EXAMEN 4th of May 2026

Rough Paths

M2 Probabilités and Modèles Aléatoires

First, some comments :

(1) While grading, I realized a problem in question 4 of exercise 2, also thanks to the doubts of some of you. Now it is corrected, I beg your pardon.

(2) One of the main aims of the course is to learn how to use the sewing bound. A frequent mistake consists in claiming that: if $|\delta R_{sut}| \lesssim |t-s|^\eta$ avec $\eta > 1$, then $|R_{st}| \lesssim |t-s|^\eta$. This is wrong: just consider $R_{st} = t - s$, to find $\delta R \equiv 0$ and

$$\sup_{0<s<t<1} \frac{t-s}{|t-s|^\eta} = +\infty.$$

The statement of the sewing bound is: IF $R_{st} = o(t-s)$, then $\|R\|_\eta \leq K_\eta \|\delta R\|_\eta$ for all $\eta > 1$. The assumption $R_{st} = o(t-s)$ must be checked separately.

In exercise 2, this is very important:

(a) In question 2, the only tool I know to prove that $\|\mathbb{B}^2\|_{2\alpha} < +\infty$ and $\|\mathbb{B}^3\|_{3\alpha} < +\infty$ is Theorem 4.1 of the lecture notes.

(b) In question 4, we want to prove $\|R^n\|_{3\alpha} \lesssim \|\delta R^n\|_{3\alpha}$, and in order to use the sewing bound we must first check that $R^n_{st} = o(t-s)$: few among you have done so (see below for a complete argument).

(c) In question 6 one can prove that $\|R^Y\|_{3\alpha} < +\infty$ without the sewing bound.

(d) In question 10, we do not need to use the sewing bound.

(e) In questions 9 and 12 we must apply the sewing bound but first we need to show that $\mathbb{B}^n_{st} = o(t-s)$, respectively $Y^{[n]}_{st} = o(t-s)$.

(3) In question 2 of exercise 2, we can't prove that $|\mathbb{B}^3_{st}| \lesssim |t-s|^{3\alpha}$ only using that $|\mathbb{B}^2_{st}| \lesssim |t-s|^{2\alpha}$ and $\mathbb{B}^3_{st} = \int_s^t \mathbb{B}^2_{sr}\, \mathrm{d}B_r$: the only proof that I know is based on point 3 of Theorem 4.1, applied to the remainder $J^{(3)}_t - J^{(3)}_s - J^{(2)}_s \mathbb{B}^1_{st} - J^{(1)}_s \mathbb{B}^2_{st}$.

Durée 3h.

Throughout the exam we consider a standard Brownian motion $B = (B_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$.

Exercise 1. Set $Z_t := \exp\left(\lambda B_t - \frac{\lambda^2 t}{2}\right)$ for $\lambda \in \mathbb{R}$ fixed.

(1) Show that Z solves the Itô SDE:

$$Z_t = 1 + \int_0^t \lambda\, Z_s\, \mathrm{d}B_s, \qquad t \geq 0. \tag{1}$$

(2) What rough equation does Z solve? Give the local development of Z with a $o(t-s)$-remainder with respect to an appropriate rough path.

Exercise 2. Set for $n \geq 0$ and $0 \leq s \leq t$

$$\mathbb{B}^0_{st} := 1, \qquad \mathbb{B}^{n+1}_{st} := \int_s^t \mathbb{B}^n_{sr}\, \mathrm{d}B_r,$$

$$J^{(0)}_t := 1, \qquad J^{(n+1)}_t := \int_0^t J^{(n)}_s\, \mathrm{d}B_s.$$

Fix $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and $T > 0$.

(1) Show (for example by recurrence) that for $n \geq 0$:

$$\mathbb{B}^n_{st} = J^{(n)}_t - \sum_{i=0}^{n-1} J^{(n-i)}_s\, \mathbb{B}^i_{st}$$

(the sum from 0 to −1 is null by definition).

