Sewing and Reconstruction, from Deterministic to Stochastic

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These slides can be downloaded from my home page

Pathwise stochastic analysis

Over the last 25 years there has been a growing activity on a new approach to stochastic analysis, said pathwise,

- first in finite dimension (T. Lyons, M. Gubinelli, ...) $\to$ ODEs, SDEs
- then in infinite dimension (M. Hairer, M. Gubinelli, ...) $\to$ PDEs, SPDEs

The 2014 Fields Medal awarded to M. Hairer has given unprecedented visibility to this topic, which however remains largely obscure to the wide mathematical community, because of its complexity.

With F. Caravenna, and then also with L. Broux, A. Bonicelli, L. Agabiti ..., I have a (rather long-term) project of re-understanding and re-formulating this theory in an accessible way.

Lecture notes and papers in collaboration with F. Caravenna and L. Broux, see my web page.

Chapter 1: Deterministic sewing

The Itô integral

Hopefully everyone in this room knows the construction of the Itô integral:

$$\int_0^t h_s \, \mathrm{d}M_s, \qquad t \geq 0,$$

where $(M_s)_{s\geq 0}$ is a continuous local martingale and $(h_s)_{s\geq 0}$ is a continuous adapted process.

The main example for us is $M = B$, a Brownian motion.

Itô integration is crucial for the theory of stochastic differential equations (SDEs):

$$X_t = x + \int_0^t \sigma(X_s) \, \mathrm{d}B_s, \qquad t \geq 0.$$

Construction

The Itô integral is constructed as a limit in $L^2(\Omega)$ or in probability of Riemann sums

$$\int_0^t h_s \, \mathrm{d}B_s = \lim_{|\mathcal{P}|\to 0} \sum_{i=1}^{k_{\mathcal{P}}} h_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}),$$

where $\mathcal{P} := \{0 = t_0 < t_1 < \cdots < t_{k_{\mathcal{P}}} = t\}$ is called a partition of $[0, t]$ and $|\mathcal{P}| := \max\{t_i - t_{i-1} : i = 1, \ldots, k_{\mathcal{P}}\}$ is the mesh of the partition.

In general, this limit in $L^2(\Omega)$ or in probability can not be improved, for example to an a.s. limit (namely for fixed $\omega \in \Omega$).

Analogously, the Itô map $B \mapsto X$, solution to the SDE, is only measurable.

Lack of pathwise definition

This means that if one has for a fixed $\omega \in \Omega$

- a trajectory $(h_s(\omega))_{s\in[0,t]}$ of h
- a trajectory $(B_s(\omega))_{s\in[0,t]}$ of B

then the value of

$$\int_0^t h_s(\omega)\,\mathrm{d}B_s(\omega)$$

is ill-defined.

In a pathwise construction, it should be possible to compute $\int_0^t h_s(\omega)\,\mathrm{d}B_s(\omega)$ knowing only $(h_s(\omega))_{s\in[0,t]}$ and $(B_s(\omega))_{s\in[0,t]}$.

Riemann sums

Let us note that

$$\int_{t_{i-1}}^{t_i} h_s\, dB_s - h_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) = \int_{t_{i-1}}^{t_i} (h_s - h_{t_{i-1}})\, \mathrm{d}B_s$$

Then

$$\int_0^t h_s\, \mathrm{d}B_s - \sum_{i=1}^{k_{\mathcal{P}}} h_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) = \sum_{i=1}^{k_{\mathcal{P}}} \int_{t_{i-1}}^{t_i} (h_s - h_{t_{i-1}})\, \mathrm{d}B_s.$$

From a pathwise perspective, it is natural to ask:

- how large/small are these two quantities?

Let us set for $0 \le s \le t$

$$R_{st} := \int_s^t (h_r - h_s)\, \mathrm{d}B_r.$$

A pathwise estimate

Note: if $|R_{st}(\omega)| \lesssim |t-s|^\eta$ with $\eta > 1$, for all $0 \le s \le t \le T$, then

$$\left| \int_0^t h_s \, \mathrm{d}B_s - \sum_{i=1}^{k_\mathcal{P}} h_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) \right| (\omega) \le \sum_{i=1}^{k_\mathcal{P}} |R_{t_{i-1},t_i}(\omega)| \lesssim |\mathcal{P}|^{\eta-1} \, t.$$

With a modified Kolmogorov-type result we can prove that

$$|R_{st}(\omega)| = \left| \int_s^t (h_r - h_s) \, \mathrm{d}B_r \right| (\omega) \lesssim |t-s|^{\alpha+\beta}, \qquad \text{a.s.}$$

if $0 < \alpha < \frac{1}{2}$, and h a.s. of class C^β, namely

$$|h_t - h_s|(\omega) \lesssim |t-s|^\beta, \qquad 0 \le s \le t \le T.$$

Here $\eta = \alpha + \beta$ may not be > 1. If $\beta = \alpha$ then $\eta = 2\alpha < 1$. Let us consider this case from now on.

A new notion

On the positive side, we have learnt that, for all $\alpha < \frac{1}{2}$, if h is adapted and a.s. of class C^α, then we have the a.s. estimate

$$|I_t - I_s - h_s(B_t - B_s)| \lesssim |t-s|^{2\alpha}, \qquad I_t := \int_0^t h_r \, \mathrm{d}B_r.$$

Now suppose that h satisfies a similar property, namely

$$\left|h_t - h_s - h^1_s(B_t - B_s)\right| \lesssim |t-s|^{2\alpha},$$

with some adapted $h^1 : [0,T] \to \mathbb{R}$, a.s. of class C^α.

