathbb{B}_{st}^2 := \int_s^t \mathbb{B}_{sr}^1 \otimes \mathrm{d}B_r, \qquad 0 \leqslant s \leqslant t \leqslant T,$$

where the integral defining $\mathbb{B}^2$ *is in the Itô sense.*

Proof. First we note that $\mathbb{B}$ is indeed a α-rough path for any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, by Theorem 6.2. Indeed, setting $h_r := \mathbb{B}_{sr}^1$, $r \in [s,T]$ and $R_{st} := \mathbb{B}_{st}^2$, then we obtain $|\mathbb{B}_{st}^2| \lesssim |t-s|^{2\alpha}$ by (6.5).

Since σ is supposed to be uniformly Lipschitz, it is well known that the SDE (6.10) satisfies existence of (probabilistically) strong solutions and pathwise uniqueness. On the other hand, since σ is of class C^3, by Theorem 4.7 the rough finite difference equation (4.13) has a unique solution. Therefore we only need to show that the solution to the SDE (6.10) is a solution to the rough finite difference equation (4.13) associated with the Itô rough path, namely that it satisfies

$$\delta y_{st} = \sigma(y_s)\mathbb{B}_{st}^1 + \sigma_2(y_s)\mathbb{B}_{st}^2 + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Let us fix $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and les us fix a sample of B of class C^{α}. Now by the Itô formula and (6.10)

$$\begin{aligned}
\sigma(y_t) &= \sigma(y_s) + \int_s^t \sum_{a=1}^k \partial_a \sigma(y_r) \mathrm{d}y_r^a + \frac{1}{2} \int_s^t \sum_{a,b=1}^k \partial_{ab} \sigma(y_r) \, \mathrm{d}\langle y^a, y^b \rangle_r \\
&= \sigma(y_s) + \int_s^t \sum_{a=1}^k \partial_a \sigma(y_r) \, \sigma^{a \cdot}(y_r) \, \mathrm{d}B_r + \\
&\quad + \frac{1}{2} \sum_{a,b=1}^k \sum_{c=1}^d \int_s^t (\partial_{ab} \sigma \, \sigma^{ac} \, \sigma^{bc})(y_r) \, \mathrm{d}r \\
&= \sigma(y_s) + \int_s^t \sigma_2(y_r) \, \mathrm{d}B_r + \int_s^t p_r \, \mathrm{d}r.
\end{aligned}$$

We obtain

$$\delta\sigma(y)_{st} - \sigma_2(y_s) \mathbb{B}_{st}^1 = \int_s^t (\sigma_2(y_r) - \sigma_2(y_s)) \, \mathrm{d}B_r + \int_s^t p_r \, \mathrm{d}r.$$

First we have trivially

$$\left| \int_s^t p_r \, \mathrm{d}r \right| \lesssim |t-s| \lesssim |t-s|^{2\alpha}.$$

Since a.s. $y \in C^{\alpha}$, we obtain that a.s. $[s,t] \ni s \mapsto \sigma_2(y_r)$ is in C^{α}; moreover a.s. $B^{\ell} \in C^{\alpha}$. By Theorem 6.2 we obtain that a.s.

$$\left| \int_s^t (\sigma_2(y_r) - \sigma_2(y_s)) \, \mathrm{d}B_r \right| \lesssim |t-s|^{2\alpha}.$$

Therefore

$$|\delta\sigma(y)_{st} - \sigma_2(y_s) \mathbb{B}_{st}^1| \lesssim |t-s|^{2\alpha}.$$

Now, if y is solution to (6.10) then

$$\delta y_{st} - \sigma(y_s) \mathbb{B}_{st}^1 - \sigma_2(y_s) \mathbb{B}_{st}^2 = \int_s^t (\delta\sigma(y)_{sr} - \sigma_2(y_s) \mathbb{B}_{sr}^1) \, \mathrm{d}B_r.$$

Then by Theorem 6.4 with $h_r = \sigma(y_r)$ and $h_r^1 = \sigma_2(y_r)$, we obtain a.s.

$$|\delta y_{st} - \sigma(y_s) \mathbb{B}_{st}^1 - \sigma_2(y_s) \mathbb{B}_{st}^2| \lesssim |t-s|^{3\alpha}.$$

The proof is complete. □

Note that here we are only assuming the bound $\|\nabla\sigma\|_{\infty} < +\infty$, which is weaker than the condition $\|\nabla\sigma\|_{\infty} + \|\nabla\sigma_2\|_{\infty} < +\infty$ needed in Theorem 4.9 for existence of a global solution on $[0,T]$ for the rough finite difference equation (4.13) associated with a generic rough path.

Let us consider now a k dimensional SDE with a non-zero drift term

$$y_t = y_0 + \int_0^t b(y_s)\,\mathrm{d}s + \int_0^t \sigma(y_s)\,\mathrm{d}B_s, \qquad t \geqslant 0, \tag{6.11}$$

where $b: \mathbb{R}^k \to \mathbb{R}^k$, $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ and $(B_t)_{t\geqslant 0}$ is a d-dimensional Brownian motion. We assume that σ and b are of class C^3 with the bound $\|\nabla\sigma\|_\infty + \|\nabla b\|_\infty < +\infty$.

We define $X: [0,T] \to \mathbb{R}^{d+1}$ as $X_t := (B_t^1, \ldots, B_t^d, t)$, $t \in [0,T]$. Note that X is a continuous semimartingale. Then we define

$$\mathbb{X}^1_{st} := \delta X_{st}, \qquad \mathbb{X}^2_{st} := \int_s^t \mathbb{X}^1_{sr} \otimes \mathrm{d}X_r, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{6.12}$$

where the integral $\int_s^t \mathbb{X}^1_{sr}\,\mathrm{d}X^i_r$ is in the Itô sense for $i \in \{1, \ldots, d\}$ while for $i = d+1$ we have a standard Riemann integral since $\mathrm{d}X^{d+1}_t = \mathrm{d}t$. Moreover we define

$$\bar{\sigma}: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^{d+1}, \qquad \bar{\sigma}(x) := (\ \sigma(x) \quad b(x)\ ),$$

in other words $\bar{\sigma}^{ij} = \mathbb{1}_{(j\leqslant d)}\sigma^{ij} + \mathbb{1}_{(j=d+1)} b^i$ for $(i,j) \in \{1,\ldots,k\} \times \{1,\ldots,d+1\}$.

Then the SDE (6.11) can be rewritten as follows

$$y_t = y_0 + \int_0^t \bar{\sigma}(y_s)\,\mathrm{d}X_s, \qquad t \geqslant 0,$$

where the integral is in the Itô sense, which is well-defined since X is a semimartingale.

THEOREM 6.7. *The unique solution to the SDE (6.11) is a.s. equal to the unique solution to the rough finite difference equation (4.13) with coefficient $\bar{\sigma}$ and associated with the Itô rough path $\mathbb{X}$ defined in (6.12) above.*

The proof is identical to that of Theorem 6.6. Again, the assumptions on σ and b (class C^3 with the bound $\|\nabla\sigma\|_\infty + \|\nabla b\|_\infty < +\infty$) are weaker than what would be necessary to have existence of a global solution on $[0,T]$ using Theorem 4.9.

6.3. A REFINED KOLMOGOROV CRITERION

In this section we prepare the ground for the proof of Lemmas 6.3 and 6.5 in Section 6.4 below. Define the set $\mathbb{D}$ of dyadic points by

$$\mathbb{D} := \bigcup_{k\geq 0} D_k, \qquad \text{where} \qquad D_k := \left\{ d_i^k := \tfrac{i}{2^k} \right\}_{0\leqslant i\leqslant 2^k}. \tag{6.13}$$

We equip $\mathbb{D}$ with a *directed graph structure*: given $d, \tilde{d} \in \mathbb{D}$, we write $d \to \tilde{d}$ if and only if $d = d_i^k$ and $\tilde{d} = d_{i+1}^k$, for some $k \geq 0$ and $0 \leqslant i \leqslant 2^k - 1$. More explicitly, $d \to \tilde{d}$ if and only if the point $\tilde{d}$ is consecutive to d in some layer D_k of $\mathbb{D}$.

Remarkably, in order to prove relation (6.7), *it is enough to have a suitable control on* $R_{d,\tilde{d}}$ *for consecutive points* $d \to \tilde{d}$ (together with a global control on δR). This is the heart of the Kolmogorov continuity criterion, but we stress that it is a deterministic statement.

Theorem 6.8. (Kolmogorov criterion: deterministic part) *Given a function* $A: \mathbb{D}^2_< \to \mathbb{R}$, *define the following quantities, for fixed* $\gamma, \rho, \sigma \in (0, \infty)$:

$$Q_\gamma := \sup_{d,\tilde{d} \in \mathbb{D}: d \to \tilde{d}} \frac{|A(d,\tilde{d})|}{|\tilde{d} - d|^\gamma}, \tag{6.14}$$

$$K_{\rho,\sigma} := \sup_{(s,u,t) \in \mathbb{D}^3_<} \frac{|\delta A(s,u,t)|}{|u-s|^\rho |t-u|^\sigma}. \tag{6.15}$$

Then there is a universal constant $C < \infty$, *depending only on* γ, ρ, σ, *such that*

$$|A(s,t)| \leqslant C(Q_\gamma |t-s|^\gamma + K_{\rho,\sigma} |t-s|^{\rho+\sigma}), \qquad \forall (s,t) \in \mathbb{D}^2_<. \tag{6.16}$$

A key tool for Theorem 6.8 is the next result, proved in Section 6.4 below, which ensures the existence of suitable *short paths* in the graph $\mathbb{D}$.

Lemma 6.9. (Dyadic paths) *For any* $s, t \in \mathbb{D}$ *with* $s < t$, *there are integers* $n, m \geq 1$ *and a path of* $(m+n+1)$ *points in* $\mathbb{D}$ *which leads from* s *to* t, *labelled as follows:*

$$s = s_m < \ldots < s_1 < s_0 = t_0 < t_1 < \ldots < t_n < t_n = t, \tag{6.17}$$

with the property that for all $i \in \{0, \ldots, m-1\}$ *and* $j \in \{0, \ldots, n-1\}$

$$s_{i+1} \to s_i, \quad t_j \to t_{j+1}; \quad |s_i - s_{i+1}| < \frac{|t-s|}{2^i}, \quad |t_{j+1} - t_j| < \frac{|t-s|}{2^j}. \tag{6.18}$$

Proof of Theorem 6.8. Fix $s, t \in \mathbb{D}$ with $s < t$. We use Lemma 6.9 with the same notation. By the definition of δA, we write

$$A(s,t) = A(s,t_0) + A(t_0,t) + \delta A(s,t_0,t)$$

In the case $m \geq 2$, we can develop $A(s,t_0) = A(s,s_0)$ as follows (recall that $s = s_m$):

$$\begin{aligned} A(s,s_0) &= A(s,s_1) + A(s_1,s_0) + \delta A(s,s_1,s_0) \\ &= (A(s,s_2) + A(s_2,s_1) + \delta A(s,s_2,s_1)) + A(s_1,s_0) + \delta A(s,s_1,s_0) \\ &= \ldots = \sum_{i=0}^{m-1} A(s_{i+1},s_i) + \sum_{i=0}^{m-2} \delta A(s,s_{i+1},s_i). \end{aligned}$$

Similarly, when $n \geq 2$, we develop

$$A(t_0,t) = \sum_{j=0}^{n-1} A(t_j,t_{j+1}) + \sum_{j=0}^{n-2} \delta A(t_j,t_{j+1},t),$$

so that

$$A(s,t) = \underbrace{\sum_{i=0}^{m-1} A(s_{i+1},s_i) + \sum_{j=0}^{n-1} A(t_j,t_{j+1}) +}_{\Xi_1}$$

$$\underbrace{+\delta A(s,t_0,t) + \sum_{i=0}^{m-2} \delta A(s,s_{i+1},s_i) + \sum_{j=0}^{n-2} \delta A(t_j,t_{j+1},t).}_{\Xi_2} \qquad (6.19)$$

By the definition of Q_γ, for any $d \to \tilde{d}$ we can bound

$$|A(d,\tilde{d})| \leqslant Q_\gamma |\tilde{d} - d|^\gamma.$$

By Lemma 6.9, this bound applies to any couple (s_{i+1},s_i) and (t_j,t_{j+1}). Then we can estimate Ξ_1 in (6.19) as follows, exploiting the bounds in (6.18):

$$\begin{aligned} &Q_\gamma \left\{ \sum_{i=0}^{m-1} |s_i - s_{(i+1)}|^\gamma + \sum_{j=0}^{n-1} |t_{j+1} - t_j|^\gamma \right\} \leqslant \\ &\leqslant Q_\gamma \left\{ \sum_{i=0}^{\infty} (2^{-i})^\gamma + \sum_{j=0}^{\infty} (2^{-j})^\gamma \right\} |t-s|^\gamma = \\ &= Q_\gamma \left\{ \frac{2}{1-2^{-\gamma}} \right\} |t-s|^\gamma, \end{aligned}$$

which agrees with (6.16).

By the definition of $K_{\rho,\sigma}$, for all $(x,y,z) \in \mathbb{D}^3_<$ we can write

$$|\delta A(x,y,z)| \leqslant K_{\rho,\sigma} |y-x|^\rho |z-y|^\sigma,$$

therefore we can estimate Ξ_2 in (6.19) by

$$K_{\rho,\sigma} \left\{ |t_0 - s|^\rho |t-t_0|^\sigma + \sum_{i=0}^{\infty} |s_{i+1} - s|^\rho |s_i - s_{i+1}|^\sigma + \sum_{j=0}^{\infty} |t_{j+1} - t_j|^\rho |t - t_{j+1}|^\sigma \right\}.$$

We now use (6.18) to bound $|s_i - s_{i+1}|$ and $|t_{j+1} - t_j|$, while we bound all other distances simply by $|t-s|$ (recall (6.17)), to get

$$\begin{aligned} &K_{\rho,\sigma} \left\{ 1 + \sum_{i=0}^{\infty} (2^{-i})^\rho + \sum_{j=0}^{\infty} (2^{-j})^\sigma \right\} |t-s|^{\rho+\sigma} \leqslant \\ &\leqslant K_{\rho,\sigma} \left\{ 1 + \frac{1}{1-2^{-\rho}} + \frac{1}{1-2^{-\sigma}} \right\} |t-s|^{\rho+\sigma}. \end{aligned}$$

□

As a simple consequence of Theorem 6.8, we show that suitable moment conditions ensure the finiteness of the constant Q_γ in (6.14), as in the classical Kolmogorov criterion.

Proposition 6.10. (Kolmogorov criterion: probabilistic part) *Let* $A = (A(s,t))_{(s,t) \in \mathbb{D}^2_<}$ *be a stochastic process which satisfies the following bound, for some* $\gamma_0, p, c \in (0, \infty)$*:*

$$\mathbb{E}[|A(s,t)|^p] \leqslant c|t-s|^{p\gamma_0}, \qquad \forall (st) \in \mathbb{D}^2_<.$$

Then, for any value of γ *such that*

$$\gamma < \gamma_0 - \frac{1}{p}, \tag{6.20}$$

the random variable $Q_\gamma = Q_\gamma(A)$ *defined in (6.14) is in* L^p*:*

$$\mathbb{E}[|Q_\gamma|^p] < \infty.$$

In particular, $Q_\gamma < \infty$ *a.s..*

Proof. By definition of Q_γ in (6.14), bounding the supremum with a sum we can write

$$|Q_\gamma|^p \leqslant \sum_{d, \tilde{d} \in \mathbb{D}: d \to \tilde{d}} \left(\frac{|A(d, \tilde{d})|}{|\tilde{d} - d|^\gamma} \right)^p = \sum_{k \geq 0} \sum_{i=0}^{2^k - 1} \frac{|A(d_i^k, d_{i+1}^k)|^p}{|d_{i+1}^k - d_i^k|^{p\gamma}}.$$

Let us write $\gamma = \gamma_0 - \frac{1+\epsilon}{p}$, for some $\epsilon > 0$. Since $d_{i+1}^k - d_i^k = \frac{1}{2^k}$ we have

$$\begin{aligned} \mathbb{E}[|Q_\gamma|^p] &\leqslant \sum_{k \geq 0} \sum_{i=0}^{2^k - 1} c |d_{i+1}^k - d_i^k|^{p(\gamma_0 - \gamma)} \\ &\leqslant \sum_{k \geq 0} \sum_{i=0}^{2^k - 1} \frac{c}{2^{(1+\epsilon)k}} = \sum_{k \geq 0} \frac{c}{2^{\epsilon k}} = \frac{c}{1 - 2^{-\epsilon}} < \infty. \end{aligned}$$

The proof is complete. □

Remark 6.11. Given a stochastic process $(X_t)_{t \in \mathbb{D}}$ defined on dyadic times, if we apply Theorem 6.8 and Proposition 6.10 to $(A(s,t) := \delta X_{st} = X_t - X_s)_{(s,t) \in \mathbb{D}^2_<}$ we obtain the classical Kolmogorov continuity criterion. Note that in this case $K_{\rho,\sigma} = 0$ because $\delta A = 0$.