(2) Show, using a result on the local developments of Itô integrals, that for $n = 1, 2, 3$, a.s.

$$|\mathbb{B}^n_{st}| \lesssim |t-s|^{n\alpha}, \qquad 0 \leq s \leq t \leq T.$$

(3) Set for $n \geq 0$ the remainders:

$$r^n_{st} := \delta J^{(n)}_{st} - J^{(n-1)}_s\, \mathbb{B}^1_{st}, \qquad 0 \leq s \leq t,$$

$$R^n_{st} := \delta J^{(n)}_{st} - J^{(n-1)}_s\, \mathbb{B}^1_{st} - J^{(n-2)}_s\, \mathbb{B}^2_{st}, \qquad 0 \leq s \leq t,$$

with $J^{(-1)} = J^{(-2)} := 0$. Show that

$$r^0 = R^0 \equiv 0, \qquad r^1 = R^1 \equiv 0, \qquad R^2 \equiv 0,$$

and compute δR^n as a function of r^{n-1} and $J^{(n-2)}$.

(4) Show that there exists a constant $C = C_{T,\alpha,\|\mathbb{B}^1\|_\alpha,\|\mathbb{B}^2\|_{2\alpha}} > 0$ such that

$$\|\delta J^{(n)}\|_\alpha + \|\delta J^{(n-1)}\|_\alpha + \|\delta J^{(n-2)}\|_\alpha + \|R^n\|_{3\alpha}$$
$$\leq C\left(\|\delta J^{(n-1)}\|_\alpha + \|\delta J^{(n-2)}\|_\alpha + \|\delta J^{(n-3)}\|_\alpha + \|R^{n-1}\|_{3\alpha}\right)$$

for all $n \geq 3$. Advice: use the notation $x \lesssim y$ instead of $x \leq Cy$ if the constant C depends only on $T, \alpha, \|\mathbb{B}^1\|_\alpha, \|\mathbb{B}^2\|_{2\alpha}$. Therefore we want to show that for all $n \geq 3$

$$\|\delta J^{(n)}\|_\alpha + \|\delta J^{(n-1)}\|_\alpha + \|\delta J^{(n-2)}\|_\alpha + \|R^n\|_{3\alpha}$$
$$\lesssim \|\delta J^{(n-1)}\|_\alpha + \|\delta J^{(n-2)}\|_\alpha + \|\delta J^{(n-3)}\|_\alpha + \|R^{n-1}\|_{3\alpha}.$$

(5) Show

$$\|\delta J^{(1)}\|_\alpha + \|\delta J^{(0)}\|_\alpha + \|R^1\|_{3\alpha} \lesssim 1,$$
$$\|\delta J^{(2)}\|_\alpha + \|\delta J^{(1)}\|_\alpha + \|\delta J^{(0)}\|_\alpha + \|R^2\|_{3\alpha} \lesssim 1,$$
$$\|\delta J^{(n)}\|_\alpha + \|\delta J^{(n-1)}\|_\alpha + \|R^n\|_{3\alpha} \lesssim C^n, \qquad n \in \mathbb{N}.$$

(6) Fix $\lambda \in \mathbb{R}$ with $|\lambda| < C^{-1}$ and set (with the convention $0^0 := 1$)

$$Y_t = \sum_{i=0}^{\infty} \lambda^i J_t^{(i)}, \qquad R^Y_{st} := \delta Y_{st} - \lambda Y_s \mathbb{B}^1_{st} - \lambda^2 Y_s \mathbb{B}^2_{st}.$$

Show that Y and R^Y are well-defined and that $\|R^Y\|_{3\alpha} < +\infty$.

(7) What rough equation is satisfied by Y? What Itô SDE is satisfied by Y? Give an explicit formula for Y.

(8) Show (for example by recurrence) the Chen relation for $(\mathbb{B}^n)_{n\geq 1}$:

$$\delta \mathbb{B}^n_{sut} = \sum_{k=1}^{n-1} \mathbb{B}^k_{su} \mathbb{B}^{n-k}_{ut}, \qquad n \geq 1,$$

(9) Show by recurrence and using the sewing bound that a.s.

$$|\mathbb{B}^n_{st}| \lesssim |t-s|^{n\alpha}, \qquad 0 \leq s \leq t \leq T, \qquad n \geq 4.$$

Discuss the difference between the case $n \leq 3$ and $n \geq 4$.