This is after all natural if we think of iterating the stochastic integral, for example in the case of Picard iterations for SDEs.

A new idea

We want to continue the expansion of $t \mapsto I_t$:

$$I_t - I_s - h_s(B_t - B_s) - h^1_s \frac{(B_t - B_s)^2 - (t-s)}{2} = \int_s^t (h_r - h_s - h^1_s(B_r - B_s))\, \mathrm{d}B_r$$

since by the Itô formula

$$\frac{(B_t - B_s)^2 - (t-s)}{2} = \int_s^t (B_r - B_s)\, \mathrm{d}B_r.$$

Let us introduce some notation:

$$\mathbb{B}^1_{st} := B_t - B_s, \qquad \mathbb{B}^2_{st} := \frac{(B_t - B_s)^2 - (t-s)}{2}.$$

Note that for any $\alpha < \frac{1}{2}$ a.s.

$$\left|\mathbb{B}^1_{st}\right| \lesssim |t-s|^\alpha, \qquad \left|\mathbb{B}^2_{st}\right| \lesssim |t-s|^{2\alpha}.$$

A pathwise estimate

We are assuming that $\alpha < \frac{1}{2}$, that h^1 is a.s. of class C^α and h satisfies

$$\left| h_t - h_s - h^1_s \, \mathbb{B}^1_{st} \right| \lesssim |t - s|^{2\alpha}.$$

We write

$$I_t - I_s - h_s \, \mathbb{B}^1_{st} - h^1_s \, \mathbb{B}^2_{st} = \int_s^t \left(h_r - h_s - h^1_s \, \mathbb{B}^1_{sr} \right) \mathrm{d}B_r =: R_{st}.$$

Now we can prove (again with a modified Kolmogorov-type result)

$$|R_{st}| \lesssim |t - s|^{3\alpha}, \qquad \text{a.s.}$$

and we obtain
$$\left| I_t - \sum_{i=1}^{k_{\mathcal{P}}} \left[h_{t_{i-1}} \mathbb{B}^1_{t_{i-1},t_i} + h^1_{t_{i-1}} \mathbb{B}^2_{t_{i-1},t_i} \right] \right| \le \sum_{i=1}^{k_{\mathcal{P}}} |R_{t_{i-1},t_i}| \lesssim |\mathcal{P}|^{3\alpha - 1} \, t.$$

If $\alpha > \frac{1}{3}$, this tends to 0 a.s. as $|\mathcal{P}| \to 0$.

A pathwise construction

Theorem (M. Gubinelli)

Let $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ *as above and* $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. *Let* $h, h^1 : [0,T] \to \mathbb{R}$ *(random) such that a.s.* h^1 *is of class* C^α *and*

$$\left| h_t - h_s - h^1_s \, \mathbb{B}^1_{st} \right| \lesssim |t-s|^{2\alpha}.$$

Then there exists a unique $I : [0,T] \to \mathbb{R}$ *such that* $I_0 = 0$ *and*

$$\left| I_t - I_s - h_s \, \mathbb{B}^1_{st} - h^1_s \, \mathbb{B}^2_{st} \right| \lesssim |t-s|^{3\alpha}.$$

Moreover
$$I_t = \lim_{|\mathcal{P}| \to 0} \sum_{i=1}^{k_{\mathcal{P}}} \left[h_{t_{i-1}} \mathbb{B}^1_{t_{i-1}, t_i} + h^1_{t_{i-1}} \mathbb{B}^2_{t_{i-1}, t_i} \right] \qquad a.s.$$

where $\mathcal{P} := \{0 = t_0 < t_1 < \cdots < t_{k_{\mathcal{P}}} = t\}$.

Theorem (M. Gubinelli)

Let $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ *as above and* $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. *Let* $h, h^1 : [0,T] \to \mathbb{R}$ *(random) such that a.s.* h^1 *is of class* C^α *and*

$$\left| h_t - h_s - h^1_s \, \mathbb{B}^1_{st} \right| \lesssim |t-s|^{2\alpha}.$$

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$$\left| I_t - I_s - h_s \, \mathbb{B}^1_{st} - h^1_s \, \mathbb{B}^2_{st} \right| \lesssim |t-s|^{3\alpha}.$$

Moreover $$I_t = \lim_{|\mathcal{P}| \to 0} \sum_{i=1}^{k_{\mathcal{P}}} \left[h_{t_{i-1}} \mathbb{B}^1_{t_{i-1}, t_i} + h^1_{t_{i-1}} \mathbb{B}^2_{t_{i-1}, t_i} \right] \qquad a.s.$$

where $\mathcal{P} := \{0 = t_0 < t_1 < \cdots < t_{k_{\mathcal{P}}} = t\}$.

