Thèse de Doctorat

Spécialité - **Mathématiques**

École doctorale de Sciences Mathématiques de Paris Centre

Présentée par

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Fonctions de Green et frontières de Martin de diffusions planaires transientes : une approche analytique.

Dirigée par **Irina Kourkova** et **Sandro Franceschi**

Soutenue le 1er Juin, devant le jury composé de :

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[0] Cette thèse a été financée par une bourse CDSN (contrat doctoral spécifique normalien) de l'Ecole Normale Supérieure de Rennes, et effectuée au sein du Laboratoire de Probabilités Statistique et Modélisation de Sorbonne Université.

Remerciements

Tout d'abord, je souhaiterais remercier mes directrice et directeur Irina Kourkova et Sandro Franceschi de m'avoir très bien encadré pendant ma thèse. Superviser une thèse est déjà un engagement important, mais l'investissement en temps, la réactivité, le dévouement et tout le soutien qu'ils m'ont apporté représentent pour moi un honneur et un privilège particulier. J'ai beaucoup aimé travailler avec eux et découvrir le monde de la recherche à travers nos collaborations. La franchise bienveillante et l'expérience d'Irina, combinées à la curiosité et au savoir de Sandro, ont formé un encadrement complet pour me permettre de m'épanouir pleinement dans ces trois années (et demie) de ma vie. Je suis fortement reconnaissant de leur bienveillance et de leur disponibilité pendant cette thèse et je ne regrette pas une seconde de leur avoir fait aveuglément confiance depuis le début de mon stage de M2.

Je suis profondément reconnaissant à Jim Dai et Reda Chhaibi d'avoir accepté de rapporter cette thèse ainsi que pour leurs retours. Je remercie aussi sincèrement François Bolley, Nicolas Fournier, Irina Ignatiouk-Robert, Antoine Lejay et Pierre Tarrago d'avoir accepté de faire partie de mon jury.

Merci également à tout le personnel administratif du LPSM sans qui tout ça ne pourrait pas avoir lieu, ainsi qu'à toutes les personnes présentes pour ce jour spécial – que ce soit les mathématicien.nes ou les non mathématicien.nes.

Je remercie également les chercheurs et chercheuses Nicolas Fournier, Irina Ignatiouk, Antoine Lejay, Kilian Raschel, Pierre Tarrago et Lorenzo Zambotti qui m'ont aidé à prendre du recul à travers des conversations particulièrement fructueuses autour de ma thèse.

C'est peut-être un peu cliché, mais je remercie également toutes les professeures et tous les professeurs de mathématiques, qui ont su donner goût à leur matière à un petit garçon parfois un peu dans la lune. J'ai une pensée particulière pour François-Xavier Manoury qui, entre deux trails, savait éveiller la curiosité de toutes et tous. J'ai également beaucoup de reconnaissance pour Melissa Bailloeuil et Rémi Souveton qui ont su graver les bases de ma passion et de mes connaissances à travers leur investissement et leur bienveillance, malgré le rythme soutenu de la prépa. Un merci particulier à Jean-Christophe Breton, Thomas Duquesne, Nicolas Fournier, Magalie Fromont, Thierry Lévy et Guillaume Poly d'avoir éveillé mon amour pour les probabilités.

Je souhaite remercier très chaleureusement Camille, Elias, François, Lucas, Nadège, Thomas L., Thomas J., Vladimir les amis du bureau pour tous les bons moments passés ensemble, en 16-26-131 et ailleurs. J'ai une pensée particulière également pour Alexandre B., Bastien, Charlie, Faël, Jules, Majid, Maxence, Sonia, Ulysse, et aux sportifs Anne-Laure, Léopold, Martin, Olivier, Simon, Titouan avec qui j'ai pu échanger des tops, des smashs et des coins de table autour des tables de ping pong à Jussieu. Un mot en particulier pour toi Elias qui m'as fait découvrir (entre autres !) les joies de Scriabin et Rachmaninov à travers nos nombreuses sessions piano. Tous ces moments de partage m'ont permis de me changer les idées et de développer une complicité ensemble qui perdurera après le doctorat.

J'ai une pensée particulière pour le cercle des crêpes normaliennes de Rennes : Antoine, Eloan, Jérôme, Matthieu et Tintin avec qui j'ai encore le plaisir de partager des bons moments, autour de crêpes, lors d'appels ou à la montagne. Je remercie également mon ami Thomas C. qui a guidé mes pas dans beaucoup de choses, que ce soit pour l'agrégation, la thèse (et même les périples d'I-prof).

Enfin, comment ne pas remercier ma fiancée Hortense et toute ma famille de m'avoir soutenu depuis toujours. Bien qu'échanger sur les mathématiques forme parfois un exercice quelque peu périlleux, c'est finalement leur écoute et leur soutien sans faille qui ont été déterminants pour m'aider à aller au bout de ces 10 années post-bac. Les concernés demanderont le sens de la phrase suivante à Hortense, qui saura la décoder : *Pas besoin de connaître un nombre non dénombrable de théorèmes sur les probabilités pour être presque sûrement des piliers de ma vie.*

Résumé de la thèse

La thèse étudie le comportement transient de certains processus de diffusion planaires avec interactions au bord, telles que des réflexions ou des interfaces poreuses. Elle porte en particulier sur le calcul des fonctions de Green, l'analyse de leurs asymptotiques et la détermination de la frontière de Martin. La théorie de la frontière de Martin permet notamment de décrire la manière dont ce type de processus s'échappe à l'infini en tenant compte de l'ensemble des trajectoires et fournit l'ensemble des fonctions excessives et harmoniques associées.

Le Chapitre 1 introduit les processus étudiés, les méthodes utilisées ainsi que les principaux résultats de la thèse. Il présente notamment les méthodes analytiques centrales des Chapitres 2, 3 et 4, fondées sur des équations fonctionnelles à noyau. Ce chapitre introductif présente également la méthode du point col, divers outils d'analyse complexe sur des courbes elliptiques, ainsi que la méthode de compensation employée pour traiter les cas de processus dégénérés. Le Chapitre 2 est consacré à un mouvement brownien réfléchi obliquement et non dégénéré dans un cône convexe de $\mathbb{R}^2$. On y établit l'asymptotique des fonctions de Green, puis on en déduit la frontière de Martin. Le Chapitre 3 étudie un mouvement brownien réfléchi obliquement dans $\mathbb{R}^2$ dans un cadre dégénéré. Les transformées de Laplace des fonctions de Green ainsi que les fonctions harmoniques y sont calculées explicitement, et une analyse asymptotique du modèle est menée. Le Chapitre 4 considère une famille de diffusions planaires avec barrière perméable. Les transformées de Laplace des fonctions de Green y sont obtenues explicitement, ainsi que leurs asymptotiques et la frontière de Martin. Dans ce modèle, la frontière de Martin et la frontière de Martin minimale sont notablement distinctes. Enfin, l'Annexe A développe de manière indépendante la théorie probabiliste de la frontière de Martin pour une classe de processus englobant ceux étudiés dans la thèse.

Mots clefs : Frontière de Martin, Equations fonctionnelles à noyau, Mouvement Brownien réfléchi obliquement, Méthode du point col, Approche par compensation, Diffusion avec barrière perméable, Transformées de Laplace.

Abstract

The thesis studies the transient behavior of certain planar diffusion processes with boundary interactions, such as reflections or porous interfaces. It focuses in particular on the computation of Green's functions, the analysis of their asymptotics, and the determination of the Martin boundary. Martin boundary theory makes it possible to describe how such processes escape to infinity by taking into account the full set of trajectories, and it provides the associated excessive and harmonic functions.

Chapter 1 introduces the processes under study, the methods employed, and the main results of the thesis. In particular, it presents the central analytical methods used in Chapters 2, 3, and 4, which are based on kernel functional equations. This introductory chapter also presents the saddle-point method, various tools from complex analysis on elliptic curves, as well as the compensation method used to handle degenerate processes. Chapter 2 is devoted to a non-degenerate obliquely reflected Brownian motion in a convex cone of $\mathbb{R}^2$. The asymptotic behavior of the Green's functions is established, and the Martin boundary is then derived. Chapter 3 studies an obliquely reflected Brownian motion in $\mathbb{R}^2$ in a degenerate setting. The Laplace transforms of the Green's functions as well as the harmonic functions are computed explicitly, and an asymptotic analysis of the model is carried out. Chapter 4 considers a family of planar diffusions with a permeable barrier. The Laplace transforms of the Green's functions are obtained explicitly, along with their asymptotics and the Martin boundary. In this model, the Martin boundary and the minimal Martin boundary are notably distinct. Finally, Appendix A independently develops the probabilistic theory of the Martin boundary for a class of processes that includes those studied in the thesis.

Keywords: Martin boundary, Kernel functional equations, Reflected Brownian motion, Obliquely reflected Brownian motion, Saddle-point method, Compensation approach, Diffusions with a permeable barrier, Laplace transforms.

Contents

Chapitre 1

Introduction

Résumé

Dans ce chapitre introductif, nous présentons les processus aléatoires et les objets étudiés dans cette thèse, ainsi que les méthodes employées et les principaux résultats des chapitres suivants. La Section 1.1 est consacrée au cadre des diffusions réfléchies – avec un accent particulier sur le cas du quadrant – ainsi qu'aux processus munis d'une barrière perméable. Nous y discutons également les conditions de transience des processus considérés. La Section 1.2 détaille ensuite les principaux outils mobilisés dans cette thèse. Elle met en particulier en évidence l'efficacité des équations fonctionnelles à noyau pour déterminer l'asymptotique des fonctions de Green, ce qui facilite par ailleurs grandement leur étude en dimension 1. Les méthodes spécifiques à la dimension 2 y sont également présentées, notamment les techniques d'analyse complexe et la méthode de compensation. Enfin, la Section 1.3 introduit les principaux résultats développés dans les Chapitres 2, 3 et 4, relatifs aux mouvements browniens réfléchis obliquement, non dégénérés et dégénérés dans des cônes, ainsi qu'au processus avec barrière perméable dans $\mathbb{R}^2$.

Abstract

In this introductory chapter, we present the stochastic processes and objects studied in this thesis, as well as the methods employed and the main results of the subsequent chapters. Section 1.1 is devoted to the framework of reflected diffusions – with a particular emphasis on the quadrant case – and to processes endowed with a permeable barrier. We also discuss the transience conditions of the processes under consideration. Section 1.2 then details the main tools used throughout this thesis. In particular, it highlights the effectiveness of kernel functional equations in determining the asymptotic behaviour of Green's functions, which also greatly facilitates their analysis in dimension 1. Methods specific to dimension 2 are also presented, notably techniques from complex analysis and the compensation method. Finally, Section 1.3 introduces the main results developed in Chapters 2, 3, and 4, concerning non-degenerate and degenerate obliquely reflected Brownian motions in cones, as well as the process with a permeable barrier in $\mathbb{R}^2$.

Contents

1.1 Présentation des diffusions planaires transientes étudiées

En Section 1.1.1, nous présentons le mouvement brownien réfléchi dans un cône convexe, ainsi que la notion de temps local. Nous consacrons ensuite la Section 1.1.2 au Skew Brownian motion et à son extension en dimension 2, étudiée plus en détail dans le Chapitre 4. Enfin, en Section 1.1.3, nous abordons les fonctions de Green et la frontière de Martin – développées ultérieurement dans l'Annexe A – qui n'ont d'intérêt que pour des processus transients. Nous y présentons également des conditions de transience pour les processus étudiés dans cette thèse.

1.1.1 Processus réfléchis dans le quadrant

1.1.1.1 Mouvement brownien réfléchi dans un domaine général

L'étude des diffusions réfléchies a été marquée par l'article de Skorokhod en 1961 [112], qui traite de la réflexion en dimension 1. Après une série de travaux, (voir [88, Introduction]), Lions et Sznitman ont établi en 1984 des conditions générales assurant l'existence et l'unicité des diffusions réfléchies dans des domaines réguliers [88]. Nous rappelons brièvement le cadre et les résultats de Lions et Sznitman pour les diffusions réfléchies.

Contexte des diffusions réfléchies

Soit Ω un domaine (ouvert connexe) de $\mathbb{R}^d$, $d \geq 1$. L'ensemble des paramètres d'une diffusion dans $\overline{\Omega}$ réfléchie obliquement sur le bord de Ω est donné par un triplet $(\sigma, \mu, (\vec{v}(x))_{x\in\partial\Omega})$ où :

1. $\sigma \ : \Omega \longrightarrow \mathcal{M}_d(\mathbb{R})$ (covariance) est mesurable, et à valeurs dans les matrices symétriques positives
2. $\mu \ : \Omega \longrightarrow \mathbb{R}^d$ (drift) est mesurable
3. $(\vec{v}(x))_{x\in\partial\Omega}$ est un champ de vecteurs pointant vers l'intérieur du domaine (si Ω est assez régulier).

Une diffusion réfléchie X dans $\overline{\Omega}$ issue de $x_0 \in \overline{\Omega}$ associée aux paramètres $(\sigma, \mu, \vec{v})$ est définie par la solution du problème de Skorokod

$$X_t = x_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t \mu(X_s)ds + \int_0^t \vec{v}(X_s)dL_s, \quad t \geq 0 \tag{1.1.1}$$

où

- $W = (W_t)_{t\geq 0}$ est un mouvement brownien $d-$dimensionnel
- $L = (L_t)_{t\geq 0}$ est un processus continu croissant issu de 0 tel que

  $$\int_0^{+\infty} \mathbb{1}_{\{X_t\in\Omega\}} dL_t = 0. \tag{1.1.2}$$

  De manière équivalente,

  $$\mathrm{supp}\{dL\} \subset \{t \geq 0, X_t \in \partial\Omega\} \tag{1.1.3}$$

  où dL est la mesure de Stieltjes associée à L, et traduit le fait que le terme $\vec{v}(X_s)dL_s$ dans (1.1.1) repousse le processus (obliquement) à l'intérieur de Ω uniquement lorsque X atteint le bord de Ω. Le processus L est appelé *temps local de X sur le bord $\partial\Omega$*.

Concrètement, le processus réfléchi se comporte comme une diffusion de type $dX_t = \sigma(X_t)dW_t + \mu(X_t)dt$ dans Ω mais un terme de réflexion $\vec{v}(X_t)dL_t$ le pousse obliquement quand il atteint le bord du domaine.

Dans l'article [88], Lions et Sznitman démontrent l'existence et l'unicité forte d'un tel problème de Skorokod lorsque Ω est un ouvert lisse, sous des hypothèses générales. Parmi celles-ci, on peut noter les conditions $v \in C^2(\partial\Omega)$ et

$$\langle v(x), n(x)\rangle \geq c > 0$$

pour tout $x \in \partial\Omega$ où $n(x)$ désigne la normale intérieure au domaine : cette dernière condition impose au champ de vecteurs $\vec{v}$ de ne pas être "trop tangent" au bord du domaine (voir [88] pour plus de précisions).

Cas unidimensionnel

Dans le cas $d = 1$, les domaines considérés sont les intervalles. On renvoie en particulier au Lemme de Skorokod [109, Section VI.2] qui donne une solution unique trajectorielle au problème (1.1.1) dans le cas $\overline{\Omega} = [0, +\infty)$ (où $\vec{v} = 1$).

Cas du quadrant ou du cône

Dans cette thèse (plus précisément dans les Chapitres 2 et 3), on considèrera Ω un cône convexe avec σ, μ constants et $\vec{v}(x)$ constant sur chaque axe. Notons que l'irrégularité du bord du cône en sa pointe et la discontinuité du champ de vecteurs en cette même pointe transgressent le cadre de Lions et Sznitman [88]. Cela soulève des questions non triviales d'existence et d'unicité dont nous discuterons en Section 1.1.1.3.

1.1.1.2 Temps local d'une semi-martingale

Dans la Section 1.1.1.1, nous avons défini le temps local comme un processus croissant permettant la réflexion d'une diffusion aléatoire. Cette approche, bien que suffisante pour l'étude des processus réfléchis dans les Chapitres 2 et 3, ne l'est pas pour définir des processus avec barrière perméable comme dans le Chapitre 4. On donne dans cette section une définition équivalente du temps local pour les semi-martingales unidimensionnelles, et on renvoie le lecteur intéressé au livre de Revuz et Yor [109, Section VI] pour plus de détails. On a en particulier le résultat suivant [109, Section VI, Corollaire 1.9].

Proposition 1.1 (Existence du temps local symétrique d'une semi-martingale en un point)**.** *Soient* $(X_t)_{t\geq 0}$ *une semi-martingale continue et* $a \in \mathbb{R}$*. Alors, il existe un processus continu et croissant* $(L_t^a)_{t\geq 0}$ *tel que, pour tout* $t \geq 0$*,*

$$L_t^a = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{(a-\epsilon, a+\epsilon)}(X_s) d\langle X\rangle_s. \tag{1.1.4}$$

L'expression précédente montre que le support de la mesure de Stieltjes associée à dL^a satisfait

$$\text{supp}(dL^a) \subset \{t \geq 0 \mid X_t = a\}.$$

Par ailleurs, on peut choisir une version de L^a de telle sorte à ce que $(a, t) \longmapsto L_t^a$ soit mesurable par rapport aux tribus boréliennes (voir [109, Section VI, Lemme 1.4]). On mentionne la formule d'occupation [109, Section VI, Corollaire 1.6] :

$$\int_0^t f(X_s) d\langle X\rangle_s = \int_{\mathbb{R}} f(a) L_t^a da \tag{1.1.5}$$

valable pour les fonctions f mesurables positives, ainsi que la formule d'Itô-Tanaka [109, Section VI, Théorème 1.5] :

$$f(X_t) = f(X_0) + \int_0^t \left(\frac{f_g'(X_s) + f_d'(X_s)}{2} \right) dX_s + \frac{1}{2} \int_{\mathbb{R}} L_t^a f''(da) \tag{1.1.6}$$

valable pour les fonctions f étant la différence de deux fonctions convexes (où f'_g et f'_d désignent respectivement les dérivées à droites et à gauche de f). Ces formules nous seront utiles pour trouver l'équation fonctionnelle qui caractérise les fonctions de Green pour un processus avec barrière perméable (Section 1.2.2.2).

Exemple 1.2 (Temps local en 0 du mouvement brownien réfléchi). Justifions l'équivalence des définitions (1.1.4) et (1.1.1) du temps local en 0 pour le mouvement brownien standard réfléchi dans $[0,+\infty)$. Soient W un mouvement brownien standard et $X = W + L$ l'écriture (1.1.1) du mouvement brownien réfléchi. La formule d'Itô-Tanaka (1.1.6) appliquée à X pour la fonction $f \ : x \longmapsto x^+ = \max(0,x)$ donne :

$$X_t = X_0 + \int_0^t \left(\frac{\mathbb{1}_{X_s>0} + 1}{2} \right) (dW_s + dL_s) + \frac{1}{2} L_t^0.$$

En utilisant la propriété (1.1.3) et le fait que $X_s > 0$ pour presque tout $s \geq 0$ (voir [109, Section VI, Théorème 2.3]), on obtient

$$X_t = X_0 + W_t + \frac{L_t + L_t^0}{2},$$

à savoir : $\forall t \geq 0,\ L_t^0 = L_t$ étant donné l'égalité $X = W + L$.

Remarque 1.3. On rencontre également dans la littérature une définition *asymétrique* du temps local [109, Section VI] donnée par

$$L_t^a = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{[a,a+\epsilon)}(X_s) d\langle X \rangle_s, \tag{1.1.7}$$

à comparer avec (1.1.4). Cette définition présente de meilleures propriétés de régularité (continuité à droite de $a \longmapsto L_t^a$), et induit une asymétrie dans le cas des processus réfléchis : typiquement, $L_t^0 = \frac{1}{2} L_t$ dans l'Exemple 1.2.

Nous étudierons dans la Section 1.1.2 le temps local via le *Skew Brownian motion*.

1.1.1.3 Mouvement brownien réfléchi dans le quadrant

Dans cette section, on présente brièvement la motivation historique du mouvement brownien réfléchi dans le quadrant et on donne les résultats sur l'existence et l'unicité de ce processus.

Motivation historique

Le mouvement brownien réfléchi obliquement dans un quadrant est historiquement né comme limite d'échelle de files d'attentes interdépendantes, voir en particulier les résultats de Harrison [60], Reiman [108] ou l'état de l'art de Lemoine [86]. Les échanges de clients entre les files d'attente induisent naturellement un processus avec covariance différente de l'identité et des réflexions obliques sur le bord du quadrant. D'autres modèles d'approximation par des diffusions réfléchies comme celui de Baccelli et Fayolle [8] (avec absorption puis “éjection”) sont également présents dans la littérature.

