Reflected Brownian motion, transient case and asymptotics of Green functions

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Reflected Brownian Motion : Transient case

Let $(Z_t)_{t\geq 0}$ be a planar reflected Brownian motion in $\mathbb{R}_+^2$ of covariance matrix Σ , of drift μ and reflection matrix R . Thanks to [1], we can establish the theorem :

Theorem : existence and transience

A such process $Z_t = z_0 + B_t + \mu t + RL_t$ exists if and only if

$$[r_{12} > 0 \text{ and } r_{21} > 0] \text{ or } [\det(R) = r_{11}r_{22} - r_{12}r_{21} > 0].$$

Where

- B is a Brownian motion of covariance Σ
- L is a a Local Time : continuous non-decreasing process, that increases only when the process touches the boundary.

Furthermore, the process is transient if and only if

$$r_{11}\mu_1 - r_{21}\mu_2^- > 0 \text{ or } r_{12}\mu_1^- - r_{22}\mu_2 > 0.$$

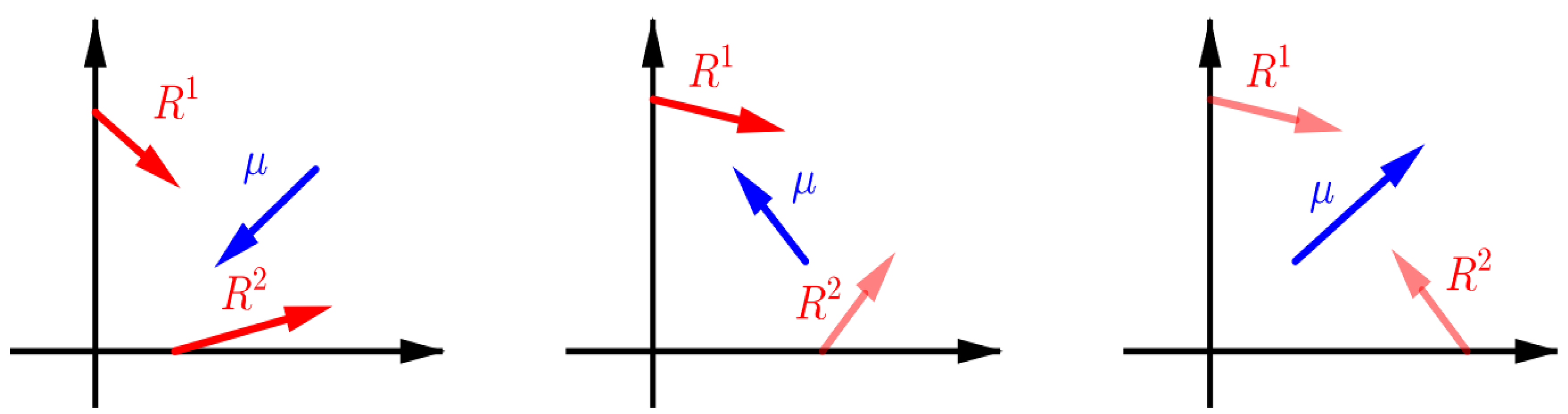
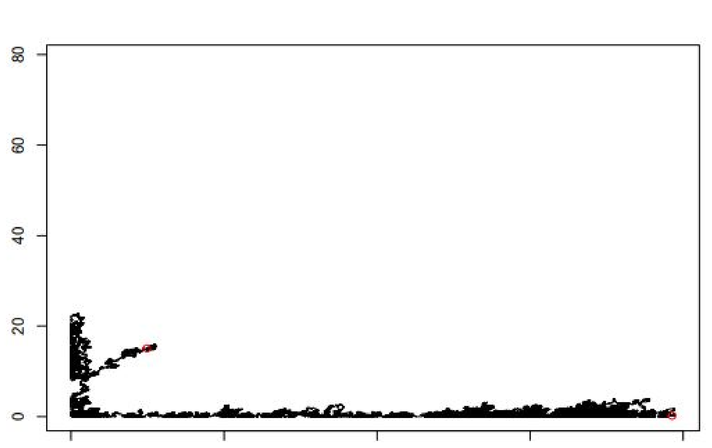
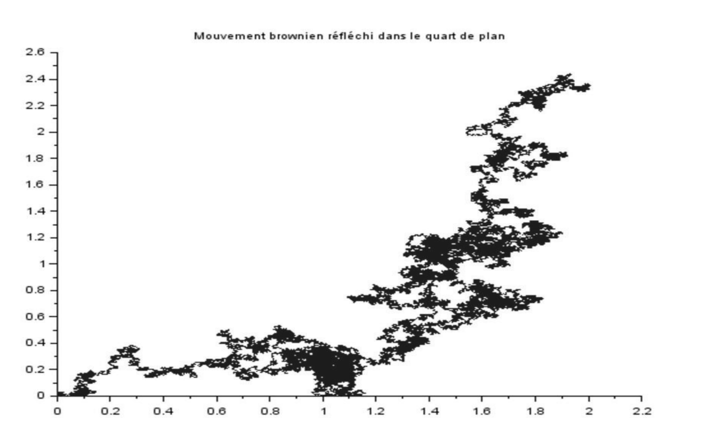