Finally, if h, h^1 *are adapted, then a.s.* $I_t = \int_0^t h_s \, \mathrm{d}B_s$ *(Itô integral).*

Comments

- The characterisation $\left|I_t - I_s - h_s\,\mathbb{B}^1_{st} - h^1_s\,\mathbb{B}^2_{st}\right| \lesssim |t-s|^{3\alpha}$ is a Taylor expansion with respect to the (random) generalised monomials $\mathbb{B}^1, \mathbb{B}^2$.
- In this context, h, h^1 are the (Gubinelli) derivatives of I with respect to $\mathbb{B}^1, \mathbb{B}^2$.
- If the expansion is long enough to obtain a remainder of order $|t-s|^\eta$ with $\eta > 1$, then we have uniqueness.
- The condition $\left|h_t - h_s - h^1_s\,\mathbb{B}^1_{st}\right| \lesssim |t-s|^{2\alpha}$ is restrictive and can not cover all possible stochastic integrals.
- For example, for two independent Brownian motions B^1, B^2, the Itô integral $\int_0^t B^1_s\,\mathrm{d}B^2_s$ can not have a pathwise construction (at least not with the above method).
- We can now define the product $h\,\dot{B} := \dot{I}$. It is unique if $\mathbb{B}^2$ is fixed (discuss).
- We define the germ $A_{st} := h_s\,\mathbb{B}^1_{st} + h^1_s\,\mathbb{B}^2_{st}$. Sewing is the application $A \mapsto I$.

Sewing: germs, integrals, remainders

Let us denote $[0,T]^2_{\leqslant} := \{(s,t) : 0 \leq s \leq t \leq T\}$.

- A continuous $A : [0,T]^2_{\leqslant} \to \mathbb{R}$ is called a germ.
- Given a germ A, a continuous $I : [0,T] \to \mathbb{R}$ is called an integral of A if, setting

$$R_{st} := I_t - I_s - A_{st}, \qquad R : [0,T]^2_{\leqslant} \to \mathbb{R},$$

  we have $R_{st} = o(t-s)$. In this case, R is called the remainder.

Easy: given a germ A, there can be at most one (I,R) with $I_0 = 0$.

If $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_m = t\}$ is a partition of $[0,t]$ and $|\mathcal{P}| := \sup_i |t_i - t_{i-1}|$, then

$$I_t = \lim_{|\mathcal{P}| \to 0} \sum_i A_{t_{i-1} t_i}.$$

Sewing is the application $A \mapsto I$.

Sewing bound

Given $I : [0,T] \to \mathbb{R}$, we set $\delta I : [0,T]^2_{\leqslant} \to \mathbb{R}$

$$\delta I_{st} := I_t - I_s, \qquad 0 \le s \le t \le T$$

Given $A : [0,T]^2_{\leqslant} \to \mathbb{R}$, we set $\delta A : [0,T]^3_{\leqslant} \to \mathbb{R}$

$$\delta A_{sut} := A_{st} - A_{su} - A_{ut}, \qquad 0 \le s \le u \le t \le T.$$

Note: $\delta \circ \delta = 0$. For $\eta > 0$ let

$$\|R\|_\eta := \sup_{s<t} \frac{|R_{st}|}{|t-s|^\eta}, \qquad \|\delta R\|_\eta := \sup_{s<u<t} \frac{|\delta R_{sut}|}{|t-s|^\eta}.$$

Theorem (Sewing bound)

Let $(R_{st})_{0\le s\le t\le T}$ *be continuous. If* $R_{st} = o(t-s)$ *and* $\eta > 1$ *then*

$$\|R\|_\eta \le K_\eta \|\delta R\|_\eta.$$

Proof of the sewing bound

For a partition $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of $[a, b]$ we define the Riemann sum

$$I_{\mathcal{P}}(R) := \sum_{i=1}^{\#\mathcal{P}} R_{t_{i-1}t_i}, \qquad (1)$$

We consider for $n \geqslant 0$ the dyadic partitions $\mathcal{P}_n := \{t_i^n := a + \frac{i}{2^n}(b - a) : \ 0 \leq i \leq 2^n\}$ of $[a, b]$. Since $\mathcal{P}_0 = \{a, b\}$, from $R_{st} = o(t - s)$ we obtain

$$\lim_{n\to\infty} I_{\mathcal{P}_n}(R) = 0$$

and

$$|R_{ab}| \leqslant \sum_{n=0}^{\infty} \left| I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R) \right| .$$

Proof of the sewing bound

If $\mathcal{P} = \{t_0 < t_1 < \ldots < t_m\}$ and $\mathcal{P}' = \mathcal{P} \setminus \{t_i\}$ then

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = R_{t_{i-1}t_{i+1}} - R_{t_{i-1}t_i} - R_{t_it_{i+1}} = \delta R_{t_{i-1}t_it_{i+1}}. \tag{2}$$

and for $\eta > 0$:

$$|I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R)| \leqslant \|\delta R\|_\eta |t_{i+1} - t_{i-1}|^\eta. \tag{3}$$

We obtain

$$\begin{aligned} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)| &\leqslant 2^n \, \|\delta R\|_\eta \left(\frac{2(b-a)}{2^{n+1}}\right)^\eta \\ &= 2^{-(\eta-1)n} \|\delta R\|_\eta (b-a)^\eta, \end{aligned}$$

and

$$|R_{ab}| \leqslant (1 - 2^{1-\eta})^{-1} \|\delta R\|_\eta (b-a)^\eta, \qquad 0 \leqslant a < b \leqslant T. \qquad \square$$

Theorem (Sewing Lemma, (Gubinelli, Feyel - de la Pradelle))

Let $A : [0,T]^2_{\leqslant} \to \mathbb{R}$ *be continuous. If* $\exists \eta > 1$ *and* $\|\delta A\|_\eta < +\infty$, *then* $\exists!(I,R)$.

If $f \in C^\alpha$, $g \in C^\beta$, with $\alpha, \beta \in (0,1)$, $\alpha + \beta > 1$, we set $A_{st} := f_s(g_t - g_s)$.