6.4. Proof of technical Lemmas

Proof of Lemma 6.3. Fix $\beta < \frac{1}{2}$. We apply Theorem 6.8 to the (random) function $A(s,t) = R_{st}$, with $\gamma = \alpha + \beta$, $\rho = \alpha$, $\sigma = \beta$ and p large enough (to be fixed later). Then relation (6.16) yields (6.7). It remains to show that a.s. $Q_{\alpha+\beta,p} < \infty$ and $K_{\alpha,\beta} < \infty$.

We recall that R_{st} is defined in (6.4). In particular, for $s < u < t$

$$\delta R_{sut} = R_{st} - R_{su} - R_{ut} = (h_u - h_s)(B_t - B_u).$$

Then by (6.15)

$$K_{\alpha,\beta}(R) \leqslant \|h\|_\alpha \|b\|_\beta < \infty \quad \text{a.s.},$$

by our assumption that $\|h\|_\alpha \in L^p$ and by the fact that b is a Brownian motion.

Next we note that, for fixed $s<t$, we have $R_{st} = \int_s^t (h_u - h_s)\,\mathrm{d}B_u$ a.s.. By the Burkholder-Davies-Gundy inequality (see ???), for any $p>1$ there is a universal constant c_p such that

$$\begin{aligned}\mathbb{E}[|R_{st}|^p] &\leqslant \mathbb{E}\bigg[\bigg(\int_s^t (h_u - h_s)^2\,\mathrm{d}u\bigg)^{\frac{p}{2}}\bigg]\\ &\leqslant c_p\,\mathbb{E}\bigg[\|h\|_\alpha^p \bigg(\int_s^t (u-s)^{2\alpha}\,\mathrm{d}u\bigg)^{\frac{p}{2}}\bigg]\\ &\leqslant c_p\,\mathbb{E}[\|h\|_\alpha^p]\,(t-s)^{p\left(\alpha+\frac{1}{2}\right)}.\end{aligned}$$

By Proposition 6.10, we have $Q_\gamma < \infty$ a.s. for any $\gamma < \alpha + \frac{1}{2} - \frac{1}{p}$. Plugging $\gamma = \alpha + \beta$ we get $\beta < \frac{1}{2} - \frac{1}{p}$, which is satisfied for p large enough, since $\beta < \frac{1}{2}$. □

Proof of Lemma 6.5. This is the same as that of Lemma 6.3, apart from the fact that

$$\delta R^2_{sut} = (\delta h_{su} - h^1_s\,\mathbb{B}^1_{su})\,\mathbb{B}^1_{ut} + \delta h^1_{su}\,\mathbb{B}^2_{ut},$$

which implies that $K_{2\alpha,\beta}(R^2) < +\infty$, and

$$\begin{aligned}\mathbb{E}[|R^2_{st}|^p] &\leqslant \mathbb{E}\bigg[\bigg(\int_s^t (\delta h_{us} - h^1_s\,\mathbb{B}^1_{su})^2\,\mathrm{d}u\bigg)^{\frac{p}{2}}\bigg]\\ &\leqslant c_p\,\mathbb{E}\bigg[\|\delta h - h^1\mathbb{B}^1\|_{2\alpha}^p \bigg(\int_s^t (u-s)^{4\alpha}\,\mathrm{d}u\bigg)^{\frac{p}{2}}\bigg]\\ &\leqslant c_p\,\mathbb{E}[\|\delta h - h^1\mathbb{B}^1\|_{2\alpha}^p]\,(t-s)^{p\left(2\alpha+\frac{1}{2}\right)}.\end{aligned}$$

The rest of the argument is identical. □

Proof of Lemma 6.9. We refer to Figure 6.1 for a graphical representation. Given $s,t \in \mathbb{D}$ with $s<t$, since $0 < |t-s| \leqslant 1$, we can define $\ell \geq 1$ as the unique integer such that

$$\frac{1}{2^\ell} < |t-s| \leqslant \frac{1}{2^{\ell-1}}. \tag{6.21}$$

We now take the smallest $k \in \{0,\ldots,2^\ell - 1\}$ for which $d^\ell_k > s$ and define

$$s_0 := t_0 := d^\ell_k.$$

Note that $0 < d^\ell_k - s \leqslant d^\ell_k - d^\ell_{k-1} = \frac{1}{2^\ell}$ and $0 < t - d^\ell_k < t - s$, by (6.21), therefore

$$0 < s_0 - s < \frac{1}{2^{\ell-1}}, \qquad 0 < t - t_0 < \frac{1}{2^{\ell-1}}. \tag{6.22}$$

Since both $s_0 - s \in \mathbb{D}$ and $t - t_0 \in \mathbb{D}$, for suitable integers $m \geq 1$ and $n \geq 1$ we have

$$s_0 - s = \frac{1}{2^{q_1}} + \frac{1}{2^{q_2}} + \ldots + \frac{1}{2^{q_m}}, \qquad t - t_0 = \frac{1}{2^{r_1}} + \frac{1}{2^{r_2}} + \ldots + \frac{1}{2^{r_n}},$$

where $q_m > q_{m-1} > \ldots > q_1 \geq \ell$ and $r_n > \ldots > r_1 \geq \ell$. We can thus write

$$\begin{aligned} s &= s_0 - \frac{1}{2^{q_1}} - \frac{1}{2^{q_2}} - \ldots - \frac{1}{2^{q_m}}, \\ t &= t_0 + \frac{1}{2^{r_1}} + \frac{1}{2^{r_2}} + \ldots + \frac{1}{2^{r_n}}. \end{aligned}$$

We can finally define

$$\begin{aligned} s_i &:= s_0 - \frac{1}{2^{q_1}} - \frac{1}{2^{q_2}} - \ldots - \frac{1}{2^{q_i}} \quad \text{for } i = 1, \ldots, m, \\ t_j &:= t_0 + \frac{1}{2^{r_1}} + \frac{1}{2^{r_2}} + \ldots + \frac{1}{2^{r_j}} \quad \text{for } j = 1, \ldots, n. \end{aligned}$$

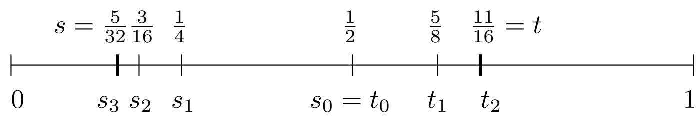

Figure 6.1. An instance of Lemma 6.9 with $s = \frac{5}{32}$ and $t = \frac{11}{16}$. Note that $\ell = 1$ (because $\frac{1}{2^1} < |t - s| = \frac{17}{32} \leq \frac{1}{2^0}$, cf. (6.21)) and $s_0 = t_0 = \frac{1}{2}$. The points $t_1, \ldots, t_n$ are built iteratively: first take the largest $\frac{1}{2^{r_1}}$ (i.e. the smallest r_1) such that $t_1 := t_0 + \frac{1}{2^{r_1}} \leq t$; if $t_1 < t$, then take the largest $\frac{1}{2^{r_2}}$ such that $t_2 := t_1 + \frac{1}{2^{r_2}} \leq t$; and so on, until $t_n = t$. Similarly for $s_1, \ldots, s_m$.

Since q_i and r_j are strictly increasing integers with $q_1 \geq \ell$ and $r_1 \geq \ell$, we have the bounds $q_i \geq \ell + (i-1)$ and $r_j \geq \ell + (j-1)$, for all $i \in \{0, \ldots, m-1\}$ and $j \in \{0, \ldots, n-1\}$, hence

$$\begin{aligned} |s_i - s_{i+1}| &= \frac{1}{2^{q_{i+1}}} \leqslant \frac{1}{2^i} \frac{1}{2^\ell} < \frac{|t-s|}{2^i}, \\ |t_{j+1} - t_j| &= \frac{1}{2^{q_{j+1}}} \leqslant \frac{1}{2^j} \frac{1}{2^\ell} < \frac{|t-s|}{2^j}. \end{aligned}$$

having used (6.21). This proves the bounds in (6.18).

We note that, for any integer $r \geq \ell$, we have the inclusion $D_\ell \subseteq D_r$. Then, given any $x \in D_\ell$, we have that $x \in D_r$, hence $x \to x + 2^{-r}$. Since $t_0 = d_k^\ell \in D_\ell$ and $r_1 \geq \ell$, this shows that $t_0 \to t_1 = t_0 + 2^{-r_1}$. Proceeding inductively, we have $t_j \to t_{j+1} = t_j + 2^{-r_{j+1}}$. A similar argument applies to the points s_i and completes the proof of (6.18). □

6.5. B-D-G inequality

We give a proof of (half of) Burkholder-Davies-Gundy inequality for $p \geq 2$.

Proposition 6.12. *For all $p \geq 2$ there is a constant $c_p < \infty$ such that for all $0 \leqslant s < t \leqslant 1$*

$$\mathbb{E}\left[\left(\int_s^t y_u \, \mathrm{d}B_u\right)^p\right] \leqslant c_p \, \mathbb{E}\left[\left(\int_s^t y_u^2 \, \mathrm{d}u\right)^{\frac{p}{2}}\right]$$

for any progressively measurable process such that $\int_0^1 y_u^2 \, \mathrm{d}u < \infty$, $\mathbb{P}$-a.s..

Proof. To simplify notation we set $s = 0$ and $m_t := \int_0^t y_u \, \mathrm{d}B_u$.

In a first time we make the additional assumptions that $\mathbb{E}\left[\int_0^1 y_u^2 \, \mathrm{d}u\right] < \infty$ and m is bounded by some deterministic constant. By the Itô formula applied to m_t, we get

$$\mathrm{d}|m_t|^p = p|m_t|^{p-1} y_t \, \mathrm{d}B_t + \frac{p(p-1)}{2} |m_t|^{p-2} y_t^2 \, \mathrm{d}t.$$

In general $(\int_0^t |m_u|^{p-1} y_u \, \mathrm{d}B_u)_t$ is a local martingale, but under our additional assumptions it is a true martingale with zero expectation, because $\mathbb{E}[\int_0^1 |m_u|^{2(p-1)} \, y_u^2 \mathrm{d}u] < \infty$ (recall that m is bounded). Consequently

$$\mathbb{E}[|m_t|^p] = \frac{p(p-1)}{2} \, \mathbb{E}\left[\int_0^t |m_u|^{p-2} y_u^2 \, \mathrm{d}u\right].$$

If we set $|\bar{m}_t| := \sup_{u \leqslant t} |m_u|$, we obtain by Hölder

$$\begin{aligned}
\mathbb{E}[|m_t|^p] &\leqslant \frac{p(p-1)}{2} \mathbb{E}\left[|\bar{m}_t|^{p-2} \int_0^t y_u^2 \, \mathrm{d}u\right] \\
&\leqslant \frac{p(p-1)}{2} \, \mathbb{E}[|\bar{m}_t|^p]^{1-\frac{2}{p}} \mathbb{E}\left[\left(\int_0^t y_u^2 \, \mathrm{d}u\right)^{\frac{p}{2}}\right]^{\frac{2}{p}}. \qquad (6.23)
\end{aligned}$$

Since $(|m_t|)_{t \geq 0}$ is submartingale bounded in L^p with continuous trajectories, by Doob L^p inequality we have: $\mathbb{E}[|\bar{m}_t|^p] \leqslant (\frac{p}{p-1})^p \, \mathbb{E}[|m_t|^p]$. Plugging the above in (6.23) we conclude:

$$\mathbb{E}\left[\left|\int_0^t y_u \, \mathrm{d}B_u\right|^p\right] \leqslant c_p \mathbb{E}\left[\left(\int_0^t y_u^2 \, \mathrm{d}u\right)^{p/2}\right].$$

As far as the general case is concerned, let us define

$$\tau^n = \inf\{t \geq 0 \colon |m(t)| > n\} \wedge \inf\left\{t \geq 0 \colon \int_0^t y_u^2 \, \mathrm{d}u > n\right\}$$

Note that τ^n is a non decreasing sequence of stopping times, with $\tau^n = \infty$ for n large enough, $\mathbb{P}$-a.s.. We denote $y_t^n := y \mathbf{1}_{[0,\tau^n]}(t)$ and $m_t^n := \int_0^t y_u^n \mathrm{d}B_u$. By construction, y^n and m^n satisfy our additional assumptions. Since $m_t^n = m_{t \wedge \tau^n}$ a.s., we have

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^{t \wedge \tau^n} y_u \, \mathrm{d}B_u\right|^p\right] &\leqslant c_p \, \mathbb{E}\left[\left(\int_0^t y_u^2 \, \mathbf{1}_{[0,\tau^n]}(u) \, \mathrm{d}u\right)^{p/2}\right] \\
&\leqslant c_p \, \mathbb{E}\left[\left(\int_0^t y_u^2 \, \mathrm{d}u\right)^{p/2}\right].
\end{aligned}$$

Finally we notice that by Fatou's Lemma

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_s^t y_u\,\mathrm{d}B_u\Big)^p\Big] &= \mathbb{E}\Big[\liminf_{n\to\infty}\Big|\int_s^{t\wedge\tau^n} y_u\,\mathrm{d}B_u|^p\Big] \\
&\leqslant \liminf_{n\to\infty}\mathbb{E}\Big[\Big|\int_s^{t\wedge\tau^n} y_u\,\mathrm{d}B_u|^p\Big] \\
&\leqslant c_p\,\mathbb{E}\Big[\Big(\int_s^t y_u^2\,\mathrm{d}u\Big)^{p/2}\Big].
\end{aligned}$$

The proof is complete. □

Chapter 7
Geometric rough paths

7.1. Geometric rough paths

We recall that the set of smooth paths C^1 is not dense in C^α, but its closure is quite large, because it contains $C^{\alpha'}$ for all $\alpha' > \alpha$. The situation is different for rough paths: the set $\mathcal{R}_{1,d}$ of canonical rough paths over smooth paths is again not dense in $\mathcal{R}_{\alpha,d}$, but its closure is a significantly smaller set, that we now describe.

Definition 7.1. *The closure of $\mathcal{R}_{1,d}$ in $\mathcal{R}_{\alpha,d}$ is denoted by $\mathcal{R}^{\mathrm{g}}_{\alpha,d}$ and its elements are called* geometric rough paths*.*

For smooth paths $f, g \in C^1$, the integration by parts formula holds:

$$\int_s^t f(u)\,\mathrm{d}g(u) = f(t)g(t) - f(s)g(s) - \int_s^t g(u)\,\mathrm{d}f(u)\,.$$

It follows that

$$\int_s^t (f(u) - f(s))\,\mathrm{d}g(u) + \int_s^t (g(u) - g(s))\,\mathrm{d}f(u) = (f(t) - f(s))(g(t) - g(s))\,.$$

We have seen in Proposition 2.8 that the same formula holds if $(f, g) \in C^\alpha \times C^\beta$ with $\alpha + \beta > 1$ and the integral is in the Young sense.

Given a smooth path $X \in C^1$, define $\mathbb{X}^2$ by (5.30) as an ordinary integral (i.e. $(\mathbb{X}^1, \mathbb{X}^2)$ is the canonical rough path over X). The previous relation for $f = X_i$ and $g = X_j$ shows that

$$\mathbb{X}^2_{ij}(s,t) + \mathbb{X}^2_{ji}(s,t) = \mathbb{X}^1_i(s,t)\,\mathbb{X}^1_j(s,t)\,. \tag{7.1}$$

This relation is called the *shuffle relation*: for $i = j$ it identifies $\mathbb{X}^2_{ii}$ in terms of X_i:

$$\mathbb{X}^2_{ii}(s,t) = \frac{1}{2}\mathbb{X}^1_i(s,t)^2\,, \tag{7.2}$$

while for $i \neq j$ it expresses $\mathbb{X}^2_{ij}$ in terms of $\mathbb{X}^1_i$, $\mathbb{X}^1_j$, $\mathbb{X}^2_{ji}$. Denoting by $\mathrm{Sym}(\mathbb{X}^2)_{ij} := \frac{1}{2}(\mathbb{X}^2_{ij} + \mathbb{X}^2_{ji})$ the symmetric part of $\mathbb{X}^2$, we can rewrite the shuffle relation more compactly as follows:

$$\mathrm{Sym}(\mathbb{X}^2) = \frac{1}{2}\,\mathbb{X}^1 \otimes \mathbb{X}^1\,. \tag{7.3}$$

DEFINITION 7.2. *Rough paths in* $\mathcal{R}_{\alpha,d}$ *that satisfy the shuffle relation (7.1)-(7.3) are called* weakly geometric *and denoted by* $\mathcal{R}_{\alpha,d}^{\mathrm{wg}}$.

