Martin Boundary of a degenerate Reflected Brownian Motion in a wedge

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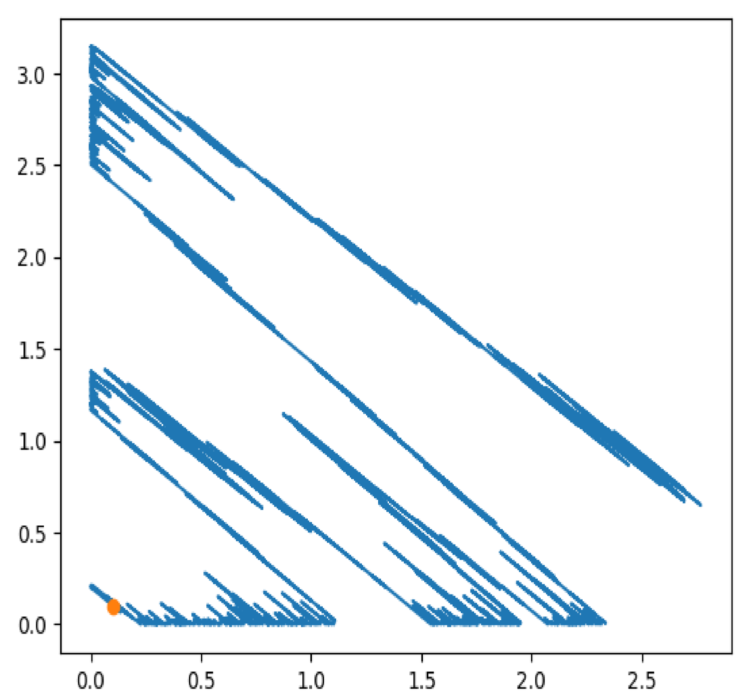
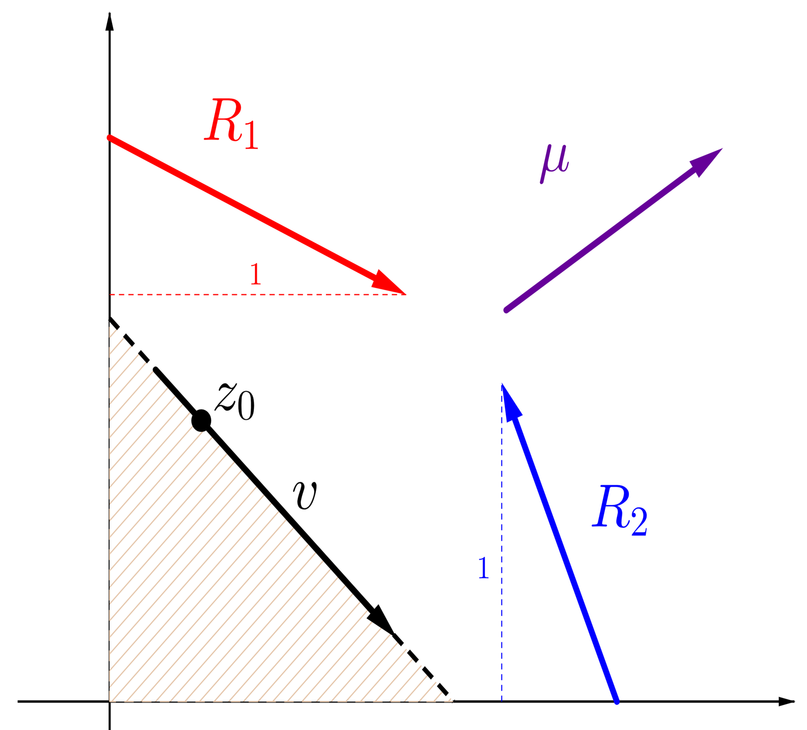
The degenerate reflected Brownian motion (DRBM)

A such process is defined as $Z_t = z_0 + vB_t + \mu t + RL_t$ where

- $(B_t)_{t\geq 0}$ is a **one dimensional** Brownian motion and $v = (v_1, v_2)$ is the **Brownian direction**
- $(L^1_t, L^2_t)_{t\geq 0}$ is the local time on the axes: it increases only when Z_t touches the boundary and $R = (R_1, R_2) = \begin{pmatrix} 1 & r_2 \\ r_1 & 1 \end{pmatrix}$ is the reflexion matrix
- $\mu = (\mu_1, \mu_2)$ is the drift.