Then

$$\delta A_{sut} = -\delta f_{su}\ \delta g_{ut}, \qquad \|\delta A\|_{\alpha+\beta} \leqslant \|\delta f\|_\alpha\ \|\delta g\|_\beta < +\infty.$$

Then there exists a unique $I : [0,T] \to \mathbb{R}$ such that

$$\delta I_{st} - A_{st} = o(t-s).$$

We note $I_t = \int_0^t f_s\, \mathrm{d}g_s$ and we call it the Young integral.

We can define canonically the Young product $f\,\dot{g} := \dot{I}$

Rough integral

Let us recall what we saw in the case of the integral: for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$

$$A_{st} = h_s \, \mathbb{B}^1_{st} + h^1_s \, \mathbb{B}^2_{st},$$

$$\mathbb{B}^1_{st} = B_t - B_s, \qquad \mathbb{B}^2_{st} = \int_s^t \mathbb{B}^1_{sr} \, \mathrm{d}B_r,$$

$$h^{[2]}_{st} := h_t - h_s - h^1_s \, \mathbb{B}^1_{st},$$

$$\left| h^{[2]}_{st} \right| \lesssim |t-s|^{2\alpha}, \qquad \left| \delta h^1_{st} \right| \lesssim |t-s|^{\alpha}$$

An explicit computation gives

$$\begin{aligned} \delta A_{sut} &= A_{st} - A_{su} - A_{ut} \\ &= -h^{[2]}_{su} \, \mathbb{B}^1_{ut} - \delta h^1_{ut} \, \mathbb{B}^2_{ut} \end{aligned}$$

$$|\delta A_{sut}| \lesssim |u-s|^{2\alpha} |t-u|^{\alpha} + |u-s|^{\alpha} |t-u|^{2\alpha} \lesssim |t-s|^{3\alpha}.$$

Therefore this germ A is in the domain of application of the Sewing bound/lemma.

Rough integral

Let us make the previous computation more explicit:

$$\begin{aligned}\delta A_{sut} &= A_{st} - A_{su} - A_{ut} \\ &= h_s \left(\mathbb{B}^1_{st} - \mathbb{B}^1_{su} \right) - h_u \, \mathbb{B}^1_{ut} \\ &\quad + h^1_s \left(\mathbb{B}^2_{st} - \mathbb{B}^2_{su} \right) - h^1_u \, \mathbb{B}^2_{ut} \\ &= - \left(h_u - h_s \right) \mathbb{B}^1_{ut} + h^1_s \delta \mathbb{B}^2_{sut} - \delta h^1_{su} \, \mathbb{B}^2_{ut}\end{aligned}$$

Now a crucial formula is:

$$\delta \mathbb{B}^2_{sut} = \int_s^t \mathbb{B}^1_{sr} \, \mathrm{d}B_r - \int_s^u \mathbb{B}^1_{sr} \, \mathrm{d}B_r - \int_u^t \mathbb{B}^1_{ur} \, \mathrm{d}B_r = \mathbb{B}^1_{su} \, \mathbb{B}^1_{ut}$$

and therefore we can conclude

$$\begin{aligned}\delta A_{sut} &= - \left(h_u - h_s \right) \mathbb{B}^1_{ut} + h^1_s \mathbb{B}^1_{su} \, \mathbb{B}^1_{ut} - \delta h^1_{su} \, \mathbb{B}^2_{ut} \\ &= - h^{[2]}_{su} \, \mathbb{B}^1_{ut} - \delta h^1_{ut} \, \mathbb{B}^2_{ut}.\end{aligned}$$

Rough integral

From this example, one can extract a possible more general setting: we need

- $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$
- functions $\mathbb{X}^1 : [0,T]^2_{\leqslant} \to \mathbb{R}$ and $\mathbb{X}^2 : [0,T]^2_{\leqslant} \to \mathbb{R}$ such that

$$|\mathbb{X}^1_{st}| \lesssim |t-s|^{\alpha}, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha},$$
$$\mathbb{X}^1_{st} = X_t - X_s, \qquad \delta\mathbb{X}^2_{sut} = \mathbb{X}^1_{su}\,\mathbb{X}^1_{ut}$$

- functions $h, h^1 : [0,T] \mapsto \mathbb{R}$ such that

$$\left|h^1_t - h^1_s\right| \lesssim |t-s|^{\alpha}, \qquad h^{[2]}_{st} := h_t - h_s - h^1_s\,\mathbb{X}^1_{st}, \qquad \left|h^{[2]}_{st}\right| \lesssim |t-s|^{2\alpha}$$

- the germ $A_{st} := h_s\,\mathbb{X}^1_{st} + h^1_s\,\mathbb{X}^2_{st}$.

Then $\delta A_{sut} = -h^{[2]}_{su}\,\mathbb{X}^1_{ut} - \delta h^1_{ut}\,\mathbb{X}^2_{ut}$ and $|\delta A_{sut}| \lesssim |t-s|^{3\alpha}$.

Rough paths

- In fact one can and must consider the vector-valued case: $d \in \mathbb{N}$, $\mathbb{X}^1 : [0,T]^2_\leqslant \to \mathbb{R}^d$, $\mathbb{X}^2 : [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$, etc.
- The pair $(\mathbb{X}^1, \mathbb{X}^2)$ is called a rough path
- The pair (h, h^1) is called a controlled path
- The (unique) I rough integral is determined by:

$$I_t - I_s - h_s \, \mathbb{X}^1_{st} - h^1_s \, \mathbb{X}^2_{st} = o(t-s).$$

- $(\mathbb{X}^1, \mathbb{X}^2)$ are generalized monomials, (h, h^1) are generalized derivatives
- The above expression is a generalized Taylor expansion of I

This construction gives a robust definition of the product

$$h \dot{X} := \dot{I},$$

where $X_t - X_s = \mathbb{X}^1_{st}$.