Lien avec les systèmes de particules en interaction

Une autre motivation pour l'étude du mouvement brownien réfléchi dans le quadrant provient des systèmes de particules en interaction sur l'axe réel, en particulier dans des contextes issus de la finance, voir Banner et al. [10]. Ces systèmes, obtenus comme limites de processus à sauts, sont étroitement liés au mouvement brownien réfléchi, voir Karatzas et al. [74]. Le modèle type correspond à $d+1$ particules diffusives $X_1, \ldots, X_{d+1}$ satisfaisant l'EDS

$$dX_t^i = \left(\sum_{k=1}^{d+1} \sigma_k \mathbb{1}_{X_i = X_{(k)}} \right) dW_t^i + \left(\sum_{k=1}^{d+1} \mu_k \mathbb{1}_{X_i = X_{(k)}} \right) dt \tag{1.1.8}$$

où $(W^1, \dots, W^{d+1})$ est un mouvement brownien standard $d-$dimensionnel, $\sigma_1, \dots \sigma_{d+1} \geq 0$ sont des réels positifs, $\mu_1, \dots \mu_{d+1}$ sont réels et où $X_{(1)} = \min\limits_{1 \leq i \leq d} X_i \leq \cdots \leq X_{(d+1)} = \max\limits_{1 \leq i \leq d+1} X_i$ sont les particules réordonnées. En considérant le processus des écarts $(X_{(d+1)} - X_{(d)}, \dots, X_{(2)} - X_{(1)})$ (ou *gap process* en anglais), on obtient un mouvement brownien réfléchi obliquement dans un quadrant : le lecteur intéressé pourra également consulter Saranstev [110, Section 4].

Définition du mouvement brownien réfléchi dans le quadrant

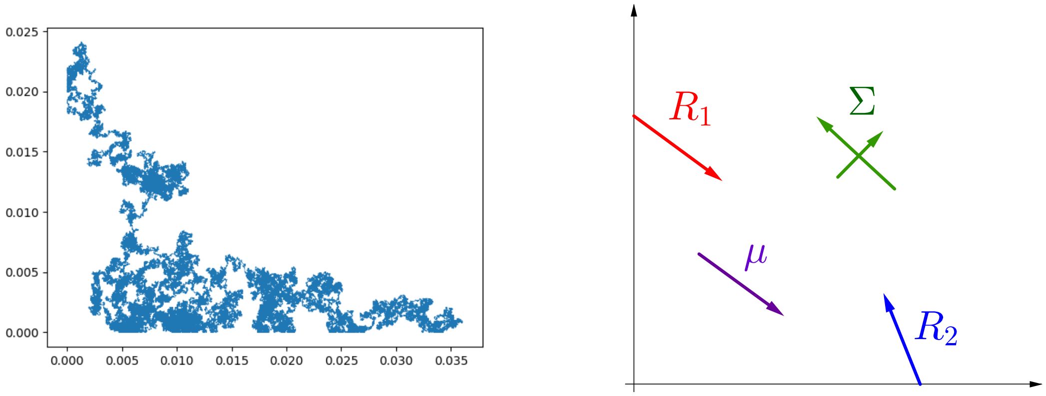
On définit le mouvement brownien réfléchi obliquement dans le quadrant en dimension $d \geq 2$ comme la solution du problème de Skorokod (1.1.1) avec drift $\mu \in \mathbb{R}^d$ et covariance $\Sigma \in \mathcal{M}_d(\mathbb{R})$ constants, et où le champ de vecteurs est constant sur chaque face du quadrant. Pour chaque face $\{x_i = 0\}$, on note R_i le vecteur de réflexion correspondant, et on pose $R = (R_1, \cdots, R_d)$. Le mouvement réfléchi X de paramètres (Σ, μ, R) associé au mouvement brownien W est alors donné (sous réserve d'existence et d'unicité) par :

$$X_t = x_0 + \sqrt{\Sigma} W_t + \mu t + R\mathbf{L}_t \in \mathbb{R}_+^d, \quad t \geq 0 \tag{1.1.9}$$

où $\mathbf{L} = (L^1, \dots, L^d)^T$ est défini par

$$L_t^i = \int_0^t \mathbb{1}_{X_t^i = 0} dL_t, \quad 1 \leq i \leq d, \quad t \geq 0.$$

Pour un panorama autour de ce processus (définition, récurrence, conditions d'existence et propriétés de la mesure invariante), on pourra se référer à Williams [121]. Notons que les éléments diagonaux de R sont strictement positifs car les R_i sont des vecteurs entrants dans le quadrant. On impose qu'ils soient égaux à 1 sans perte de généralité, quitte à dilater les L^i.



(a) Trajectoire type.

(b) Paramètres du processus.

FIGURE 1.1 : Paramètres et trajectoire type du Mouvement brownien réfléchi obliquement dans le quadrant. Les flèches vertes représentent les vecteurs propres de la matrice de covariance Σ.

Existence et unicité

Lorsque le mouvement brownien réfléchi est vu comme limite d'échelle de files d'attente, la matrice de réflexion $R = I - Q$ satisfait naturellement $\rho(Q) < 1$ [108] : dans ce cas, on a existence et unicité trajectorielle [63, Théorème 1]. Cependant, on peut définir plus généralement le mouvement brownien réfléchi obliquement pour une plus grande classe de vecteurs de réflexion.

Théorème 1.4 ([115] Taylor, Williams (1993))**.** *Soient* Σ *une matrice de covariance (avec* $\det(\Sigma) > 0$*),* $\mu \in \mathbb{R}^2$ *et* $R = (R_1, \cdots, R_d)$ *une matrice dont la diagonale ne contient que des* 1*. Alors, l'existence et l'unicité en loi du mouvement brownien réfléchi de paramètres* (Σ, μ, R) *est équivalente à la propriété suivante :*

$$\exists \alpha_1, \dots \alpha_d > 0, \quad \alpha_1 R_1 + \dots \alpha_d R_d \in (0, +\infty)^d. \tag{1.1.10}$$

On remarquera qu'on n'a, en général, pas unicité forte du problème de Skorokod associé à l'EDS réfléchie (1.1.9), voir Bass et Burdzy [11].

Dans la littérature, une matrice satisfaisant la condition (1.1.10) est appelée une $\mathcal{S}$-matrice. Cette condition signifie que le mouvement brownien "peut s'échapper du coin s'il s'en approche trop" en suivant les directions de réflexion. Si cette condition n'est pas satisfaite, alors le mouvement brownien réfléchi peut être absorbé en 0, et le processus n'est plus une semi-martingale. On sort alors du cadre de la définition énoncée. On pourra consulter Varadhan et Williams [117] pour plus de détails.

Cas du cône convexe

Un mouvement brownien réfléchi dans un cône convexe (transformation linéaire de $\mathbb{R}_+^d$) avec drift constant, covariance constante (non dégénérée) et vecteurs de réflexions constants sur chaque face peut être identifié à la transformation linéaire d'un mouvement brownien réfléchi dans un quadrant. Pour cette raison, on ramène donc l'étude au cas du quadrant dans le cadre de cette thèse. On a en fait une bijection entre cette famille de processus dans le quadrant avec covariance quelconque, la famille de processus dans les cônes avec covariance quelconque, et la famille de processus dans les cônes avec covariance identité. La transformation linéaire adéquate est décrite en détails dans [56, Annexe] pour le cas $d = 2$. Ces équivalences entre processus dans les quadrants et dans les cônes sont propres au cas continu car la géométrie de l'espace d'états dans un cône discret dépend du réseau choisi.

L'analogue de la condition (1.1.10) pour la définition des mouvements browniens dans des cônes est décrite dans [117] en dimension deux, et s'écrit $\alpha := \frac{\varepsilon + \delta - \pi}{\beta} < 1$ (pour le cas où le processus est une semi-martingale), où β désigne l'angle du cône et δ, ϵ désignent les angles formés entre les axes et les vecteurs de réflexion (voir Figure 1.2).

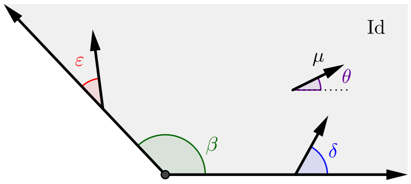
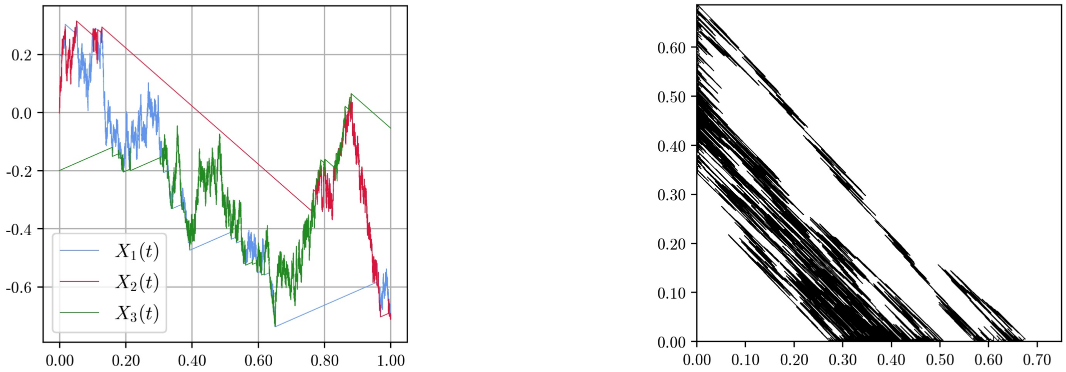


FIGURE 1.2 : Paramètres du mouvement brownien réfléchi dans un cône.

Mouvement brownien dégénéré réfléchi dans le quadrant

Dans la définition du mouvement brownien réfléchi introduite en Section 1.1.1.1, rien n'interdit à la matrice de covariance σ d'être dégénérée, c'est-à-dire de vérifier $\det(\sigma) = 0$. Une telle dégénérescence apparaît naturellement dans les articles de Ichiba et Karatzas [71] et de Franceschi et al. [51], dans le cadre des processus des écarts (*gap process*) associés à des systèmes de particules dont certaines suivent des trajectoires balistiques. C'est par exemple le cas lorsque $d + 1 = 3$ et $\sigma_1 = \sigma_3 = 0$ dans (1.1.8), comme illustré à la Figure 1.5. Lorsque le processus est récurrent, la classe algébrique de la transformée de Laplace de la mesure invariante a été étudiée par Dreyfus et al. [35]. Le Chapitre 3 est quant à lui consacré à l'étude de ce processus dans le régime transient.

La construction de ces processus repose sur la résolution d'un problème de Skorokhod (voir Harrison et Reiman [63, Théorème 1]), qui définit le mouvement brownien réfléchi comme une fonction trajectorielle du mouvement brownien initial, qu'il soit dégénéré ou non.



(a) Exemple de trajectoire pour $d+1=3$ et $\sigma_1 = \sigma_3 = 0$.

(b) Processus des écarts $(X_{(3)} - X_{(2)}, X_{(2)} - X_{(1)})$ associé.

FIGURE 1.3 : Lien entre système de particules et mouvement brownien réfléchi. Ces graphiques sont issus de l'article de Dreyfus et al. [35].

1.1.2 Processus avec barrière perméable

1.1.2.1 Le Skew Brownian motion

Le *Skew Brownian motion*, ou mouvement brownien asymétrique, apparaît dans de nombreux domaines d'application, notamment en dynamique des populations (voir Cantrell et Cosner [18]), en astrophysique (voir Zhang [123]) et en finance (voir Decamps et al. [29]). Pour une liste plus exhaustive d'applications, on pourra se référer à Lejay [84]. Ce processus constitue une variante du mouvement brownien classique, introduit afin de modéliser des phénomènes faisant intervenir une frontière perméable. Introduit dans les années 1970, le mouvement brownien asymétrique a fait l'objet d'une étude de synthèse détaillée dans Lejay [84].

On dit que X est un Skew Brownian motion lorsque X est solution de l'EDS avec temps local :

$$dX_t = dW_t + q dL_t^0(X) \tag{1.1.11}$$

où W est un mouvement brownien, $L^0(X)$ est le temps local symétrique de X en 0 (voir Section 1.1.1.2) et où $q \in [-1, 1]$. L'équation (1.1.11) admet une unique solution forte. En posant $p = \frac{q+1}{2}$, une construction classique du *Skew Brownian motion* consiste à recoller des excursions browniennes issues de 0, orientées vers les positifs avec probabilité p et vers les négatifs avec probabilité $1-p$ (voir Lejay [84]). Ce processus peut également être obtenu comme limite d'échelle de marches aléatoires simples présentant un comportement asymétrique en l'origine, comme l'ont montré Harrison et Shepp [64]. Lorsque $q = 0$, on retrouve le mouvement brownien standard, tandis que les cas $q = 1$ (resp. $q = -1$) correspondent au mouvement brownien réfléchi sur $\mathbb{R}_+$ (resp. sur $\mathbb{R}_-$). En revanche, lorsque $|q| > 1$, l'équation (1.1.11) n'admet pas de solution [64].

Remarque 1.5 (Drift local). Le terme $q dL_t^0(X)$ peut être également interprété comme un “drift local”, qui agit sur le processus uniquement quand il atteint 0, en le poussant vers la droite ou vers la gauche selon le signe de q. Cette interprétation guidera l'extension que nous proposerons en dimension supérieure.

Plus généralement, si σ, b sont mesurables bornés avec $\sigma \geq \epsilon > 0$ et si ν est une mesure finie sur $\mathbb{R}$ telle que $\nu(\{x\}) < 1$ pour tout $x \in \mathbb{R}$, alors l'existence et l'unicité forte de solutions du type

$$X_t = X_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds + \int_{\mathbb{R}} L_t^x d\nu(x) \tag{1.1.12}$$

tient toujours, voir Legall [83] et Lejay [84]. On pourra se référer à Lenotre [87] pour des méthodes de simulation, ou à Fernholz et al. [44] pour l'étude d'un processus SBBBM (skew bang-bang Brownian motion) où deux particules sur $\mathbb{R}$ interagissent entre elles (skew), dont les coefficients de diffusion dépendent de leur rang (bang-bang) et dont la différence satisfait une équation du type (1.1.12).

1.1.2.2 Barrière perméable et milieu stratifié

Des généralisations du Skew Brownian motion ont été développées en dimension $d \geq 2$, notamment dans les travaux de Portenko [104, 105], qui définissent un processus avec un drift singulier le long d'une surface fermée régulière. On peut également citer les travaux d'Atar [7], Lejay [85] et Zaitseva [122] concernant la construction et l'étude de processus avec drift local le long d'un hyperplan. L'extension au cas général d'une diffusion avec drift local oblique, où les coefficients sont non constants, reste cependant un problème difficile [6, 103]. Il convient de noter que, pour $d \geq 2$, la notion de temps local n'a de sens que le long d'une hypersurface : l'analogue d-dimensionnel de la formule (1.1.4) est en général mal défini, sauf dans des cas très particuliers comme dans Baxter [12].

Dans [85], Lejay considère un processus bidimensionnel dans un milieu stratifié. Plus précisément, le processus est gouverné par

- Deux matrices diagonales D^+ et D^- (coefficients de diffusions) dans les demi-plans supérieurs et inférieurs de $\mathbb{R}^2$;
- Un drift local sur l'axe choisi normal à l'axe $\{y = 0\}$.

L'orientation du drift local est choisie de sorte que le générateur s'écrive sous forme divergence (cf. prochaine section). Un tel modèle s'écarte néanmoins des cadres proposés par Atar, Portenko et Zaitseva où le coefficient de diffusion est choisi continu.

Nous proposons une généralisation de ce processus avec drift local *oblique* sur l'axe et covariances non nécessairement diagonales, qui sera étudiée dans le Chapitre 4. Soient deux matrices de covariance Σ^+ et Σ^- (non dégénérées), deux drifts $\mu^+, \mu^- \in \mathbb{R}^2$, et une direction $q \in \mathbb{R} \times (-1, 1)$. Posons enfin

$$\Sigma(y) = \Sigma^- \mathbb{1}_{y<0} + \Sigma^+ \mathbb{1}_{y \geq 0} \quad \text{et} \quad \mu(y) = \mu^- \mathbb{1}_{y<0} + \mu^+ \mathbb{1}_{y \geq 0}. \tag{1.1.13}$$

On définit alors un processus de type *skew bang-bang* grâce au temps local de la deuxième coordonnée sur l'axe $\{y = 0\}$.

Définition 1.6 (Skew bang-bang diffusion avec paramètres constants par morceaux)**.** *On dit qu'un processus continu et adapté $(Z_t)_{t \geq 0} = (A_t, B_t)_{t \geq 0}$ sur $\mathbb{R}^2$ est une skew bang-bang diffusion associée aux paramètres $(\Sigma^+, \Sigma^-, \mu^+, \mu^-, q)$ issu de $z_0 \in \mathbb{R}^2$ s'il est solution de :*

$$Z_t = z_0 + \int_0^t \sqrt{\Sigma(B_s)}\, dW_s + \int_0^t \mu(B_s)\, ds + qL_t^0(B), \quad t \geq 0, \tag{1.1.14}$$

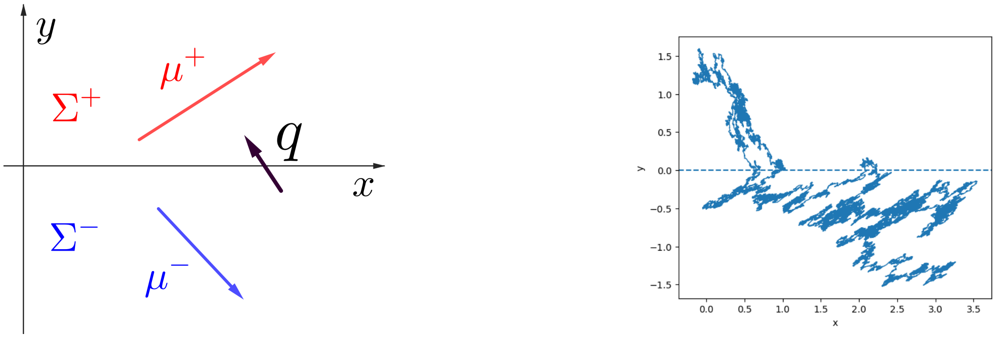
où

(i) $(W_t)_{t \geq 0}$ est un mouvement brownien bidimensionnel adapté

(ii) $\mathbb{P}_{z_0}$-presque sûrement,

$$L_t^0(B) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{[-\epsilon,\epsilon]}(B_s)\, ds \tag{1.1.15}$$

existe dans $\mathbb{R}$ pour tout $t \geq 0$.



(a) Paramètres du processus.

(b) Trajectoire typique.

FIGURE 1.4 : Localement, Z se comporte comme un mouvement brownien de covariance Σ^+ de drift μ^+ dans le demi-plan supérieur (et de même pour le demi-plan inférieur pour les données Σ^-, μ^-). Le processus est localement “drifté” dans la direction q quand il touche l'axe $\{y = 0\}$.

Ce modèle peut être interprété comme un processus de type *bang-bang*, en ce sens qu'il alterne entre deux régimes distincts. Une telle structure apparaît naturellement dans les modèles de contrôle stochastique (voir Shreve [118]). En raison de l'ajout d'un drift localisé sur l'axe, on parle alors de *skew bang-bang*. L'existence et l'unicité d'un tel processus sont assurées, tant au sens trajectoriel qu'en loi, et celui-ci constitue un processus de Markov.

Proposition 1.7. *Soit* $(W_t)_{t\geq 0}$ *mouvement brownien bidimensionnel adapté. Alors, l'équation* (1.1.14) *admet une unique solution trajectorielle* Z *associée à* $(W_t)_{t\geq 0}$ *pour les paramètres* $(\Sigma^+, \Sigma^-, \mu^+, \mu^-, q)$. *L'unicité est également valable en loi. De plus,* Z *définit un processus de Markov fort.*

La preuve est donnée en Section 4.2.1, Chapitre 4.

1.1.2.3 Opérateurs sous forme divergence et barrière perméable

Dans l'étude de processus physiques avec barrière perméable (voir Etoré [39, Introduction]), il est naturel de considérer des générateurs sous forme divergence, c'est-à-dire de la forme

$$\mathcal{L} = \frac{1}{2}\nabla(\Sigma(x)\nabla) + \mu(x)\nabla.$$

On pourra se référer à l'article de Stroock [113] pour une définition précise des processus dont le générateur est sous forme divergence (notamment le Théorème II.3.8. pour l'existence et l'unicité). Dans cette partie, on choisit une direction du drift local q sur l'axe, de sorte que le processus défini dans la Section 1.1.2.2 admette un générateur sous forme divergence.