Figure 1. Examples of transient cases.

In facts, there is two possibilites to go to infinity. For the rest of the poster, we focus on the case (b) where $\mu_1, \mu_2 > 0$.



(a) Escape along an axis



(b) Both coordinates tend to infinity

Laplace inverse and functional equation

Defintion : Green measures and Laplace transforms

- Green measure inside G and its Laplace transform :

$$G(z_0, A) := \mathbb{E}_{z_0}\left[\int_0^\infty \mathbb{1}_A(Z_t)dt\right], \quad \varphi(w) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{w\cdot Z_t}dt\right].$$

- Green measures on sides H_i and its Laplace transforms : for $i \in \{1, 2\}$,

$$H_i(z_0, A) = \mathbb{E}_{z_0}\left[\int_0^\infty \mathbb{1}_A(Z_t)dL_t^i\right], \quad \varphi_i(w) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{w\cdot Z_t}dL_t^i\right].$$

The Ito formula, a sign argue and methods of [3] and [2] give the :

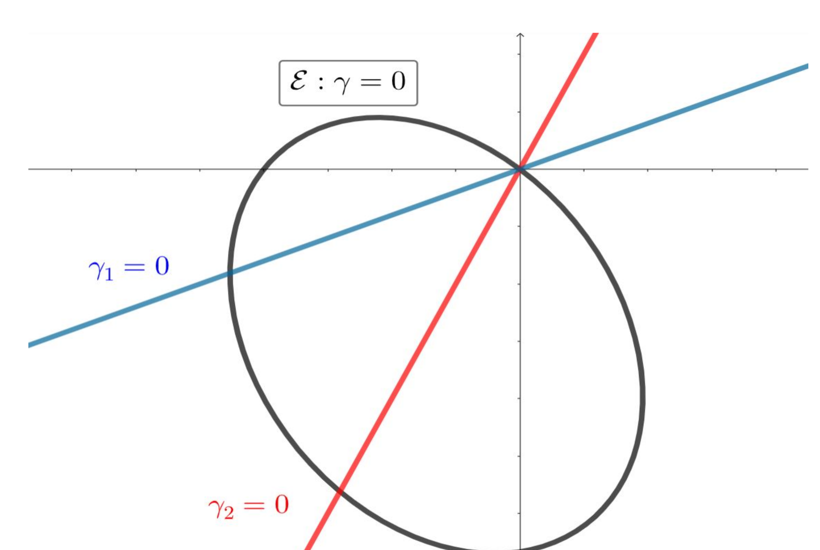
Theorem : functional equation

For $w = (x, y) \in \mathbb{C}^2$ satisfying $\Re(x) < 0$ and $\Re(y) < 0$,

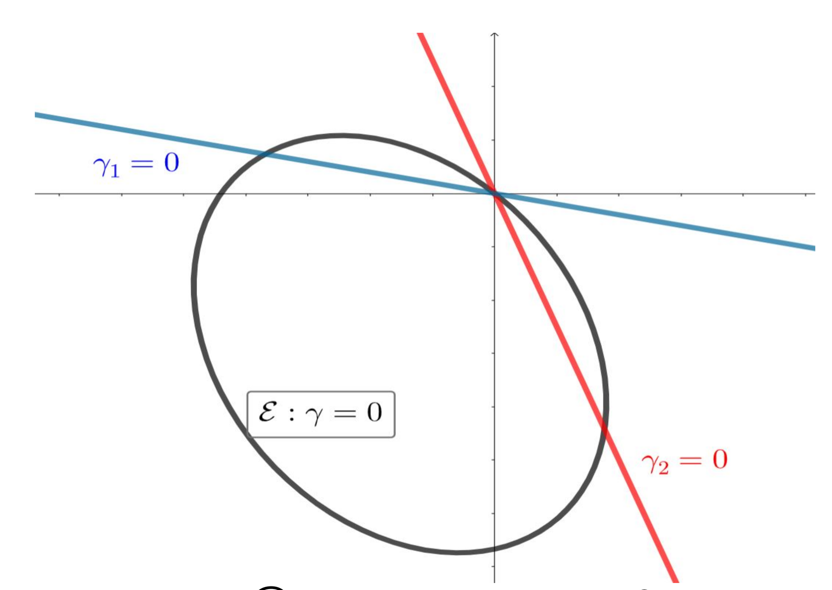