Controlled differential equations

Let $X : [0,T] \to \mathbb{R}^d$ of class C^1 and $\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ be smooth so that the ODE

$$Z_t = Z_0 + \int_0^t \sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s$$

has a unique solution for all $Z_0 \in \mathbb{R}^k$.

We want to give a meaning to this equation for X only Hölder and cover the Itô and Stratonovich cases. Note that we have the generalized expansion

$$Z_t - Z_s - \sigma(Z_s)(X_t - X_s) = \int_s^t (\sigma(Z_r) - \sigma(Z_s))\,\dot{X}_r\,\mathrm{d}r,$$

which is at least $o(t-s)$. If we want to continue the expansion then we note that:

Controlled differential equations

$$\sigma(Z_t) - \sigma(Z_s) = \int_s^t \frac{\mathrm{d}}{\mathrm{d}r}\sigma(Z_r)\,\mathrm{d}r = \int_s^t \nabla\sigma(Z_r)\frac{\mathrm{d}}{\mathrm{d}r}Z_r\,\mathrm{d}r$$
$$= \int_s^t \nabla\sigma(Z_r)\sigma(Z_r)\,\dot{X}_r\,\mathrm{d}r = \int_s^t \sigma_2(Z_r)\,\dot{X}_r\,\mathrm{d}r,$$

where $\sigma_2 := \nabla\sigma\sigma$, and therefore

$$\sigma(Z_t) - \sigma(Z_s) - \sigma_2(Z_s)(X_t - X_s) = \int_s^t (\sigma_2(Z_r) - \sigma_2(Z_s))\,\dot{X}_r\,\mathrm{d}r.$$

Finally this suggests:

$$Z_t - Z_s - \sigma(Z_s)(X_t - X_s) - \sigma_2(Z_s)\int_s^t (X_r - X_s)\otimes\dot{X}_r\,\mathrm{d}r$$
$$= \int_s^t (\sigma(Z_r) - \sigma(Z_s) - \sigma_2(Z_s)(X_r - X_s))\,\dot{X}_r\,\mathrm{d}r.$$

Taylor expansion

If we introduce the family of generalised monomials

$$\mathbb{X}^1_{st} := X_t - X_s, \qquad \mathbb{X}^2_{st} := \int_s^t (X_r - X_s) \otimes \dot{X}_r \, \mathrm{d}r,$$

then the controlled equation implies the generalised Taylor expansion

$$Z_t - Z_s = \sigma(Z_s)\, \mathbb{X}^1_{st} + \sigma_2(Z_s)\, \mathbb{X}^2_{st} + o(t-s).$$

The pair $(\mathbb{X}^1, \mathbb{X}^2)$ satisfies for $s \le u \le t$

$$\underbrace{\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut},}_{\text{Chen relation}} \qquad |\mathbb{X}^1_{st}| \lesssim |t-s|, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^2.$$

If X is not differentiable, one can choose a $\mathbb{X}^2$.

Rough paths

We fix a Rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ where

$$\underbrace{\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}}_{\text{Chen relation}}, \qquad |\mathbb{X}^1_{st}| \lesssim |t-s|^{\alpha}, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}$$

and we study (following Davie) the finite difference equation

$$Z_t - Z_s = \sigma(Z_s)\,\mathbb{X}^1_{st} + \sigma_2(Z_s)\,\mathbb{X}^2_{st} + o(t-s), \qquad \sigma_2 := \nabla\sigma\,\sigma.$$

With the Sewing Bound one can prove

- A priori estimates on Z
- Uniqueness of Z
- Local / global existence of Z
- Continuity of the map $(Z_0, \mathbb{X}^1, \mathbb{X}^2) \mapsto Z$.
- Continuous extension of the map $C^1 \ni X \mapsto Z$ to include the Itô SDE $(\mathbb{B}^1, \mathbb{B}^2) \mapsto Z$.

Itô SDEs

Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d$ and $\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ be smooth, so that the SDE

$$Z_t = Z_0 + \underbrace{\int_0^t \sigma(Z_s)\, \mathrm{d}B_s}_{\text{Itô integral}}$$

has a unique solution for all $Z_0 \in \mathbb{R}^k$.

If we differentiate both sides of the equation, we realize that it can be written as

$$\dot{Z}_t = \sigma(Z_t)\, \dot{B}_t$$

where however the product $\sigma(Z_t)\, \dot{B}_t$ is ill-defined.

Classically, one needs that the sum of the regularities of the factors in a product to be positive in order for the product to be well-defined. Here we have $\frac{1}{2} - \kappa - \frac{1}{2} - \kappa = -2\kappa < 0$.

Pathwise Itô SDEs

$$Z_t = Z_0 + \underbrace{\int_0^t \sigma(Z_s)\, \mathrm{d}B_s}_{\text{Itô integral}}$$

Theorem (Lyons 98, Davie 07)

A.s. $(Z_t)_{t\geq 0}$ *is the only continuous function* $Y : [0,\infty) \to \mathbb{R}^k$ *such that* $Y_0 = Z_0$ *and*

$$Y_t - Y_s - \sigma(Y_s)(B_t - B_s) - \sigma_2(Y_s) \underbrace{\int_s^t (B_r - B_s) \otimes \mathrm{d}B_r}_{\textit{Itô integral}} = o(t-s)$$

(where $\sigma_2 := \nabla\sigma \cdot \sigma$*) uniformly over* $0 \leq s \leq t \leq T$ *for any* $T \geq 0$.