Exemple 1.8 (Opérateur sous forme divergence en dimension 1). Cet exemple est issu de la thèse d'Etoré [39, Section 2.1]. Supposons que σ soit continu sauf en un ensemble discret $\mathcal{I}$ de $\mathbb{R}$ où elle présente des discontinuités de première espèce, et qu'il existe $\lambda > 0$ tel que $\frac{1}{\lambda} \leq \sigma(x) \leq \lambda$ pour tout $x \in \mathbb{R}$. Alors, l'unique processus de Markov d'opérateur $\mathcal{L} = \frac{1}{2}\nabla(\sigma^2(x)\nabla) + \mu(x)\nabla$ est la diffusion X solution de

$$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt + \sum_{i\in\mathcal{I}} \frac{\sigma^2(x_i^+) - \sigma^2(x_i^-)}{\sigma^2(x_i^+) + \sigma^2(x_i^-)} L_t^{x_i}(X) \tag{1.1.16}$$

où W est un mouvement brownien standard. Lorsque $\sigma(x) = \sigma_+ \mathbb{1}_{x\geq 0} + \sigma_- \mathbb{1}_{x<0}$ où $\sigma_-, \sigma_+ > 0$, l'équation (1.1.16) se réécrit

$$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt + \frac{\sigma_+^2 - \sigma_-^2}{\sigma_+^2 + \sigma_-^2} L_t^0(X).$$

On obtient ainsi un processus de type (1.1.14), avec une valeur de q donnée par $\frac{\sigma_+^2-\sigma_-^2}{\sigma_+^2+\sigma_-^2} \in (-1,1)$.

Dans l'article [85], le processus avec des matrices de covariance diagonales est choisi pour avoir un opérateur sous forme divergence. Dans ce cadre, la direction q du drift local sur l'axe est orthogonale à l'axe. Nous généralisons ce résultat en considérant une matrice de covariance quelconque.

Proposition 1.9 (Opérateur sous forme divergence et barrière perméable en dimension 2)**.** *Soient* Σ *et* μ *définies par* (1.1.13)*. Alors, l'unique processus de Markov de générateur* $\mathcal{L} = \frac{1}{2}\nabla(\Sigma(y)\nabla) + \mu(x)\nabla$ *est la diffusion* Z *solution de* (1.1.14) *avec* $q = q_0 := \frac{1}{\Sigma_{22}^+ + \Sigma_{22}^-}\begin{pmatrix}\Sigma_{12}^+ - \Sigma_{12}^- \\ \Sigma_{22}^+ - \Sigma_{22}^-\end{pmatrix} \in \mathbb{R}\times(-1,1)$.

On présente les grandes lignes de la démonstration dans la Section 4.2.1 du Chapitre 4.

1.1.3 Transience et frontière de Martin

Dans cette thèse, nous nous placerons dans le cadre de processus transients à l'infini. La théorie du potentiel et la théorie de la frontière de Martin pour les processus transients occupent une place importante dans la littérature, mais restent relativement dispersées lorsqu'on se place en temps et espace continus. Pour cette raison, l'Annexe A est consacrée à une synthèse de ces résultats, dans un cadre suffisamment général pour englober les processus étudiés dans ce manuscrit. Nous présentons néanmoins dans cette section les objets et théorèmes fondamentaux, ainsi que les conditions de transience propres aux processus considérés.

1.1.3.1 Fonctions de Green et EDP

On rappelle qu'un processus est dit *transient* si, presque sûrement, il quitte définitivement tout compact en un temps fini : voir la Section A.2.2 pour des définitions équivalentes. Lorsque le processus de Markov X est transient, il est naturel de s'intéresser au temps moyen passé par le processus dans un ensemble donné. On note E l'espace d'état, supposé localement compact et à base dénombrable.

Définition 1.10 (Mesure de Green et fonctions de Green)**.** *Soit* $x_0 \in E$*. On définit la mesure de Green issue de* x_0 *par*

$$G(x_0, A) = \mathbb{E}_{x_0}\left[\int_0^{+\infty} \mathbb{1}_{X_t\in A}, dt\right] = \int_0^{+\infty} \mathbb{P}_{x_0}(X_t \in A), dt, \qquad A \in \mathcal{B}(E). \tag{1.1.17}$$

Cette mesure est également appelée mesure d'occupation. Lorsqu'elle admet une densité par rapport à une mesure m*, on la note* $g(x_0,\cdot)$*. On appelle alors fonction de Green issue de* x_0 *pour la mesure* m *la densité ainsi obtenue. Lorsque* g *est bien définie, la loi de* X_t *issue de* x_0 *admet une densité* $p_t(x_0,x)$ *par rapport à la mesure* m *pour tout* $t>0$*, et on a*

$$g(x_0,x) = \int_0^{+\infty} p_t(x_0,x)dt. \tag{1.1.18}$$

Lorsque X est transient, la mesure de Green est finie sur les compacts (voir la Section A.2.2 dans l'Annexe A). Sous une hypothèse de dualité appropriée (voir l'Hypothèse A.27 en Section A.3.3.2 de l'Annexe A), la fonction $g(x_0,\cdot)$ peut être définie de manière non arbitraire en tout point de E, et non seulement m-presque partout. Dans la plupart des situations considérées, la mesure m est la mesure de Lebesgue.

En pratique, lorsque le processus est une diffusion sur un domaine de $\mathbb{R}^d$, les fonctions de Green satisfont des équations aux dérivées partielles, dites *rétrogrades* (ou *backward* en anglais) relativement à la première variable et *directe* (ou *forward* en anglais) relativement à la seconde. Dans ce travail, nous nous concentrons sur les équations rétrogrades, qui seront reliées aux fonctions harmoniques dans la Remarque 1.13.

Pour le mouvement brownien tué au bord d'un domaine, la fonction de Green satisfait l'équation (A.3.1) avec conditions de Dirichlet nulles au bord ; on renvoie aux Sections A.3.1 et A.3.2 de l'Annexe A.

Dans le cas du mouvement brownien réfléchi suivant la normale au bord d'un domaine D, la densité de transition $p_t(x,y)$ satisfait l'équation rétrograde suivante (voir [17, Théorème 2.5]) :

$$\begin{cases} \partial_t p_t(x,y) - \frac{1}{2}\Delta_x p_t(x,y) = 0, & x \in D \\ \partial_n p_t(x,y) = 0, & x \in \partial D. \end{cases} \tag{1.1.19}$$

Ainsi, si D est non borné et que le processus est transient, l'équation (1.1.18) conduit formellement à :

$$\begin{cases} \frac{1}{2}\Delta_x g(x,y) = -\delta_y(dx), & x \in D \\ \partial_n^x g(x,y) = 0, & x \in \partial D. \end{cases} \tag{1.1.20}$$

Pour le mouvement brownien réfléchi *obliquement* dans un cône bidimensionnel, Harrison et Reiman [62] ont montré que la densité de transition satisfait l'équation rétrograde

$$\begin{cases} \partial_t p_t(x,y) - \mathcal{L}_x p_t(x,y) = 0, & x \in D \\ \partial_{R_i} p_t(x,y) = 0, & \text{sur la face } \{x_i = 0\} \end{cases} \tag{1.1.21}$$

où $\mathcal{L}_x = \frac{1}{2}\sum_{1\le i,j\le d} a_{i,j}\partial_{x_i}\partial_{x_j} + \sum_{1\le i\le d}\mu_i\partial_{x_i}$, $\Sigma = (a_{i,j})_{1\le i,j\le d}$ désigne la matrice de covariance, μ le drift du processus, et R_i la direction de réflexion sur la face $x_i = 0$ du quadrant. On en déduit alors heuristiquement l'équation rétrograde analogue à (1.1.20) pour les fonctions de Green $g(x,y)$, munie des conditions aux frontières obliques :

$$\begin{cases} \mathcal{L}_x g(x,y) = -\delta_y(dx), & x \in D \\ \partial_{R_i}^x g(x,y) = 0, & x_i = 0. \end{cases} \tag{1.1.22}$$

1.1.3.2 Frontière de Martin

Dans l'étude de processus de Markov transients, il est naturel de s'intéresser aux fonctions harmoniques positives (voir Sections A.2 et A.5 en annexe). Ces fonctions peuvent être déterminées à l'aide de la théorie de la frontière de Martin, en se basant sur le comportement asymptotique des fonctions de Green définies précédemment. Commençons par rappeler la notion de fonction harmonique.

Définition 1.11 (Fonction harmonique)**.** *Soit $h : E \to [0, +\infty]$ une fonction mesurable et positive. On dit que h est harmonique sur un ouvert $G \subset E$ pour le processus de Markov X si, pour tout ouvert U relativement compact dans G et pour tout $x \in E$, on a*

$$\mathbb{E}_x[h(X_{\tau_{U^c}})\mathbb{1}_{\tau_{U^c}<\zeta}] = h(x) \tag{1.1.23}$$

où ζ est le temps de vie du processus. La fonction h est dite harmonique si elle est harmonique sur tout E.

Cette définition est présentée via le prisme de la propriété de la moyenne, on pourra trouver différentes caractérisations en Section A.2.2.

On peut montrer (Proposition A.30) que les fonctions de Green $g(\cdot,y)$ sont harmoniques sur $E\setminus\{y\}$. Cela conduit naturellement à considérer le quotient $\frac{g(x,y)}{g(x_0,y)}$ lorsque $y \to \infty$, avec x_0 un point fixé de E, dans l'espoir d'obtenir une fonction harmonique. La normalisation par $g(x_0,y)$ est nécessaire car, en général, $g(x,y) \to 0$ lorsque $y \to \infty$. De manière plus générale, on introduit une mesure de référence r (voir Définition A.32) et on définit le noyau de Martin K par

$$K(x,y) = \frac{g(x,y)}{\int_E g(u,y)r(du)}. \tag{1.1.24}$$

Le noyau K coïncide avec le quotient $\frac{g(x,y)}{g(x_0,y)}$ lorsque $r = \delta_{x_0}$ est une mesure de référence.

On s'intéresse ensuite à l'asymptotique de $K(x,y)$ lorsque $y \to \infty$. Pour formaliser les fonctions limites, on considère l'ensemble F des suites $(y_n)_{n\geq 0}$ de E qui sortent de tout compact et telles que $(K(x,y_n))_{n\geq 0}$ converge pour tout $x \in E$. On définit la frontière de Martin par $\partial M = F/\sim$ où $\sim$ est la relation d'équivalence sur F définie par

$$(y_n)_{n\geq 0} \sim (z_n)_{n\geq 0} \Leftrightarrow \forall x \in E, \quad |K(x,y_n) - K(x,z_n)| \underset{n\to+\infty}{\longrightarrow} 0.$$

À chaque point $\eta \in \partial M$ correspond canoniquement une fonction $K(\cdot,\eta)$. Cette construction peut être vue comme une compactification de E par rapport à une certaine métrique (voir Section A.4.1).

Avant d'énoncer le théorème de représentation des fonctions harmoniques, on définit l'ensemble ∂M_{min} par

$$\partial M_{min} = \{\eta \in \partial M \mid K(\cdot,\eta) \text{ est une fonction harmonique minimale}\}.$$

La notion de fonction minimale (ou extrémale) correspond à la définition classique dans les ensembles convexes (voir Définition A.36). Le théorème de représentation des fonctions harmoniques positives s'écrit alors comme suit.

Théorème 1.12 (Représentation des fonctions harmoniques positives)**.** *Soit $h : E \to \mathbb{R}$ une fonction harmonique positive sur E. Il existe une unique mesure borélienne finie ν^h sur $\partial M_{\min}$ telle que, pour tout $x \in E$,*

$$h(x) = \int_{\partial M_{min}} K(x,\eta) d\nu^h(\eta).$$

Réciproquement, toute fonction exprimée sous cette forme est harmonique.

On obtient en réalité un théorème plus général de représentation des fonctions excessives (généralisation des fonctions harmoniques), voir Section A.4.3.

Remarque 1.13 (EDP satisfaites par les fonctions harmoniques). Si $(\mathcal{G}, D(\mathcal{G}))$ est le générateur du processus, toute fonction harmonique h dans $D(\mathcal{G})$ satisfait $\mathcal{G}h = 0$ (voir Proposition A.12 dans l'Annexe A). Lorsque X se comporte comme une diffusion dans un domaine D et que h est régulière, cette harmonicité se traduit par une EDP, dont la condition au bord dépend du comportement de X sur le bord de D. Cette condition frontière apparaît également dans les EDP satisfaites par les fonctions de Green. Par exemple, pour un mouvement brownien réfléchi dans le quadrant (cf. (1.1.25)), les fonctions harmoniques obtenues par l'asymptotique de $g(x,y)$ lorsque $y \to \infty$ vérifient informellement :

$$\begin{cases} \mathcal{L}h(x) = 0, & x_1 > 0, x_2 > 0 \\ \partial^x_{R_i} h(x) = 0, & x_1 = 0 \end{cases} \tag{1.1.25}$$

1.1.3.3 Conditions de transience pour les processus étudiés

Examinons la transience des processus étudiés dans les Chapitres 2, 3 et 4.

Pour le mouvement brownien réfléchi non dégénéré en dimension 2, les conditions de transience sont bien établies (voir Williams [119] et Hobson et Rogers [68]). La transience résulte d'une compétition entre le drift et les rebonds : ces deux effets peuvent se compenser, ramenant le processus (ou non) près du coin du quadrant. Dans les Chapitres 2 et 3, nous supposerons que le drift $\mu = (\mu_1, \mu_2)^T$ satisfait (dans le quadrant) :

$$\mu_1 > 0 \quad \text{et} \quad \mu_2 > 0,$$

ce qui rend le processus transient peu importe les vecteurs de rebonds. Des conditions nécessaires ou suffisantes ont été établies en dimensions $d \geq 3$ [14, 15, 19, 23], mais le problème d'une caractérisation simple de la transience reste encore un problème ouvert.

Pour le cas du mouvement brownien avec barrière perméable dans le Chapitre 4, nous supposerons que les drifts $\mu^+ = (\mu_1^+, \mu_2^+)^T$ et $\mu^- = (\mu_1^-, \mu_2^-)^T$ de part et d'autre de la barrière perméable $\{y = 0\}$ satisfont

$$\mu_2^+ > 0 \quad \text{et} \quad \mu_2^- < 0,$$

ce qui rend également le processus transient.

1.2 Méthodes analytiques et équations fonctionnelles

1.2.1 Contexte historique et plan

L'étude de la frontière de Martin repose principalement sur l'analyse du comportement asymptotique des fonctions de Green et sur l'identification des fonctions harmoniques associées (voir Section A.4.2). Cette section présente les principales méthodes analytiques utilisées dans la thèse pour atteindre ces objectifs, en les replaçant dans leur contexte historique.

Les méthodes analytiques employées pour l'étude de l'asymptotique des fonctions de Green ont été initialement développées dans un cadre discret, principalement pour l'analyse des marches aléatoires dans le quadrant. L'approche de Malyshev [91], consacrée à l'asymptotique de la distribution stationnaire, a ouvert la voie à une approche reposant sur l'étude d'équations fonctionnelles associées aux fonctions génératrices. Cette ligne de recherche a été poursuivie et enrichie par Kurkova et Malyshev [79, 81], qui ont appliqué ces techniques à l'étude de la frontière de Martin et à des modèles de files d'attente, ou encore Kurkova et Raschel [80] pour l'étude de marches aléatoires absorbées sur le bord d'un quadrant. Ces méthodes pour les marches aléatoires ont été synthétisées et unifiées dans l'ouvrage de référence de Fayolle, Iasnogorodski et Malyshev [41], désigné comme le « livre jaune ». Plus récemment, ces méthodes analytiques ont été étendues au cadre continu par Ernst, Franceschi et al. [37, 50, 52] pour l'étude de diffusions réfléchies dans le demi-plan et dans le quadrant ou tuées dans le quadrant.

Une littérature abondante s'est développée dans le cas récurrent, portant aussi bien sur l'asymptotique de la densité stationnaire [24, 25, 52, 61, 98, 110], que sur des méthodes numériques pour son calcul par Dai et Harrison [21, 22], ou encore sur l'obtention de formules explicites [8, 13, 31, 45, 47, 56, 57, 66].

En revanche, le cas transient a été beaucoup moins étudié dans le cadre continu à l'aide de ces méthodes analytiques. Quelques travaux récents s'intéressent à des quantités spécifiques, telles que les probabilités d'échappement le long des axes par Fomichov et al. [46], la probabilité d'absorption au sommet du cône ou encore les fonctions de Green [37, 38, 49, 50, 55]. En particulier, l'article de Ernst et Franceschi [37] est, à notre connaissance, le premier à étudier l'asymptotique des fonctions de Green dans le cas transient à l'aide de ces outils analytiques.

Parallèlement, une autre famille de méthodes, dite *approche par compensation*, a été introduite par Adan et al. [2] pour déterminer la mesure stationnaire de certaines marches aléatoires dégénérées dans le quadrant. Cette approche a ensuite été développée par Adan et al. [1, 3, 4] pour obtenir les fonctions génératrices pour des marches aléatoires à petits pas ainsi que dans le contexte des systèmes de files d'attente. Cette méthode a été également utilisée par Hoang et al. [67] pour déterminer les fonctions harmoniques pour des marches singulières dans le quadrant. Plus récemment, elle a également été adaptée au cadre continu, notamment par Franceschi et al. pour obtenir des expressions explicites de distributions stationnaires [51] et pour l'étude de la frontière de Martin de mouvements browniens dégénérés tués dans des cônes bidimensionnels [50].

L'un des apports centraux et l'une des originalités de cette thèse résident dans le développement systématique de ces méthodes analytiques dans le cas transient pour des processus continus, en s'appuyant sur des outils initialement conçus pour le cas récurrent. Cette approche permet une étude approfondie de l'asymptotique des fonctions de Green et de la frontière de Martin dans des situations jusqu'ici peu explorées.

Dans cette optique, la Section 1.2.2 présente d'abord plusieurs exemples en dimension 1, illustrant l'intérêt de l'introduction d'équations fonctionnelles pour obtenir des expressions explicites des fonctions de Green. La Section 1.2.3 est ensuite consacrée à la description des outils théoriques permettant d'en déterminer l'asymptotique en dimension 2. Enfin, la Section 1.2.4 présente deux méthodes pour expliciter les fonctions harmoniques.

1.2.2 Quelques exemples unidimensionnels

1.2.2.1 Exemple du mouvement brownien 1D avec drift et tué en 0.

On détermine ici l'expression explicite des fonctions de Green d'un mouvement brownien avec drift et tué en 0, ce qui permet en particulier de déduire la frontière de Martin. Soit W un mouvement brownien standard unidimensionnel, $\mu > 0$, et $X_t = x_0 + W_t + \mu t$, $t \geq 0$ où $x_0 > 0$. Posons $T_0 = \inf\{t \geq 0 \mid X_t = 0\}$ (avec la convention $\inf(\varnothing) = +\infty$) et Y le processus (standard) défini par

$$Y_t = \begin{cases} X_t & \text{si } 0 \leq t < T_0 \\ \partial & \text{si } t \geq T_0. \end{cases}$$

Par la formule d'Itô, on a pour $p < 0$:

$$e^{pX_t} = e^{px_0} + p\int_0^t e^{pX_s}dW_s + \left(\frac{1}{2}p^2 + \mu p\right)\int_0^t e^{pX_s}ds.$$

Comme l'intégrale brownienne est une vraie martingale, le théorème d'arrêt donne :

$$\mathbb{E}_{x_0}[e^{pX_{t\wedge T_0}}] = e^{px_0} + \left(\frac{1}{2}p^2 + \mu p\right)\mathbb{E}_{x_0}\left[\int_0^{t\wedge T_0} e^{pX_s}ds\right].$$

Comme $p < 0$, on peut faire tendre t vers $+\infty$ dans l'équation précédente :

$$\mathbb{P}_{x_0}(T_0 < +\infty) = e^{px_0} + \left(\frac{1}{2}p^2 + \mu p\right)\underbrace{\mathbb{E}_{x_0}\left[\int_0^{T_0} e^{pX_s}ds\right]}_{=:\varphi^{x_0}(p)}$$

où φ^{x_0} est la transformée de Laplace de la fonction de Green $g(x_0, \cdot)$ (par rapport à la mesure de Lebesgue) du processus Y issu de x_0. En prenant $p = -2\mu$, on obtient en particulier $\mathbb{P}_{x_0}(T_0 < +\infty) = e^{-2\mu x_0}$ puis

$$\varphi^{x_0}(p) = \frac{2(e^{-2\mu x_0} - e^{px_0})}{p(p+2\mu)}, \quad p < 0.$$

En inversant explicitement la transformée de Laplace, on obtient :

$$\begin{aligned} g(x_0, x) &= \frac{4}{\mu}\left[e^{-2\mu x_0}\left(e^{2\mu x} - 1\right)\mathbb{1}_{x>0} + \left(1 - e^{2\mu(x-x_0)}\right)\mathbb{1}_{x>x_0}\right] \\ &= \frac{4}{\mu}\begin{cases} e^{-2\mu x_0}(e^{2\mu x} - 1) & \text{si } 0 < x \leq x_0 \\ 1 - e^{-2\mu x_0} & \text{si } x > x_0. \end{cases} \end{aligned}$$

En considérant la limite du quotient $\frac{g(x_0,x)}{g(1,x)}$ lorsque $x \to 0^+$ et $x \to +\infty$, on en déduit immédiatement la proposition suivante.