(10) Set for $n \geq 0$:

$$Y^{[n]}_{st} := Y_t - Y_s \sum_{i=0}^{n-1} \lambda^i \mathbb{B}^i_{st}$$

Show that for $n = 1, 2, 3$, a.s.

$$|Y^{[n]}_{st}| \lesssim |t-s|^{n\alpha}, \qquad 0 \leq s \leq t \leq T.$$

(11) Show (for example by recurrence) that for all $n \geq 1$

$$\delta Y^{[n]}_{sut} = \sum_{i=1}^{n-1} Y^{[n-i]}_{su} \lambda^i \mathbb{B}^i_{ut}.$$

(12) Show using the sewing bound that for all $n \geq 4$, a.s.

$$|Y^{[n]}_{st}| \lesssim |t-s|^{n\alpha}, \qquad 0 \leq s \leq t \leq T.$$

(13) Interpret and discuss the latter result.

Correction of Exercise 1.

(1) This is a classical application of the Itô formula.

(2) Equation (1) is a Itô SDE with diffusion coefficient $\sigma(z) = \lambda z$. By the course the rough equation is based on the coefficients σ and $\sigma_2 := \sigma'\sigma$. Here $\sigma_2(z) = \lambda^2 z$. We have therefore

$$Z_t - Z_s - \lambda Z_s(B_t - B_s) - \lambda^2 Z_s \frac{(B_t - B_s)^2 - (t-s)}{2} = o(t-s).$$

In one dimension, the Itô *rough path* is given by $\mathbb{B}^1_{st} = B_t - B_s$ and $\mathbb{B}^2_{st} = \frac{(B_t - B_s)^2 - (t-s)}{2}$.

Correction of Exercise 2.

(1) The equality holds for $n = 0$; now

$$\begin{aligned}\mathbb{B}^{n+1}_{st} &= \int_s^t \mathbb{B}^n_{sr}\,\mathrm{d}B_r = \int_s^t \left(J^{(n)}_r - \sum_{i=0}^{n-1} J^{(n-i)}_s\,\mathbb{B}^i_{sr}\right)\mathrm{d}B_r\\ &= J^{(n+1)}_t - J^{(n+1)}_s - \sum_{i=0}^{n-1} J^{(n-i)}_s\,\mathbb{B}^{i+1}_{st} = J^{(n+1)}_t - \sum_{i=0}^{n} J^{(n+1-i)}_s\,\mathbb{B}^i_{st}.\end{aligned}$$

(2) For $n = 1$ we have $|\mathbb{B}^1_{st}| = |B_t - B_s| \lesssim |t-s|^{\alpha}$. For $n = 2$ we have

$$\mathbb{B}^2_{st} = \int_s^t (B_r - B_s)\,\mathrm{d}B_r = J^{(2)}_t - J^{(2)}_s - J^{(1)}_s(B_t - B_s)$$

and by point 2 of Theorem 4.1 of the lecture notes we obtain $|\mathbb{B}^2_{st}| \lesssim |t-s|^{2\alpha}$. Finally

$$\begin{aligned}\mathbb{B}^3_{st} &= \int_s^t \left(J^{(2)}_r - J^{(2)}_s - J^{(1)}_s(B_r - B_s)\right)\mathrm{d}B_r\\ &= J^{(3)}_t - J^{(3)}_s - J^{(2)}_s\,\mathbb{B}^1_{st} - J^{(1)}_s\,\mathbb{B}^2_{st}\end{aligned}$$

and we obtain by point 3 of Theorem 4.1 of the lecture notes that a.s. $|\mathbb{B}^3_{st}| \lesssim |t-s|^{3\alpha}$.

(3) For all $n \geq 0$

$$\begin{aligned}\delta R^n_{sut} &= \left(\delta J^{(n-1)}_{su} - J^{(n-2)}_s\,\mathbb{B}^1_{su}\right)\mathbb{B}^1_{ut} + \delta J^{(n-2)}_{su}\mathbb{B}^2_{ut}\\ &= r^{n-1}_{su}\mathbb{B}^1_{ut} + \delta J^{(n-2)}_{su}\mathbb{B}^2_{ut}.\end{aligned}$$

In particular

$$\|\delta R^n\|_{3\alpha} \lesssim \|r^{n-1}\|_{2\alpha} + \|\delta J^{(n-2)}\|_{\alpha}.$$