Pathwise Stratonovich SDEs

Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d$ and $\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ smooth, so that the SDE

$$\bar{Z}_t = \bar{Z}_0 + \underbrace{\int_0^t \sigma(\bar{Z}_s) \circ \mathrm{d}B_s}_{\text{Stratonovich integral}}$$

has a unique solution for all $\bar{Z}_0 \in \mathbb{R}^k$.

Theorem (Lyons 98, Davie 07)

A.s. $(\bar{Z}_t)_{t\geq 0}$ is the only continuous function $\bar{Y} : [0,\infty) \to \mathbb{R}^k$ such that $\bar{Y}_0 = \bar{Z}_0$ and

$$\bar{Y}_t - \bar{Y}_s - \sigma(\bar{Y}_s)(B_t - B_s) - \sigma_2(\bar{Y}_s)\underbrace{\int_s^t (B_r - B_s) \otimes \circ\, \mathrm{d}B_r}_{\textit{Stratonovich integral}} = o(t-s)$$

uniformly over $0 \leq s \leq t \leq T$ for any $T \geq 0$.

Itô vs Stratonovich

Note that

$$Z_t = Z_0 + \underbrace{\int_0^t \sigma(Z_s)\,\mathrm{d}B_s}_{\text{Itô integral}},$$

$$\bar Z_t = \bar Z_0 + \underbrace{\int_0^t \sigma(\bar Z_s)\circ \mathrm{d}B_s}_{\text{Stratonovich integral}} = \bar Z_0 + \underbrace{\int_0^t \sigma(\bar Z_s)\,\mathrm{d}B_s}_{\text{Itô integral}} + \underbrace{\frac{1}{2}\int_0^t \mathrm{Tr}[\sigma_2(\bar Z_s)]\,\mathrm{d}s}_{\text{Itô-Stratonovich correction}}$$

but

$$Z_t - Z_s - \sigma(Z_s)(B_t - B_s) - \sigma_2(Z_s)\int_s^t (B_r - B_s)\otimes \mathrm{d}B_r = o(t-s),$$

$$\bar Z_t - \bar Z_s - \sigma(\bar Z_s)(B_t - B_s) - \sigma_2(\bar Z_s)\int_s^t (B_r - B_s)\otimes \circ\,\mathrm{d}B_r = o(t-s).$$

We have a map Λ_σ s.t. $\quad Z = \Lambda_\sigma(\mathbb{B}^{\mathrm{Ito}}), \qquad \bar Z = \Lambda_\sigma(\mathbb{B}^{\mathrm{Strato}}).$

Wong-Zakai

Let us mollify B setting $B^\varepsilon := \rho_\varepsilon \star B$ with $(\rho_\varepsilon)_{\varepsilon>0}$ a family of mollifiers, and let us set

$$Z_t^\varepsilon = Z_0 + \int_0^t \sigma(Z_s^\varepsilon)\,\dot{B}_s^\varepsilon\,\mathrm{d}s.$$

Then Z^ε converges (in probability) to the solution Z of the Stratonovich SDE

$$Z_t = Z_0 + \int_0^t \sigma(Z_s) \circ \mathrm{d}B_s = Z_0 + \int_0^t \sigma(Z_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \mathrm{Tr}[\sigma_2(Z_s)]\,\mathrm{d}s,$$

the famous Itô-Stratonovich correction.

This convergence can be proved and explained with the above theory, showing that

$$\int_s^t (B_r^\varepsilon - B_s^\varepsilon) \otimes \dot{B}_r^\varepsilon\,\mathrm{d}r \to \int_s^t (B_r - B_s) \otimes \circ\,\mathrm{d}B_r.$$

Chapter 2: Stochastic sewing

Sewing and stochastic processes

Until now we have worked pathwise, namely fixing $\omega \in \Omega$.

The Sewing bound/lemma hold also for germs/integrals/remainders taking values in a Banach space, for example $L^p(\Omega)$.

However there is a more interesting idea, contained in the Stochastic Sewing Lemma by Khoa Lê (2020).

We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and we denote $\mathbb{E}^s := \mathbb{E}(\cdot \,|\, \mathcal{F}_t)$.

We use the heat semigroup $(P_t)_{t\geq 0}$ on $\mathbb{R}^d$: for a bounded measurable $f : \mathbb{R}^d \to \mathbb{R}$

$$P_t f(x) = \int_{\mathbb{R}^d} f(y)\, \mathcal{N}(x, t\,\mathrm{Id})(\mathrm{d}y), \qquad x \in \mathbb{R}^d,\ t \geq 0,$$

and Hölder-Besov spaces C^γ for $\gamma \in \mathbb{R}$, such that for $\gamma < \alpha$

$$\|P_t f\|_{C^\alpha} \lesssim t^{\frac{\gamma-\alpha}{2}} \|f\|_{C^\gamma}, \qquad t > 0.$$

The stochastic sewing lemma

Theorem

Let $p \geq 2$*. Suppose* $A : \Omega \times [0,T]^2_{\leqslant} \to \mathbb{R}$ *satisfies*