Exercise 7.1. For $\alpha > \frac{1}{2}$ we have $\mathcal{R}_{\alpha,d} = \mathcal{R}_{\alpha,d}^{\mathrm{wg}}$ (every rough path is weakly geometric).

We can now show that the closure of $\mathcal{R}_{1,d}$ in $\mathcal{R}_{\alpha,d}$ is included in $\mathcal{R}_{\alpha,d}^{\mathrm{wg}}$.

LEMMA 7.3. *Geometric rough paths are weakly geometric:* $\mathcal{R}_{\alpha,d}^{\mathrm{g}} \subset \mathcal{R}_{\alpha,d}^{\mathrm{wg}}$ *for any* $\alpha \in (\frac{1}{3}, 1)$*, with a strict inclusion.*

Proof. Canonical rough paths $(\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{1,d}$ over smooth paths satisfy the shuffle relation (7.1)-(7.3). Geometric rough paths are by definition limits in $\mathcal{R}_{\alpha,d}$ of smooth paths in $\mathcal{R}_{1,d}$. Since convergence in $\mathcal{R}_{\alpha,d}$ implies pointwise convergence, geometric rough paths satisfy the shuffle relation too. This shows that $\mathcal{R}_{\alpha,d}^{\mathrm{g}} \subset \mathcal{R}_{\alpha,d}^{\mathrm{wg}}$.

To prove that the inclusion $\mathcal{R}_{\alpha,d}^{\mathrm{g}} \subset \mathcal{R}_{\alpha,d}^{\mathrm{wg}}$ is strict, it suffices to consider a weakly geometric rough path $(\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{\alpha,d}^{\mathrm{wg}}$ which lies above a path $X \in C^\alpha$ which is not in the closure of C^1. Such a path is not geometric (recall that $(\mathbb{X}_n^1, \mathbb{X}_n^2) \to (\mathbb{X}^1, \mathbb{X}^2)$ in $\mathcal{R}_{\alpha,d}$ implies $\mathbb{X}_n^1 \to \mathbb{X}^1$ in C_2^α).

To prove the existence of such a rough path, in the one-dimensional case $d = 1$ it is enough to consider the one provided by (5.23), which is by construction weakly geometric, since the shuffle relation reduces to $\delta\mathbb{X}_{st}^2 := \frac{1}{2}(\mathbb{X}_{st}^1)^2$. □

Although the inclusion $\mathcal{R}_{\alpha,d}^{\mathrm{g}} \subset \mathcal{R}_{\alpha,d}^{\mathrm{wg}}$ is strict, what is left out turns out to be not so large. More precisely, recalling that $\mathcal{R}_{\alpha,d}^{\mathrm{g}}$ is the closure of $\mathcal{R}_{1,d}$ in $\mathcal{R}_{\alpha,d}$, we have a result which is similar to what happens for Hölder spaces, with the important difference that the whole space $\mathcal{R}_{\alpha,d}$ is replaced by $\mathcal{R}_{\alpha,d}^{\mathrm{wg}}$. The proof is non-trivial and we omit it.

PROPOSITION 7.4. *For any* $\frac{1}{3} < \alpha' < \alpha < 1$ *one has* $\mathcal{R}_{\alpha,d}^{\mathrm{wg}} \subseteq \mathcal{R}_{\alpha',d}^{\mathrm{g}}$. *This means that for any* $\mathbb{X} \in \mathcal{R}_{\alpha,d}^{\mathrm{wg}}$ *there is a sequence* $\mathbb{X}_n \in \mathcal{R}_{1,d}$ *such that* $\mathbb{X}_n \to \mathbb{X}$ *in* $\mathcal{R}_{\alpha',d}$.

We stress that the notion of "weakly geometric" rough path depends only on the function $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$, but the notion of "geometric" rough path depends also on the chosen space $\mathcal{R}_{\alpha,d}$. Given a weakly geometric rough path $\mathbb{X} \in \mathcal{R}_{\alpha,d}$, even though $\mathbb{X}$ may fail to be geometric in $\mathcal{R}_{\alpha,d}$, it is certainly geometric in $\mathcal{R}_{\alpha',d}$ for all $\alpha' < \alpha$. In this sense, *every weakly geometric rough path is a geometric rough path, of a possibly slightly lower regularity.*

7.2. NON-GEOMETRIC ROUGH PATHS

We next consider generic rough paths. These cannot be approximated by canonical rough paths over smooth paths. However we have

Lemma 7.5. *Given an arbitrary rough path* $(\mathbb{X}^1,\mathbb{X}^2)\in\mathcal{R}_{\alpha,d}$ *lying above* X*, there is always a weakly geometric rough path* $(\mathbb{X}^1,\tilde{\mathbb{X}}^2)\in\mathcal{R}^{\mathrm{wg}}_{\alpha,d}$ *lying above the same path* X*.*

Proof. It suffice to define $\tilde{\mathbb{X}}^2_{ij}:=\mathbb{X}^2_{ij}$ for all $i>j$ and use the shuffle relation to define the remaining entries of $\tilde{\mathbb{X}}^2$, i.e. $\tilde{\mathbb{X}}^2_{ii}:=\frac{1}{2}(\mathbb{X}^1_i)^2$ and $\tilde{\mathbb{X}}^2_{ij}:=\mathbb{X}^1_i\,\mathbb{X}^1_j-\mathbb{X}^2_{ji}$ for all $i<j$. In this way $(\mathbb{X}^1,\tilde{\mathbb{X}}^2)$ satisfies the shuffle relation by construction and it is easy to check that $\tilde{\mathbb{X}}^2\in C_2^{2\alpha}$.

It remains to prove that the Chen relation (5.25) holds for $(\mathbb{X}^1,\tilde{\mathbb{X}}^2)$, that is

$$\delta\tilde{\mathbb{X}}^2_{ij}(s,u,t)=\mathbb{X}^1_i(s,u)\,\mathbb{X}^1_j(u,t)\,.$$

If $i>j$ this holds because $\tilde{\mathbb{X}}^2_{ij}=\mathbb{X}^2_{ij}$, so we only need to consider $i=j$ and $i<j$. Note that if we define $A_{st}:=\delta f_{st}\,\delta g_{st}$, for arbitrary $f,g\colon[a,b]\to\mathbb{R}$, we have

$$\begin{aligned}\delta A_{sut} &= \delta f_{st}\,\delta g_{st}-\delta f_{su}\,\delta g_{su}-\delta f_{ut}\,\delta g_{ut}\\ &= (\delta f_{su}+\delta f_{ut})\,\delta g_{st}-\delta f_{su}\,\delta g_{su}-\delta f_{ut}\,\delta g_{ut}\\ &= \delta f_{su}\,\delta g_{ut}+\delta g_{su}\delta f_{ut}.\end{aligned}$$

Applying this to $f=X^i$ and $g=X^j$ yields, for $i<j$,

$$\begin{aligned}\delta\tilde{\mathbb{X}}^2_{ij}(s,u,t) &= \delta(\mathbb{X}^1_i\,\mathbb{X}^1_j-\mathbb{X}^2_{ji})(s,u,t)\\ &= \mathbb{X}^1_i(s,u)\,\mathbb{X}^1_j(u,t)+\mathbb{X}^1_j(s,u)\,\mathbb{X}^1_i(u,t)-\mathbb{X}^1_j(s,u)\,\mathbb{X}^1_i(u,t)\\ &= \mathbb{X}^1_i(s,u)\,\mathbb{X}^1_j(u,t)\,.\end{aligned}$$

Similarly, choosing $f=g=X_i$ gives $\delta\tilde{\mathbb{X}}^2_{ii}(s,u,t)=\mathbb{X}^1_i(s,u)\mathbb{X}^1_i(u,t)$. □

As a corollary, we obtain a useful approximation result.

Proposition 7.6. *For any rough path* $(\mathbb{X}^1,\mathbb{X}^2)\in\mathcal{R}_{\alpha,d}$*, there is a function* $f\in C^{2\alpha}([0,T];\mathbb{R}^d\otimes\mathbb{R}^d)$ *and a sequence of canonical rough paths over smooth paths* $(\mathbb{X}^1_n,\mathbb{X}^2_n)\in\mathcal{R}_{1,d}$ *such that*

$$(\mathbb{X}^1_n,\mathbb{X}^2_n+\delta f)\to(\mathbb{X}^1,\mathbb{X}^2)\qquad \text{in } \mathcal{R}_{\alpha',d}\,,\qquad \forall\alpha'<\alpha\,.$$

Proof. By Lemma 7.5 there is a weakly geometric rough path $(\mathbb{X}^1,\tilde{\mathbb{X}}^2)$ lying above the same path X. Then $\mathbb{X}^2-\tilde{\mathbb{X}}^2=\delta f$ for some $f\in C^{2\alpha}([0,T];\mathbb{R}^d\otimes\mathbb{R}^d)$, by Lemma 5.17. By Proposition 7.4, there is a sequence $(\mathbb{X}^1_n,\mathbb{X}^2_n)\in\mathcal{R}_{1,d}$ such that $(\mathbb{X}^1_n,\mathbb{X}^2_n)\to(\mathbb{X}^1,\tilde{\mathbb{X}}^2)$ in $\mathcal{R}_{\alpha',d}$, for any $\alpha'<\alpha$. It follows that $(\mathbb{X}^1_n,\mathbb{X}^2_n+\delta f)\to(\mathbb{X}^1,\tilde{\mathbb{X}}^2+\delta f)=(\mathbb{X}^1,\mathbb{X}^2)$. □

7.3. Pure area rough paths

Given $X\in C^\alpha$, we denote by $\mathcal{R}_{\alpha,d}(X)$ the subset of rough paths $(\mathbb{X}^1,\mathbb{X}^2)\in\mathcal{R}_{\alpha,d}$ lying above X, i.e. such that $\mathbb{X}^1=\delta X$. Here is a special case.

DEFINITION 7.7. *The elements of* $\mathcal{R}_{\alpha,d}(0)$*, i.e. those of the form* $\mathbb{X} = (0, \mathbb{X}^2)$*, are called* pure area *rough paths.*

Pure area rough paths are very explicit. Let us denote by $(\mathbb{R}^{d\times d})^{\mathrm{a}}$ the subspace of $\mathbb{R}^{d\times d}$ given by antisymmetric matrices.

LEMMA 7.8. $X = (0, \mathbb{X}^2)$ *is a pure area* α*-rough path if and only if* $\mathbb{X}^2 = \delta f$*, for some* $f \in C^{2\alpha}([0,T]; \mathbb{R}^{d\times d})$*. Such rough path is weakly geometric if and only if* $\mathbb{X}^2_{st} \in (\mathbb{R}^{d\times d})^{\mathrm{a}}$*, i.e. is an antisymmetric matrix, for all* $s, t \in [0,T]^2_{\leqslant}$*; equivalently, we can take* $f \in C^{2\alpha}([0,T]; (\mathbb{R}^{d\times d})^{\mathrm{a}})$*.*

Proof. Since $(0,0)$ is a rough path, it follows by Lemma 5.17 that for all (pure area) rough paths $(0, \mathbb{X}^2)$ we have $\mathbb{X}^2 = \delta f$ for some $f \in C^{2\alpha}$. We may assume that $f(a) = 0$ (just redefine $f(t)$ as $f(t) - f(0)$). Since $x = 0$, the shuffle relation (7.3) becomes $\mathrm{Sym}(\mathbb{X}^2) = 0$, i.e. $\mathbb{X}^2_{st}$ is an antisymmetric matrix. Then $f(t) = f(t) - f(0) = \mathbb{X}^2_{0t}$ is antisymmetric too. □

Note that the set $\mathcal{R}_{\alpha,d}(0)$ of pure area rough paths is a *vector space*, because the Chen relation (5.25) reduces to the linear relation $\delta\mathbb{X}^2 = 0$. Here is the link with general rough paths.

PROPOSITION 7.9. *The set* $\mathcal{R}_{\alpha,d}(X)$ *of rough paths laying above a given path* X *is an affine space, with associated vector space* $\mathcal{R}_{\alpha,d}(0)$*, the space of pure area rough paths.*

Proof. Given rough paths $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ and $\bar{\mathbb{X}} = (\mathbb{X}^1, \bar{\mathbb{X}}^2)$ lying above the same path X, their difference $\mathbb{X} - \bar{\mathbb{X}} = (0, \mathbb{X}^2 - \bar{\mathbb{X}}^2)$ is a pure area rough path, because it satisfies the Chen relation $\delta(\mathbb{X}^2 - \bar{\mathbb{X}}^2) = 0$ (since $\delta\mathbb{X}^2 = \mathbb{X}^1 \otimes \mathbb{X}^1 = \delta\,\bar{\mathbb{X}}^2$).

Alternatively, Lemma 5.17 yields $\mathbb{X}^2 - \bar{\mathbb{X}}^2 = \delta f$ for some $f \in C^{2\alpha}$, hence $(0, \mathbb{X}^2 - \bar{\mathbb{X}}^2)$ is a pure area rough path by Lemma 7.8. □

We have seen in Section 5.8 how pure area rough paths can arise concretely.

7.4. WONG-ZAKAI

(to be completed)

We choose any $\rho\colon \mathbb{R} \to \mathbb{R}$ of class C^1 such that $\rho(x) = \rho(-x)$ for all $x \in \mathbb{R}$ and $\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x = 1$. We define, for $\varepsilon > 0$, $\rho_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ by

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon}\,\rho\Big(\frac{x}{\varepsilon}\Big), \qquad x \in \mathbb{R}.$$

Let $(B_t)_{t\geqslant 0}$ be a d-dimensional Brownian motion, extended to $B\colon \mathbb{R} \to \mathbb{R}^d$ by setting $B_t := 0$ for $t < 0$. Then we set for $\varepsilon > 0$

$$B^\varepsilon_t := (B * \rho_\varepsilon)(t) = \int_{\mathbb{R}} B_s\,\rho_\varepsilon(t-s)\,\mathrm{d}s, \qquad t \in \mathbb{R}.$$

We set the following random ODE

$$\mathrm{y}_t^\varepsilon = \mathrm{y}_0 + \int_0^t b(\mathrm{y}_s^\varepsilon)\,\mathrm{d}s + \int_0^t \sigma(\mathrm{y}_s^\varepsilon)\,\dot{B}_s^\varepsilon\,\mathrm{d}s,$$

where $\dot{B}_s^\varepsilon = \frac{\mathrm{d}}{\mathrm{d}s} B_s^\varepsilon$ which is a continous function since ρ is C^1.

Then Wong-Zakai's result states that a.s. y_t^ε converges to the solution y_t of

$$\begin{aligned} \mathrm{y}_t &= \mathrm{y}_0 + \int_0^t b(\mathrm{y}_s)\,\mathrm{d}s + \int_0^t \sigma(\mathrm{y}_s) \circ \mathrm{d}B_s \\ &= \mathrm{y}_0 + \int_0^t b(\mathrm{y}_s)\,\mathrm{d}s + \int_0^t \sigma(\mathrm{y}_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \mathrm{Tr}_{\mathbb{R}^d}\,\sigma_2(\mathrm{y}_s)\,\mathrm{d}s, \end{aligned}$$

where $(\mathrm{Tr}_{\mathbb{R}^d}\,\sigma_2(y))^i := \sum_{j,m=1}^d (\sigma_2(y))^{ijm}$.

7.5. Doss-Sussmann

In this section we suppose that σ is such that for all $i \in \{1, \ldots, k\}$ the $d \times d$ matrix $(\sigma_2^{ijm})_{jm}$ is symmetric, namely

$$(\sigma_2(y))^{ijm} = (\sigma_2(y))^{imj}, \qquad \forall y \in \mathbb{R}^k, i \in \{1, \ldots, k\}, j, m \in \{1, \ldots, d\}. \tag{7.4}$$

For example, if $k = d = 2$ and we consider

$$\sigma(y) = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, \qquad y = (y_1, y_2) \in \mathbb{R}^2,$$

then

$$\partial_a \sigma^{ij}(y) = \mathbb{1}_{\{i=1, j=1, a=1\}} + \mathbb{1}_{\{i=2, j=2, a=2\}},$$

and

$$\sigma_2^{ijm}(y) = \sum_{a=1}^2 \partial_a \sigma^{ij}(y)\,\sigma^{am}(y) = \mathbb{1}_{\{i=1,j=1,m=1\}} y_1 + \mathbb{1}_{\{i=2,j=2,m=2\}} y_2,$$

which is clearly symmetric in (j, m).