(a) Degenerate RBM



(b) Parameters of DRBM

Hypothesis

- $\mu_1, \mu_2 > 0$: transient Markov process
- $r_1 > \frac{v_2}{v_1}$, $\quad r_2 > \frac{v_1}{v_2}$: $(Z_t)_{t\geq 0}$ can't go back to the origin

Harmonic functions

A function h is harmonic if for any $t > 0$ and z_0,

$$h(z_0) = \mathbb{E}_{z_0}[h(Z_t)].$$

Equivalently, h is harmonic if it satisfies the Boundary value problem:

$$\begin{cases} (H_0) & \mathcal{G}h = 0 \quad \text{on} \quad (0, +\infty)^2 \\ (H_1) & \partial_{R_1} h(0, y) = 0, \quad y \geq 0 \\ (H_2) & \partial_{R_2} h(x, 0) = 0, \quad x \geq 0 \end{cases} \tag{1}$$

where $\mathcal{G} = (\partial_v)^2 + \mu\nabla$.

Compensation method

Note that a function $(x, y) \longmapsto e^{ax+by}$ satisfies (H_0) if and only if $\mathcal{G}e^{ax+by} = \gamma(a, b)e^{ax+by} = 0$ i.e. $(a, b) \in \mathcal{P}$ where

$$\mathcal{P} := \{(x, y) \in \mathbb{R}^2, \gamma(x, y) := (v_1 x + v_2 y)^2 + \mu_1 x + \mu_2 y = 0\}.$$