1. A_{st} *is* $\mathcal{F}_t$*-adapted and in* L^p *for all* $0 \leq s \leq t \leq T$
2. $\|\delta A_{sut}\|_{L^p} \leq \Gamma_1 |t-s|^{\gamma_1}$, $\gamma_1 \in \left(\frac{1}{2}, 1\right]$
3. $\|\mathbb{E}^s \delta A_{sut}\|_{L^p} \leq \Gamma_2 |t-s|^{\gamma_2}$, $\gamma_1 > 1$

Then among the infinitely many adapted processes $I : \Omega \times [0,T] \to \mathbb{R}$ *in* L^p *such that*

$$\|I_t - I_s - A_{st}\|_{L^p} \lesssim |t-s|^{\gamma_1},$$

there is one and only one choice satisfying moreover

$$\|\mathbb{E}^s(I_t - I_s - A_{st})\|_{L^p} \lesssim |t-s|^{\gamma_2}.$$

Moreover

$$\|I_t - I_s\|_{L^p} \lesssim \|A_{st}\|_{L^p} + \Gamma_1 |t-s|^{\gamma_1} + \Gamma_2 |t-s|^{\gamma_2}.$$

An example

Let us consider $A_{st} = h_s(B_t - B_s)$ where B is a BM and h is adapted, with

$$\|h\|_{C^\alpha([0,T];L^p)} := \sup_{0 \le s < t \le T} \frac{\|h_t - h_s\|_{L^p}}{|t-s|^\alpha} < +\infty,$$

for $\alpha \in (0,1)$. We know that

$$\delta A_{sut} = -\delta h_{su}\, \delta B_{ut}$$

so that by independence of δB_{ut} and δh_{su}

$$\|\delta A_{sut}\|_{L^p} = \|\delta h_{su}\|_{L^p} \|\delta B_{ut}\|_{L^p} \lesssim |u-s|^\alpha |t-u|^{\frac{1}{2}} \le |t-s|^{\frac{1}{2}+\alpha},$$

$$\mathbb{E}^s(\delta A_{sut}) = 0.$$

Therefore $\gamma_1 = \frac{1}{2} + \alpha$ and $\gamma_2 = +\infty$. Then $\mathbb{E}^s(I_t - I_s - A_{st}) = 0$, and

$$\mathbb{E}^s(I_t - I_s) = \mathbb{E}^s(A_{st}) = 0,$$

namely I is a martingale. The stochastic rough integral is equal to the Itô integral.

An application

Let B be a BM with values in $\mathbb{R}^d$ and f a distribution on $\mathbb{R}^d$. We want to give a meaning to the process

$$\int_0^t f(B_s)\,\mathrm{d}s, \qquad t \geq 0.$$

By differentiating with respect to time, this allows to give a meaning to the random distribution $(f(B_t))_{t\geq 0}$.

The essential property is that the conditional variance of B_t given $\mathcal{F}_s$ is positive for all $s < t$. It is crucial that noise is injected in the system all along the trajectory of B.

Now we can set

$$A_{st} = \int_s^t \mathbb{E}^s(f(B_r))\,\mathrm{d}r = \int_s^t P_{r-s}f(B_s)\,\mathrm{d}r$$

where P is the heat semigroup. Then

$$\delta A_{sut} = \int_u^t \left(P_{r-s}f(B_s) - P_{r-u}f(B_u)\right)\,\mathrm{d}r.$$

An application

Recall the estimate

$$\|P_t f\|_{C^\alpha} \lesssim t^{\frac{\gamma-\alpha}{2}} \|f\|_{C^\gamma}, \qquad t > 0.$$

Then we have for $\alpha = 0$

$$\begin{aligned} |\delta A_{sut}| &\lesssim \int_u^t \left[(r-s)^{\frac{\gamma}{2}} + (r-u)^{\frac{\gamma}{2}} \right] \mathrm{d}r \, \|f\|_{C^\gamma} \\ &\lesssim (t-s)^{1+\frac{\gamma}{2}} \|f\|_{C^\gamma} \end{aligned}$$

while

$$(\delta A_{sut}) = \int_u^t \mathbb{E}^s \left[\mathbb{E}^s(f(B_r)) - \mathbb{E}^u(f(B_r)) \right] \mathrm{d}r = 0.$$

If $\gamma > -1$, then $\gamma_1 := 1 + \frac{\gamma}{2} > \frac{1}{2}$. Moreover $\gamma_2 = +\infty$.

Note that for $d = 1$, $\gamma > -1$ does not cover $f = \delta_0$, which would lead to local times. On the other hand, this construction holds for any dimension d.

The previous construction can be used to solve SDEs with distributional coefficients and additive noise. Again, B is a BM in $\mathbb{R}^d$ and b is a distribution on $\mathbb{R}^d$.

$$X_t = X_0 + \int_0^t b(X_s)\,\mathrm{d}s + B_t.$$

Note that without noise the equation makes no sense.

Again, the essential property is that the conditional variance of B_t given $\mathcal{F}_s$ is positive for all $s < t$. It is crucial that noise is injected in the system all along the trajectory of B.

This phenomenon is called regularisation by noise.

Noise is your friend.