In this case, if $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ is a weakly geometric α-rough path, we obtain

$$\begin{aligned} (\sigma_2(y)\,\mathbb{X}^2)^i &= \frac{1}{2}\left\{ \sum_{a,b=1}^2 \sigma_2^{iab} (\mathbb{X}^2)^{ba} + \sum_{a,b=1}^2 \sigma_2^{iba} (\mathbb{X}^2)^{ab} \right\} \\ &= \frac{1}{2} \sum_{a,b=1}^2 \sigma_2^{iab} \{(\mathbb{X}^2)^{ba} + (\mathbb{X}^2)^{ab}\} \\ &= \frac{1}{2} \sum_{a,b=1}^2 \sigma_2^{iab}(y)\,(\mathbb{X}^1)^a (\mathbb{X}^1)^b. \end{aligned} \tag{7.5}$$

In this case the solution y to the finite difference equation is a function of $\mathbb{X}^1$ alone since (4.13) is equivalent to

$$|\mathrm{y}^3_{st}| \lesssim |t-s|^{\zeta}, \qquad \mathrm{y}^3_{st} = \delta \mathrm{y}_{st} - \sigma(\mathrm{y}_s)\, \mathbb{X}^1_{st} - \sigma_2(\mathrm{y}_s)\, (\mathbb{X}^1_{st} \otimes \mathbb{X}^1_{st}). \tag{7.6}$$

It can be seen that the map $(\mathrm{y}_0, \mathbb{X}^1) \mapsto \mathrm{y}$ is continuous.

Proposition 7.10. *Let $M > 0$ and let us suppose that $\mathbb{X}$ is a weakly geometric rough path and σ satisfies the Frobenius condition (7.4). If*

$$\max \{|\sigma(\mathrm{y}_0)| + |\sigma(\bar{\mathrm{y}}_0)| + |\sigma_2(\bar{\mathrm{y}}_0)|, \|\mathbb{X}^1\|_{\alpha}, \|\bar{\mathbb{X}}^1\|_{\alpha}\} \leqslant M,$$

then for every $T > 0$ there are $\hat{\tau}_{M,D,T}, C_{M,D,T} > 0$ such that for $\tau \in \left]0, \hat{\tau}_{M,D,T}\right]$

$$\|\mathrm{y} - \bar{\mathrm{y}}\|_{\infty,\tau} + \|\delta \mathrm{y} - \delta \bar{\mathrm{y}}\|_{\alpha,\tau} + \|\mathrm{y}^2 - \bar{\mathrm{y}}^2\|_{2\alpha,\tau} \leqslant \\ \leqslant C_{M,D,T}\, (|\mathrm{y}_0 - \bar{\mathrm{y}}_0| + \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_{\alpha}).$$

Proof. The proof is identical to the proof of Proposition 4.10. □

Remark 7.11. Doss and Sussmann prove a continuity result in the sup-norm.

7.6. Lack of continuity (again)

In section 7.5 we have seen that, under appropriate conditions on σ, the map $\mathbb{X}^1 \mapsto \mathrm{y}$ is continuous if $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ varies in the class of weakly geometric rough paths. In this section we show that this is not a general fact, and the continuity result of Proposition 4.10 can not be improved in general.

More precisely, we show that, for a suitably chosen σ, one has a sequence $\mathbb{X}_n = (\mathbb{X}^1_n, \mathbb{X}^2_n)$ such that $\mathbb{X}^1_n \to 0$, $\mathbb{X}^2_n \to \mathbb{X}^2 \neq 0$, and the associated solutions y^n converge to …

For $y_1, y_2 \in \mathbb{R}$, $\sigma \colon \mathbb{R}^2 \to \mathbb{R}^2 \otimes \mathbb{R}^2$, we set

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \sigma(y) := \begin{pmatrix} y_2 & 0 \\ 0 & y_1 \end{pmatrix}.$$

In coordinates,

$$\sigma^{ij}(y) = \mathbb{1}_{\{i=1,j=1\}}\, y_2 + \mathbb{1}_{\{i=2,j=2\}}\, y_1$$

then we compute the partial derivative,

$$\frac{\partial \sigma^{ij}(y)}{\partial y_m} = \mathbb{1}_{\{i=1,j=1,m=2\}} + \mathbb{1}_{\{i=2,j=2,m=1\}}$$

From chapter 4 we have the expression for σ_2 in coordinates,

$$\sigma_2^{ijm}(y) = \sum_{a=1}^{2} \partial_a \sigma^{ij}(y)\, \sigma^{am}(y) = \mathbb{1}_{\{i=1,j=1,m=2\}}\, y_2 + \mathbb{1}_{\{i=2,j=2,m=1\}}\, y_1.$$

Note that σ_2 is not symmetric with respect to (j,m) i.e. $\sigma_2^{ijm} \neq \sigma_2^{imj}$ (which means that it does not satisfy the Frobenius's condition in Doss). By taking $\mathbb{X}^2$ from Section 5.8, we compute

$$(\sigma_2(y)\,\mathbb{X}^2)^i = \sum_{a,b=1}^{2} \sigma_2^{iab}(y)(\mathbb{X}^2)^{ba} = \frac{t-s}{2}\,(\mathbb{1}_{\{i=2\}}\,y_1 - \mathbb{1}_{\{i=1\}}\,y_2).$$

Since we have already shown that $\mathbb{X}^1 \to 0$, we get

$$\dot{y} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y,$$

we can conclude that the solution y is in the form of exponential different from 0.

Chapter 8

Rough integration

8.1. Controlled paths

Given a vector space V, we can canonically identify $V\otimes\mathbb{R}^d$ with the space of linear maps from $\mathbb{R}^d$ to V, namely

$$V\otimes\mathbb{R}^d = L(\mathbb{R}^d, V), \qquad (v\otimes x)y = \langle x, y\rangle\, v, \qquad v\in V, x, y\in\mathbb{R}^d, \tag{8.1}$$

where $\langle\cdot,\cdot\rangle$ is the canonical scalar product on $\mathbb{R}^d$. This justifies the notation $(A, B)\mapsto AB\in\mathbb{R}^m\otimes\mathbb{R}^n$ where

$$(AB)^{ab} = \sum_{k=1}^{d} A^{ak}B^{kb}, \qquad A\in\mathbb{R}^m\otimes\mathbb{R}^d, B\in\mathbb{R}^d\otimes\mathbb{R}^n. \tag{8.2}$$

Note also that on $\mathbb{R}^m\otimes\mathbb{R}^n$ we have the natural scalar product $\langle A, B\rangle = \mathrm{Tr}(AB^T)$.

We fix $\alpha\in\,]1/3, 1/2]$, $X\in\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$. We recall that fixing a α-rough path $\mathbb{X}$ over X as in Definition 5.11 is equivalent to choosing a solution $(I, \mathbb{X}^2)$ to (5.21), with I and $\mathbb{X}^2$ representing our choices of the integrals, respectively,

$$I_t =: \int_0^t X_r\otimes \mathrm{d}X_r, \qquad \mathbb{X}^2_{st} =: \int_s^t (X_r - X_s)\otimes \mathrm{d}X_r = I_t - I_s - X_s\otimes(X_t - X_s).$$

The key point is that, having fixed a choice of $\mathbb{X}^2$, it is now possible to give *a canonical definition of the integral* $\int_0^\cdot Y\,\mathrm{d}X$ for a wide class of $Y\in\mathcal{C}^\alpha([0, T];\mathbb{R}^k\otimes\mathbb{R}^d)$, namely those *paths Y which are controlled by* $\mathbb{X}$. In order to motivate this notion, let us recall that, given $X\in\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ and $Y\colon[0, T]\to\mathbb{R}^k\otimes\mathbb{R}^d$, we look now for $J\colon[0,T]\to\mathbb{R}^k$ and $R^J\colon[0,T]^2\to\mathbb{R}^k$ such that, in analogy with (5.4),

$$J_0 = 0, \qquad \delta J_{st} = Y_s\,\delta X_{st} + R^J_{st}, \qquad |R^J_{st}| \lesssim |t-s|^{2\alpha}.$$

In order to make this operation *iterable*, it is natural to require that *each component of* Y has an analogous property. This is exactly the motivation for the next

DEFINITION 8.1. *Let* $\alpha \in]1/3, 1/2]$ *and* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *an* α*-rough path on* $\mathbb{R}^d$*. A pair* $\boldsymbol{y} = (\mathrm{y}, \mathrm{y}^1) \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^k) \times \mathcal{C}^\alpha([0,T]; \mathbb{R}^k \otimes \mathbb{R}^d)$ *is* a path controlled *by* $\mathbb{X}$ *if*

$$\delta \mathrm{y}_{st} = \mathrm{y}^1_s \, \mathbb{X}^1_{st} + \mathrm{y}^2_{st}, \qquad |\mathrm{y}^2_{st}| \lesssim |t-s|^{2\alpha}, \qquad (s,t) \in [0,T]^2_{\leqslant}. \tag{8.3}$$

The function y^1 *is called a* derivative *of* y *with respect to* $\mathbb{X}$ *and* y^2 *is the* remainder *of the couple* $(\mathrm{y}, \mathrm{y}^1)$*. Note that* $\mathrm{y}^2 \in C_2^{2\alpha}$ *is* defined *by the first identity in (8.3).*

For a fixed α*-rough path* $\mathbb{X}$ *on* $\mathbb{R}^d$*, we denote by* $\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k)$ *the space of paths controlled by* $\mathbb{X}$ *with values in* $\mathbb{R}^k$*.*

Note that in general y^1 is *not* determined by $(\mathrm{y}, \mathbb{X}^1)$, so that we say that y^1 is *a* derivative rather than *the* derivative of y.

It is now clear from the definitions that, unlike rough paths, controlled paths have a natural linear structure, in particular as a linear subspace of $\mathcal{C}^\alpha \times \mathcal{C}^\alpha$.

8.2. THE ROUGH INTEGRAL

Now we can finally show how to modify the germ $Y_s(X_t - X_s)$ in order to obtain a well-defined integration theory.

PROPOSITION 8.2. *Let* $\alpha \in]1/3, 1/2]$ *and* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *a* α*-rough path on* $\mathbb{R}^d$*. If* $\boldsymbol{y} = (\mathrm{y}, \mathrm{y}^1)$ *is controlled by* $\mathbb{X}$ *with values in* $\mathbb{R}^k \otimes \mathbb{R}^d$ *as in Definition 8.1, then the germ*

$$A_{st} = \mathrm{y}_s \, \mathbb{X}^1_{st} + \mathrm{y}^1_s \, \mathbb{X}^2_{st}$$

satisfies $\delta A \in C_3^{3\alpha}$ *with* $3\alpha > 1$*.*

Therefore we can canonically *define* J_t=“$\int_0^t \mathrm{y} \, \mathrm{d}\mathbb{X}$” *as the unique function* $J\colon [0,T] \to \mathbb{R}$ *such that* J_0=0 *and* $\delta \mathrm{J} - A \in C_2^{3\alpha}$*, namely*

$$|\mathrm{J}_t - \mathrm{J}_s - \mathrm{y}_s \, \mathbb{X}^1_{st} - \mathrm{y}^1_s \, \mathbb{X}^2_{st}| \lesssim |t-s|^{3\alpha}.$$

Finally we have

$$\mathrm{J}_t = \lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{\#\mathcal{P}-1} (\mathrm{y}_{t_i} \mathbb{X}^1_{t_i t_{i+1}} + \mathrm{y}^1_{t_i} \mathbb{X}^2_{t_i t_{i+1}})$$

along arbitrary partitions $\mathcal{P}$ *of* $[0,t]$ *with vanishing mesh* $|\mathcal{P}| \to 0$*.*

Proof. We compute by (5.24)

$$\begin{aligned} \delta A_{sut} &= -\delta \mathrm{y}_{su} \, \mathbb{X}^1_{ut} + \mathrm{y}^1_s \, \delta \mathbb{X}^2_{sut} - \delta \mathrm{y}^1_{su} \, \mathbb{X}^2_{ut} \\ &= -(\delta \mathrm{y}_{su} - \mathrm{y}^1_s \, \mathbb{X}^1_{su}) \, \mathbb{X}^1_{ut} - \delta \mathrm{y}^1_{su} \, \mathbb{X}^2_{ut} \\ &= -\mathrm{y}^2_{su} \, \mathbb{X}^1_{ut} - \delta \mathrm{y}^1_{su} \, \mathbb{X}^2_{ut}, \end{aligned} \tag{8.4}$$

where we remark that for $i=1,\ldots,k$

$$
\begin{aligned}
(\mathrm{y}^1_s\,\delta\mathbb{X}^2_{sut})^i &= \sum_{a,b=1}^d (\mathrm{y}^1_s)^{iab}(\delta\mathbb{X}^2_{sut})^{ba}\\
&= \sum_{a,b=1}^d (\mathrm{y}^1_s)^{iab}\,(\mathbb{X}^1_{su})^b\,(\mathbb{X}^1_{ut})^a\\
&= [(\mathrm{y}^1_s\,\mathbb{X}^1_{su})\mathbb{X}^1_{ut}]^i.
\end{aligned}
$$

Then by (1.32)

$$
\begin{aligned}
|\delta A_{sut}| \;\leqslant\; & \|\mathrm{y}^2\|_{2\alpha}|u-s|^{2\alpha}\|\mathbb{X}^1\|_\alpha|t-u|^\alpha+\|\delta\mathrm{y}^1\|_\alpha|u-s|^\alpha\|\mathbb{X}^2\|_{2\alpha}|t-u|^{2\alpha}\\
& (\|\mathrm{y}^2\|_{2\alpha}\|\mathbb{X}^1\|_\alpha+\|\delta\mathrm{y}^1\|_\alpha\|\mathbb{X}^2\|_{2\alpha})|t-s|^{3\alpha} \qquad (8.5)
\end{aligned}
$$

Since $\delta A\in C_3^{3\alpha}$, we can apply the Sewing Lemma and define $\mathrm{J}^3{:=}-\Lambda(\delta A)$ and $\mathrm{J}\colon[0,T]\to\mathbb{R}$ such that $\mathrm{J}_0{=}0$ and $\delta\mathrm{J}=A+\mathrm{J}^3$ where Λ is the Sewing Map of Theorem 1.13, so that

$$
\mathrm{J}_0=0\,,\qquad \delta\mathrm{J}_{st}=\mathrm{y}_s\,\mathbb{X}^1_{st}+\mathrm{y}^1_s\,\mathbb{X}^2_{st}+\mathrm{J}^3_{st},\qquad |\mathrm{J}^3_{st}|\lesssim|t-s|^{3\alpha}\,. \qquad (8.6)
$$

The last assertion on the convergence of the generalised Riemann sums follows from (1.16). □

We have in particular proved by (1.19) and (8.5) that

$$
\begin{aligned}
\|\mathrm{J}^3\|_{3\alpha} &\leqslant K_{3\alpha}\,(\|\mathrm{y}^2\|_{2\alpha}\|\mathbb{X}^1\|_\alpha+\|\delta\mathrm{y}^1\|_\alpha\|\mathbb{X}^2\|_{2\alpha}), \qquad (8.7)\\
\mathrm{J}^3_{st} &= \mathrm{J}_t-\mathrm{J}_s-\mathrm{y}_s\,\mathbb{X}^1_{st}-\mathrm{y}^1_s\,\mathbb{X}^2_{st}.
\end{aligned}
$$

We stress that the function J depends on $(\boldsymbol{y},\mathbb{X})$, in particular on y^1 as well. We use the following notations

$$
\boldsymbol{J}:=(\mathrm{J},\mathrm{y}),\qquad \int_0^t \boldsymbol{y}\,\mathrm{d}\mathbb{X}:=(\mathrm{J}_t,\mathrm{y}_t)=\boldsymbol{J}_t. \qquad (8.8)
$$

We shall see in Proposition 8.4 below that $\boldsymbol{J}\colon[0,T]\to\mathbb{R}^k\times(\mathbb{R}^k\otimes\mathbb{R}^d)$ is controlled by $\mathbb{X}$.