Proposition 1.14. *La frontière de Martin du mouvement brownien Y avec drift $\mu > 0$ et tué en 0 est donnée par $\{0, +\infty\}$, et est minimale. De plus, les fonctions harmoniques (minimales) associées sont*

- $x_0 \longmapsto e^{-2\mu x_0}$

- $x_0 \longmapsto 1 - e^{-2\mu x_0}$.

Remarquons que les fonctions harmoniques obtenues dans la proposition précédente coïncident avec les fonctions $x_0 \longmapsto \mathbb{P}_{x_0}(T_0 < +\infty)$ et $x_0 \longmapsto \mathbb{P}_{x_0}(Y_t \underset{t\to+\infty}{\longrightarrow} \infty)$ respectivement. Cette identification découle directement des calculs effectués pour $\mathbb{P}_{x_0}(T_0 < +\infty)$, ainsi que de la relation $\mathbb{P}_{x_0}(T_0 < +\infty) + \mathbb{P}_{x_0}(Y_t \underset{t\to+\infty}{\longrightarrow} \infty) = 1$.

1.2.2.2 Exemple de barrière perméable en 1D.

Cette section est consacrée à la détermination de la frontière de Martin d'un processus unidimensionnel avec barrière perméable en 0 (voir Section 1.1.2), où les drifts et covariances sont constants de part et d'autre de 0. On considère le processus unidimensionnel X satisfaisant l'EDS

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t + q\mathrm{d}L_t^0(X)$$

où $q \in (-1, 1)$,

$$\sigma(x) = \sigma^- \mathbb{1}_{x<0} + \sigma^+ \mathbb{1}_{x\geq 0}, \;\; b(x) = \mu^- \mathbb{1}_{x<0} + \mu^+ \mathbb{1}_{x\geq 0},$$

$\sigma^-, \sigma^+ > 0$, où W est un mouvement brownien standard et où $L_t^0(X)$ désigne le temps local symétrique en 0. On choisit les paramètres μ_- et μ_+ de façon à assurer la transience de X. Dans un premier temps, on suppose $\mu_- > 0$ et $\mu_+ > 0$, ce qui assure que $X_t \to +\infty$ p.s. Dans un second temps, on prend $\mu_- < 0$ et $\mu_+ > 0$ de façon à ce que les événements $\{X_t \to +\infty\}$ et $\{X_t \to -\infty\}$ aient tous deux une probabilité strictement positive.

(a) Cas $\mu_- > 0$ et $\mu_+ > 0$. On définit les transformées de Laplace des mesures de Green restreintes à $(-\infty, 0)$ et $(0, +\infty)$: pour $x_0 \in \mathbb{R}$ et $y, z \in \mathbb{R}$, on pose

$$\varphi_-^{x_0}(y) = \mathbb{E}_{x_0}\left[\int_0^{+\infty} e^{X_s y} \mathbb{1}_{X_s<0} ds\right], \quad \varphi_+^{x_0}(z) = \mathbb{E}_{x_0}\left[\int_0^{+\infty} e^{X_s z} \mathbb{1}_{X_s>0} ds\right]. \tag{1.2.1}$$

On rappelle que le processus passe un temps de mesure nulle sur le bord. Posons également

$$\gamma_-(y) = \frac{\sigma_-^2 y^2}{2} + \mu_- y, \quad \gamma_+(z) = \frac{\sigma_+^2 z^2}{2} + \mu_+ z$$

et notons

$$z^* = -\frac{2\mu_+}{\sigma_+^2}, \quad y^* = -\frac{2\mu_-}{\sigma_-^2} \tag{1.2.2}$$

de sorte à avoir $\gamma_-(y) = \frac{\sigma_-^2}{2} y(y - y^*)$ et $\gamma_+(z) = \frac{\sigma_+^2}{2} z(z - z^*)$. La proposition suivante fournit une équation fonctionnelle satisfaite par les transformées de Laplace φ_- et φ_+, ce qui permet d'obtenir leurs expressions explicites.

Proposition 1.15. *Supposons $\mu_- > 0$ et $\mu_+ > 0$. Alors, $\mathbb{E}_{x_0}[L_\infty^0] < +\infty$ et, pour $z < 0$ et $y \geq 0$, les quantités $\varphi_-^{x_0}(y)$, $\varphi_+^{x_0}(z)$ convergent et sont reliées par l'équation fonctionnelle suivante*

$$\gamma_-(y)\varphi_-^{x_0}(y) + \gamma_+(z)\varphi_+^{x_0}(z) + \frac{y(q-1) + z(1+q)}{2}\mathbb{E}_{x_0}[L_\infty^0] = -(e^{x_0 z}\mathbb{1}_{x_0\geq 0} + e^{x_0 y}\mathbb{1}_{x_0<0}). \tag{1.2.3}$$

De plus, pour $z < 0$,

$$\varphi_+^{x_0}(z) = \frac{-1}{\gamma_+(z)}\left(\mathbb{1}_{x_0<0}\left(1 - \frac{z}{z^*}\right) + \mathbb{1}_{x_0\geq 0}\left(e^{x_0 z} - \frac{z}{z^*}e^{x_0 z^*}\right)\right) \tag{1.2.4}$$

et pour $y \geq 0$,

$$\varphi_-^{x_0}(y) = \frac{1}{\gamma_-(y)}\left(\mathbb{1}_{x_0<0}\left(1 - e^{x_0 y} + \frac{y(1-q)}{z^*(1+q)}\right) + \mathbb{1}_{x_0\geq 0} e^{x_0 z^*}\frac{y(1-q)}{z^*(1+q)}\right). \tag{1.2.5}$$

Démonstration. La preuve est basée sur la formule d'Itô-Tanaka (1.1.6) et la formule d'occupation (1.1.5). Soit f la fonction définie par $f(x) = e^{xz}\mathbb{1}_{x\geq 0} + e^{xy}\mathbb{1}_{x<0}$. Notons que $f''(dx) = (y^2 e^{yx}\mathbb{1}_{x<0} + z^2 e^{zx}\mathbb{1}_{x>0}) + (z-y)\delta_0$. Par la formule d'Itô-Tanaka,

$$f(X_t) - f(x_0) = \int_0^t (y e^{yX_s}\mathbb{1}_{X_s<0} + z e^{zX_s}\mathbb{1}_{X_s\geq 0})\sigma(X_s)dW_s \tag{1.2.6}$$

$$+ \mu_- y \int_0^t e^{yX_s}\mathbb{1}_{X_s<0}ds \; + \mu_+ z \int_0^t e^{zX_s}\mathbb{1}_{X_s\geq 0}ds + \frac{(y+z)q}{2}L_t^0 \tag{1.2.7}$$

$$+ \frac{1}{2}\int_{\mathbb{R}} L_t^x (y^2 e^{yX_s}\mathbb{1}_{X_s<0} + z^2 e^{zX_s}\mathbb{1}_{X_s\geq 0})dx + \frac{(z-y)}{2}L_t^0. \tag{1.2.8}$$

De plus, on a $d\langle X\rangle_t = \sigma(X_t)^2 dt$: en appliquant la formule d'occupation à (1.2.8), on obtient

$$\frac{1}{2}\int_{\mathbb{R}} L_t^x (y^2 e^{yX_s}\mathbb{1}_{X_s<0} + z^2 e^{zX_s}\mathbb{1}_{X_s\geq 0})dx = \frac{1}{2}\left(\sigma_-^2 y^2 \int_0^t e^{yX_s}\mathbb{1}_{X_s<0}ds \; + \sigma_+^2 z^2 \int_0^t e^{zX_s}\mathbb{1}_{X_s\geq 0}ds\right).$$

Notons également que l'espérance de l'intégrale brownienne (1.2.6) est nulle. Ainsi,

$$\mathbb{E}_{x_0}[f(X_t)] - f(x_0) = \gamma_-(y)\mathbb{E}_{x_0}\left[\int_0^t e^{yX_s}\mathbb{1}_{X_s<0}ds\right] + \gamma_-(z)\mathbb{E}_{x_0}\left[\int_0^t e^{zX_s}\mathbb{1}_{X_s\geq 0}ds\right] + \frac{y(q-1)+z(1+q)}{2}\mathbb{E}_{x_0}[L_t^0]. \tag{1.2.9}$$

Rappelons que $X_t \underset{t\to+\infty}{\longrightarrow} +\infty$ presque sûrement car $\mu_- > 0, \mu_+ > 0$. Comme $z < 0$ et $y \geq 0$, on a $|f(x)| \leq 1$. Ainsi, par convergence dominée, on obtient l'asymptotique $\mathbb{E}_{x_0}[f(X_t)] \underset{t\to+\infty}{\longrightarrow} 0$. En injectant $y = 0$ et $z = z^*$ dans l'équation (1.2.9), on a alors

$$\mathbb{E}_{x_0}[L_t^0] = \frac{2}{(1+q)z^*}(\mathbb{E}_{x_0}[f(X_t)] - f(x_0)) \underset{t\to+\infty}{\longrightarrow} \frac{-2}{(1+q)z^*}(e^{x_0 z^*}\mathbb{1}_{x_0\geq 0} + \mathbb{1}_{x_0<0}) = \mathbb{E}_{x_0}[L_\infty^0]$$

par convergence monotone. De la même manière, en injectant $y = 0$ et $z < 0$ (resp. $y \geq 0$ et $z = z^* < 0$) dans l'équation (1.2.9), on obtient la finitude de $\varphi_+(z)$ (resp. $\varphi_-^{x_0}(y)$). L'équation fonctionnelle (1.2.3) est alors une conséquence de (1.2.9) passée à l'asymptotique $t \to +\infty$. Déterminons maintenant les expressions explicites des transformées de Laplace. En injectant $y = 0$ et $z < 0$ dans (1.2.3), on obtient

$$-(e^{x_0 z}\mathbb{1}_{x_0\geq 0} + \mathbb{1}_{x_0<0}) = \gamma_+(z)\varphi_+^{x_0}(z) + \frac{z(1+q)}{2}\frac{-2}{(1+q)z^*}(e^{x_0 z^*}\mathbb{1}_{x_0\geq 0} + \mathbb{1}_{x_0<0}) \tag{1.2.10}$$

ce qui prouve l'égalité (1.2.4). La démonstration de (1.2.5) est parfaitement analogue. □

Notons $g(x_0, x)$ la fonction de Green du processus (par rapport à la mesure de Lebesgue), donnée par $g(x_0, x) = \int_0^{+\infty} p_t(x_0, x)dt$ où p_t est la densité de transition du processus. Les transformées de Laplace φ_+ et φ_- étant les transformées de Laplace de $g(x_0, \cdot)$ respectivement sur $\mathbb{R}^+$ et $\mathbb{R}^-$, on obtient une formule explicite de g par inversion de Laplace.

Corollaire 1.16. *Pour $x_0, x \in \mathbb{R}$, on a :*

$$g(x_0,x) = \begin{cases} \frac{1}{\mu_+} e^{\frac{2\mu_+}{\sigma_+^2}(x_0-x)_+} & \text{si } x > 0, \\ \frac{1}{\mu_-} e^{\frac{2\mu_-}{\sigma_-^2}x}\left(e^{\frac{-2\mu_-}{\sigma_-^2}x_0} - 1 + \frac{\sigma_+^2(1-q)\mu_-}{\sigma_-^2(1+q)\mu_+}\right) & \text{si } x \leq x_0 \leq 0, \text{ et } x < 0 \\ \frac{1}{\mu_-}\left(1 + e^{\frac{2\mu_-}{\sigma_-^2}x}\left(\frac{\sigma_+^2(1-q)\mu_-}{\sigma_-^2(1+q)\mu_+} - 1\right)\right) & \text{si } x_0 \leq x < 0, \\ \frac{\sigma_+^2(1-q)}{\sigma_-^2(1+q)\mu_+} e^{-x_0\frac{2\mu_+}{\sigma_+^2}} e^{\frac{2\mu_-}{\sigma_-^2}x}, & \text{si } x < 0 \leq x_0. \end{cases}$$

En particulier, la frontière de Martin est égale à deux points isolés $\{-\infty, +\infty\}$, est minimale et les deux fonctions harmoniques positives de Martin sont :

- *La fonction constante* 1.
- *La fonction*

$$x_0 \in \mathbb{R} \longmapsto \begin{cases} e^{\frac{-2\mu_+}{\sigma_+^2} x_0} & \text{si } x_0 \geq 0, \\ \left(\frac{\sigma_+^2(1-q)\mu_-}{\sigma_-^2(1+q)\mu_+}\right)^{-1} \left(e^{\frac{-2\mu_-}{\sigma_-^2} x_0} + \frac{\sigma_+^2(1-q)\mu_-}{\sigma_-^2(1+q)\mu_+} - 1\right) & \text{si } x_0 < 0, \end{cases}$$

Ebauche de preuve. L'expression de g vient d'un calcul direct de transformée de Laplace inverse. Ici, on ne prend pas mesure $m = dx$ (cf. Hypothèse A.27 dans l'Annexe A) mais

$$m = \left((\sigma_+^2(1-q))^{-1}\mathbb{1}_{x<0} + (\sigma_-^2(1+q))^{-1}\mathbb{1}_{x\geq 0}\right) dx.$$

On peut montrer que, via cette mesure m, le processus est en dualité avec le processus avec barrière perméable en 0 dont les paramètres sont donnés par $(\sigma^+, \sigma^-, -\mu^+, -\mu^-, -q)$. La mesure de Green satisfait ainsi $G(x_0, dx) = \left((\sigma_+^2(1-q))g(x_0,x)\mathbb{1}_{x<0} + (\sigma_-^2(1+q))g(x_0,x)\mathbb{1}_{x\geq 0}\right) dx =: \widetilde{g}(x_0,x)dx$ où $\widetilde{g}(x_0,\cdot)$ est continue sur $\mathbb{R}$ car $\sigma_-^2(1+q)g(x_0,0^+) = \sigma_+^2(1-q)g(x_0,0^-)$. La mesure δ_0 est alors une mesure de référence. En considérant l'asymptotique de $\frac{\widetilde{g}(x_0,x)}{\widetilde{g}(0,x)} = \frac{g(x_0,x)}{g(0,x)}$ quand $x \to \pm\infty$, on obtient la structure de la frontière de Martin ainsi que l'ensemble des fonctions harmoniques positives. □

Remarque 1.17 (EDP satisfaite par les fonctions harmoniques)*.* On peut vérifier que les fonctions obtenues sont solutions de l'EDP

$$\begin{cases} \frac{\sigma^2(x)}{2}h''(x) + b(x)h'(x) = 0 \quad \text{si} \ \ x \neq 0 \\ (1+q)h'(0^+) = (1-q)h'(0^-) \end{cases}$$

(b) Cas $\mu_- < 0$ et $\mu_+ > 0$. Avec la même analyse, on obtient l'expression explicite de la fonction de Green. Plutôt que de réécrire l'expression de g dans ce cas, on donne ici simplement l'asymptotique en $\pm\infty$.

Proposition 1.18. *Supposons $\mu_- < 0$ et $\mu_+ > 0$. Alors, pour $x_0 \in \mathbb{R}$, on a*

$$g(x_0,x) \underset{x\to\pm\infty}{\sim} h_\pm(x_0), \tag{1.2.11}$$

où

$$h_+(x_0) = \frac{-2(1+q)}{\sigma_+^2(z^*(1+q)+y^*(q-1))}e^{x_0 y^*}\mathbb{1}_{x_0<0} + \frac{1}{\mu_+}\left(1 - \frac{y^*(q-1)}{z^*(1+q)+y^*(q-1)}e^{x_0 z^*}\right)\mathbb{1}_{x_0\geq 0}, \tag{1.2.12}$$

$$h_-(x_0) = \frac{1}{\mu_-}\left(1 - \frac{z^*(1+q)}{z^*(1+q)+y^*(q-1)}e^{x_0 y^*}\right)\mathbb{1}_{x_0<0} + \frac{-2(1+q)}{\sigma_-^2(z^*(1+q)+y^*(q-1))}e^{x_0 z^*}\mathbb{1}_{x_0\geq 0} \tag{1.2.13}$$

et où y^*, z^* sont donnés par (1.2.2)*. Par conséquent, la frontière de Martin est réduite à deux points isolés $\{-\infty, +\infty\}$, et est minimale. Les fonctions harmoniques associées sont données par h_+ et h_-.*

Dans le cas $\mu_- < 0$, $\mu_+ > 0$, on peut identifier grâce au théorème de représentation des fonctions harmoniques (Théorème A.37 de l'Annexe A) les fonctions $h_\pm$ aux fonctions $h_1 \ : x \longmapsto \mathbb{P}_x[X_t \underset{t\to\infty}{\longrightarrow} +\infty]$ et $h_2 \ : x \longmapsto \mathbb{P}_x[X_t \underset{t\to\infty}{\longrightarrow} -\infty]$ (à constante multiplicative près).

Corollaire 1.19. *Pour tout $x_0 \in \mathbb{R}$, on a*

$$\mathbb{P}_{x_0}[X_t \underset{t\to\infty}{\longrightarrow} +\infty] = 1 - \mathbb{P}_{x_0}[X_t \underset{t\to\infty}{\longrightarrow} -\infty] = \mu^+ h_+(x_0) \tag{1.2.14}$$

où h_+ est donné par (1.2.12)*.*

Démonstration. Tout d'abord, remarquons que les fonctions h_1 et h_2 sont harmoniques. En effet, si A est un ouvert relativement compact de $\mathbb{R}$, alors par la propriété de Markov forte, on a pour $x \in \mathbb{R}$:

$$\mathbb{E}_x\left[h_1(X_{\tau_A})\right] = \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_A}}\left[\mathbb{1}_{X_t \xrightarrow[t\to\infty]{} +\infty}\right]\right] = \mathbb{E}_x\left[\mathbb{1}_{X_t \xrightarrow[t\to\infty]{} +\infty}\circ\theta_{\tau_A}\right] = h_1(x)$$

car $\tau_A < +\infty$ presque sûrement. Par harmonicité de ces fonctions et par le Théorème A.37 de représentation des fonctions harmoniques positives, il existe des constantes A et B positives telles que pour tout $x \in \mathbb{R}$, $\mathbb{P}_x[X_t \xrightarrow[t\to\infty]{} +\infty] = Ah_+(x) + Bh_-(x)$. Or, pour $x \geq 0$,

$$\mathbb{P}_x[X_t \xrightarrow[t\to\infty]{} +\infty] \geq \mathbb{P}_x\left[\{\sigma_+ W_t + \mu_+ t \xrightarrow[t\to\infty]{} +\infty\} \cap \{\sigma_+ W_t + \mu_+ t > 0 \quad \forall t \geq 0\}\right] \qquad (1.2.15)$$

où $(W_t)_{t\geq 0}$ est un mouvement brownien standard issu de x sous $\mathbb{P}_x$. Le membre de droite de (1.2.15) tend vers 1 quand $x \to +\infty$ par propriété du mouvement brownien. Ainsi, $1 = \frac{A}{\mu_+} + B.0 = \frac{A}{\mu_+}$ i.e. $A = \mu^+$. De même, en considérant l'asymptotique $x \to -\infty$, on obtient $B = 0$ et donc le résultat. ☐

1.2.2.3 Exemple du brownien réfléchi 1D avec drift.

Soient $\sigma > 0$, $\mu > 0$ et X un mouvement brownien réfléchi sur $\mathbb{R}_+$ de paramètres σ et μ, c'est à dire :

$$dX_t = \sigma dW_t + \mu dt + dL_t^0(X), \quad t \geq 0$$

où W est un mouvement brownien standard. Le processus est alors transient car $\mu > 0$. Des considérations similaires (voir [48, Proposition 27]) fournissent l'expression explicite des fonctions de Green par rapport à la mesure de Lebesgue :

$$g(x_0, x) = \frac{e^{2\frac{\mu(x-x_0)}{\sigma}}}{\mu}\mathbb{1}_{0\leq x<x_0} + \frac{1}{\mu}\mathbb{1}_{x\geq x_0}.$$

En particulier, la frontière de Martin se réduit à un point (car $g(x_0,x) \xrightarrow[x\to+\infty]{} \frac{1}{\mu}$) et les fonctions harmoniques positives pour ce processus sont constantes.

1.2.3 Méthodes analytiques pour l'asymptotique en dimension 2

1.2.3.1 Plan d'attaque

En dimension 2, les fonctions de Green sont plus difficiles à étudier et il n'est pas toujours possible d'obtenir des formules explicites. Cependant, à partir d'une équation fonctionnelle, il est tout de même possible de déterminer leurs asymptotiques. La démarche adoptée dans cette thèse est la suivante.