(4) For $n \geq 3$ we have $J^{(n-1)}_0 = J^{(n-2)}_0 = 0$ and therefore

$$\begin{aligned}\|\delta J^{(n)}\|_{\alpha} &\leq \|J^{(n-1)}\|_{\infty}\|\mathbb{B}^1\|_{\alpha} + \|J^{(n-2)}\|_{\infty}\|\mathbb{B}^2\|_{\alpha} + \|R^n\|_{\alpha}\\ &\leq T^{\alpha}\|\delta J^{(n-1)}\|_{\alpha}\|\mathbb{B}^1\|_{\alpha} + T^{\alpha}\|\delta J^{(n-2)}\|_{\alpha}\|\mathbb{B}^2\|_{\alpha} + T^{2\alpha}\|R^n\|_{3\alpha}\\ &\lesssim \|\delta J^{(n-1)}\|_{\alpha} + \|\delta J^{(n-2)}\|_{\alpha} + \|R^n\|_{3\alpha}.\end{aligned}$$

We obtain

$$\begin{aligned}&\|\delta J^{(n)}\|_{\alpha} + \|\delta J^{(n-1)}\|_{\alpha} + \|\delta J^{(n-2)}\|_{\alpha} + \|R^n\|_{3\alpha}\\ &\lesssim \|\delta J^{(n-1)}\|_{\alpha} + \|\delta J^{(n-2)}\|_{\alpha} + \|R^n\|_{3\alpha}.\end{aligned}$$

Now we want to apply the sewing bound in order to bound $\|R^n\|_{3\alpha}$ from above. For this we need to show that a.s. $R^n_{st} = o(t-s)$. This follows from Theorem 4.1 of the lecture notes, with $I = J^{(n)}$, $h = J^{(n-1)}$ and $h^1 = J^{(n-2)}$, which yields $|R^n_{st}| \lesssim |t-s|^{3\alpha}$ almost surely; we also need $J^{(n-2)} \in C^{\alpha}$, which follows by recurrence.

We obtain

$$\begin{aligned}&\|\delta J^{(n)}\|_\alpha + \|\delta J^{(n-1)}\|_\alpha + \|R^n\|_{3\alpha}\\ &\lesssim \|\delta J^{(n-1)}\|_\alpha + \|\delta J^{(n-2)}\|_\alpha + \|R^n\|_{3\alpha}\\ &\lesssim \|\delta J^{(n-1)}\|_\alpha + \|\delta J^{(n-2)}\|_\alpha + \|\delta R^n\|_{3\alpha}\\ &\lesssim \|\delta J^{(n-1)}\|_\alpha + \|\delta J^{(n-2)}\|_\alpha + \|r^{n-1}\|_{2\alpha}.\end{aligned}$$

Now $\|r^{n-1}\|_{2\alpha} \le \|R^{n-1}\|_{2\alpha} + T^\alpha \|\delta J^{(n-2)}\|_\alpha \|\mathbb{B}^2\|_{2\alpha}$, therefore

$$\begin{aligned}&\|\delta J^{(n)}\|_\alpha + \|\delta J^{(n-1)}\|_\alpha + \|r^n\|_{3\alpha}\\ &\lesssim \|\delta J^{(n-1)}\|_\alpha + \|\delta J^{(n-2)}\|_\alpha + \|R^{n-1}\|_{2\alpha}.\end{aligned}$$

(5) For $n = 1, 2$

$$\|\delta J^{(0)}\|_\alpha = \|R^1\|_{2\alpha} = \|R^2\|_{2\alpha} = 0, \qquad \|\delta J^{(1)}\|_\alpha = \|\mathbb{B}^1\|_\alpha \lesssim 1,$$

$$\begin{aligned}\|\delta J^{(2)}\|_\alpha &\le T^\alpha \|\delta J^{(1)}\|_\alpha \|\mathbb{B}^1\|_\alpha + \|\mathbb{B}^2\|_\alpha\\ &\le T^\alpha (\|\mathbb{B}^1\|_\alpha)^2 + T^\alpha \|\mathbb{B}^2\|_{2\alpha} \lesssim 1.\end{aligned}$$