$$-\gamma(w)\varphi(w) = \gamma_1(w)\varphi_1(y) + \gamma_2(w)\varphi_2(x) + e^{w\cdot z_0} \quad (1)$$

where $\gamma(w) = \frac{1}{2}w\cdot\Sigma w + w\cdot\mu$, $\gamma_1(w) = R^1\cdot w$, $\gamma_2(w) = R^2\cdot w$.

- **By vanishing γ in (1) φ_2 can be meromorphically extended on $\mathbb{C}\setminus[x_{max}, +\infty[$ (same for φ_1, with y_{max} instead).**



(a) Case $r_{12}, r_{21} < 0$



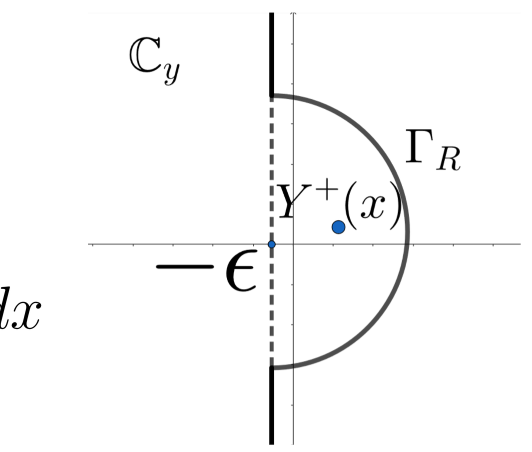
(b) Case $r_{12}, r_{21} > 0$

- By Laplace inverse formula, if g denotes the density of G according to the Lebesgue measure, we get : $g(a, b) =$

$$\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \frac{\gamma_1(x,y)\varphi_1(y) + \gamma_2(x,y)\varphi_2(x) + e^{-(x_0x+y_0y)}}{\gamma(x,y)} e^{-ax-by}\frac{dxdy}{(2i\pi)^2}$$

$$= I_1 + I_2 + I_3$$

- **Residue theorem : from double integral to simple integral.**



$$I_1 = \frac{1}{2i\pi}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \frac{\gamma_2(x, Y^+(x))\varphi_2(x)}{\partial_y\gamma(x, Y^+(x))}e^{-ax-bY^+(x)}dx$$

(same for I_2, I_3).