(i) Etablissement d'une équation fonctionnelle à noyau reliant les transformées de Laplace des mesures de Green à l'intérieur du domaine, et sur les bords dans le cas du brownien réfléchi ou sur l'interface dans le cas d'un brownien dans un milieu stratifié.

(ii) Prolongement analytique des transformées de Laplace sur la (les) surface(s) de Riemann générée(s) par le(s) noyau(x) et détermination de leurs singularités.

(iii) Inversion de Laplace pour exprimer la fonction de Green sous forme d'une double intégrale complexe.

(iv) Annulation du (ou des) noyau(x) γ avec des branches $X(y), Y(x)$ satisfaisant $\gamma(X(y), y) = \gamma(x, Y(x)) = 0$. Utilisation du théorème des résidus pour transformer l'intégrale double en intégrale simple.

(v) Application de la méthode du point col pour obtenir le développement asymptotique des fonctions de Green.

(vi) Détermination des fonctions harmoniques via l'asymptotique du quotient des fonctions de Green.

L'approche suivie pour l'asymptotique a été développée dans [24, 37, 41, 52, 79, 80, 91] pour diverses applications : asymptotiques de la mesure invariante, asymptotiques des fonctions de Green, frontière de Martin. L'idée de prolonger analytiquement les fonctions génératrices ou les transformées de Laplace sur une surface de Riemann engendrée par le noyau provient de Malyshev [41, 91]. Une synthèse de la méthode du noyau (ou *kernel method* en anglais), basée sur une équation fonctionnelle, peut être trouvée dans Zhao [124].

L'un des apports majeurs de cette thèse, dans le cadre de cette méthode analytique, consiste à introduire une variable supplémentaire dans l'équation fonctionnelle pour étudier les diffusion à plusieurs domaines d'homogénéité introduite dans la Section 4.2.3 (voir Chapitre 4). Cette approche est illustrée dans l'exemple de la Section 1.2.2.2 : nous avons utilisé deux variables en dimension 1, y et z, pour exploiter la constance des paramètres dans chaque zone de l'espace d'états. Trois variables seront utilisées en dimension 2. Pour des méthodes différentes, l'utilisation d'une variable supplémentaire a été appliquée pour la première fois par Ignatyuk et al. [73] dans l'analyse des grandes déviations des marches aléatoires sur $\mathbb{Z}^{d+1}$ présentant une discontinuité sur un hyperplan. Elle a également été utilisée par Miyazawa [97] pour l'étude de la distribution stationnaire où trois zones d'homogénéité des paramètres dans le demi-plan émanent naturellement d'un modèle de files d'attente.

Dans les sections suivantes, cette méthode sera illustrée en l'appliquant à l'exemple classique du mouvement brownien planaire avec drift.

1.2.3.2 Equation fonctionnelle et transformée de Laplace

Dans cette section, nous établissons une équation satisfaite par la transformée de Laplace des fonctions de Green associées à un mouvement brownien avec drift (voir le point (i)). Plus précisément, nous démontrons la proposition suivante.

Proposition 1.20 (Équation fonctionnelle pour le mouvement brownien avec drift)**.** *Soit* $(Z_t)_{t\geq 0}$ *un mouvement brownien bidimensionnel de covariance* $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$*, de drift* $\mu = (\mu_1, \mu_2)^\top \in \mathbb{R}^2 \setminus \{0\}$*, et issu de* $z_0 \in \mathbb{R}^2$*. On définit*

$$\varphi^{z_0}(x,y) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} e^{(x,y)\cdot Z_t}\, dt\right],$$

la transformée de Laplace de la fonction de Green associée à Z*. Alors, pour tout* $(x,y) \in \mathbb{R}^2$ *vérifiant* $(x,y)\cdot\mu < 0$*, on a*

$$\gamma(x,y)\,\varphi^{z_0}(x,y) = -e^{(x,y)\cdot z_0}, \tag{1.2.16}$$

où

$$\gamma(x,y) = \frac{1}{2}\left(\sigma_{11}x^2 + 2\sigma_{12}xy + \sigma_{22}y^2\right) + \mu_1 x + \mu_2 y. \tag{1.2.17}$$

L'équation (1.2.16) est qualifiée de *fonctionnelle* bien qu'elle implique une seule fonction inconnue, à savoir φ^{z_0} (ce qui simplifie considérablement l'analyse ultérieure).

Démonstration. Sans perte de généralité, identifions Z comme la solution de l'EDS $dZ_t = \sqrt{\Sigma}\, dW_t + \mu\, dt$, où $(W_t)_{t\geq 0}$ est un mouvement brownien bidimensionnel standard. Par la formule d'Itô, pour tout $(x,y) \in \mathbb{R}^2$ et tout $t \geq 0$:

$$e^{(x,y)\cdot Z_t} = e^{(x,y)\cdot z_0} + \int_0^t e^{(x,y)\cdot Z_s} \begin{pmatrix} x \\ y \end{pmatrix} \cdot \sqrt{\Sigma}\, dW_s + \gamma(x,y)\int_0^t e^{(x,y)\cdot Z_s}\, ds.$$

Comme l'intégrale stochastique est une vraie martingale, on obtient, en prenant l'espérance :

$$\mathbb{E}_{z_0}\left[e^{(x,y)\cdot Z_t}\right] = e^{(x,y)\cdot z_0} + \gamma(x,y)\,\mathbb{E}_{z_0}\left[\int_0^t e^{(x,y)\cdot Z_s}\, ds\right].$$

Sous l'hypothèse $(x, y) \cdot \mu < 0$, un calcul direct montre que

$$\mathbb{E}_{z_0}\left[e^{(x,y)\cdot Z_t}\right] \underset{t\to+\infty}{\longrightarrow} 0$$

(l'expression de la densité de transition de Z étant explicite). Par le théorème de convergence monotone, on obtient alors l'équation (1.2.16), ce qui conclut la preuve. □

1.2.3.3 Inversion de Laplace

Nous présentons maintenant la formule d'inversion de Laplace inverse bidimensionnelle (voir le point (iii)).

Lemme 1.21 (Formule d'inversion de Laplace)**.** *Soit $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ une fonction mesurable et soit φ sa transformée de Laplace définie par*

$$\varphi(x, y) = \int_{\mathbb{R}\times\mathbb{R}} e^{ax+by} f(a, b) da db \tag{1.2.18}$$

pour les complexes x, y tels que l'intégrale ci-dessus converge. Soient $A < B$ et $C < D$ les abscisses de convergence de φ, c'est-à-dire les réels pour lesquels l'intégrale converge absolument pour tout $(x, y) \in (A, B) \times (C, D)$. Soit $(\epsilon_1, \epsilon_2) \in (A, B) \times (C, D)$. Alors, pour tout $(a, b) \in \mathbb{R}^2$ où f est continue,

$$f(a, b) = \frac{1}{(2i\pi)^2} \int_{\epsilon_1 - i\infty}^{\epsilon_1 + i\infty} \int_{\epsilon_2 - i\infty}^{\epsilon_2 + i\infty} e^{-ax-by} \varphi(x, y) dy dx \tag{1.2.19}$$

sous réserve de la convergence de cette intégrale au sens de la valeur principale

Le lecteur intéressé pourra se référer à Doetsch [33, Theorems 24.3 et 24.4] pour la preuve de ce résultat (en dimension 1 et dont la preuve s'adapte à la dimension deux), et à Brychkov [16] qui traite directement le cas multidimensionnel.

Exemple du mouvement brownien avec drift. Vérifions que les conditions du Lemme 1.21 sont satisfaites pour le mouvement brownien avec drift, afin que la formule d'inversion puisse s'appliquer.

Lemme 1.22 (Inversion de Laplace, cas du mouvement brownien avec drift)**.** *Soit $g^{(a_0,b_0)}(a, b)$ la fonction de Green d'un mouvement brownien de covariance Σ muni d'un drift μ, et $\varphi^{(a_0,b_0)}$ sa transformée de Laplace (voir (1.2.16)). Alors, pour $(a, b) \neq (a_0, b_0)$ et (ϵ_1, ϵ_2) vérifiant $(\epsilon_1, \epsilon_2) \cdot \mu < 0$, on a*

$$g^{(a_0,b_0)}(a, b) = \frac{1}{(2i\pi)^2} \int_{\epsilon_1 - i\infty}^{\epsilon_1 + i\infty} \int_{\epsilon_2 - i\infty}^{\epsilon_2 + i\infty} e^{-ax-by} \varphi^{(a_0,b_0)}(x, y) dy dx \tag{1.2.20}$$

où l'intégrale du terme de droite converge en valeur principale.

Démonstration. En considérant l'expression explicite de la densité de transition du mouvement brownien, on peut montrer de manière élémentaire la continuité de la fonction $g^{(a_0,b_0)}$ sur $\mathbb{R}^2 \setminus \{(a_0, b_0)\}$. Il suffit donc, par le Lemme 1.21, de démontrer la convergence en valeur principale de l'intégrale (1.2.20). Il s'agit ici de démontrer que la quantité

$$\lim_{\substack{R\to+\infty \\ S\to+\infty}} \frac{1}{(2i\pi)^2} \int_{\epsilon_1 - iR}^{\epsilon_1 + iR} \int_{\epsilon_2 - iS}^{\epsilon_2 + iS} e^{-ax-by} \varphi(x, y) dy dx$$

existe et que la limite ne dépend pas de la manière dont R et S tendent vers $+\infty$. Comme l'intégrale n'est pas absolument convergente, nous effectuons une intégration par parties : par (1.2.16), on a

$$\begin{aligned} -\int_{\epsilon_1 - iR}^{\epsilon_1 + iR} e^{-ax-by} \varphi(x, y) dx &= \int_{\epsilon_1 - iR}^{\epsilon_1 + iR} \frac{e^{(a_0-a)x+(b_0-b)y}}{\gamma(x, y)} dx \\ &= \underbrace{e^{(b_0-b)y} \left[\frac{e^{(a_0-a)x}}{i(a_0 - a)\gamma(x, y)}\right]_{x=\epsilon_1 - iR}^{x=\epsilon_1 + iR}}_{=:A_R(y)} - \underbrace{e^{(b_0-b)y} \int_{\epsilon_1 - iR}^{\epsilon_1 + iR} \frac{e^{(a_0-a)x}}{i(a_0 - a)} \frac{d}{dx} [\gamma(x, y)] dx}_{=:B_R(y)} \end{aligned}$$

Tout d'abord, montrons que

$$\sup_{S>0}\left|\int_{\epsilon_2-iS}^{\epsilon_2+iS} A_R(y)dy\right| \underset{R\to+\infty}{\longrightarrow} 0. \tag{1.2.21}$$

Pour cela, remarquons que

$$\int_{t\in\mathbb{R}} \frac{1}{|\gamma(-\epsilon_1+iR,\epsilon_2+it)|}dt = R\int_{|s|\geq 0}\frac{1}{|\gamma(-\epsilon_1+iR,\epsilon_2+isR)|}ds.$$

Or,

$$\gamma(-\epsilon_1+iR,\epsilon_2+isR) = -\frac{R^2}{2}\big(\sigma_{11}+O(R^{-1})+2(\sigma_{12}+O(R^{-1}))s+(\sigma_{22}+O(R^{-1}))s^2\big)$$

où les termes $O(R^{-1})$ sont indépendants de s. En prenant R assez grand pour que la matrice $(\sigma_{i,j}+O(R^{-1}))_{1\leq i,j\leq 2}$ soit strictement elliptique, on obtient R_0 et $C>0$ tels que

$$\int_{t\in\mathbb{R}} \frac{1}{|\gamma(-\epsilon_1+iR,\epsilon_2+it)|}dt \leq \frac{C}{R^2}\int_{s\geq 0}\frac{ds}{1+s^2}$$

pour tout $R\geq R_0$. Ceci implique (1.2.21). Traitons à présent le terme B_R. Il suffit de montrer que

$$\int_{\epsilon_1-i\infty}^{\epsilon_1+i\infty}\int_{\epsilon_2-i\infty}^{\epsilon_2+i\infty}\left|\frac{d}{dx}\left[\frac{1}{\gamma(x,y)}\right]\right||dy||dx| < +\infty. \tag{1.2.22}$$

(où, si $x=\epsilon_1+is$, on utilise la notation $|dx|=ds$ et de même pour $|dy|$). Considérons $M>0$ que nous fixerons plus tard. On a :

$$\int_{\epsilon_1-i\infty}^{\epsilon_1+i\infty}\int_{\epsilon_2-i\infty}^{\epsilon_2+i\infty}\left|\frac{d}{dx}\left[\frac{1}{\gamma(x,y)}\right]\right||dy||dx| \leq 2M\sup_{-M\leq t\leq M}\int_{\epsilon_1-i\infty}^{\epsilon_1+i\infty}\left|\frac{\sigma_{11}x+\sigma_{12}(\epsilon_2+it)+\mu_1}{\gamma(x,\epsilon_2+it)^2}\right||dx| \tag{1.2.23}$$

$$+\int_{|t|\geq M}\int_{\epsilon_1-i\infty}^{\epsilon_1+i\infty}\left|\frac{\sigma_{11}x+\sigma_{12}(\epsilon_2+it)+\mu_1}{\gamma(x,\epsilon_2+it)^2}\right||dx|dt. \tag{1.2.24}$$

Le terme de droite dans (1.2.23) est fini par l'expression explicite de γ. Pour le terme (1.2.24), on fait le changement de variables $x=\epsilon_1+its$ de sorte à avoir

$$\int_{|t|\geq M}\int_{\epsilon_1-i\infty}^{\epsilon_1+i\infty}\left|\frac{\sigma_{11}x+\sigma_{12}(\epsilon_2+it)+\mu_1}{\gamma(x,\epsilon_2+it)^2}\right||dx|dt = \int_{|t|\geq M} t\int_{s\in\mathbb{R}}\left|\frac{\sigma_{11}(\epsilon_1+its)+\sigma_{12}(\epsilon_2+it)+\mu_1}{\gamma(\epsilon_1+its,\epsilon_2+it)^2}\right|dsdt.$$

Or,

$$\gamma(\epsilon_1+its,\epsilon_2+it) = -\frac{t^2}{2}\big(\sigma_{11}+O(t^{-1})+2(\sigma_{12}+O(t^{-1}))s+(\sigma_{22}+O(t^{-1}))s^2\big)$$

où les termes $O(t^{-1})$) ne dépendent pas de s. En choisissant M tel que la matrice $(\sigma_{i,j}+O(t^{-1}))_{1\leq i,j\leq 2}$ soit strictement elliptique pour tout $t\geq M$, on obtient $C>0$ tel que

$$\int_{|t|\geq M}\int_{\epsilon_1-i\infty}^{\epsilon_1+i\infty}\left|\frac{\sigma_{11}x+\sigma_{12}(\epsilon_2+it)+\mu_1}{\gamma(x,\epsilon_2+it)^2}\right||dx|dt \leq C\int_{|t|\geq M}|t|\int_{s\in\mathbb{R}}\frac{|ts|+|t|+1}{t^4(1+s^2)^2}dsdt < +\infty$$

ce qui conclut sur la convergence en valeur principale de l'intégrale (1.2.19) pour le mouvement brownien de covariance Σ et de drift μ. □

1.2.3.4 Etude du noyau et passage à l'intégrale simple

Etudions à présent le point (iv). Nous détaillons les propriétés du noyau $\gamma(x,y)$ défini par (1.2.17), où Σ est définie positive.

L'équation elliptique $\gamma(x,y)=0$ définit une surface de Riemann $\mathcal{E} := \{(x,y)\in\mathbb{C}^2 \mid \gamma(x,y)=0\}$, paramétrable par la sphère de Riemann à l'aide de fonctions homographiques [13, Section 3.2]. Comme les équations $\gamma(x,\cdot)=0$ et $\gamma(\cdot,y)=0$ sont de degré 2, elles admettent chacune exactement deux solutions $Y^{\pm}(x)$ et $X^{\pm}(y)$ comptées avec multiplicité, données par

$$Y^{\pm}(x) = \frac{1}{\sigma_{22}}\Big(-\sigma_{12}x - \mu_2 \pm \sqrt{(\sigma_{12}^2-\sigma_{11}\sigma_{22})x^2 + 2(\mu_2\sigma_{12}-\mu_1\sigma_{22})x + \mu_2^2}\Big),$$

et une expression symétrique pour $X^{\pm}$. Les fonctions $Y^{\pm}$ et $X^{\pm}$ – appelées branches de γ – possèdent des points de branchement x_{min}, x_{max} et y_{min}, y_{max} respectivement, donnés par

$$x_{min} = \frac{\mu_2\sigma_{12}-\mu_1\sigma_{22}-\sqrt{D_1}}{\det(\Sigma)}, \quad x_{max} = \frac{\mu_2\sigma_{12}-\mu_1\sigma_{22}+\sqrt{D_1}}{\det(\Sigma)},$$

$$y_{min} = \frac{\mu_1\sigma_{12}-\mu_2\sigma_{11}-\sqrt{D_2}}{\det(\Sigma)}, \quad y_{max} = \frac{\mu_1\sigma_{12}+\mu_2\sigma_{11}-\sqrt{D_2}}{\det(\Sigma)},$$

où

$$D_1 = (\mu_2\sigma_{12}-\mu_1\sigma_{22})^2 + \mu_2^2\det(\Sigma), \quad D_2 = (\mu_1\sigma_{12}-\mu_2\sigma_{11})^2 + \mu_1^2\det(\Sigma).$$

Les points x_{min}, x_{max} et y_{min}, y_{max} représentent les abscisses et ordonnées extrémales de l'ellipse réelle d'équation $\gamma(x,y)=0$ (voir Figure 1.5a). De plus, $x_{min}<0<x_{max}$ et $y_{min}<0<y_{max}$.

À chaque point $(x,y)\in\mathcal{E}$ correspondent canoniquement deux autres points $(x,y')\in E$ et $(x',y)\in E$ obtenus via les racines des polynômes de degré deux $\gamma(x,\cdot)$ et $\gamma(\cdot,y)$; avec $y=y'$ lorsque $x\in\{x_{min},x_{max}\}$ et $x=x'$ lorsque $y\in\{y_{min},y_{max}\}$. On définit alors les automorphismes de Galois η et ζ sur $\mathcal{E}$ par $\zeta(x,y)=(x,y')$ et $\eta(x,y)=(x',y)$. Ces automorphismes s'expriment explicitement via les branches $Y^{\pm}(x)$ et $X^{\pm}(y)$ caractérisées par $\gamma(x,Y^{\pm}(x))=0$ et $\gamma(X^{\pm}(y),y)=0$ (voir Figure 1.5a). Enfin, les branches de $Y^{\pm}(x)$ et $X^{\pm}(y)$ sont réelles si et seulement si $x\in[x_{min},x_{max}]$ et $y\in[y_{min},y_{max}]$ (voir Figure 1.5a).

Exemple du mouvement brownien avec drift.

Grâce au travail d'analyse complexe effectué précédemment et au théorème des résidus, nous transformons la représentation de la fonction de Green (1.2.19) d'intégrale double en intégrale simple. Illustrons cela avec mouvement brownien avec drift.

Lemme 1.23 (Intégrale double à intégrale simple)**.** *Soit g la fonction de Green du mouvement brownien de covariance Σ et de drift μ (non nul). Alors, si $b>b_0$, on a :*

$$g^{(a_0,b_0)}(a,b) = \frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \exp((a_0-a)x+(b_0-b)Y^{+}(x))\frac{dx}{\partial_y\gamma(x,Y^{+}(x))}. \tag{1.2.25}$$

On dispose de représentations analogues dans les cas $b<b_0$, $a>a_0$ et $a<a_0$.

Démonstration. Comme l'expression (1.2.16) de φ est explicite, l'égalité $\gamma(x,Y^{+}(x))=0$ donne que l'intégrande (1.2.19) pour x fixé, admet un pôle en $y=Y^{+}(x)$. L'application du théorème des résidus permet de conclure, on renvoie à la preuve du Lemme 2.27 pour plus de détails. ☐

1.2.3.5 Prolongement méromorphe des transformées de Laplace

Étudions ici le point (ii). Cette étape est nécessaire pour présenter la méthode dite du *point col* dans la section suivante, qui requiert de modifier les contours d'intégration jusqu'à des domaines où les transformées de Laplace ne sont pas initialement définies.

La méthode consiste à itérer l'équation fonctionnelle afin de prolonger les transformées de Laplace sur toute la surface $\mathcal{E}$ définie dans la section précédente. Cette idée vient de Malyshev [91]. A partir de l'ensemble des points $(x, y) \in \mathcal{E}$ où les transformées de Laplace convergent, on applique les automorphismes de Galois η et ζ présentés en Section 1.2.3.4) à (x, y) en justifiant que $\eta(x, y)$ et $\zeta(x, y)$ peuvent être injectés dans l'équation fonctionnelle. Cela fournit des expressions méromorphes des transformées de Laplace sur un domaine étendu. En répétant ce procédé, on peut espérer les prolonger sur toute la surface $\mathcal{E}$. Cette difficulté n'apparaît pas dans le cas du mouvement brownien avec drift, pour lequel les membres de l'équation fonctionnelle (1.2.16) sont déjà explicites.