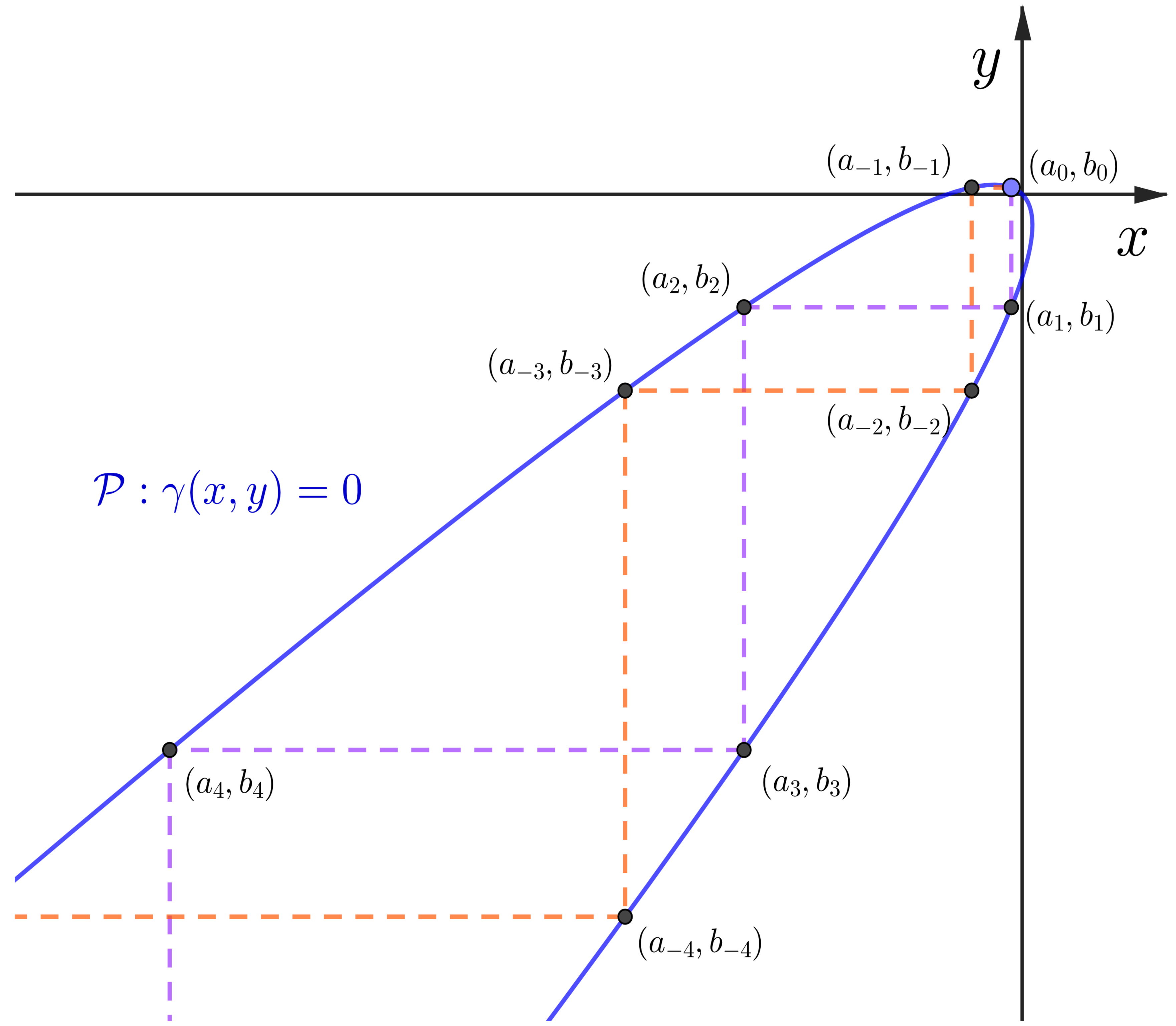


Figure 2. Parabola $\mathcal{P}$ and sequence $(a_n, b_n), n \in \mathbb{Z}$.

$$h(x, y) = \overbrace{\ldots + \underbrace{c_{-1}e^{a_{-1}x+b_{-1}y} + c_0 e^{a_0 x+b_0 y}}_{\in(H_1)}}^{\in(H_2)} + \overbrace{\underbrace{c_1 e^{a_1 x+b_1 y} + c_2 e^{a_2 x+b_2 y}}_{\in(H_1)} + \ldots}^{\in(H_2)}$$

Harmonic functions from the compensation method

Every point $(a_0, b_0) \in \mathcal{P}$ corresponds to a harmonic function :

$$h(x, y) = \overbrace{\ldots + \underbrace{c_{-1}e^{a_{-1}x+b_{-1}y} + c_0 e^{a_0 x+b_0 y}}_{\in(H_1)}}^{\in(H_2)} + \overbrace{\underbrace{c_1 e^{a_1 x+b_1 y} + c_2 e^{a_2 x+b_2 y}}_{\in(H_1)} + \ldots}^{\in(H_2)}$$

where $(c_n)_{n\in\mathbb{Z}}$ are adjusted to fulfill (H_1) and (H_2).

We index those harmonic function $(h_\alpha)_{\alpha\in[0,\pi/2]}$ taking $(a_0, b_0) = z(\alpha)$