We define a norm $\|\cdot\|_{\mathcal{D}^{2\alpha}_{\mathbb{X}}}$ and a seminorm $[\cdot]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}}$ on the space $\mathcal{D}^{2\alpha}_{\mathbb{X}}$ of paths controlled by $\mathbb{X}$, defined as follows:

$$
\begin{aligned}
\|\boldsymbol{y}\|_{\mathcal{D}^{2\alpha}_{\mathbb{X}}} &:= |\mathrm{y}_0|+|\mathrm{y}^1_0|+[\boldsymbol{y}]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}}, &\qquad \boldsymbol{y}=(\mathrm{y},\mathrm{y}^1) \qquad (8.9)\\
[\boldsymbol{y}]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}} &:= \|\delta\mathrm{y}^1\|_\alpha+\|\mathrm{y}^2\|_{2\alpha}, &\qquad \mathrm{y}^2_{st}=\delta\mathrm{y}_{st}-\mathrm{y}^1_s\,\mathbb{X}^1_{st},
\end{aligned}
$$

as in (8.3). Recall that we defined the standard norm $\|f\|_{\mathcal{C}^\alpha}=\|f\|_\infty+\|\delta f\|_\alpha$ in (2.5).

Lemma 8.3. *We have the equivalence of norms for all* $\boldsymbol{y}=(\mathrm{y},\mathrm{y}^1)\in\mathcal{D}^{2\alpha}_{\mathbb{X}}$

$$
\|\boldsymbol{y}\|_{\mathcal{D}^{2\alpha}_{\mathbb{X}}}\leqslant\|\mathrm{y}\|_{\mathcal{C}^\alpha}+\|\mathrm{y}^1\|_{\mathcal{C}^\alpha}+\|\mathrm{y}^2\|_{2\alpha}\leqslant C\|\boldsymbol{y}\|_{\mathcal{D}^{2\alpha}_{\mathbb{X}}}, \qquad (8.10)
$$

where $C>0$ is an explicit constant which depends only on $(\mathbb{X},T,\alpha)$. In particular, $(\mathcal{D}_{\mathbb{X}}^{2\alpha},\|\cdot\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}})$ is a Banach space.

Proof. The first inequality in (8.10) is obvious by the definition of the norm $\|\cdot\|_{\mathcal{C}^\alpha}$. In order to prove the second one, first we note that by (1.24)-(1.25) and by (1.29)

$$\begin{aligned}\|f\|_{\mathcal{C}^\alpha} &= \|f\|_\infty + \|\delta f\|_\alpha \leqslant |f_0| + (1+T^\alpha)\|\delta f\|_\alpha \\ &\leqslant (1+T^\alpha)(|f_0| + \|\delta f\|_\alpha).\end{aligned}$$

This shows that $\|\mathrm{y}^1\|_{\mathcal{C}^\alpha} \lesssim \|\boldsymbol{y}\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}$ for $(\mathrm{y},\mathrm{y}^1)\in\mathcal{D}_{\mathbb{X}}^{2\alpha}$. Now, since $\delta\mathrm{y}_{st} = \mathrm{y}_s^1\,\mathbb{X}_{st}^1 + \mathrm{y}_{st}^2$ by (8.3),

$$\|\delta \mathrm{y}\|_\alpha \leqslant \|\mathrm{y}^1\|_\infty \|\mathbb{X}^1\|_\alpha + \|\mathrm{y}^2\|_\alpha \leqslant C_{T,\alpha}(|\mathrm{y}_0^1| + \|\delta \mathrm{y}^1\|_\alpha)\|\mathbb{X}^1\|_\alpha + T^\alpha\|\mathrm{y}^2\|_{2\alpha},$$

namely $\|\mathrm{y}\|_{\mathcal{C}^\alpha} \lesssim \|\boldsymbol{y}\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}} + \|\mathrm{y}^2\|_{2\alpha}$. Finally $\|\mathrm{y}^2\|_{2\alpha} \leqslant \|\boldsymbol{y}\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}$. The proof is complete. □

8.3. Continuity properties of the rough integral

We wrote before Definition 8.1 that the notion of controlled path aimed at making the rough integral map $(\mathrm{y},\mathrm{y}^1)\mapsto(\mathrm{J},\mathrm{y})$ iterable, where we use the notation of Proposition 8.2. In order to make this precise, we need the following important

PROPOSITION 8.4. *Let $\mathbb{X}$ be a α-rough path on $\mathbb{R}^d$ with $\alpha\in\,]1/3,1/2]$ and $\boldsymbol{y}\in\mathcal{D}_{\mathbb{X}}^{2\alpha}$ a path controlled by $\mathbb{X}$. Then, in the notation of (8.8),*

- *$\boldsymbol{J} = \int_0^\cdot \boldsymbol{y}\,\mathrm{d}\mathbb{X}$ is controlled by $\mathbb{X}$*
- *the map $\mathcal{D}_{\mathbb{X}}^{2\alpha}\ni\boldsymbol{y}\mapsto\boldsymbol{J}\in\mathcal{D}_{\mathbb{X}}^{2\alpha}$ is linear and for all $\boldsymbol{y}\in\mathcal{D}_{\mathbb{X}}^{2\alpha}$*

$$[\boldsymbol{J}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}} \leqslant 2(1+K_{3\alpha})(1+\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}})[|\mathrm{y}_0^1| + T^\alpha[\boldsymbol{y}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}]. \qquad (8.11)$$

Proof. Recall first (8.6), so that in particular $\|\mathrm{J}^3\|_{3\alpha} < +\infty$. Now $\mathrm{J}_{st}^2 = \mathrm{y}_s^1\,\mathbb{X}_{st}^2 + \mathrm{J}_{st}^3$ satisfies

$$\|\mathrm{J}^2\|_{2\alpha} \leqslant \|\mathrm{y}^1\|_\infty\|\mathbb{X}^2\|_{2\alpha} + \|\mathrm{J}^3\|_{2\alpha} \leqslant \|\mathrm{y}^1\|_\infty\|\mathbb{X}^2\|_{2\alpha} + T^\alpha\|\mathrm{J}^3\|_{3\alpha}. \qquad (8.12)$$

Finally $\delta\mathrm{J}_{st} = \mathrm{y}_s\,\mathbb{X}_{st}^1 + \mathrm{J}_{st}^2$ and therefore

$$\|\delta\mathrm{J}\|_\alpha \leqslant \|\mathrm{y}\|_\infty\|\mathbb{X}^1\|_\alpha + \|\mathrm{y}^1\|_\infty\|\mathbb{X}^2\|_{2\alpha} + T^{2\alpha}\|\mathrm{J}^3\|_{3\alpha}.$$

Therefore $(\mathrm{J},\mathrm{y},\mathrm{J}^2)\in\mathcal{C}^\alpha\times\mathcal{C}^\alpha\times C_2^{2\alpha}$ and we obtain that (J,y) is controlled by $\mathbb{X}$.

We prove now the second assertion. Since $\delta\mathrm{y}_{st} = \mathrm{y}_s^1\,\mathbb{X}_{st}^1 + \mathrm{y}_{st}^2$, by (1.29)

$$\begin{aligned}\|\delta\mathrm{y}\|_\alpha &\leqslant \|\mathrm{y}^1\|_\infty\|\mathbb{X}^1\|_\alpha + T^\alpha\|\mathrm{y}^2\|_{2\alpha} \\ &\leqslant (\|\mathbb{X}^1\|_\alpha + 1)(|\mathrm{y}_0^1| + T^\alpha[\boldsymbol{y}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}).\end{aligned}$$

Now, analogously to (8.12), again by (1.29)

$$\begin{aligned}\|\mathrm{J}^2\|_{2\alpha} &\leqslant \|\mathrm{y}^1\|_\infty \|\mathbb{X}^2\|_{2\alpha} + \|\mathrm{J}^3\|_{2\alpha}\\ &\leqslant T^\alpha \|\mathrm{J}^3\|_{3\alpha} + \|\mathbb{X}^2\|_{2\alpha}(|\mathrm{y}_0^1| + T^\alpha \|\delta \mathrm{y}^1\|_\alpha).\end{aligned}$$

Therefore, since $\|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} = \|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}}$, recall (5.27),

$$\|\delta \mathrm{y}\|_\alpha + \|\mathrm{J}^2\|_{2\alpha} \leqslant T^\alpha \|\mathrm{J}^3\|_{3\alpha} + (1 + \|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}})[|\mathrm{y}_0^1| + T^\alpha [\boldsymbol{y}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}].$$

By (8.11) we obtain

$$\begin{aligned}[\boldsymbol{J}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}} &= \|\delta \mathrm{y}\|_\alpha + \|\mathrm{J}^2\|_{2\alpha}\\ &\leqslant 2\,(1 + \|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}})[|\mathrm{y}_0^1| + (1 + K_{3\alpha}) T^\alpha [\boldsymbol{y}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}]\end{aligned}$$

Since $\mathrm{J}_0 = 0$ and $\mathrm{J}_0^1 = \mathrm{y}_0$, we obtain

$$\|\boldsymbol{J}\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}} = |\mathrm{y}_0| + [\boldsymbol{J}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}} \leqslant 2(1 + K_{3\alpha})(1 + \|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}})[|\mathrm{y}_0| + |\mathrm{y}_0^1| + T^\alpha [\boldsymbol{y}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}].$$

The proof is complete. □

By 8.4, the operator d is linear and continuous. In fact a much stronger property holds: we have continuity of the map $(\mathbb{X}, \boldsymbol{y}) \mapsto \int_0^\cdot \boldsymbol{y}\, \mathrm{d}\mathbb{X}$. In order to prove this, we need to introduce the following space

$$\mathcal{S}_\alpha := \{(\mathbb{X}, \boldsymbol{y}) : \mathbb{X} \text{ is a } \alpha\text{-rough path}, \boldsymbol{y} \in \mathcal{D}_{\mathbb{X}}^{2\alpha}\},$$

and the following quantity for $\boldsymbol{y} \in \mathcal{D}_{\mathbb{X}}^{2\alpha}$ and $\bar{\boldsymbol{y}} \in \mathcal{D}_{\bar{\mathbb{X}}}^{2\alpha}$

$$[\boldsymbol{y}; \bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha} := \|\delta \mathrm{y}^1 - \delta \bar{\mathrm{y}}^1\|_\alpha + \|\mathrm{y}^2 - \bar{\mathrm{y}}^2\|_{2\alpha},$$

where $\mathrm{y}^2 = \delta \mathrm{y} - \mathrm{y}^1 \mathbb{X}^1$ and $\bar{\mathrm{y}}^2 = \delta \bar{\mathrm{y}} \text{-} \bar{\mathrm{y}}^1 \bar{\mathbb{X}}^1$, recall (8.9). We endow $\mathcal{S}_\alpha$ with a family of distances (see (5.28) for the definition of $d_{\mathcal{R}_{\alpha,d}}$)

$$\mathrm{d}_\alpha((\mathbb{X}, \boldsymbol{y}), (\bar{\mathbb{X}}, \bar{\boldsymbol{y}})) = d_{\mathcal{R}_{\alpha,d}}(\mathbb{X}, \bar{\mathbb{X}}) + |\mathrm{y}_0 - \bar{\mathrm{y}}_0| + |\mathrm{y}_0^1 - \bar{\mathrm{y}}_0^1| + [\boldsymbol{y}; \bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha}.$$

Let us note that in the case $\mathbb{X} = \bar{\mathbb{X}}$, we have

$$[\boldsymbol{y}; \bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha} = [\boldsymbol{y} - \bar{\boldsymbol{y}}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}, \qquad \mathrm{d}_\alpha((\mathbb{X}, \boldsymbol{y}), (\mathbb{X}, \bar{\boldsymbol{y}})) = \|\boldsymbol{y} - \bar{\boldsymbol{y}}\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}},$$

see the definition (8.9) of the norm $\|\cdot\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}$. Note that $[\boldsymbol{y}; \bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha}$ is *not* a function of $\boldsymbol{y} - \bar{\boldsymbol{y}}$ when $\mathbb{X} \neq \bar{\mathbb{X}}$.

PROPOSITION 8.5. (LOCAL LIPSCHITZ ESTIMATE) *Let* $\alpha \in]1/3, 1/2]$. *The function d is continuous with respect to* d_α.

More precisely, for every $M \geq 0$ *there is* $K_{M,\alpha} \geq 0$ *such that for all* $(\mathbb{X}, \boldsymbol{y}), (\bar{\mathbb{X}}, \bar{\boldsymbol{y}}) \in \mathcal{S}_\alpha$ *satisfying*

$$1 + T^\alpha + \|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}} + \|\bar{\boldsymbol{y}}\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}} \leqslant M,$$

setting $\boldsymbol{J} := \int_0^\cdot \boldsymbol{y}\, \mathrm{d}\mathbb{X}$ *and* $\bar{\boldsymbol{J}} := \int_0^\cdot \bar{\boldsymbol{y}}\, \mathrm{d}\bar{\mathbb{X}}$ *we have*

$$\begin{aligned}&\mathrm{d}_\alpha((\mathbb{X},\boldsymbol{J}),(\bar{\mathbb{X}},\bar{\boldsymbol{J}})) \leqslant \\ &\leqslant 2M^2(1+K_{3\alpha})[d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}}) + |\mathrm{y}_0-\bar{\mathrm{y}}_0| + |\mathrm{y}_0^1-\bar{\mathrm{y}}_0^1| + T^\alpha[\boldsymbol{y};\bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha}] \\ &\leqslant 2M^3(1+K_{3\alpha})\,\mathrm{d}_\alpha((\mathbb{X},\boldsymbol{y}),(\bar{\mathbb{X}},\bar{\boldsymbol{y}})).\end{aligned}$$

Proof. Let $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ and $\bar{\mathbb{X}} = (\bar{\mathbb{X}}^1, \bar{\mathbb{X}}^2)$ be α-rough paths with $\alpha \in]1/3, 1/2]$ and $\boldsymbol{y} \in \mathcal{D}_{\mathbb{X}}^{2\alpha}$, $\bar{\boldsymbol{y}} \in \mathcal{D}_{\bar{\mathbb{X}}}^{2\alpha}$. We argue as in the proof of (8.11), using furthermore a number of times the simple estimate

$$|a b - \bar{a}\,\bar{b}| \leqslant |a-\bar{a}|\,|b| + |\bar{a}|\,|b-\bar{b}|. \tag{8.13}$$

We set for notational convenience $\varepsilon := T^\alpha$. Then, since $\delta \mathrm{y}_{st} = \mathrm{y}_s^1\, \mathbb{X}_{st}^1 + \mathrm{y}_{st}^2$, by (1.29)

$$\begin{aligned}\|\delta\mathrm{y} - \delta\bar{\mathrm{y}}\|_\alpha &\leqslant \|\mathrm{y}^1 - \bar{\mathrm{y}}^1\|_\infty \|\mathbb{X}^1\|_\alpha + \|\bar{\mathrm{y}}^1\|_\infty \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \varepsilon\|\mathrm{y}^2 - \bar{\mathrm{y}}^2\|_{2\alpha} \\ &\leqslant (\|\mathbb{X}^1\|_\alpha + 1)(|\mathrm{y}_0^1 - \bar{\mathrm{y}}_0^1| + \varepsilon[\boldsymbol{y};\bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha}) + M^2\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha,\end{aligned}$$

since by assumption

$$\|\bar{\mathrm{y}}^1\|_\infty \leqslant |\bar{\mathrm{y}}_0^1| + \varepsilon\|\delta\bar{\mathrm{y}}^1\|_\alpha \leqslant (1+\varepsilon)(|\bar{\mathrm{y}}_0^1| + \|\delta\bar{\mathrm{y}}^1\|_\alpha) \leqslant M^2.$$

Now $\mathrm{J}_{st}^2 = \mathrm{y}_s^1\, \mathbb{X}_{st}^2 + \mathrm{J}_{st}^3$, so that arguing similarly

$$\begin{aligned}&\|\mathrm{J}^2 - \bar{\mathrm{J}}^2\|_{2\alpha} \leqslant \|\mathrm{J}^3 - \bar{\mathrm{J}}^3\|_{2\alpha} + \|\mathrm{y}^1\,\mathbb{X}^2 - \bar{\mathrm{y}}^1\,\bar{\mathbb{X}}^2\|_{2\alpha} \leqslant \\ &\leqslant \varepsilon\|\mathrm{J}^3 - \bar{\mathrm{J}}^3\|_{3\alpha} + \|\mathbb{X}^2\|_{2\alpha}(|\mathrm{y}_0^1 - \bar{\mathrm{y}}_0^1| + \varepsilon\|\delta\mathrm{y}^1 - \delta\bar{\mathrm{y}}^1\|_\alpha) + M^2\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}.\end{aligned}$$