Illustrons ce procédé pour le mouvement brownien réfléchi dans le quadrant. Dans ce cas, l'équation fonctionnelle suivante est valable pour $\Re(x) < 0$ et $\Re(y) < 0$ (voir Chapitre 2, Section 2.2) :

$$-\gamma(x, y)\varphi^{z_0}(x, y) = \gamma_1(x, y)\varphi_1^{z_0}(y) + \gamma_2(x, y)\varphi_2^{z_0}(x) + e^{(x,y)\cdot z_0}, \tag{1.2.26}$$

où $\varphi_1^{z_0}(y)$ et $\varphi_2^{z_0}(x)$ sont les transformées de Laplace des temps locaux sur les axes $\{x = 0\}$ et $\{y = 0\}$ respectivement et où

$$\gamma_1(x, y) = R^1 \cdot (x, y) = x + r_1 y, \qquad \gamma_2(x, y) = R^2 \cdot (x, y) = r_2 x + y.$$

En restreignant l'équation fonctionnelle à $\mathcal{E} \cap \{\Re(x),\ \Re(y) < 0\}$, cela donne

$$0 = \gamma_1(x, y)\varphi_1^{z_0}(y) + \gamma_2(x, y)\varphi_2^{z_0}(x) + e^{(x,y)\cdot z_0}, \quad (x, y) \in \mathcal{E} \cap \{\Re(x),\ \Re(y) < 0\}.$$

Dans le cas du Chapitre 2, nous supposons $\mu_1 > 0$ et $\mu_2 > 0$, ce qui implique $\Re(Y^-(x)) < 0$ pour $\Re(x) \leq x_{max}$ (voir Lemme 2.17 Chapitre 2). Cela permet d'obtenir directement le prolongement méromorphe de $\varphi_2(x)$ sur $\{\Re(x) < x_{max}\}$ par

$$\varphi_2(x) = \frac{\gamma_1(x, Y^-(x))\varphi_1^{z_0}(Y^-(x)) + e^{(x,Y^-(x))\cdot z_0}}{\gamma_2(x, Y^-(x))}, \quad \Re(x) < x_{max}$$

car $\varphi_1(y)$ est holomorphe sur $\{\Re(y) < 0\}$. Le raisonnement est analogue pour le prolongement de φ_1 sur $\{\Re(y) < y_{max}\}$. Ces prolongements suffisent à l'étude de l'asymptotique des fonctions de Green. Le cas $\mu_1 < 0, \mu_2 < 0$ étudié par Bousquet-Mélou et al. [13] (pour les mesures stationnaires) est plus technique. Une fois φ_1 et φ_2 prolongés par l'étape précédente, on réinjecte les automomorphismes dans l'équation fonctionnelle (valables sur un domaine étendu) pour prolonger les transformées de Laplace à un domaine encore plus étendu (et ainsi de suite).

1.2.3.6 Méthode du point col

Nous présentons à présent le point (v). Avant d'introduire la méthode du point col, rappelons brièvement le principe de la méthode de Laplace.

Heuristique de la méthode de Laplace

La méthode du point col s'inspire de la méthode de Laplace, utilisée pour déterminer l'asymptotique de certaines intégrales. La méthode de Laplace permet d'obtenir l'asymptotique d'intégrales du type

$$\int_a^b g(x)e^{tf(x)}dx$$

lorsque $t \to +\infty$ où $-\infty < a < b < +\infty$, $g : [a, b] \longrightarrow \mathbb{R}$ est une fonction continue et où $f\ : [a, b] \longrightarrow \mathbb{R}$ est une fonction réelle de classe C^2. Supposons que f admette un maximum global en $x_0 \in (a, b)$, que $g(x_0) \neq 0$ et que $f''(x_0) < 0$. Heuristiquement, lorsque $t \to +\infty$, la contribution principale à l'intégrale provient d'un voisinage de x_0. Puisque

$$f(x) \underset{x \to x_0}{\sim} f(x_0) + \frac{f''(x_0)}{2}(x - x_0)^2, \tag{1.2.27}$$

on obtient formellement

$$\int_a^b g(x)e^{tf(x)}dx \underset{t\to+\infty}{\sim} e^{tf(x_0)}g(x_0)\int_{x_0-\epsilon}^{x_0+\epsilon} e^{t\frac{f''(x_0)}{2}(x-x_0)^2}dx$$

$$\underset{t\to+\infty}{\sim} \sqrt{\frac{2\pi}{|f''(x_0)|}}e^{tf(x_0)}g(x_0).$$

Pour une démonstration rigoureuse, on pourra se référer par exemple à [32, Chapitre IV].

Méthode du point col

La méthode du point col repose sur les mêmes principes que la méthode de Laplace. Etant donnée une intégrande contenant une exponentielle dont certains paramètres tendent vers $\pm\infty$ (typiquement (1.2.25)), on déforme le chemin d'intégration de manière à ce que l'exponentielle atteigne un extremum le long d'un chemin *de plus grande pente*.

Exemple du mouvement brownien avec drift

Nous illustrons à présent cette technique dans le cadre du mouvement brownien avec drift. En posant $(a,b) = (r\cos(\alpha), r\sin(\alpha))$ (où r est voué à tendre vers l'infini et α à tendre vers $\alpha_0 \in [0,2\pi]$), l'intégrande de (1.2.25) se réécrit

$$\frac{e^{a_0x+b_0Y^+(x)}}{\partial_y\gamma(x,Y^+(x))}e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}.$$

Le principe de la méthode est le suivant.

- **(A)** Maximiser la fonction définie par $x\in[x_{min},x_{max}] \longmapsto \cos(\alpha)x+\sin(\alpha)Y^+(x)$ (ou, de manière plus générale, maximiser l'expression apparaissant dans l'exponentielle).
- **(B)** Construire dans $\mathbb{C}$ un chemin de plus grande pente à l'aide du Lemme 2.A.1 (lemme de Morse à paramètre), de sorte que l'approximation (1.2.27) soit exacte le long de ce chemin.
- **(C)** Déformer le contour d'intégration à l'aide du théorème des résidus afin de remplacer le contour initial par celui construit en (B).
- **(D)** Vérifier enfin que le terme asymptotique dominant correspond bien à celui issu de la contribution correspondante au chemin de plus grande pente.

On prouve la proposition suivante.

Proposition 1.24 (Asymptotiques des fonctions de Green pour le mouvement brownien avec drift)**.** *Si $g^{(a_0,b_0)}(\cdot)$ désigne la fonction de Green du mouvement brownien de covariance Σ et de drift μ issu de (a_0,b_0), on a l'asymptotique suivante : pour $\alpha_0\in[0,2\pi]$,*

$$g^{(a_0,b_0)}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to+\infty\\ \alpha\to\alpha_0}}{\sim} A^{(a_0,b_0)}(\alpha)\frac{e^{-r(\cos(\alpha)x(\alpha)+y(\alpha)\sin(\alpha))}}{\sqrt{r}} \tag{1.2.28}$$

où $A^{(a_0,b_0)}(\alpha)$ est définie par :

$$A^{(a_0,b_0)}(\alpha) = \frac{C(\alpha)}{\sqrt{2\pi(\sigma_{11}\sin^2(\alpha)-2\sigma_{12}\sin(\alpha)\cos(\alpha)+\sigma_{22}\cos^2(\alpha))}}e^{a_0x(\alpha)+b_0y(\alpha)} \tag{1.2.29}$$

avec

$$C(\alpha) = \sqrt{\frac{\sin(\alpha)}{\partial_y\gamma(x(\alpha),y(\alpha))}} = \sqrt{\frac{\cos(\alpha)}{\partial_x\gamma(x(\alpha),y(\alpha))}}$$

Démonstration. Afin d'alléger la présentation, nous nous limitons au cas $\alpha_0 \in (0,\pi)$, de sorte que l'expression (1.2.25) reste valable (car $b \to +\infty$).

(A) Pour $\alpha \in [0, 2\pi)$, on définit

$$(x(\alpha), y(\alpha)) \ := \text{argmax}_{(x,y):\gamma(x,y)=0}(x \cos\alpha + y \sin\alpha), \tag{1.2.30}$$

voir la Figure 1.5b pour une interprétation géométrique. L'application $\alpha \in [0, 2\pi]/\{0 = 2\pi\} \longmapsto (x(\alpha), y(\alpha)) \in \{(x, y) \ : \gamma(x, y) = 0\}$ est un C^∞ difféomorphisme. Pour $\alpha \in (0, \pi)$, la fonction $x \longmapsto \cos(\alpha)\, x + \sin(\alpha)\, Y^+(x)$ atteint son maximum sur $[x_{min}, x_{max}]$ en l'unique point $x(\alpha)$, et $(x(\alpha), Y^+(x(\alpha))) = (x(\alpha), y(\alpha))$. Les valeurs de $x(\alpha)$ et $y(\alpha)$ sont explicites, voir (2.5.5) et (2.5.6) Chapitre 2. Définissons la fonction :

$$F(x, \alpha) = -\cos(\alpha)x - \sin(\alpha)Y^+(x) + \cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha). \tag{1.2.31}$$

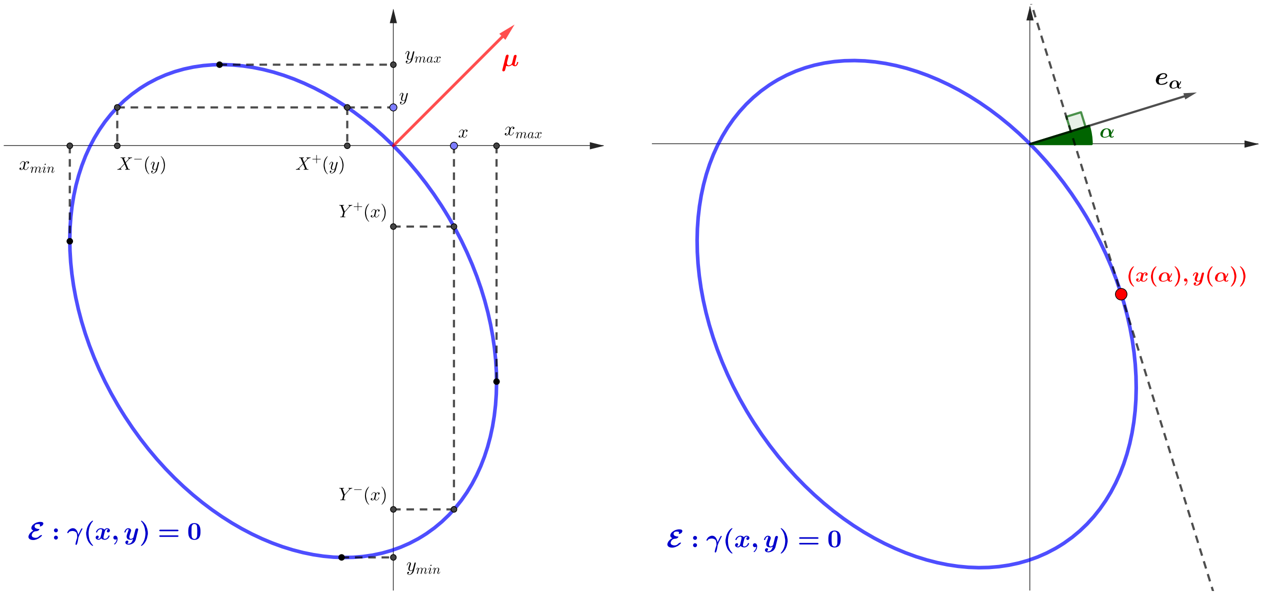
Par définition de $x(\alpha)$ on a, pour tout $\alpha \in (0, \pi)$,

$$F(x(\alpha), \alpha) = 0, \quad \text{et} \quad F'_x(x(\alpha), \alpha) = 0.$$

Après quelques calculs (voir Section 2.5 Chapitre 2), on obtient :

$$F''_x(x(\alpha), \alpha) = \frac{\sigma_{11} \sin^2(\alpha) - 2\sigma_{12} \sin(\alpha) \cos(\alpha) + \sigma_{22} \cos^2(\alpha)}{\partial_y \gamma(x(\alpha), y(\alpha)) \sin(\alpha)} > 0, \tag{1.2.32}$$

l'inégalité stricte résultant du caractère défini positif de Σ.



(a) Branches réelles $X^\pm$ et $Y^\pm$.

(b) Illustration du point col $(x(\alpha), y(\alpha))$.

FIGURE 1.5 : Branches et point col sur l'ellipse réelle $\mathcal{E}$ d'équation $\gamma(x, y) = 0$. Le vecteur e_α désigne $(\cos(\alpha), \sin(\alpha))$.

(B) Fixons $\alpha_0 \in (0, \pi)$. On applique le Lemme 2.A.1 : soient K, η, $\Omega(0, \alpha_0)$ et $x \ : \Omega(0, \alpha_0) \longrightarrow \mathbb{C}$ les quantités qui y sont introduites. Fixons $\epsilon \in (0, K)$ et posons :

$$\Gamma_{x,\alpha} = \{x(it, \alpha) \mid t \in [-\epsilon, \epsilon]\}$$

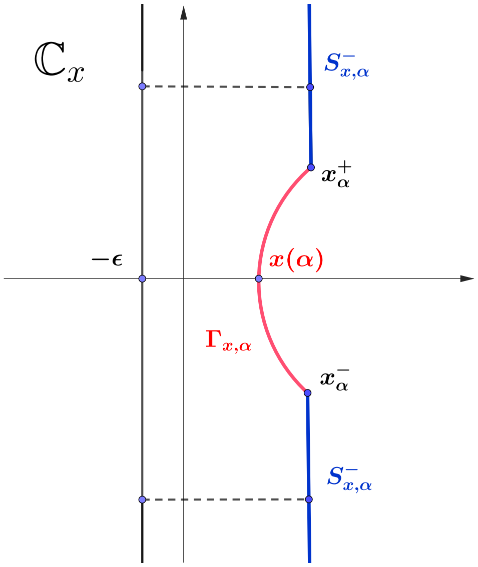


FIGURE 1.6 : Changement du contour d'intégration

(voir Figure 1.6). Le chemin $\Gamma_{x,\alpha}$ est appelé chemin de plus grande pente, appellation justifiée par l'égalité

$$F(x(it,\alpha),\alpha) = -t^2, \quad t \in (-\epsilon,\epsilon).$$

(C) Nous modifions maintenant le contour d'intégration. Introduisons les notations

$$x^-_\alpha := x(-i\epsilon,\alpha), \quad x^+_\alpha := x(i\epsilon,\alpha)$$

ainsi que les contours suivants définis par

$$S^+_{x,\alpha} = \{x^+_\alpha + it \mid t \geq 0\ \}, \quad S^-_{x,\alpha} = \{x^-_\alpha - it \mid t \geq 0\}.$$

On remarque que $Im(x^+_\alpha) > 0$ et $Im(x^-_\alpha) < 0$ (quitte à réduire ϵ) car $x'_\omega(0,\alpha) > 0$, voir (2.A.1). En appliquant le théorème des résidus (sans pôle) on obtient, pour $b > b_0$,

$$g((a_0,b_0)^T,(a,b)^T) = \frac{1}{2\pi i}\int_{S^-_{x,\alpha}\cup\Gamma_{x,\alpha}\cup S^+_{x,\alpha}} \frac{e^{a_0x+b_0Y^+(x)}}{\gamma'_y(x,Y^+(x))} e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}dx \tag{1.2.33}$$

(voir la Figure 1.6). La justification de ce changement de chemin est détaillée au Chapitre 2 dans la preuve de la Proposition 2.6.

(D) Précisons tout d'abord l'asymptotique de l'intégrale le long du chemin $\Gamma_{x,\alpha}$. Par définition de $\Gamma_{x,\alpha}$, on a

$$\frac{1}{2\pi i}\int_{\Gamma_{x,\alpha}} \frac{e^{a_0x+b_0Y^+(x)}}{\partial_y\gamma(x,Y^+(x))} e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}dx$$
$$= \frac{e^{-r(\cos(\alpha)x(\alpha)+y(\alpha)\sin(\alpha))}}{2\pi}\int_{-\epsilon}^{\epsilon} \frac{e^{a_0x(it,\alpha)+b_0Y^+(x(it,\alpha))}}{\partial_y\gamma(x(it,\alpha),Y^+(x(it,\alpha)))}\partial_\omega x(it,\alpha)e^{-rt^2}dt.$$

À l'aide d'un développement de Taylor, ainsi qu'en utilisant les équations (2.A.1), (1.2.32) et l'égalité $Y^+(\alpha) = y(\alpha)$ (cette dernière est justifiée par le fait que $\alpha \in (0,\pi)$), on montre que la quantité précédente est équivalente à :

$$\frac{e^{-r(\cos(\alpha)x(\alpha)+y(\alpha)\sin(\alpha))}}{2\pi}\frac{e^{a_0x(\alpha)+b_0Y^+(x(\alpha))}}{\partial_y\gamma(x(\alpha),Y^+(x(\alpha)))}\partial_\omega x(0,\alpha)\int_{-\epsilon}^{\epsilon}e^{-rt^2}dt \tag{1.2.34}$$

$$\sim A^{(a_0,b_0)}(\alpha)\frac{e^{-r(\cos(\alpha)x(\alpha)+y(\alpha)\sin(\alpha))}}{\sqrt{r}} \tag{1.2.35}$$

lorsque $r\to+\infty$ et $\alpha\to\alpha_0$ où $A^{(a_0,b_0)}(\alpha)$ est défini par (1.2.29). L'approximation de Taylor (1.2.34) est justifiée dans la preuve du Lemme 2.33 du Chapitre 2.

En ce qui concerne les intégrales le long de $S^{\pm}_{x,\alpha}$ dans l'écriture (1.2.33), on peut montrer (c.f. preuve du Lemme 2.29 du Chapitre 2) l'inégalité suivante :

$$\left|\int_{S^{\pm}_{x,\alpha}}\frac{e^{a_0x+b_0Y^+(x)}}{\partial_y\gamma(x,Y^+(x))}e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}dx\right|\leq\frac{D}{b-b_0}e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha)-\epsilon^2)}$$

où D est une constante positive. Ainsi, les intégrales le long des chemins $S^{\pm}_{x,\alpha}$ dans la représentation (1.2.33) sont asymptotiquement négligeables devant celle le long de $\Gamma_{x,\alpha}$. On obtient finalement l'asymptotique (1.2.28). □

On remarque que l'asymptotique coïncide avec celle donnée dans l'Exemple A.35 de l'Annexe A dans le cas particulier où $\Sigma = Id$. Par ailleurs, la preuve du Lemme 2.33 du Chapitre 2 permet d'obtenir un développement asymptotique complet des fonctions de Green.

En conséquence de l'asymptotique obtenue en (1.2.29), on identifie la frontière de Martin à l'ellipse réelle d'équation $\gamma(x,y)=0$. Les fonctions harmoniques associées sont alors données par $(a_0,b_0)\longmapsto e^{a_0x(\alpha)+b_0y(\alpha)},\quad\alpha\in[0,2\pi)$.

1.2.3.7 Cas particuliers : directions limites

Lors de l'étude du point (vi), certaines directions nécessiteront une attention particulière.

Pour les mouvements browniens réfléchis dans le quadrant (Chapitres 2 et 3), l'étude asymptotique des fonctions de Green le long des bords implique certaines considérations supplémentaires. En particulier, l'asymptotique lorsque $\alpha=0$ et $r\to+\infty$ fait intervenir un théorème Taubérien faisant le lien entre l'asymptotique d'une fonction à l'infini et l'asymptotique de sa transformée de Laplace en sa première singularité, voir Dai [24, Annexe C], Doetsch [33] ou la Section 4.A pour le cas bivarié. Le principe consiste ensuite à relier l'asymptotique obtenue lorsque $\alpha>0,\alpha\to0$ par la méthode de point col et l'asymptotique lorsque $\alpha=0$. Un raisonnement analogue s'appliquera aux directions le long de la barrière perméable dans le Chapitre 4.

Par ailleurs, l'intégrande étudiée peut contenir un pôle contrairement au cas du mouvement brownien avec drift. Prenons l'exemple du mouvement brownien réfléchi (Chapitre 2) : typiquement, la représentation intégrale de la fonction de Green fait intervenir la quantité

$$\frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\frac{e^{-ax-bY^+(x)}}{\partial_y\gamma(x,Y^+(x))}\varphi_2^{z_0}(x)dx$$

où $\varphi_2^{z_0}(x)$ peut avoir un pôle réel positif x^*. L'analyse asymptotique dans le cas où le point col $x(\alpha)$ rencontre x^* (cf. Figure 1.6) requiert une analyse plus fine.