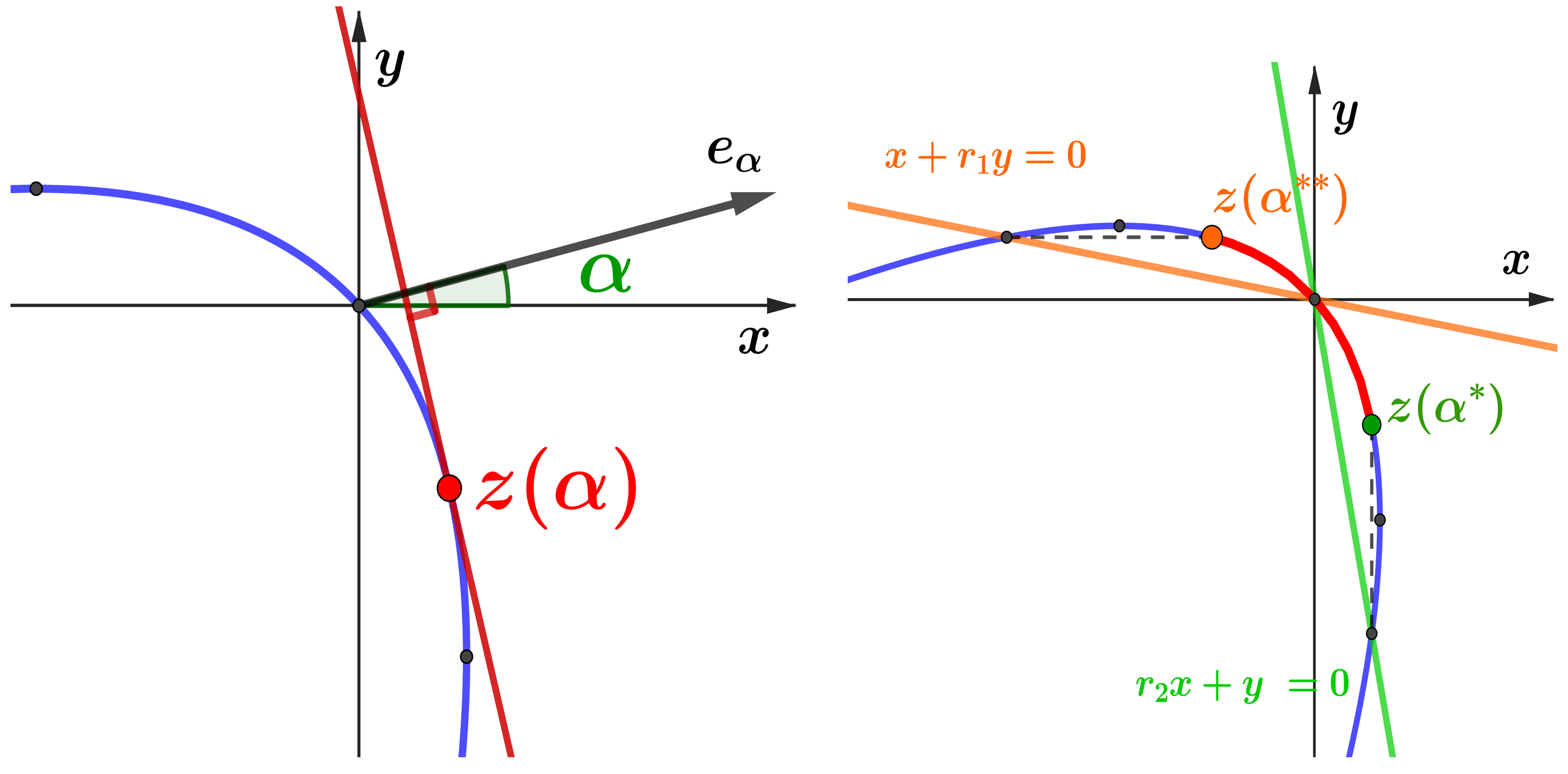


Figure 3. Definition of $z(\alpha)$, α^* and α^{**}.