SDEs with distributional coefficients

$$X_t = X_0 + \int_0^t b(X_s)\,\mathrm{d}s + B_t.$$

Setting $Y_t := X_t - B_t$

$$Y_t = X_0 + \int_0^t b(B_s + Y_s)\,\mathrm{d}s.$$

We define the germ

$$A_{st} = \int_s^t \mathbb{E}^s b(B_r + Y_s)\,\mathrm{d}r = \int_s^t P_{r-s} b(B_s + Y_s)\,\mathrm{d}r,$$

and we compute
$$\delta A_{sut} = \int_u^t \left[P_{r-s} b(B_s + Y_s) - P_{r-u} b(B_u + Y_u)\right]\,\mathrm{d}r,$$

$$\mathbb{E}^s \delta A_{sut} = \int_u^t \mathbb{E}^s \left[\mathbb{E}^u b(B_r + Y_s) - \mathbb{E}^u b(B_r + Y_u)\right]\,\mathrm{d}r.$$

SDEs with distributional coefficients

Now we must estimate: since $A_{st} = \int_s^t P_{r-s} b(B_s + Y_s)\, \mathrm{d}r$,

$$\begin{aligned}
\|A_{st}\|_{L^p} &\lesssim \int_s^t \|P_{r-s} b(B_s + Y_s)\|_{C^0}\, \mathrm{d}r \\
&\lesssim \int_s^t |r-s|^{\frac{\gamma}{2}}\, \mathrm{d}r\, \|b\|_{C^\gamma} \\
&\lesssim |t-s|^{1+\frac{\gamma}{2}} \|b\|_{C^\gamma}.
\end{aligned}$$

Analogously, since

$$\begin{aligned}
\delta A_{sut} &= \int_u^t [P_{r-s} b(B_s + Y_s) - P_{r-u} b(B_u + Y_u)]\, \mathrm{d}r, \\
\|\delta A_{sut}\|_{L^p} &\lesssim \int_s^t |r-s|^{\frac{\gamma}{2}}\, \mathrm{d}r\, \|b\|_{C^\gamma} \lesssim |t-s|^{1+\frac{\gamma}{2}} \|b\|_{C^\gamma},
\end{aligned}$$

at least if $\gamma > -2$. Here $\gamma_1 = 1 + \frac{\gamma}{2}$.

Since

$$\mathbb{E}^s \delta A_{sut} = \int_u^t \mathbb{E}^s \left[P_{r-u} b(B_u + Y_s) - P_{r-u} b(B_u + Y_u) \right] \mathrm{d}r,$$

we have

$$\begin{aligned} |\mathbb{E}^s \delta A_{sut}| &\lesssim \int_u^t \|P_{r-u} b\|_{C^1} \,\mathrm{d}r \, \mathbb{E}^s |Y_u - Y_s| \\ &\lesssim |t-u|^{1+\frac{\gamma-1}{2}} \, \mathbb{E}^s |Y_u - Y_s|, \end{aligned}$$

and for $\alpha \in (0,1)$ to be chosen later,

$$\begin{aligned} \|\mathbb{E}^s \delta A_{sut}\|_{L^p} &\lesssim |t-u|^{1+\frac{\gamma-1}{2}} \, \|Y_u - Y_s\|_{L^p} \\ &\leq |t-u|^{1+\frac{\gamma-1}{2}} |u-s|^{\alpha} \, \frac{\|Y_u - Y_s\|_{L^p}}{|u-s|^{\alpha}} \\ &\lesssim |t-s|^{\frac{1+\gamma}{2}+\alpha} \, \|Y\|_{C^\alpha([0,T];L^p)}. \end{aligned}$$

In this case $\gamma_2 = \frac{1+\gamma}{2} + \alpha < +\infty$.

SDEs with distributional coefficients

Now we need $\gamma_1 = 1 + \frac{\gamma}{2} > \frac{1}{2}$ and $\gamma_2 = \frac{1+\gamma}{2} + \alpha > 1$, namely $\gamma > -1$ and $\alpha > \frac{1-\gamma}{2}$.

Then by the Stochastic Sewing Lemma:

$$\|I_t - I_s\|_{L^p} \lesssim \|A_{st}\|_{L^p} + \Gamma_1|t-s|^{\gamma_1} + \Gamma_2|t-s|^{\gamma_2},$$

which gives

$$\|Y_t - Y_s\|_{L^p} \leq \Gamma\left(|t-s|^{1+\frac{\gamma}{2}} + |t-s|^{\frac{1+\gamma}{2}+\alpha}\,\|Y\|_{C^\alpha([0,T];L^p)}\right).$$

In particular

$$\|Y\|_{C^{1+\frac{\gamma}{2}}([0,T];L^p)} \leq \Gamma\left(1 + T^{\frac{1+\gamma}{2}}\,\|Y\|_{C^\alpha([0,T];L^p)}\right).$$

SDEs with distributional coefficients

$$\|Y\|_{C^{1+\frac{\gamma}{2}}([0,T];L^p)} \leq \Gamma\left(1 + T^{\frac{1+\gamma}{2}}\,\|Y\|_{C^{\alpha}([0,T];L^p)}\right).$$

Now we choose $\alpha = 1 + \frac{\gamma}{2}$. Note that $1 + \frac{\gamma}{2} > \frac{1-\gamma}{2}$ is equivalent to $\gamma > -\frac{1}{2}$, a further restriction which is needed.

Then

$$\|Y\|_{C^{1+\frac{\gamma}{2}}([0,T];L^p)} \leq \Gamma\left(1 + T^{\frac{1+\gamma}{2}}\,\|Y\|_{C^{1+\frac{\gamma}{2}}([0,T];L^p)}\right).$$

If $\Gamma T^{\frac{1+\gamma}{2}} \leq \frac{1}{2}$, we obtain the (local) a priori estimate

$$\|Y\|_{C^{1+\frac{\gamma}{2}}([0,T];L^p)} \leq 2\Gamma.$$