By the previous point

$$\|\delta J^{(i)}\|_\alpha + \|\delta J^{(i-1)}\|_\alpha + \|R^i\|_{3\alpha} \lesssim C^i, \qquad i \in \mathbb{N}.$$

(6) By the previous point

$$\|Y\|_\infty \le 1 + T^\alpha \sum_{i=1}^{\infty} \lambda^i \|\delta J^{(i)}\|_\alpha \le 1 + T^\alpha \sum_{i=1}^{\infty} \lambda^i C^i < +\infty.$$

Now:

$$\begin{aligned}R^Y_{st} &= \delta Y_{st} - \lambda Y_s\, \mathbb{B}^1_{st} - \lambda^2 Y_s\, \mathbb{B}^2_{st}\\ &= \sum_{i\ge 0} \lambda^i \left(\delta J^{(i)}_{st} - J^{(i-1)}_s\, \mathbb{B}^1_{st} - J^{(i-2)}_s\, \mathbb{B}^2_{st}\right) = \sum_{i\ge 0} \lambda^i R^i_{st}\end{aligned}$$

so that

$$\|R^Y\|_{3\alpha} \lesssim \sum_{i\ge 0} \lambda^i \|R^i\|_{3\alpha} \le \sum_{i\ge 0} (\lambda C)^i < +\infty.$$

(7) We have obtained in particular

$$Y_t - Y_s - \lambda Y_s\, \mathbb{B}^1_{st} - \lambda^2 Y_s\, \mathbb{B}^2_{st} = o(t-s)$$

which corresponds to a Itô *rough* equation, with coefficient $\sigma(y) = \lambda y$, so that $\sigma_2(y) = \sigma'\sigma(y) = \lambda^2 y$. The corresponding SDE is

$$Y_t = 1 + \int_0^t \lambda Y_s\, \mathrm{d}B_s$$

which coincides with the SDE (1) and therefore has Z as unique solution.

(8) The case $n = 0$ is $\delta \mathbb{B}^1 = 0$, for the recurrence we find:

$$\begin{aligned}
\delta \mathbb{B}^{n+1}_{sut} &= \int_u^t (\mathbb{B}^n_{sr} - \mathbb{B}^n_{ur}) \, \mathrm{d}B_r \\
&= \int_u^t \delta \mathbb{B}^n_{sur} \, \mathrm{d}B_r + \mathbb{B}^n_{su} \, \mathbb{B}^1_{ut} \\
&= \sum_{k=1}^{n-1} \mathbb{B}^k_{su} \int_u^t \mathbb{B}^{n-k}_{ur} \, \mathrm{d}B_r + \mathbb{B}^n_{su} \, \mathbb{B}^1_{ut} \\
&= \sum_{k=1}^{n-1} \mathbb{B}^k_{su} \, \mathbb{B}^{n+1-k}_{ut} + \mathbb{B}^n_{su} \, \mathbb{B}^1_{ut} = \sum_{k=1}^{n} \mathbb{B}^k_{su} \, \mathbb{B}^{n+1-k}_{ut}.
\end{aligned}$$

(9) For $n = 4$ we obtain and $Y^{[3]}_{st} = o(t-s)$, $\mathbb{B}^3_{st} = o(t-s)$ by point 6. For the general case, we argue by recurrence. First for $n \geq 3$

$$\delta \mathbb{B}^{n+1}_{sut} = \sum_{k=1}^{n} \mathbb{B}^k_{su} \, \mathbb{B}^{n+1-k}_{ut}$$

et by recurrence l'expression à droite est $\lesssim |t-s|^{(n+1)\alpha}$. For utiliser la sewing bound il faut vérifier that $\mathbb{B}^{n+1}_{st} = o(t-s)$. For cela on sait that

$$\begin{aligned}
\mathbb{B}^{n+1}_{st} &= \delta J^{(n+1)}_{st} - J^{(n)}_s \, \mathbb{B}^1_{st} - J^{(n-1)}_s \, \mathbb{B}^2_{st} - \sum_{i=3}^{n} J^{(n+1-i)}_s \, \mathbb{B}^i_{st} \\
&= R^{n+1}_{st} - \sum_{i=3}^{n} J^{(n+1-i)}_s \, \mathbb{B}^i_{st}.
\end{aligned}$$

Or $R^{n+1}_{st} = o(t-s)$ by point 5 and $\|\mathbb{B}^i\|_{3\alpha} < +\infty$ for $3 \leq i \leq n$ by recurrence sur n.