Therefore, since $1 + \|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} = 1 + \|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}} \leqslant M$,

$$\begin{aligned}&\|\delta\mathrm{y} - \delta\bar{\mathrm{y}}\|_\alpha + \|\mathrm{J}^2 - \bar{\mathrm{J}}^2\|_{2\alpha} \leqslant \\ &\leqslant \varepsilon\|\mathrm{J}^3 - \bar{\mathrm{J}}^3\|_{3\alpha} + M^2(|\mathrm{y}_0^1 - \bar{\mathrm{y}}_0^1| + \varepsilon[\boldsymbol{y};\bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha} + d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})).\end{aligned}$$

Since ut, we can estimate in the same way

$$\begin{aligned}\|\delta A - \delta\bar{A}\|_{3\alpha} &\leqslant \|\mathrm{y}^2 - \bar{\mathrm{y}}^2\|_{2\alpha}\|\mathbb{X}^1\|_\alpha + \|\bar{\mathrm{y}}^2\|_{2\alpha}\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \\ &\quad + \|\delta\mathrm{y}^1 - \delta\bar{\mathrm{y}}^1\|_\alpha\|\mathbb{X}^2\|_{2\alpha} + \|\delta\bar{\mathrm{y}}^1\|_\alpha\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} \\ &\leqslant [\boldsymbol{y};\bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha}\,\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}} + [\bar{\boldsymbol{y}}]_{\mathcal{D}_{\bar{\mathbb{X}}}^{2\alpha}}\, d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}}) \\ &\leqslant M([\boldsymbol{y};\bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha} + d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})).\end{aligned}$$

By the Sewing Lemma (1.33), and since $\varepsilon \leqslant M$,

$$\varepsilon\|\mathrm{J}^3 - \bar{\mathrm{J}}^3\|_{3\alpha} \leqslant K_{3\alpha} M(\varepsilon[\boldsymbol{y};\bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha} + M d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})).$$

We obtain

$$\begin{aligned}&[\boldsymbol{J};\bar{\boldsymbol{J}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha} = \|\delta\mathrm{y} - \delta\bar{\mathrm{y}}\|_\alpha + \|\mathrm{J}^2 - \bar{\mathrm{J}}^2\|_{2\alpha} \leqslant \\ &\qquad \leqslant M^2(1+K_{3\alpha})[|\mathrm{y}_0^1 - \bar{\mathrm{y}}_0^1| + d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}}) + \varepsilon[\boldsymbol{y};\bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha}].\end{aligned}$$

Since $\mathrm{J}_0 - \bar{\mathrm{J}}_0 = 0$, $\mathrm{J}_0^1 - \bar{\mathrm{J}}_0^1 = \mathrm{y}_0 - \bar{\mathrm{y}}_0$, we obtain

$$\begin{aligned}&\mathrm{d}_\alpha((\mathbb{X},\boldsymbol{J}),(\bar{\mathbb{X}},\bar{\boldsymbol{J}})) = d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}}) + |\mathrm{y}_0 - \bar{\mathrm{y}}_0| + [\boldsymbol{J};\bar{\boldsymbol{J}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha}\\ &\leqslant 2M^2(1+K_{3\alpha})[|\mathrm{y}_0-\bar{\mathrm{y}}_0| + |\mathrm{y}_0^1 - \bar{\mathrm{y}}_0^1| + d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}}) + \varepsilon[\boldsymbol{y};\bar{\boldsymbol{y}}]_{\mathbb{X},\bar{\mathbb{X}},2\alpha}].\end{aligned}$$

The second estimate follows since we have assumed that $1+\varepsilon \leqslant M$. □

8.4. Stochastic and rough integrals

8.5. Properties in the geometric case

We have seen in Proposition 2.8 that the Young integral satisfies the classical integration by parts formula. We consider now a weakly geometric rough path $\mathbb{X}$ and two paths $\boldsymbol{f} = (f, f^1)$, $\boldsymbol{g} = (g, g^1)$ controlled by $\mathbb{X}$. We set

$$F_t := F_0 + \int_0^t f_s \,\mathrm{d}\mathbb{X}_s, \qquad G_t := G_0 + \int_0^t g_s \,\mathrm{d}\mathbb{X}_s, \qquad t \geqslant 0.$$

We want to show that, under the assumption that $\mathbb{X}$ is geometric, an analogous integration by parts formula holds, namely:

$$F_t G_t = F_0 G_0 + \underbrace{\int_0^t F_s g_s \,\mathrm{d}\mathbb{X}_s + \int_0^t G_s f_s \,\mathrm{d}\mathbb{X}_s}_{I_t}.$$

We start by showing that $(F_s g_s, F_s g_s^1 + f_s g_s)_{s\in[0,T]}$ is controlled by $\mathbb{X}$:

$$\begin{aligned}F_t g_t - F_s g_s &= F_t \delta g_{st} + g_s \delta F_{st}\\ &= F_s \delta g_{st} + g_s \delta F_{st} + \delta F_{st} \delta g_{st}\\ &= (F_s g_s^1 + f_s g_s)\mathbb{X}_{st}^1 + O(|t-s|^{2\alpha}).\end{aligned}$$

The same holds of course for $(f_s G_s, G_s f_s^1 + f_s g_s)_{s\in[0,T]}$. Now we know that I_t is the integral uniquely associated with the germ

$$A_{st} = (F_s g_s + G_s f_s)\mathbb{X}_{st}^1 + (F_s g_s^1 + G_s f_s^1 + 2 f_s g_s)\mathbb{X}_{st}^2.$$

By the geometric condition, we have $2\mathbb{X}_{st}^2 = (\mathbb{X}_{st}^1)^2$ and therefore we obtain

$$A_{st} = (F_s g_s + G_s f_s)\mathbb{X}_{st}^1 + (F_s g_s^1 + G_s f_s^1)\mathbb{X}_{st}^2 + f_s g_s (\mathbb{X}_{st}^1)^2.$$

Now we write

$$\begin{aligned}\delta(FG)_{st} &= \delta F_{st} G_t + F_s \delta G_{st}\\ &= G_s \delta F_{st} + F_s \delta G_{st} + \delta F_{st} \delta G_{st}\\ &= (F_s g_s + G_s f_s)\mathbb{X}_{st}^1 + (F_s g_s^1 + G_s f_s^1)\mathbb{X}_{st}^2 + \delta F_{st}\delta G_{st} + O(|t-s|^{3\alpha}).\end{aligned}$$

Now

$$\begin{aligned}\delta F_{st}\delta G_{st} &= (f_s \mathbb{X}_{st}^1 + f_s^1 \mathbb{X}_{st}^2)(g_s \mathbb{X}_{st}^1 + g_s^1 \mathbb{X}_{st}^2) + O(|t-s|^{3\alpha})\\ &= f_s g_s (\mathbb{X}_{st}^1)^2 + O(|t-s|^{3\alpha}).\end{aligned}$$

Then we obtain that

$$\delta(FG)_{st} \;=\; A_{st} + O(|t-s|^{3\alpha}).$$

Since $3\alpha > 1$, it follows that $F_t G_t - F_0 G_0 = I_t$ for all $t \geqslant 0$.

Example 8.6. It is well known that the Stratonovich stochastic integral satisfies the above integration by parts formula. This section extends this result to all (weakly) geometric rough paths.

Chapter 9

Rough integral equations

In this chapter we go back to the finite difference equation (4.13) in the rough setting, and we discute its integral formulation that we already mentioned in Section 4.9. Now that we have studied the rough integral in Chapter 8, we can indeed show that the equation

$$|\mathrm{y}^3_{st}| \lesssim |t-s|^{3\alpha}, \qquad \mathrm{y}^3_{st} = \delta \mathrm{y}_{st} - \sigma(\mathrm{y}_s)\, \mathbb{X}^1_{st} - \sigma_2(\mathrm{y}_s)\, \mathbb{X}^2_{st}, \tag{9.1}$$

recall Lemma 4.3, can be interpreted in the context of controlled paths. Indeed, (9.1) suggests that, for any candidate solution y, the pair $(\mathrm{y}, \sigma(\mathrm{y}))$ should be controlled by $\mathbb{X}$. At the same time, in order to apply Proposition 8.2 and interpret (9.1) as an integral equation, we need existence of $h \in \mathcal{C}^\alpha$ such that $(\sigma(\mathrm{y}), h)$ is controlled by $\mathbb{X}$. This is guaranteed by the following

Lemma 9.1. *Let $\phi: \mathbb{R}^k \to \mathbb{R}^\ell$ be of class C^2 and $\boldsymbol{f} = (\mathrm{f}, \mathrm{f}^1) \in \mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k)$. Set*

$$\phi(\boldsymbol{f}) := (\phi(\mathrm{f}), \nabla \phi(\mathrm{f})\, \mathrm{f}^1),$$

where $\phi(\mathrm{f}): [0,T] \to \mathbb{R}^\ell$ is defined by $\phi(\mathrm{f})_t := \phi(\mathrm{f}_t)$ and

$$\nabla \phi(\mathrm{f})\, \mathrm{f}^1 : [0,T] \to \mathbb{R}^\ell \otimes \mathbb{R}^d, \qquad (\nabla \phi(\mathrm{f})\, \mathrm{f}^1)^{ab}_t = \sum_{j=1}^k \nabla_j \phi^a(\mathrm{f}_t) \cdot (\mathrm{f}^1_t)^{jb}.$$

Then $\phi(\boldsymbol{f}) \in \mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^\ell)$.

Proof. Analogously to (4.16) we have for $\boldsymbol{f} = (\mathrm{f}, \mathrm{f}^1) \in \mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k)$, setting $\mathrm{f}^2_{st} := \delta \mathrm{f}_{st} - \mathrm{f}^1_s\, \mathbb{X}^1_{st}$ as in (8.3),

$$\begin{aligned} \phi(\boldsymbol{f})^2_{st} &:= \phi(\mathrm{f}_t) - \phi(\mathrm{f}_s) - \nabla \phi(\mathrm{f}_s)\, \mathrm{f}^1_s\, \mathbb{X}^1_{st} \\ &= \nabla \phi(\mathrm{f}_s)\, \mathrm{f}^2_{st} + \int_0^1 [\nabla \phi(\mathrm{f}_s + r\, \delta \mathrm{f}) - \nabla \phi(\mathrm{f}_s)]\, \mathrm{d}r\, \delta \mathrm{f}_{st} \\ &= \nabla \phi(\mathrm{f}_s)\, \mathrm{f}^2_{st} + \int_0^1 (1-u)\, \nabla^2 \phi(\mathrm{f}_s + u\, \delta \mathrm{f}_{st})\, \mathrm{d}u\ \delta \mathrm{f}_{st} \otimes \delta \mathrm{f}_{st}. \end{aligned} \tag{9.2}$$

Then we can write using the estimate $|a\, b - \bar{a}\, \bar{b}| \leqslant |a - \bar{a}|\, |b| + |\bar{a}|\, |b - \bar{b}|$

$$\begin{aligned} |\nabla \phi(\mathrm{f}_t)\, \mathrm{f}^1_t - \nabla \phi(\mathrm{f}_s)\, \mathrm{f}^1_s| &\leqslant c^{(1)}_{\phi, f}\, |\mathrm{f}^1_t - \mathrm{f}^1_s| + c^{(2)}_{\phi, f}\, |\mathrm{f}_t - \mathrm{f}_s|\, \|\mathrm{f}^1\|_\infty, \\ |\phi(\boldsymbol{f})^2_{st}| &\leqslant c^{(1)}_{\phi, f}\, |\mathrm{f}^2_{st}| + c^{(2)}_{\phi, f}\, |\delta \mathrm{f}_{st}|^2, \end{aligned} \tag{9.3}$$

where

$$c^{(1)}_{\phi,f} := \sup_{s\in[0,T]} |\nabla\phi(\mathrm{f}_s)|, \qquad c^{(2)}_{\phi,f} := \sup_{s,t\in[0,T],u\in[0,1]} |\nabla^2\phi(\mathrm{f}_s + u\,\delta\mathrm{f}_{st})|. \tag{9.4}$$

Therefore $(\phi(\mathrm{f}), \nabla\phi(\mathrm{f})\,\mathrm{f}^1)$ is controlled by $\mathbb{X}$. □

This suggests that we can reinterpret the finite difference equation (9.1) as follows: we look for $\mathrm{y}: [0,T] \to \mathbb{R}^k$ such that $\boldsymbol{y} = (\mathrm{y}, \sigma(\mathrm{y}))$ is controlled by $\mathbb{X}$ (namely it belongs to $\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k)$) and

$$\boldsymbol{y}_t = (\mathrm{y}_0, 0) + \int_0^t \sigma(\boldsymbol{y})\,\mathrm{d}\mathbb{X}, \qquad \forall\, t \in [0,T]. \tag{9.5}$$

By Lemma 9.1, $\sigma(\mathrm{y}) = (\sigma(\mathrm{y}), \nabla\sigma(\mathrm{y})\,\mathrm{y}^1)$, but here $\mathrm{y}^1 = \sigma(\mathrm{y})$, so that

$$\sigma(\mathrm{y}) = (\sigma(\mathrm{y}), \nabla\sigma(\mathrm{y})\,\sigma(\mathrm{y})) = (\sigma(\mathrm{y}), \sigma_2(\mathrm{y})),$$

where we use the notation $\sigma_2: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d \otimes \mathbb{R}^d$

$$\sigma_2(y) := \nabla\sigma(y)\,\sigma(y), \qquad [\sigma_2(y)]^{ijm} := \sum_{a=1}^{k} \nabla_a \sigma^{ij}(y)\,\sigma^{am}(y).$$

By Proposition 8.2, the integral equation in (9.5) is equivalent to

$$|\mathrm{y}^3_{st}| \lesssim |t-s|^{3\alpha}, \qquad \mathrm{y}^3_{st} = \delta\mathrm{y}_{st} - \sigma(\mathrm{y}_s)\,\mathbb{X}^1_{st} - \sigma_2(\mathrm{y}_s)\,\mathbb{X}^2_{st}. \tag{9.6}$$

Viceversa, if $\mathrm{y} \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^k)$ is such that $\mathrm{y}^3 \in C^{3\alpha}_2$, then setting $\mathrm{y}^1 := \sigma(\mathrm{y})$ the path $\boldsymbol{y} = (\mathrm{y}, \mathrm{y}^1)$ is controlled by $\mathbb{X}$ and satisfies (9.5). Therefore, the integral equation (9.5) is equivalent to the finite difference equation (9.6).