Asymptotics of Green's functions and Martin Boundary

Defintion : Green's measure $G(z_0, \cdot)$ and Green's function $g(z_0, z)$

$$G(z_0, A) := \mathbb{E}_{z_0}\left[\int_0^\infty \mathbb{1}_A(Z_t)dt\right] = \iint_A g(z_0, z)dz.$$

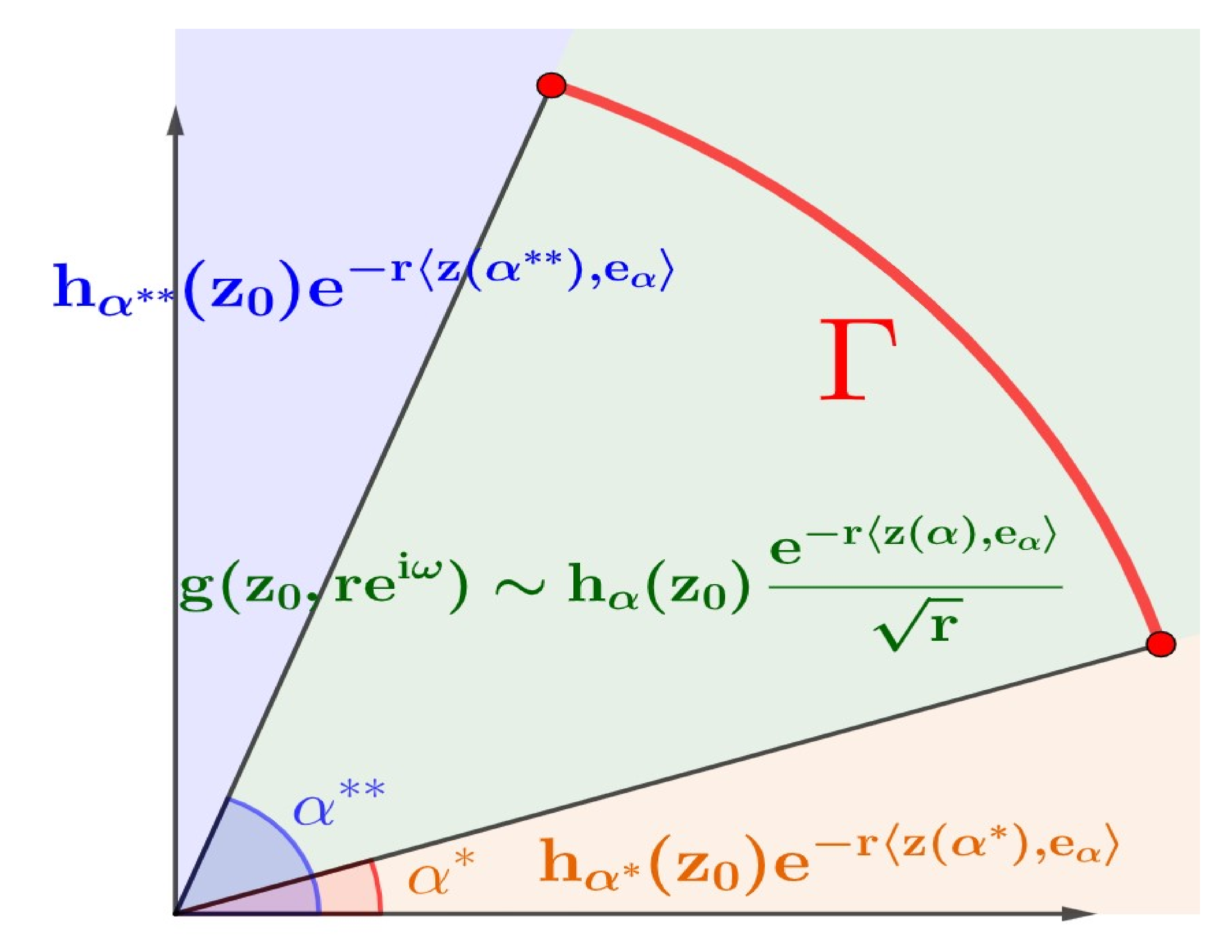


Figure 4. Asymptotics of $g(z_0, re^{i\omega})$ as $r \to \infty, \omega \to \alpha$.

Corollary

The (minimal) Martin Boundary Γ is given by $[\alpha^*, \alpha^{**}]$. Every non-negative harmonic function can be uniquely written as

$$h(z) = \int_{[\alpha^*, \alpha^{**}]} h_\alpha(z)d\mu(\alpha).$$