Un dernier cas, propre au Chapitre 4, présente deux ellipses donc quatre points de branchement x^+_{min}, x^-_{min}, x^+_{max}, x^-_{max} au lieu de deux pour le cas du mouvement brownien avec drift (réfléchi ou non). Ce phénomène empêche alors de faire passer le chemin d'intégration $\Gamma_{x,\alpha}$ par l'axe réel comme dans la Figure 1.6 au-delà des coupures. L'asymptotique des fonctions de Green lorsque $x(\alpha)$ rencontre ces coupures sera également étudiée en détails.

1.2.3.8 Recherche du terme principal : non nullité des constantes principales

Nous soulignons un dernier point technique concernant l'étape (v) (Section 1.2.3.1). Grâce aux techniques présentées précédemment, on obtient typiquement des asymptotiques de la forme

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) = \Big(h_{\alpha_0}(z_0) + o_{\substack{r\to+\infty \\ \alpha\to\alpha_0}}(1)\Big) e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))} r^{-1/2}$$

(éventuellement avec des vitesses différentes, voir la Section 1.3 pour une présentation détaillée des résultats). Dans le cas du mouvement brownien avec drift, $h_{\alpha_0}(z_0)$ est donnée par l'expression (1.2.29) qui est évidemment non nulle. Cependant, dans le cas général, il n'est pas évident que cette constante soit non nulle (en particulier dans les Chapitres 2 et 3). Afin que le développement asymptotique soit consistant et pour identifier cette fonction à une fonction harmonique, il est alors nécessaire d'établir la non nullité de cette constante. Pour ce faire, on considère z_0 asymptotiquement "loin" de l'origine pour justifier que $h_{\alpha_0}(z_0) \neq 0$, puis on utilise une propriété de Markov pour montrer la non nullité de $h_\alpha(z_0)$ pour les autres z_0.

Dans le cas particulier où $\alpha_0 = 0$ pour les Chapitres 2, 3 et 4, on montre que le premier terme du développement asymptotique issu de la méthode du point col est nul et qu'il est nécessaire de pousser le développement jusqu'au second terme.

1.2.4 Méthodes pour expliciter les transformées de Laplace en dimension 2

Jusqu'ici, les fonctions harmoniques étudiées en dimension 1 (Section 1.2.2) ou pour le mouvement brownien avec drift dans $\mathbb{R}^d$ (Exemple A.35) sont explicites. Il en est de même dans l'article de Ernst et Franceschi [37] pour le mouvement brownien réfléchi obliquement dans le demi-plan (avec drift) et dans le Chapitre 4.

Cependant, les formules obtenues ne sont pas toujours closes (voir par exemple Kourkova et Malyshev [79, Section 2.5]). Dans les Chapitres 2 et 3, on obtient des fonctions harmoniques s'écrivant sous la forme

$$z_0 \longmapsto \gamma_1(x(\alpha), y(\alpha))\varphi_1^{z_0}(y(\alpha)) + \gamma_2(x(\alpha), y(\alpha))\varphi_2^{z_0}(x(\alpha)) + e^{z_0\cdot(x(\alpha),y(\alpha))} \tag{1.2.36}$$

où les fonctions $\gamma_i(x(\alpha), y(\alpha))$, $i = 1, 2$ sont explicites, et où $\varphi_i^{z_0}$, $i = 1, 2$ sont les transformées de Laplace des mesures d'occupation sur les bords du quadrant lorsque le processus est issu de z_0. Nous présentons ici deux méthodes permettant d'expliciter ces transformées de Laplace, la seconde sera illustrée dans le Chapitre 3.

1.2.4.1 Forme intégrale par la théorie de Riemann-Hilbert-Carleman

En adoptant une approche par les problèmes aux frontières de type Riemann-Hilbert-Carleman, on peut exprimer certaines transformées de Laplace sous la forme d'une intégrale explicite. Cette approche est présentée dans Litvinchuk [90], Muskhelishvili [100] et Gakhov [58], voir également Franceschi [48, Annexe C] pour une brève introduction. L'idée principale consiste à montrer que les transformées de Laplace satisfont un problème aux limites, puis de les identifier aux formes intégrales proposées par la théorie générale de Riemann-Hilbert-Carleman. Cette méthode a d'abord été exploitée dans le cas discret par Fayolle et al. [41], Bacelli et Fayolle [8], Kourkova et Raschel [80], Raschel [107] puis également dans le cas continu, voir Franceschi [49, 57].

Cependant, cette méthode ne sera pas utilisée dans la suite : en particulier, dans le cas du mouvement brownien réfléchi dégénéré (Chapitre 3), nous utiliserons plutôt la *méthode par compensation* pour obtenir directement les transformées de Laplace sous la forme de séries de fonctions exponentielles. Dans le cas du processus avec barrière perméable dans le Chapitre 4, l'équation fonctionnelle sera simple à résoudre et donnera directement l'expression des transformées de Laplace sous une forme explicite.

1.2.4.2 Méthode par compensation pour le cas dégénéré

Dans les années 1990, Adan et al. [2, 3, 4] ont développé une approche appelée *méthode par compensation* pour expliciter la mesure stationnaire de processus bidimensionnels discrets avec des applications en théorie des files d'attente. La mesure stationnaire peut alors s'exprimer comme une somme infinie alternée. Cette méthode, qui s'avère également utile pour expliciter des fonctions génératrices de marches aléatoires à petits pas (voir Adan et al. [1]), a été utilisée plus récemment pour donner l'expression de fonctions harmoniques pour des marches aléatoires singulières dans le quart de plan par Hoang et al. [67]. Elle a également porté ses fruits pour expliciter des mesures stationnaires pour des systèmes de particules dans Franceschi et al. [51] et pour déterminer la frontière de Martin d'un mouvement brownien dégénéré tué sur le bord d'un cône dans Franceschi [50].

Nous présentons ici cette approche de manière heuristique pour le mouvement brownien dégénéré réfléchi obliquement dans le quadrant (Chapitre 3). Le principe consiste ici à relier la notion de fonction harmonique à une EDP avec conditions frontières et d'effectuer une construction géométrique sur une parabole pour résoudre l'EDP en question. Soit

$$(Z_t)_{t\geq 0} = (z_0 + vW_t + \mu t + RL_t)_{t\geq 0}$$

un mouvement brownien réfléchi dégénéré dans $\mathbb{R}_+^2$, où $(W_t)_{t\geq 0}$ est un mouvement brownien standard unidimensionnel, $(L_t)_{t\geq 0} = \begin{pmatrix} L_t^1 \\ L_t^2 \end{pmatrix}_{t\geq 0}$ est le temps local du processus sur le bord, $v = (1,-1), R = \begin{pmatrix} R_1 & R_2 \end{pmatrix} = \begin{pmatrix} 1 & r_2 \\ r_1 & 1 \end{pmatrix}$ et où $\mu = \mu_1, \mu_2$ satisfait $\mu_1 > 0$ et $\mu_2 > 0$. Soit h une fonction régulière qui satisfait l'équation différentielle partielle suivante, avec conditions aux bords :

$$\begin{cases} (H_0) & \mathcal{G}h = 0 \qquad\qquad \text{sur} \quad (0,+\infty)^2, \\ (H_1) & \partial_{R_1} h(0,y) = 0, \quad y \geq 0, \\ (H_2) & \partial_{R_2} h(x,0) = 0, \quad x \geq 0, \end{cases} \tag{1.2.37}$$

où $\mathcal{G} = \frac{1}{2}\nabla \cdot \Sigma \nabla + \mu \cdot \nabla$. Montrons que h est une fonction harmonique (ou voir [37, Section 6]). En appliquant la formule d'Itô, on obtient :

$$h(Z_t) = h(z_0) + \int_0^t v \cdot \nabla h(Z_s)\, dW_s + \int_0^t \mathcal{G}h(s)\, ds + \sum_{i=1}^{2} \int_0^t R_i \cdot \nabla h(Z_s)\, dL_s^i.$$

Si h satisfait (1.2.37), alors

$$h(Z_t) = h(z_0) + \int_0^t \nabla h(Z_s)\, dW_s.$$

Ainsi, pour tout ouvert A relativement compact de $\mathbb{R}_+^2$, si T_A désigne le temps d'atteinte de A pour le processus, on a par le théorème d'arrêt : $\mathbb{E}_{z_0}[h(Z_{T_A})] = h(z_0)$. Ainsi, h est harmonique.

Le principe de la méthode par compensation consiste à construire des fonctions de la forme

$$h(x,y) = \sum_{n\in\mathbb{Z}} c_n e^{a_n x + b_n y},$$

telles que chaque terme exponentiel satisfasse la condition (H_0), et à “compenser” les exponentielles avec les constantes $(c_n)_{n\in\mathbb{Z}}$ de manière à satisfaire les conditions (H_1) et (H_2) de la manière suivante :

$$h(x,y) = \ldots + \rlap{$\displaystyle\overbrace{\phantom{c_{-2}e^{a_{-2}x+b_{-2}y} + c_{-1}e^{a_{-1}x+b_{-1}y}}}^{\in(H_2)}$} c_{-2}e^{a_{-2}x+b_{-2}y} + \underbrace{c_{-1}e^{a_{-1}x+b_{-1}y} + \rlap{$\displaystyle\overbrace{\phantom{e^{a_0x+b_0y} + c_1e^{a_1x+b_1y}}}^{\in(H_2)}$} e^{a_0x+b_0y}}_{\in(H_1)} + \underbrace{c_1e^{a_1x+b_1y} + c_2e^{a_2x+b_2y}}_{\in(H_1)} + \ldots. \tag{1.2.38}$$

Tout d'abord, on remarque que la condition $\mathcal{G}e^{a_n x + b_n y} = 0$ s'écrit $\gamma(a_n, b_n) = 0$ où $\gamma(x,y) = \frac{1}{2}(x-y)^2 + \mu_1 x + \mu_2 y$. L'équation $\gamma(x,y) = 0$ définit une parabole $\mathcal{P}$, voir Figure 1.7. Fixons $(a_0, b_0) \in \mathcal{P}$

et cherchons $(a_1, b_1) \in \mathcal{P}$ de telle sorte que l'on ait $e^{a_0x+b_0y} + c_1e^{a_1x+b_1y} \in (H_2)$. Cette condition est satisfaite si et seulement si

$$(r_2a_0 + b_0)e^{a_0x} + c_1(r_2a_1 + b_1)e^{a_1x}$$

pour tout $x \geq 0$ ce qui impose

$$a_0 = a_1, \quad \text{et} \quad c_1 = -\frac{r_2a_0 + b_0}{r_2a_1 + b_1}$$

(cf. Figure 1.7 pour l'unique choix du point (a_1, b_1) sur la parabole). De la même manière, choisir $c_2 \in \mathbb{R}$ et $(a_2, b_2) \in \mathcal{P}$ tels que $c_1e^{a_1x+b_1y} + c_2e^{a_2x+b_2y} \in (H_0) \cap (H_1)$ revient à imposer

$$a_1 = a_2, \quad \text{et} \quad c_2 = -\frac{a_1 + r_1b_1}{a_2 + r_2b_1}c_1.$$

Sous réserve que les quotients soient bien définis à chaque étape, on peut construire récursivement les suites $(c_n)_{n\in\mathbb{Z}}$ et $(a_n, b_n)_{n\in\mathbb{Z}} \subset \mathcal{P}$ "en escalier" (voir Figure 1.7) de sorte que les conditions (H_1) et (H_2) dans l'équation (1.2.38) soient satisfaites. Les conditions $(H_0), (H_1), (H_2)$ étant linéaires, on obtient ainsi formellement une famille de fonctions harmoniques indexée par $(a_0, b_0) \in \mathcal{P}$.

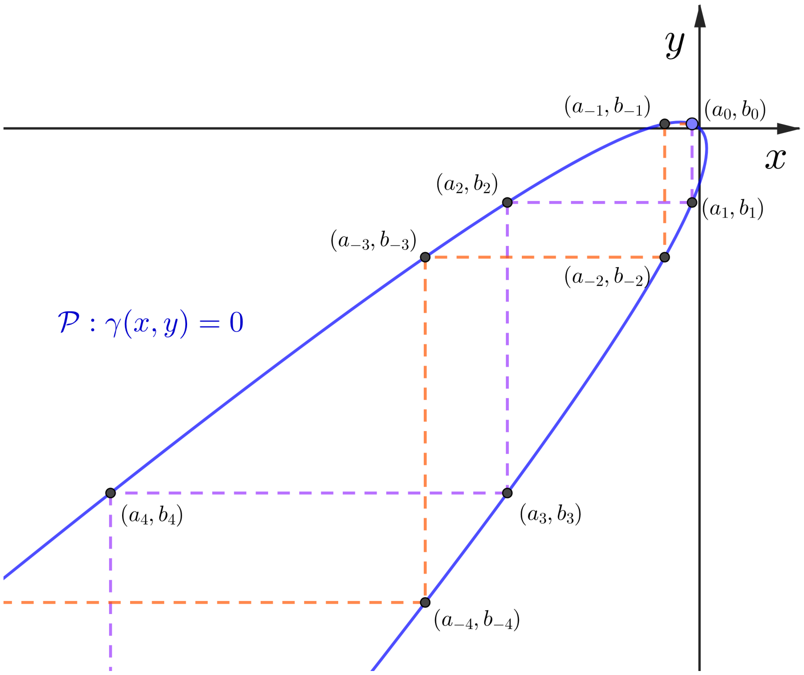


FIGURE 1.7 : Parabole $\mathcal{P}$ et les points (a_n, b_n) définis récursivement depuis (a_0, b_0).

Dans le Chapitre 3, nous obtiendrons les fonctions harmoniques comme limite du noyau de Martin sous la forme (1.2.36). En itérant l'équation fonctionnelle sur chacun des points (a_n, b_n) de la parabole $\mathcal{P}$, on pourra identifier $\varphi_2^{(x,y)}(x(\alpha))$ à $c_1e^{a_1x+b_1y} + c_2e^{a_2x+b_2y} + \ldots$ et $\varphi_1^{z}(y(\alpha))$ à ... $+ c_{-2}e^{a_{-2}x+b_{-2}y} + c_{-1}e^{a_{-1}x+b_{-1}y}$ où $(a_0, b_0) = (x(\alpha), y(\alpha))$.

1.3 Principaux résultats

Cette section présente les principaux résultats du manuscrit. La Section 1.3.1 esquisse les résultats sur le mouvement brownien non dégénéré dans un cône convexe (Chapitre 2), tandis que la Section 1.3.2 présente les résultats sur le mouvement brownien dégénéré réfléchi obliquement dans le quadrant $\mathbb{R}_+^2$

(Chapitre 3). Enfin la Section 1.3.3 décrit les résultats relatifs au processus dans $\mathbb{R}^2$ avec barrière perméable (Chapitre 4). Pour le processus avec barrière perméable, nous considérons uniquement le cas de “flux continu” ($q = q_0$, Section 1.1.2.3). Le Chapitre 4 (Section 4.6) propose une généralisation à tout $q \in (-1, 1)$, sous réserve de certaines conjectures sur la densité de transition.

Pour chaque processus, on présente l’équation fonctionnelle liant les quantités importantes, on décrit les fonctions harmoniques de Martin (quand elles peuvent être explicitées, cf. Sections 1.3.2 et 1.3.3) et on donne la structure de la frontière de Martin. L’ensemble des fonctions harmoniques positives découle alors du théorème de représentations des fonctions harmoniques positives (Théorème 1.12).

1.3.1 Brownien non dégénéré réfléchi obliquement dans un cône

Soit Z un mouvement brownien non dégénéré réfléchi obliquement dans un cône d’angle $\beta \in (0, \pi)$, dont les vecteurs de réflexion forment des angles δ, ε par rapport aux axes et dont l’orientation du drift μ effectue un angle θ par rapport à l’axe des abscisses (voir Figure 1.8). On suppose sans perte de généralité que la covariance vaut $\Sigma = Id$ (voir Section 1.1.1.3).

Hypothèses. On suppose $\delta + \varepsilon < \beta + \pi$, ce qui garantit que le processus est bien défini : Z est alors une semi-martingale (voir Section 1.1.1.3). L’unique hypothèse supplémentaire concerne le drift μ :

$$\theta \in (0, \beta),$$

voir Figure 1.8. Le processus est alors transient.

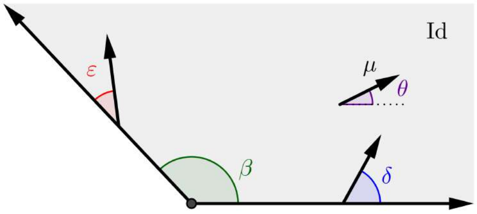


FIGURE 1.8 : Paramètres du mouvement brownien réfléchi dans un cône.

Equation fonctionnelle. L’analyse dans le Chapitre 2 est effectuée sur le cas du quadrant $\mathbb{R}_+^2$ avec covariance quelconque, et les résultats sont transposés sur le cas du cône par une transformation linéaire. Décrivons l’équation fonctionnelle établie dans le cas du quadrant. Notons $R = \begin{pmatrix} 1 & r_2 \\ r_1 & 1 \end{pmatrix}$ la matrice de réflexion. On rappelle que la fonction de Green $g^{z_0}(z) := g(z_0, z)$ par rapport à la mesure de Lebesgue est caractérisée par

$$G(z_0, A) \ := \int_0^{+\infty} \mathbb{P}_{z_0}(Z_t \in A) dt = \int_A g(z_0, z) dz.$$

On définit également les mesures de Green sur les bords du quadrant comme

$$H_i(z_0, A) \ := \mathbb{E}_{z_0}\left[\int_0^{\infty} \mathbb{1}_A(Z_t) dL_t^i\right], \quad i = 1, 2$$

où $(L_t^1)_{t\geq 0}$ et $(L_t^2)_{t\geq 0}$ sont les temps locaux du processus sur les axes $\{x = 0\}$ et $\{y = 0\}$, respectivement. La mesure H_1 (resp. H_2) est supportée sur l’axe $\{x = 0\}$ (resp. $\{y = 0\}$). Si $\varphi^{z_0}(x, y)$, $\varphi_1^{z_0}(y)$ et $\varphi_2^{z_0}(x)$ désignent les transformées de Laplace respectives de $g(z_0, \cdot)$, $H_1(z_0, \cdot)$ et $H_2(z_0, \cdot)$, alors on prouve l’équation fonctionnelle suivante

$$-\gamma(x, y)\varphi^{z_0}(x, y) = \gamma_1(x, y)\varphi_1^{z_0}(y) + \gamma_2(x, y)\varphi_2^{z_0}(x) + e^{(x,y)\cdot z_0}, \quad \Re(x) < 0, \Re(y) < 0, \qquad (1.3.1)$$

où $\gamma(x,y), \gamma_1(x,y)$ et $\gamma_2(x,y)$ sont définis par

$$\gamma(x,y) = \frac{1}{2}(\sigma_{11}x^2 + 2\sigma_{12}xy + \sigma_{22}y^2) + \mu_1 x + \mu_2 y,$$

$$\gamma_1(x,y) = R^1 \cdot (x,y) = x + r_1 y, \qquad \gamma_2(x,y) = R^2 \cdot (x,y) = r_2 x + y.$$

Asymptotiques des fonctions de Green. On donne à présent les résultats principaux du Chapitre 2. Pour cela, on définit les angles ω^* et ω^{**} par :

$$\omega^* := \theta - 2\delta \quad \text{et} \quad \omega^{**} := \theta + 2\varepsilon. \tag{1.3.2}$$

Notons que $\omega^* < \theta < \omega^{**}$. Alors, la fonction de Green $g^{z_0}(\rho\cos\omega, \rho\sin\omega)$ de ce processus admet les développements asymptotiques suivants lorsque $\omega \to \omega_0 \in (0,\beta)$ et $\rho \to \infty$: pour tout $n \in \mathbb{N}$,

- Si $\omega^* < \omega_0 < \omega^{**}$, la fonction de Green admet une asymptotique de type point col :

$$g^{z_0}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to\omega_0}}{\sim} e^{-2\rho|\mu|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{1}{\sqrt{\rho}} \sum_{k=0}^{n} \frac{c_k^{z_0}(\omega)}{\rho^k}. \tag{1.3.3}$$

- Si $\omega_0 < \omega^*$, l'asymptotique est de type pôle :

$$g^{z_0}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to\omega_0}}{\sim} c_*^{z_0} e^{-2\rho|\mu|\sin^2(\omega+\delta-\theta)} + e^{-2\rho|\mu|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{1}{\sqrt{\rho}} \sum_{k=0}^{n} \frac{c_k^{z_0}(\omega)}{\rho^k}. \tag{1.3.4}$$

- Si $\omega^{**} < \omega_0$, l'asymptotique est de type pôle :

$$g^{z_0}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to\omega_0}}{\sim} c_{**}^{z_0}(z_0) e^{-2\rho|\mu|\sin^2(\omega-\varepsilon-\theta)} + e^{-2\rho|\mu|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{1}{\sqrt{\rho}} \sum_{k=0}^{n} \frac{c_k^{z_0}(\omega)}{\rho^k}. \tag{1.3.5}$$

Les constantes $c_*^{z_0}$ et $c_{**}^{z_0}$ dans les asymptotiques précédentes sont strictement positives, $c_0^{z_0}(\omega) > 0$ dans (1.3.3), et les $c_k^{z_0}(\omega)$, $k \geq 0$ sont des constantes telles que $c_k^{z_0}(\omega) \underset{\omega\to\omega_0}{\longrightarrow} c_k^{z_0}(\omega_0)$. On a quatre cas possibles selon l'appartenance, ou non, des angles ω^*, ω^{**} à $(0,\beta)$: ceux-ci sont illustrés dans la Figure 1.9.