(10) Remarquons that $Y^{[3]} \equiv R^Y$, therefore a.s. $\|Y^{[3]}\|_{3\alpha} < +\infty$. Or

$$Y^{[1]}_{st} = R^Y_{st} + \lambda Y_s \, \mathbb{B}^1_{st} + \lambda^2 Y_s \, \mathbb{B}^2_{st},$$

therefore for $n = 1, 2$

$$\|Y^{[n]}\|_{n\alpha} \leq \|R^Y\|_{n\alpha} + \sum_{i=n}^{2} \lambda^i \|Y\|_\infty \|\mathbb{B}^i\|_{n\alpha} < +\infty.$$

(11)

$$\begin{aligned}
\delta Y^{[n+1]}_{sut} &= Y_s \sum_{i=1}^{n} \lambda^i (\mathbb{B}^i_{su} - \mathbb{B}^i_{st}) + Y_u \sum_{i=1}^{n} \lambda^i \mathbb{B}^i_{ut} \\
&= -Y_s \sum_{i=1}^{n} \lambda^i \delta\mathbb{B}^i_{sut} + \delta Y_{su} \sum_{i=1}^{n} \lambda^i \mathbb{B}^i_{ut} \\
&= -Y_s \sum_{i=1}^{n} \lambda^i \sum_{k=1}^{i-1} \mathbb{B}^k_{su} \mathbb{B}^{i-k}_{ut} + \delta Y_{su} \sum_{i=1}^{n} \lambda^i \mathbb{B}^i_{ut} \\
&= -Y_s \sum_{k=1}^{n-1} \mathbb{B}^k_{su} \sum_{i=k+1}^{n} \lambda^i \mathbb{B}^{i-k}_{ut} + \delta Y_{su} \sum_{i=1}^{n} \lambda^i \mathbb{B}^i_{ut} \\
&= -Y_s \sum_{k=1}^{n-1} \mathbb{B}^k_{su} \lambda^k \sum_{i=1}^{n-k} \lambda^i \mathbb{B}^i_{ut} + \delta Y_{su} \sum_{i=1}^{n} \lambda^i \mathbb{B}^i_{ut} \\
&= \sum_{i=1}^{n} Y^{[n+1-i]}_{su} \lambda^i \, \mathbb{B}^i_{ut}
\end{aligned}$$

(12) For $n = 4$ we obtain

$$\delta Y^{[4]}_{sut} = \sum_{i=1}^{3} Y^{[4-i]}_{su} \lambda^i \, \mathbb{B}^i_{ut}$$

et dans ce cas

$$|\delta Y^{[4]}_{sut}| \lesssim |u-s|^{(4-i)\alpha} |t-u|^{i\alpha} \lesssim |t-s|^{4\alpha}.$$

La sewing bound we dit that, si $Y^{[4]}_{st} = o(t-s)$, alors

$$\|Y^{[4]}\|_{4\alpha} \leq K \|\delta Y^{[4]}\|_{4\alpha} < +\infty.$$

For prouver that $Y^{[4]}_{st} = o(t-s)$, we voyons that

$$Y^{[4]}_{st} = Y^{[3]}_{st} - \lambda^3 \, \mathbb{B}^3_{st}$$

and $Y^{[3]}_{st} = o(t-s)$ by point 10, $\mathbb{B}^3_{st} = o(t-s)$ by point 2. For the general case, we argue by recurrence similarly. By point 7 :

$$\delta Y^{[n+1]}_{sut} = \sum_{i=1}^{n} Y^{[n+1-i]}_{su} \lambda^i \, \mathbb{B}^i_{ut}.$$

By recurrence, $\|Y^{[n+1-i]}\|_{(n+1-i)\alpha} < +\infty$ and $\|\mathbb{B}^i\|_{i\alpha} < +\infty$ by point 9. We show that $Y^{[n+1]}_{st} = o(t-s)$ arguing like for $Y^{[4]}$.

(13) We have obtained a.s. a generalized Taylor expansion of the solution to the SDE (1) with an arbitrarily small remainder.