Saddle point method

Now that we have a simple integral with an exponential, we use the saddle point method.

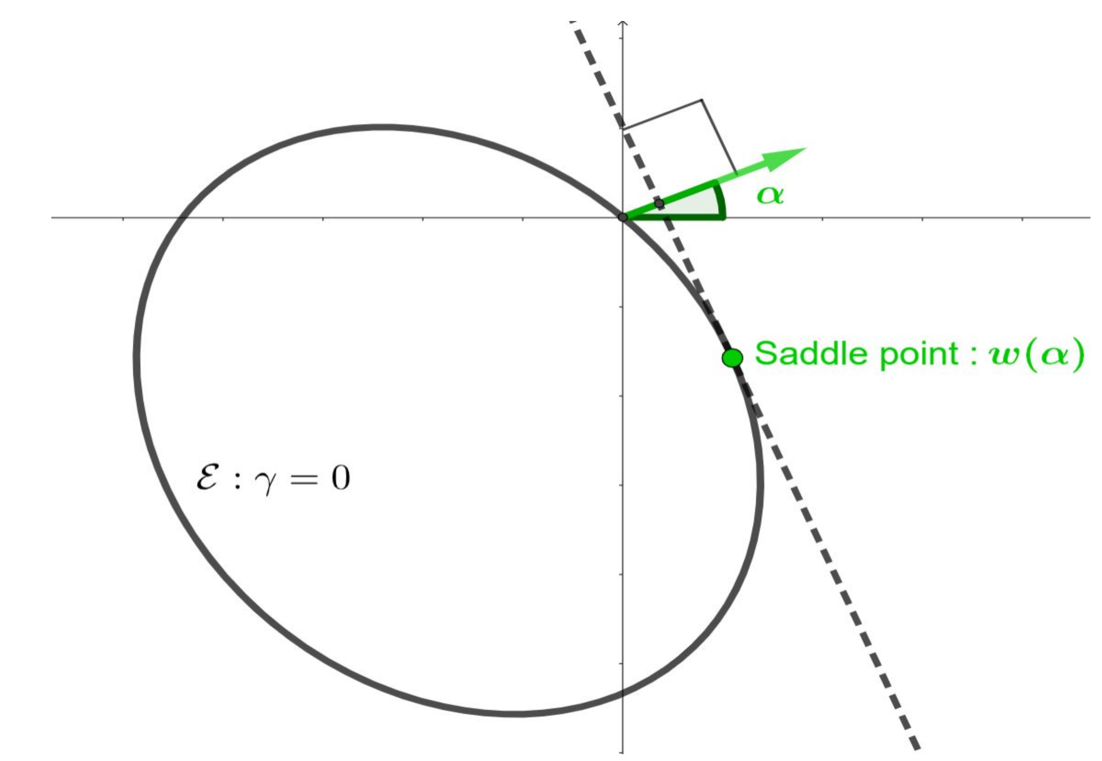


Figure 4. Graphic representation of the Saddle point $w(\alpha)$.

Expected results

- If there is no pole coming from the saddle point method,

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{r\to\infty}{\sim} \frac{c(\alpha_0)}{\sqrt{r}}e^{-r\langle w(\alpha), e_\alpha\rangle}.$$

- If there is a pole,

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{r\to\infty}{\sim} de^{-r\langle \tilde{w}(\alpha), e_\alpha\rangle}.$$

where $e_\alpha = (\cos(\alpha), \sin(\alpha))$.

References

[1] Recurrence Hobson D.G. Rogers L.C.G. and transience of reflecting Brownian motion in the quadrant.

[2] Asymptotic expansion of stationary distribution for reflected Brownian motion in the quarter plane via analytic approach Stochastic Systems Volume 7 Number 1 pages 32-94 2017 Irina Kourkova, Sandro Franceschi.

[3] Journal of Theoretical Probability Springer Sandro Franceschi, Green's Functions with Oblique Neumann Boundary Conditions in the Quadrant.