9.1. Localization argument

Proposition 9.2. *If we can prove local existence for the rough differential equation (9.6) under the assumption that σ is of class C^3 and $\sigma, \nabla\sigma, \nabla^2\sigma, \nabla^3\sigma$ are bounded, then we can prove local existence for (9.6) assuming only that σ is of class C^3.*

Proof. Let σ be of class C^3. Note that σ and its derivatives are bounded on the closed unit ball $B := \{z \in \mathbb{R}^k : |z - \mathrm{y}_0| \leq 1\}$, which is a compact subset of $\mathbb{R}^k$. Then we can find a function $\hat{\sigma}$ of class C^3 which is bounded with all its derivatives up to the third on the whole $\mathbb{R}^k$ and coincides with σ on B. By local existence for $\hat{\sigma}$, there is a solution $\hat{\mathrm{y}}: [0,T] \to \mathbb{R}^k$ of the RDE (9.6) with σ replaced by $\hat{\sigma}$. Since y is continuous with $\mathrm{y}_0 \in B$, we can find $T' > 0$ such that $\mathrm{y}_t \in B$ for all $t \in [0,T']$. Then $\sigma(\mathrm{y}_t) = \hat{\sigma}(\mathrm{y}_t)$ and $\sigma_2(\mathrm{y}_t) = \hat{\sigma}_2(\mathrm{y}_t)$ for all $t \in [0,T']$, so that y is a solution of the original RDE (9.6) on the shorter time interval $[0,T']$. We have proved *local existence* assuming only that σ is of class C^3. □

9.2. INVARIANCE

In this section we prepare the ground for a contraction argument to be proved in the next section. We start with an estimate of $[\phi(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^\ell)}$ in terms of $[\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}$, under the assumption that ϕ is of class C^2 with bounded first and second derivative. We fix $D > 0$ such that

$$D \geq \max\{\|\nabla\sigma\|_\infty, \|\nabla^2\sigma\|_\infty\}.$$

LEMMA 9.3. *Let $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ be of class C^2 with $\|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty \leqslant D$, for some $D < +\infty$. Then for some $C > 0$ and any $\boldsymbol{f} = (\mathrm{f}, \mathrm{f}^1) \in \mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$*

$$[\sigma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^\ell)} \leqslant D([\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} + \|\mathrm{f}^1\|_\infty \|\delta\mathrm{f}\|_\alpha + \|\delta\mathrm{f}\|_\alpha^2). \tag{9.7}$$

Proof. By (9.3) we have

$$\|\delta(\nabla\sigma(\mathrm{f})\,\mathrm{f}^1)\|_\alpha \leqslant D(\|\delta\mathrm{f}^1\|_\alpha + \|\mathrm{f}^1\|_\infty \|\delta\mathrm{f}\|_\alpha),$$

$$\|\sigma(\boldsymbol{f})^2\|_{2\alpha} \leqslant D(\|\mathrm{f}^2\|_{2\alpha} + \|\delta\mathrm{f}\|_\alpha^2).$$

Therefore, recalling (8.9),

$$\begin{aligned}[\sigma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^\ell)} &= \|\delta(\nabla\sigma(\mathrm{f})\,\mathrm{f}^1)\|_\alpha + \|\sigma(\boldsymbol{f})^2\|_{2\alpha} \\ &\leqslant D([\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} + \|\mathrm{f}^1\|_\infty \|\delta\mathrm{f}\|_\alpha + \|\delta\mathrm{f}\|_\alpha^2).\end{aligned}$$

where, in the last inequality, we apply (8.10). □

We define $\Gamma: \mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k) \to \mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$

$$\Gamma(\boldsymbol{f}) := (\mathrm{y}_0, 0) + \int_0^\cdot \sigma(\boldsymbol{f})\,\mathrm{d}\mathbb{X},$$

(we know that indeed Γ maps $\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$ into $\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$ by Lemma 9.1). In other words, $\Gamma(\mathrm{f}, \mathrm{f}^1)$ is equal to the only $(\mathrm{J}, \mathrm{J}^1) \in \mathcal{D}_{\mathbb{X}}^{2\alpha}$ such that

$$\mathrm{J}_0 = \mathrm{y}_0, \quad \mathrm{J}_s^1 = \sigma(\mathrm{f}_s), \quad \delta\mathrm{J}_{st} - \sigma(\mathrm{f}_s)\,\mathbb{X}_{st}^1 - \nabla\sigma(\mathrm{f}_s)\,\mathrm{f}_s^1\,\mathbb{X}_{st}^2 \in C_2^{3\alpha}. \tag{9.8}$$

We want to construct solutions to (9.6) by a Schauder fixed point argument for T small enough. Let $M > 0$ and $\mathbb{X}$ such that $\|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} \leqslant M$ and

$$\mathcal{B} := \{\boldsymbol{f} = (\mathrm{f}, \mathrm{f}^1) \in \mathcal{D}_{\mathbb{X}}^{2\alpha}: (\mathrm{f}_0, \mathrm{f}_0^1) = (\mathrm{y}_0, \sigma(\mathrm{y}_0)),\ [\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \leqslant 4C\}, \tag{9.9}$$

where

$$C := (1 + M) D \|\sigma\|_\infty. \tag{9.10}$$

LEMMA 9.4. *If $T^\alpha \leqslant \varepsilon_0$ given by*

$$\varepsilon_0 := \frac{1}{8(1 + K_{3\alpha})(1 + D)(1 + \|\sigma\|_\infty)(1 + M)^2}, \tag{9.11}$$

then $\Gamma(\mathcal{B}) \subseteq \mathcal{B}$. Moreover, setting

$$L := 2(1 + M)\|\sigma\|_\infty = \frac{2C}{D}, \tag{9.12}$$

for any $\boldsymbol{f} = (\mathrm{f}, \mathrm{f}^1) \in \mathcal{B}$ *we have*

$$\max\{\|\delta \mathrm{f}\|_\alpha, \|\mathrm{f}^1\|_\infty\} \leqslant L.$$

Proof. Let $\boldsymbol{f} \in \mathcal{B}$. Setting $\varepsilon := T^\alpha$, if $\varepsilon \leqslant \varepsilon_0$ then in particular

$$\varepsilon C \leqslant \frac{\|\sigma\|_\infty}{8(1+K_{3\alpha})(1+\|\sigma\|_\infty)(1+M)} \leqslant \frac{\|\sigma\|_\infty}{8}.$$

We obtain

$$\|\mathrm{f}^1\|_\infty \leqslant |\sigma(\mathrm{y}_0)| + \varepsilon\|\delta \mathrm{f}^1\|_\alpha \leqslant \|\sigma\|_\infty + \varepsilon[\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \leqslant 2\|\sigma\|_\infty \leqslant L.$$

Similarly

$$\begin{aligned} \|\delta \mathrm{f}\|_\alpha &\leqslant \varepsilon\|\mathrm{f}^2\|_{2\alpha} + \|\mathrm{f}^1\|_\infty \|\mathbb{X}^1\|_\alpha \leqslant \varepsilon C + (\|\sigma\|_\infty + \varepsilon C)M \\ &\leqslant \frac{\|\sigma\|_\infty}{8}(1+M) + \|\sigma\|_\infty M \leqslant 2(1+M)\|\sigma\|_\infty = L. \end{aligned}$$

We recall that $\Gamma(\boldsymbol{f}) = (\mathrm{J}, \sigma(\mathrm{f}))$, where J is uniquely determined by (9.8). By (8.11) and (9.7)

$$\begin{aligned} [\Gamma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} &\leqslant 2(1+M)(|\nabla\sigma(\mathrm{y}_0)\,\sigma(\mathrm{y}_0)| + \varepsilon(1+K_{3\alpha})[\sigma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}) \\ &\leqslant 2(1+M)(D\|\sigma\|_\infty + \varepsilon(1+K_{3\alpha})D([\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} + 2L^2)). \end{aligned}$$

Now $(1+M)D\|\sigma\|_\infty = C$, and

$$D([\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} + 2L^2) \leqslant D\left(4C + 2\frac{4C^2}{D^2}\right) \leqslant 8C\left(D + \frac{C}{D}\right).$$

Note that

$$D + \frac{C}{D} = D + (1+M)\|\sigma\|_\infty \leqslant (1+M)(1+D)(1+\|\sigma\|_\infty), \tag{9.13}$$

so that

$$[\Gamma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \leqslant 2C + 2C = 4C.$$

Therefore, $\Gamma(\boldsymbol{f}) \in \mathcal{B}$. □

9.3. LOCAL LIPSCHITZ CONTINUITY

We suppose that σ is of class C^3, with $\|\sigma\|_\infty + \|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty < +\infty$ and we fix $D > 0$ such that

$$D \geqslant \|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty.$$

LEMMA 9.5. (LOCAL LIPSCHITZ ESTIMATE) *We have for* $\boldsymbol{f}, \bar{\boldsymbol{f}} \in \mathcal{B}$, *where* $\mathcal{B}$ *is defined in (9.9), the local Lipschitz estimate*

$$[\sigma(\boldsymbol{f}) - \sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k\otimes\mathbb{R}^d)} \leqslant (2 + D + \|\sigma\|_\infty)\,[\boldsymbol{f} - \bar{\boldsymbol{f}}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \tag{9.14}$$

Proof. By Lemma 9.4 we have for $\boldsymbol{f}=(\mathrm{f},\mathrm{f}^1)$, $\bar{\boldsymbol{f}}=(\bar{\mathrm{f}},\bar{\mathrm{f}}^1)$

$$\max\{\|\delta \mathrm{f}\|_\alpha, \|\delta\bar{\mathrm{f}}\|_\alpha, \|\bar{\mathrm{f}}^1\|_\infty\} \leqslant L,$$

with L as in (9.12). Now, we want to estimate

$$[\sigma(\boldsymbol{f})-\sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k\otimes\mathbb{R}^d)} = \underbrace{\|\delta(\nabla\sigma(\mathrm{f})\,\mathrm{f}^1-\nabla\sigma(\bar{\mathrm{f}})\,\bar{\mathrm{f}}^1)\|_\alpha}_{A} + \underbrace{\|\sigma(\boldsymbol{f})^2-\sigma(\bar{\boldsymbol{f}})^2\|_{2\alpha}}_{B}.$$

We first estimate A:

$$\begin{aligned}
&|\delta(\nabla\sigma(\mathrm{f})\,\mathrm{f}^1-\nabla\sigma(\bar{\mathrm{f}})\,\bar{\mathrm{f}}^1)_{st}| =\\
&=|\delta(\nabla\sigma(\mathrm{f}))_{st}\,\mathrm{f}^1_t+\nabla\sigma(\mathrm{f}_s)\delta\mathrm{f}^1_{st}-\delta(\nabla\sigma(\bar{\mathrm{f}}))_{st}\,\bar{\mathrm{f}}^1_t-\nabla\sigma(\bar{\mathrm{f}}_s)\delta\bar{\mathrm{f}}^1_{st}|\\
&\leqslant|\delta(\nabla\sigma(\mathrm{f})-\nabla\sigma(\bar{\mathrm{f}}))_{st}\,\mathrm{f}^1_t|+|\delta(\nabla\sigma(\bar{\mathrm{f}}))_{st}\,(\mathrm{f}^1_t-\bar{\mathrm{f}}^1_t)|+\\
&\quad+|(\nabla\sigma(\mathrm{f}_s)-\nabla\sigma(\bar{\mathrm{f}}_s))\delta\mathrm{f}^1_{st}|+|\nabla\sigma(\bar{\mathrm{f}}_s)(\delta\mathrm{f}-\delta\bar{\mathrm{f}})_{st}|.
\end{aligned}$$

By Lemma 3.5 and (1.27) we have for $\varepsilon=T^\alpha$

$$\begin{aligned}
A &\leqslant D[\|\mathrm{f}^1\|_\infty(\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\alpha+(\|\delta\mathrm{f}\|_\alpha+\|\delta\bar{\mathrm{f}}\|_\alpha)\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty)+\|\delta\bar{\mathrm{f}}\|_\alpha\|\mathrm{f}^1-\bar{\mathrm{f}}^1\|_\infty+\\
&\quad+\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty\|\delta\mathrm{f}^1\|_\alpha+\|\delta\mathrm{f}^1-\delta\bar{\mathrm{f}}^1\|_\alpha]\\
&\leqslant D[((\|\delta\mathrm{f}\|_\alpha+\|\delta\bar{\mathrm{f}}\|_\alpha)\|\mathrm{f}^1\|_\infty+\|\delta\mathrm{f}^1\|_\alpha)\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty+\|\mathrm{f}^1\|_\infty\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\alpha+\\
&\quad+(1+\varepsilon\|\delta\bar{\mathrm{f}}\|_\alpha)\|\delta\mathrm{f}^1-\delta\bar{\mathrm{f}}^1\|_\alpha]\\
&\leqslant D[(2L^2+\|\delta\mathrm{f}^1\|_\alpha)\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty+L\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\alpha+(1+\varepsilon L)\|\delta\mathrm{f}^1-\delta\bar{\mathrm{f}}^1\|_\alpha]
\end{aligned}$$

We show now that

$$\begin{aligned}
&\|\sigma(\boldsymbol{f})^2-\sigma(\bar{\boldsymbol{f}})^2\|_{2\alpha}\leqslant \qquad (9.15)\\
&\leqslant D((\|\mathrm{f}^2\|_{2\alpha}+3\|\delta\mathrm{f}\|^2_\alpha)\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty+(\|\delta\mathrm{f}\|_\alpha+\|\delta\bar{\mathrm{f}}\|_\alpha)\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\alpha+\|\mathrm{f}^2-\bar{\mathrm{f}}^2\|_{2\alpha})\\
&\leqslant D[(\|\mathrm{f}^2\|_{2\alpha}+3L^2)\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty+2L\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\alpha+\|\mathrm{f}^2-\bar{\mathrm{f}}^2\|_{2\alpha}].
\end{aligned}$$

We have by (9.2)

$$\begin{aligned}
&\|\sigma(\boldsymbol{f})^2-\sigma(\bar{\boldsymbol{f}})^2\|_{2\alpha}\leqslant\|\nabla\sigma(\mathrm{f})\,\mathrm{f}^2-\nabla\sigma(\bar{\mathrm{f}})\,\bar{\mathrm{f}}^2\|_{2\alpha}+\\
&+\int_0^1\|\nabla^2\sigma\,(\mathrm{f}+u\delta\mathrm{f})\,\delta\mathrm{f}\otimes\delta\mathrm{f}-\nabla^2\sigma\,(\bar{\mathrm{f}}+u\delta\bar{\mathrm{f}})\,\delta\bar{\mathrm{f}}\otimes\delta\bar{\mathrm{f}}\|_{2\alpha}\,\mathrm{d}u.
\end{aligned}$$

With the usual estimate $|a\,b-\bar{a}\,\bar{b}|\leqslant|a-\bar{a}|\,|b|+|\bar{a}|\,|b-\bar{b}|$ we can write

$$\begin{aligned}
&\|\nabla\sigma(\mathrm{f})\,\mathrm{f}^2-\nabla\sigma(\bar{\mathrm{f}})\bar{\mathrm{f}}^2\|_{2\alpha}\leqslant\\
&\leqslant\|\nabla\sigma(\mathrm{f})-\nabla\sigma(\bar{\mathrm{f}})\|_\infty\|\mathrm{f}^2\|_{2\alpha}+\|\nabla\sigma(\bar{\mathrm{f}})\|_\infty\|\mathrm{f}^2-\bar{\mathrm{f}}^2\|_{2\alpha}\\
&\leqslant\|\nabla^2\sigma\|_\infty\,\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty\|\mathrm{f}^2\|_{2\alpha}+\|\nabla\sigma\|_\infty\,\|\mathrm{f}^2-\bar{\mathrm{f}}^2\|_{2\alpha}\\
&\leqslant D(\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty\|\mathrm{f}^2\|_{2\alpha}+\|\mathrm{f}^2-\bar{\mathrm{f}}^2\|_{2\alpha}).
\end{aligned}$$

For the other term

$$\begin{aligned}
&\int_0^1\|\nabla^2\sigma(f+u\delta\mathrm{f})\cdot\delta\mathrm{f}\otimes\delta\mathrm{f}-\nabla^2\sigma(\bar{f}+u\delta\bar{\mathrm{f}})\cdot\delta\bar{\mathrm{f}}\otimes\delta\bar{\mathrm{f}}\|_{2\alpha}\,\mathrm{d}u\leqslant\\
&\leqslant\|\nabla^3\sigma\|_\infty\,\|\delta\mathrm{f}\|^2_\alpha(\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty+\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\infty)+\|\nabla^2\sigma\|_\infty(\|\delta\mathrm{f}\|_\alpha+\|\delta\bar{\mathrm{f}}\|_\alpha)\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\alpha\\
&\leqslant D(\|\delta\mathrm{f}\|^2_\alpha(\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty+\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\infty)+(\|\delta\mathrm{f}\|_\alpha+\|\delta\bar{\mathrm{f}}\|_\alpha)\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\alpha).
\end{aligned}$$

Recalling that $\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\infty \leqslant 2\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty$, we have finished the proof of (9.15).

Since $\mathrm{f}_0-\bar{\mathrm{f}}_0=0$, we have $\|\mathrm{f}-\bar{\mathrm{f}}\|_\infty \leqslant \varepsilon\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\alpha$. Summing up, we obtain

$$
\begin{aligned}
&[\sigma(\boldsymbol{f})-\sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k\otimes\mathbb{R}^d)} = A+B\\
&\leqslant \{(3L+\varepsilon(5L^2+[\boldsymbol{f}]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k)}))\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\alpha + (1+\varepsilon L)\,[\boldsymbol{f}-\bar{\boldsymbol{f}}]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k)}\}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\|\delta\mathrm{f}-\delta\bar{\mathrm{f}}\|_\alpha &\leqslant \varepsilon\|\mathrm{f}^2-\bar{\mathrm{f}}^2\|_{2\alpha}+\|\mathrm{f}^1-\bar{\mathrm{f}}^1\|_\infty\|\mathbb{X}^1\|_\alpha\\
&\leqslant \varepsilon\|\mathrm{f}^2-\bar{\mathrm{f}}^2\|_{2\alpha}+\varepsilon M\,\|\delta\mathrm{f}^1-\delta\bar{\mathrm{f}}^1\|_\alpha\\
&\leqslant \varepsilon(1+M)\,[\boldsymbol{f}-\bar{\boldsymbol{f}}]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k)}.
\end{aligned}
$$

Therefore

$$
[\sigma(\boldsymbol{f})-\sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k\otimes\mathbb{R}^d)} \leqslant (\varepsilon(1+M)c_1+c_2)\,[\boldsymbol{f}-\bar{\boldsymbol{f}}]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k)},
$$

where we set

$$
c_1 := D\,(3L+\varepsilon([\boldsymbol{f}]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k)}+3L^2)), \qquad c_2 := D(1+\varepsilon L).
$$

Since $[\boldsymbol{f}]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k)} \leqslant 4C$ we obtain, recalling that $DL=2C$ by (9.12),

$$
\begin{aligned}
c_1 &\leqslant D\,(3L+\varepsilon(4C+5L^2)) \leqslant 6C+20\varepsilon C\left(D+\frac{C}{D}\right)\\
&\leqslant 6C+20\varepsilon C(1+D)(1+\|\sigma\|_\infty)(1+M)\\
&\leqslant 6C+3C=9C,
\end{aligned}
$$

where we have used first (9.13) and then (9.10)-(9.11). Similarly

$$
\varepsilon(1+M)c_1 \leqslant 9\varepsilon C(1+M) = 9\varepsilon D\|\sigma\|_\infty(1+M)^2 \leqslant 2,
$$

and

$$
c_2 = D+\varepsilon DL = D+2\varepsilon C \leqslant D+\|\sigma\|_\infty.
$$

Therefore

$$
\varepsilon(1+M)c_1+c_2 \leqslant 2+D+\|\sigma\|_\infty.
$$

The proof is finished. □

9.4. Contraction

In this section we prove local existence by means of a Banach fixed point, assuming σ to be of class C^3 and bounded with its first, second and third derivatives, namely $\|\sigma\|_\infty+\|\nabla\sigma\|_\infty+\|\nabla^2\sigma\|_\infty+\|\nabla^3\sigma\|_\infty<+\infty$. Therefore the assumptions are stronger than for the Schauder fixed point argument of Section 9.2 or for the discrete approximation of Section 4.8. However this method has the advantage of not requiring compactness of the image of Γ and therefore this approach works also for rough equations with values in infinite-dimensional spaces.

Let us fix $D>0$ such that

$$D\geqslant \max\{\|\nabla\sigma\|_\infty, \|\nabla^2\sigma\|_\infty, \|\nabla^3\sigma\|_\infty\}.$$

Recalling that $\mathcal{B}$ was defined in (9.9), we can now show the following

LEMMA 9.6. *If $T^\alpha\in\,]0,\varepsilon_0]$ where ε_0 is as in (9.11), then $\Gamma\colon\mathcal{B}\to\mathcal{B}$ is a contraction for $\|\cdot\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}$.*

Proof. Let $\boldsymbol{f}=(\mathrm{f},\mathrm{f}^1)$ and $\bar{\boldsymbol{f}}=(\bar{\mathrm{f}},\bar{\mathrm{f}}^1)$ be in $\mathcal{B}$. Since $\mathrm{f}_0=\bar{\mathrm{f}}_0$ and $\mathrm{f}_0^1=\bar{\mathrm{f}}_0^1$, by the definitions, see in particular (8.9),

$$\|\Gamma(\boldsymbol{f})-\Gamma(\bar{\boldsymbol{f}})\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \;=\; [\Gamma(\boldsymbol{f})-\Gamma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}.$$

We set $\varepsilon:=T^\alpha$. By (8.11)

$$[\Gamma(\boldsymbol{f})-\Gamma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \;\leqslant\; \varepsilon 2(1+M)(1+K_{3\alpha})\,[\sigma(\boldsymbol{f})-\sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}.$$

Now by Lemma 9.5

$$[\sigma(\boldsymbol{f})-\sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k\otimes\mathbb{R}^d)} \;\leqslant\; (2+D+\|\sigma\|_\infty)\,[\boldsymbol{f}-\bar{\boldsymbol{f}}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}.$$

Now $2+D+\|\sigma\|_\infty\leqslant 2(1+D)(1+\|\sigma\|_\infty)$. Therefore

$$[\Gamma(\boldsymbol{f})-\Gamma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \;\leqslant c_4\,[\boldsymbol{f}-\bar{\boldsymbol{f}}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)},$$

with

$$c_4=\varepsilon 2(1+M)(1+K_{3\alpha})2(1+D)(1+\|\sigma\|_\infty)\leqslant\frac{1}{2}$$

by (9.11). This concludes the proof. □

Chapter 10

Algebra

Let us recall that a d-dimensional α-rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ with $\alpha > \frac{1}{3}$ is such that $\mathbb{X}_{st}$ takes values in $G := \mathbb{R}^d \times (\mathbb{R}^d \otimes \mathbb{R}^d)$ for all $0 \leqslant s \leqslant t \leqslant T$. We want to show that the Chen relation (5.25) has a very natural algebraic interpretation if we endow G with a suitable group structure.

10.1. A non-commutative group

We denote in the following generic elements $x \in G = \mathbb{R}^d \times (\mathbb{R}^d \otimes \mathbb{R}^d)$ by $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^d$ and $x_2 \in \mathbb{R}^d \otimes \mathbb{R}^d$. We define an operation $*$: $G \times G \to G$ as follows: for $x, y \in G$ with $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we set

$$x * y := z = (z_1, z_2), \qquad z_1 := x_1 + y_1, \qquad z_2 := x_2 + y_2 + x_1 \otimes y_1.$$

It is simple to see that $(G, *, \mathbf{1})$, is a group, where $\mathbf{1} := (0, 0)$. First associativity of the product:

$$\begin{aligned}(x * y) * z &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2 + x_1 \otimes y_1 + (x_1 + y_1) \otimes z_1)\\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2 + x_1 \otimes (y_1 + z_1) + y_1 \otimes z_1)\\ &= x * (y * z).\end{aligned}$$

Now the fact that $\mathbf{1}$ is the neutral element is obvious. Finally the inverse is given explicitly by

$$x^{*(-1)} = (-x_1, -x_2 + x_1 \otimes x_1). \tag{10.1}$$

Let us note that $(G, *, \mathbf{1})$ is non-commutative for $d \geqslant 2$, since in general $x_1 \otimes y_1 \neq y_1 \otimes x_1$.

Now we want to interpret the Chen relation (5.25) in this setting. Given a α-rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$, we write

$$\mathbb{X}\colon [0, T]^2_{\leqslant} \to G, \qquad \mathbb{X}_{st} := (\mathbb{X}^1_{st}, \mathbb{X}^2_{st}).$$

Then the Chen formula (5.25) yields

$$\mathbb{X}_{st} = \mathbb{X}_{su} * \mathbb{X}_{ut}, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Indeed it is enough to note that for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\mathbb{X}^1_{st} = \mathbb{X}^1_{su} + \mathbb{X}^1_{ut}, \qquad \mathbb{X}^2_{st} = \mathbb{X}^2_{su} + \mathbb{X}^2_{ut} + \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}.$$

Note that we also have, by the analytical estimates $|\mathbb{X}^i_{st}| \lesssim |t - s|^{i\alpha}$ that $\mathbb{X}_{tt} = \mathbf{1}$.

10.2. Shuffle group

We can consider the subset $H \subset G$ given by

$$H := \{x = (x_1, x_2) \in G: \quad x_2 + x_2^T = x_1 \otimes x_1\},$$

where $(a \otimes b)^T := b \otimes a$ for $a, b \in \mathbb{R}^d$.

We can see that H is a subgroup of G: if $x, y \in H$ then $z := x * y$ satisfies

$$\begin{aligned} z_2 + z_2^T &= x_2 + y_2 + x_1 \otimes y_1 + x_2^T + y_2^T + y_1 \otimes x_1 \\ &= x_1 \otimes x_1 + y_1 \otimes y_1 + x_1 \otimes y_1 + y_1 \otimes x_1 \\ &= (x_1 + y_1) \otimes (x_1 + y_1) = z_1 \otimes z_1. \end{aligned}$$

Moreover if $x \in H$ then its inverse $y = x^{*(-1)} \in G$ satisfies

$$\begin{aligned} y_2 + y_2^T &= -x_2 + x_1 \otimes x_1 - x_2^T + x_1 \otimes x_1 \\ &= -x_1 \otimes x_1 + 2x_1 \otimes x_1 \\ &= (-x_1) \otimes (-x_1) = y_1 \otimes y_1 \end{aligned}$$

so that $x^{*(-1)} \in H$. Finally $\mathbf{1} \in H$. Therefore H is indeed a (proper) subgroup of G. Moreover by (10.1) and the relation defining elements of H we have the simpler expression for the inverse

$$x^{*(-1)} = (-x_1, x_2^T), \qquad x \in H. \tag{10.2}$$

Therefore we have the following

Lemma 10.1. *A rough path $\mathbb{X}$ is weakly geometric if and only if the associated map $\mathbb{X}: [0,T]_{\leqslant}^2 \to G$ takes values in H.*

10.3. Algebra and integral

As we explained at the beginning of Chapter 5, given $\mathbb{X}^1 = \delta X \in C_2^{\alpha}$, a choice of $\mathbb{X}^2$ is equivalent to a choice of an integral $I_t = \int_0^t X_s \otimes \mathrm{d}X_s$, $t \in [0,T]$, namely

$$I: [0,T] \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad I_0 = 0, \quad \delta I_{st} - X_s \otimes \delta X_{st} = \mathbb{X}_{st}^2, \quad \mathbb{X}^2 \in C_2^{2\alpha}.$$

Given $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$, we set now

$$\mathbb{X}: [0,T] \to G, \qquad \mathbb{X}_t := (X_t, I_t), \qquad t \in [0,T].$$

Then for $0 \leqslant s \leqslant t$

$$\begin{aligned} \mathbb{X}_s^{*(-1)} * \mathbb{X}_t &= (-X_s, -I_s + X_s \otimes X_s) * (X_t, I_t) \\ &= (X_t - X_s, I_t - I_s + X_s \otimes X_s - X_s \otimes X_t) \\ &= (\mathbb{X}_{st}^1, \delta I_{st} - X_s \otimes (X_t - X_s)) \\ &= (\mathbb{X}_{st}^1, \mathbb{X}_{st}^2) = \mathbb{X}_{st} \end{aligned} \tag{10.3}$$

again by the Chen relation (5.25).

Remark 10.2. This definition of $\mathbb{X}: [0,T] \to G$ is not the only possible one. For example, if $\mathbb{X}_t := \mathbb{X}_{0t}$, then we also have $\mathbb{X}_s^{*(-1)} * \mathbb{X}_t = \mathbb{X}_{st}$.

10.4. Unordered times

Given the relation (10.3) $\mathbb{X}_s^{*(-1)} * \mathbb{X}_t = \mathbb{X}_{st}$ for $s \leqslant t$, it is natural to wonder whether we have an expression for $\mathbb{X}_s^{*(-1)} * \mathbb{X}_t$ when $s > t$. In fact, this turns out to be equivalent to having an expression for $\mathbb{X}_{st}$ when $s > t$.

The definition of $\mathbb{X}_{st}^1$ is simple:

$$\mathbb{X}_{st}^1 := -\mathbb{X}_{ts}^1, \qquad 0 \leqslant t < s \leqslant T.$$

In particular, if X of class C^α is such that $\mathbb{X}^1 = \delta X$, then we obtain

$$\mathbb{X}_{st}^1 = X_t - X_s, \qquad |\mathbb{X}_{st}^1| \lesssim |t-s|^\alpha, \qquad \forall s, t \in [0,T].$$

We want now to extend $\mathbb{X}^2$ to $[0,T]^2$ so that for all $s, u, t \in [0,T]$

$$\delta \mathbb{X}_{sut}^2 = \mathbb{X}_{su}^1 \otimes \mathbb{X}_{ut}^1, \qquad |\mathbb{X}_{st}^2| \lesssim |t-s|^{2\alpha}.$$

We set for $0 \leqslant t < s \leqslant T$

$$\mathbb{X}_{st}^2 := -\mathbb{X}_{ts}^2 + \mathbb{X}_{st}^1 \otimes \mathbb{X}_{st}^1 = \mathbb{X}_{ts}^{*(-1)}.$$

Note that then we clearly have $|\mathbb{X}_{st}^2| \lesssim |t-s|^{2\alpha}$ for all $s, t \in [0,T]$.

With these choices, we have by (10.1)

$$\mathbb{X}_{st} = \mathbb{X}_{ts}^{*(-1)}, \qquad \forall s, t \in [0,T].$$

Then by (10.3), for $0 \leqslant t < s \leqslant T$

$$\mathbb{X}_{st} = \mathbb{X}_{ts}^{*(-1)} = \left(\mathbb{X}_t^{*(-1)} * \mathbb{X}_s\right)^{*(-1)} = \mathbb{X}_s^{*(-1)} * \mathbb{X}_t,$$

namely (10.3) holds for all $s, t \in [0,T]$.

Now, suppose that we have a general germ $A: [0,T]^2 \to \mathbb{R}$. We suppose that it satisfies for some $\eta > 1$

$$|A_{st} - A_{su} - A_{ut}| \leqslant C_A(|u-s| \vee |t-u|)^\eta, \qquad s, u, t \in [0,T].$$

In particular, the restriction $A: [0,T]_\leqslant^2 \to \mathbb{R}$ is such that $\delta A: [0,T]_\leqslant^3 \to \mathbb{R}$ belongs to C_3^η. By the Sewing Lemma, we have a unique choice for (I, R) such that

$$I_0 = 0, \qquad \delta I_{st} = A_{st} + R_{st}, \qquad |R_{st}| \lesssim |t-s|^\eta, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

We want to extend R to a function on $[0,T]^2$ in such a way that the previous formula holds over $[0,T]^2$. We set

$$R_{st} = -A_{st} - A_{ts} - R_{ts}, \qquad 0 \leqslant t \leqslant s \leqslant T. \tag{10.4}$$

Since $\delta I_{ts} = -\delta I_{st}$, we have for $t \leqslant s$

$$R_{st} = -A_{st} - (\delta I_{ts} - R_{ts}) - R_{ts} = -A_{st} - \delta I_{ts} = \delta I_{st} - A_{st},$$

so that $\delta I = A + R$ on $[0,T]^2$. Moreover, since $A_{ss} = 0$ by Remark 1.8,

$$|R_{st}| \leqslant |(\delta A)_{sts}| + |R_{ts}| \leq (C_\eta + 1)\, C_A\, |t-s|^\eta, \qquad 0 \leqslant t \leqslant s \leqslant T. \tag{10.5}$$

10.5. An example: the Brownian case

Let consider the Itô Brownian rough paths in $\mathbb{R}^d$

$$\mathbb{B}^1_{st} = B_t - B_s, \qquad \mathbb{B}^2_{st} = \int_s^t (B_r - B_s) \otimes \mathrm{d}B_r, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Then we obtain from the definitions of the previous section for $0 \leqslant t < s \leqslant T$

$$\begin{aligned} \mathbb{B}^1_{st} &= B_t - B_s, \\ \mathbb{B}^2_{st} &= -\int_t^s (B_r - B_t) \otimes \mathrm{d}B_r + (B_s - B_t) \otimes (B_s - B_t) \\ &= \int_t^s \mathrm{d}B_r \otimes (B_r - B_t) + (s-t) I, \end{aligned}$$

where I is the identity matrix of $\mathbb{R}^d$.

Note that we can not write the latter expression as $\int_t^s (B_s - B_r) \otimes \mathrm{d}B_r$ since the integrand is not adapted to the filtration of B. Here the one-parameter function $\mathbb{B}\colon [0,T] \to G$ such that $\mathbb{B}_{st} = \mathbb{B}_s^{*(-1)} * \mathbb{B}_t$ is given by

$$\mathbb{B}_t = \left(B_t, \int_0^t B_s \otimes \mathrm{d}B_s \right), \qquad t \geqslant 0.$$

Let us consider now the Stratonovich case:

$$\bar{\mathbb{B}}^1_{st} = B_t - B_s, \qquad \bar{\mathbb{B}}^2_{st} = \int_s^t (B_r - B_s) \otimes \circ \mathrm{d}B_r, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Then we obtain from the definitions of the previous section for $0 \leqslant t < s \leqslant T$

$$\bar{\mathbb{B}}^1_{st} = B_t - B_s,$$

and if one applies (10.2) then we have for $0 \leqslant t < s \leqslant T$

$$\bar{\mathbb{B}}^2_{st} = (\bar{\mathbb{B}}^2_{ts})^T = \int_t^s \circ \mathrm{d}B_r \otimes (B_r - B_t).$$

Here the one-parameter function $\bar{\mathbb{B}}\colon [0,T] \to G$ such that $\bar{\mathbb{B}}_{st} = \bar{\mathbb{B}}_s^{*(-1)} * \bar{\mathbb{B}}_t$ is given by

$$\bar{\mathbb{B}}_t = \left(B_t, \int_0^t B_s \otimes \circ \mathrm{d}B_s \right), \qquad t \geqslant 0.$$

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