On donne également l'asymptotique des fonctions de Green lorsque $\omega \to \omega_0 \in \{0, \omega^*, \omega^{**}, \beta\}$. Pour $\omega_0 = 0$, on a $c_0^{z_0}(\omega) \underset{\omega\to 0}{\sim} h_0(z_0)\omega$ et $c_1(\omega) \underset{\omega\to 0}{\sim} \kappa h_0(z_0)$ où κ et $h_0(z_0)$ sont des constantes strictement positives (lorsque $\omega^* < 0$). On montre alors les asymptotiques suivantes.

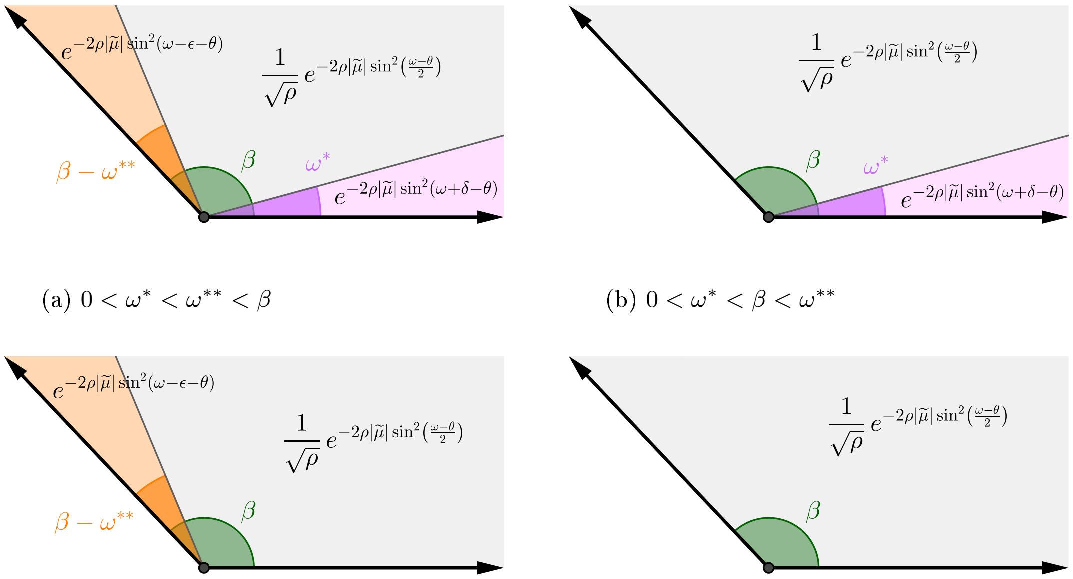
- Si $\omega_0 = 0$ et $\omega^* < 0$, on a une compétition entre les deux premiers termes de la somme $\sum_{k=0}^{n} \frac{c_k^{z_0}(\omega)}{\rho^k}$, ce qui fournit l'asymptotique suivante :

$$g^{z_0}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to 0}}{\sim} e^{-2\rho|\mu|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{h_0(z_0)}{\sqrt{\rho}} \left(\omega + \frac{\kappa}{\rho}\right). \tag{1.3.6}$$

- Si $\omega_0 = 0$ et $\omega^* > 0$, l'asymptotique (1.3.4) reste valable.

De même, pour $\omega_0 = \beta$, le résultat symétrique s'applique. Nous expliquerons dans les Propositions 2.46 et 2.47 que ω^* et ω^{**} correspondent en un certain sens aux pôles des transformées de Laplace des fonctions de Green, et que ω correspond au point col obtenu lors de l'inversion de la transformée de Laplace.

Il reste à énoncer les asymptotiques des fonctions de Green lorsque $\omega \to \omega^*$ ou $\omega \to \omega^{**}$, c'est à dire lorsque le point col *rencontre* les pôles.



(a) $0 < \omega^* < \omega^{**} < \beta$

(b) $0 < \omega^* < \beta < \omega^{**}$

(c) $\omega^* < 0 < \omega^{**} < \beta$

(d) $\omega^* < 0 < \beta < \omega^{**}$

FIGURE 1.9 : Asymptotiques de la fonction de Green selon la direction ω_0. Lorsque ω_0 est dans la région grise, l'asymptotique est donnée par (1.3.3), dans la région violette par (1.3.4), et dans la région orange par (1.3.5).

- Lorsque $\rho(\omega-\omega^*)^2 \to 0$, l'asymptotique est donnée par (1.3.4), mais la constante $c_*^{z_0}$ du premier terme doit être remplacée par $\frac{1}{2}c_*^{z_0}$.
- Lorsque $\rho(\omega-\omega^*)^2 \to c > 0$ pour une constante c :
  - Si $\omega < \omega^*$, l'asymptotique est encore donnée par (1.3.4), mais la constante $c_*^{z_0}$ du premier terme doit être remplacée par $\frac{1}{2}c_*^{z_0}(1+\Phi(\sqrt{c}A))$ pour une certaine constante A.
  - Si $\omega > \omega^*$, l'asymptotique est encore donnée par (1.3.4), mais la constante $c_*^{z_0}$ du premier terme doit être remplacée par $\frac{1}{2}c_*^{z_0}(1-\Phi(\sqrt{c}A))$ pour une certaine constante A.

  Dans les points précédents, Φ désigne la fonction :

$$\Phi(z) := \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)\,dt.$$

- Lorsque $\rho(\omega-\omega^*)^2 \to \infty$:
  - Si $\omega < \omega^*$, l'asymptotique est donnée par (1.3.4).
  - Si $\omega > \omega^*$, l'asymptotique est donnée par (1.3.3) et on a :

$$c_*^{z_0}(\omega) \underset{\omega\to\omega^*}{\sim} \frac{c^{z_0}}{\omega-\omega^*}$$

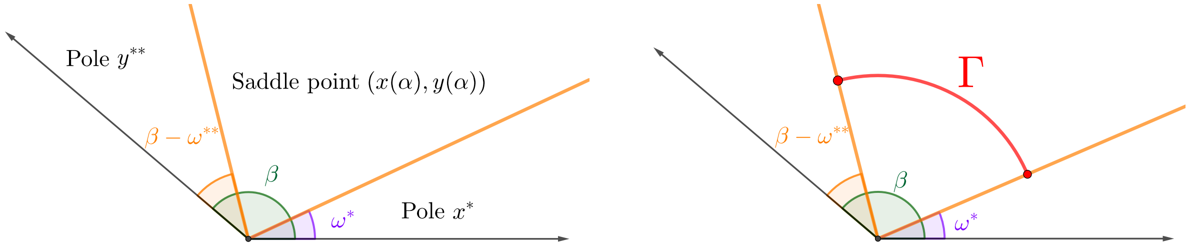
    pour une certaine constante c^{z_0}.

De même, pour $\omega_0 = \omega^{**}$, le résultat symétrique s'applique.

Fonctions harmoniques et frontière de Martin. Le terme principal de chaque asymptotique décrite précédemment fournit une fonction harmonique. Si $\omega \in (0,\beta) \cap (\omega^*, \omega^{**})$, on note $h_\omega(z_0) = c_0^{z_0}(\omega)$ la constante du terme principal dans l'asymptotique (1.3.3). On note également $h_{\omega^*}(z_0) = c_*^{z_0}$ si $\omega^* > 0$ et $h_{\omega^{**}}(z_0) = c_{**}^{z_0}$ (cf. asymptotiques (1.3.4) et (1.3.5)). On rappelle la notation $h_0(z_0)$ issue de l'asymptotique en $\omega_0 = 0$ donnée par (1.3.6), et on note $h_\beta(z_0)$ son analogue en $\omega_0 = \beta$. La frontière de Martin est alors minimale et donnée par

$$\Gamma = [0,\beta] \cap [\omega^*, \omega^{**}]$$

(voir Figure 1.10). La famille de fonctions $(h_\omega)_{\omega\in\Gamma}$ constitue l'ensemble des fonctions harmoniques obtenues par compactification de Martin.



(a) Trois zones pour trois types d'asymptotiques.

(b) Frontière de Martin Γ.

Figure 1.10 : Types d'asymptotiques et frontière de Martin pour le mouvement brownien réfléchi dans un cône. Dans les zones orange et violette, le type d'asymptotique correspond aux pôles (lorsqu'ils existent) des transformées de Laplace des temps locaux sur les axes adjacents.

Remarque 1.25. En intégrant formellement l'équation rétrograde [62, Equation (8.2)] (voir également la Remarque 1.13), on obtient l'EDP avec conditions frontière satisfaites par les fonctions harmoniques :

$$\begin{cases} \left(\frac{1}{2}\nabla \cdot \Sigma\nabla + \mu \cdot \nabla\right) h = 0 & \\ \partial_{R_1} h(0,y) = 0, & y \geq 0 \\ \partial_{R_2} h(x,0) = 0, & x \geq 0. \end{cases}$$

1.3.2 Brownien dégénéré réfléchi obliquement dans le quadrant

Soit Z un mouvement brownien dégénéré réfléchi obliquement dans le quadrant $\mathbb{R}_+^2$ de covariance Σ (où $\det(\Sigma) = 0$) avec les directions de réflexion $R_1 = \begin{pmatrix} 1 \\ r_1 \end{pmatrix}$ sur l'axe $\{x = 0\}$ et $R_2 = \begin{pmatrix} r_2 \\ 1 \end{pmatrix}$ sur l'axe $\{y = 0\}$ (cf. Section 1.1.1.3).

Hypothèses. Les hypothèses concernant le processus sont les suivantes :

- Le drift μ satisfait :
  $$\mu_1 > 0, \quad \mu_2 > 0.$$
  Le processus est alors transient.
- La matrice Σ est de rang 1, et la direction brownienne $v = (v_1, v_2)$ (vecteur propre associé à la valeur propre non nulle) du processus est antidiagonale, c'est à dire $v_1 v_2 < 0$.
- La matrice de réflexion $R = \begin{pmatrix} 1 & r_2 \\ r_1 & 1 \end{pmatrix}$ satisfait
  $$r_1 > -\frac{\sigma_2}{\sigma_1}, \quad r_2 > -\frac{\sigma_1}{\sigma_2},$$
  assurant que le processus s'éloigne du coin du quadrant (voir Figure 1.11).

Cette dernière condition sur R n'est pas nécessaire pour les résultats concernant les asymptotiques des fonctions de Green, mais est requise pour que les expressions explicites des fonctions harmoniques soient valides.

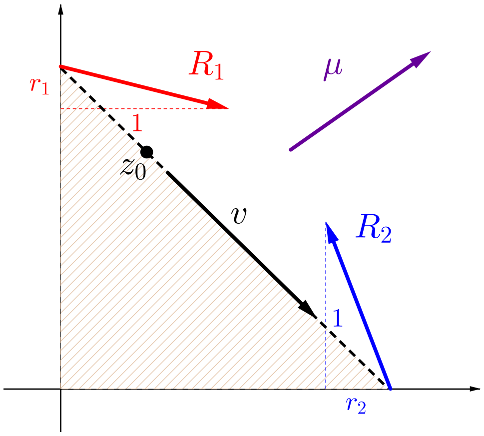


FIGURE 1.11 : Réflexions R_1, R_2 sur les bords, le drift μ, et la direction v du mouvement brownien dégénéré. Le processus issu de z_0 n'atteint jamais la zone hachurée.

Equation fonctionnelle. En définissant, comme dans la Section 1.3.1, les quantités $\varphi^{z_0}(x,y)$, $\varphi_1^{z_0}(y)$ et $\varphi_2^{z_0}(x)$, on obtient l'équation fonctionnelle (1.3.1). La principale différence est que l'équation $\gamma(x,y)=0$ définit ici une parabole, plutôt qu'une ellipse : on a (à une dilatation des axes près),

$$\gamma(x,y) = \frac{1}{2}(x-y)^2 + \mu_1 x + \mu_2 y. \tag{1.3.7}$$

Asymptotiques des fonctions de Green. Par des techniques analogues au cas non dégénéré, on définit les angles limites $\alpha^* < \alpha^{**}$ délimitant une, deux ou trois zones où les asymptotiques diffèrent (on renvoie à (3.1.13) et (3.1.14), Chapitre 3 pour leur définition). On définit le point col $(x(\alpha), y(\alpha)$ sur la parabole $\mathcal{P} := \{(x,y)\in\mathbb{R}^2 \mid \gamma(x,y)=0\}$ par :

$$(x(\alpha), y(\alpha)) := \underset{\gamma(x,y)=0}{\operatorname{argmax}} \quad (x\cos(\alpha) + y\sin(\alpha)),$$

voir (3.5.5) et (3.4.1) pour les expressions explicites. Alors, pour $z_0 \in \mathbb{R}_+^2$, on a les asymptotiques suivantes.

- Si $\alpha^* < \alpha < \alpha^{**}$, la fonction de Green admet une asymptotique de type point col :

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to+\infty \\ \alpha\to\alpha_0}}{\sim} C(\alpha) h_\alpha(z_0) \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}, \tag{1.3.8}$$

- Si $0 < \alpha_0 < \alpha^*$, l'asymptotique de type pôle :

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to+\infty \\ \alpha\to\alpha_0}}{\sim} C^* h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x(\alpha^*)+\sin(\alpha)y(\alpha^*))}, \tag{1.3.9}$$

- Si $\alpha^{**} < \alpha < \pi/2$, l'asymptotique de type pôle :

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to+\infty \\ \alpha\to\alpha_0}}{\sim} C^{**} h_{\alpha^{**}}(z_0) e^{-r(\cos(\alpha)x(\alpha^{**})+\sin(\alpha)y(\alpha^{**}))} \tag{1.3.10}$$

où C, C^*, C^{**} sont des constantes strictement positives et où $h_\alpha(z_0), h_{\alpha^*}(z_0), h_{\alpha^{**}}(z_0)$ sont strictement positives et harmoniques en z_0. On montre en particulier que pour $\alpha^* < \alpha < \alpha^{**}$, on a

$$h_\alpha(z_0) = \gamma_1(x(\alpha), y(\alpha))\varphi_1^{z_0}(y(\alpha)) + \gamma_2(x(\alpha), y(\alpha))\varphi_2^{z_0}(x(\alpha)) + e^{(x(\alpha),y(\alpha))\cdot z_0}. \tag{1.3.11}$$

Plus précisément, comme dans le cas non dégénéré, un développement asymptotique des fonctions de Green avec un nombre arbitraire de termes peut être établi, mais n'est pas écrit ici. On établit également les asymptotiques limites, lorsque $\alpha \to \alpha_0 \in \{0, \alpha^*, \alpha^{**}, \pi/2\}$, analogues à celles du cas non dégénéré en Section 1.3.1. On renvoie au Chapitre 3 pour des énoncés plus précis.

Fonctions harmoniques et frontière de Martin. Comme annoncé dans la Section 1.2.4.2, $\varphi_1^{z_0}(y(\alpha))$ et $\varphi_2^{z_0}(x(\alpha))$ peuvent être exprimées sous forme de séries alternées, ce qui permet de calculer $h_\alpha(z_0)$ par l'équation (1.3.11). On pose $(a_0, b_0) = (x(\alpha), y(\alpha))$ et on définit la suite $((a_n, b_n))_{n\in\mathbb{Z}}$ par

$$a_{2k} = -2k^2 + 2(a_0 - b_0 - \mu_2)k + a_0, \quad a_{2k+1} = a_{2k}, \quad k \in \mathbb{Z},$$

$$b_{2k} = -2k^2 + 2(a_0 - b_0 + \mu_1)k + b_0, \quad b_{2k+1} = b_{2k+2}, \quad k \in \mathbb{Z}.$$

Les points $((a_n, b_n))_{n\in\mathbb{Z}}$ correspondent aux points sur la parabole $\mathcal{P}$ de la Figure 1.7. Alors, pour $\alpha^* < \alpha < \alpha^{**}$, on a :

$$\varphi_1^{z_0}(y(\alpha)) = \frac{1}{\gamma_1(x(\alpha), y(\alpha))} \sum_{m=-\infty}^{-1} \kappa_m(\alpha) e^{z_0\cdot(a_m, b_m)}, \quad \varphi_2^{z_0}(x(\alpha)) = \frac{1}{\gamma_2(x(\alpha), y(\alpha))} \sum_{m=1}^{+\infty} \kappa_m(\alpha) e^{z_0\cdot(a_m, b_m)}$$

où

$$\kappa_m(\alpha) = \begin{cases} (-1)^m \left[\prod_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \frac{\frac{\gamma_1}{\gamma_2}(a_{2k+1}, b_{2k+1})}{\frac{\gamma_1}{\gamma_2}(a_{2k+2}, b_{2k+2})}\right] \frac{\gamma_2(a_0, b_0)}{\gamma_2(a_m, b_m)} & \text{si } m > 0 \\ (-1)^m \left[\prod_{k=0}^{\lfloor \frac{-m}{2} \rfloor - 1} \frac{\frac{\gamma_2}{\gamma_1}(a_{-2k-1}, b_{-2k-1})}{\frac{\gamma_2}{\gamma_1}(a_{-2k-2}, b_{-2k-2})}\right] \frac{\gamma_1(a_0, b_0)}{\gamma_1(a_m, b_m)} & \text{si } m < 0. \end{cases}$$

En particulier, pour $\alpha \in (\alpha^*, \alpha^{**})$, la fonction harmonique h_α s'écrit :

$$h_\alpha(z_0) = \sum_{m=-\infty}^{+\infty} \kappa_m(\alpha) e^{z_0\cdot(a_m, b_m)}$$

par (1.3.11), où $\kappa_0(\alpha) = 1$. Des expressions analogues s'appliquent pour $\alpha_0 \in \{\alpha^*, \alpha^{**}\}$. On en conclut que la frontière de Martin Γ du mouvement brownien dégénéré avec drift est minimale et donnée par

$$\Gamma = [\alpha^*, \alpha^{**}] \cap [0, \pi/2]$$

(cf. Figure 1.10 où $\beta = \pi/2$, et où ω^*, ω^{**} jouent le rôle de α^*, α^{**}). De plus, les fonctions harmoniques positives associées sont données par h_α, $\alpha \in \Gamma$.

1.3.3 Diffusion planaire avec barrière perméable.

On étudie ici le processus à barrière perméable décrit à la Section 1.1.2.2 (Proposition 1.7), de paramètres $(\Sigma^+, \Sigma^-, \mu^+, \mu^-, q = q_0)$. On suppose que $\Sigma^\pm$ sont non dégénérées, c'est à dire $\det(\Sigma^\pm) > 0$. Contrairement aux mouvements browniens réfléchis dans le quadrant, ici les fonctions harmoniques positives s'expriment simplement, généralement comme une somme de deux exponentielles. Par ailleurs, la discontinuité des paramètres sur l'axe $\{y = 0\}$ entraîne la non minimalité de la frontière de Martin.

Hypothèse. On ne fait ici qu'une seule hypothèse, portant sur les drifts :

$$\mu_1^\pm > 0, \ \mu_2^+ > 0, \ \mu_2^- < 0.$$

Autrement dit, les drifts s'éloignent de l'axe $\{y = 0\}$ et pointent vers la droite (voir Figure 1.4b).

Equation fonctionnelle. On rappelle la notation

$$G(z_0, S) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} \mathbb{1}_S(Z_t)\,dt\right] = \int_S g^{z_0}(z)\,dz, \quad S \in \mathcal{B}(\mathbb{R}^2)$$

pour le processus $Z = (A_t, B_t)_{t\geq 0}$. On définit les transformées de Laplace de $g(z_0, \cdot)$ restreintes à $\mathbb{R} \times (-\infty, 0)$ et $\mathbb{R} \times (0, +\infty)$ par

$$\varphi_+^{z_0}(x, y) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} e^{xA_t + yB_t}\mathbb{1}_{B_t>0}\,dt\right], \quad \varphi_-^{z_0}(x, z) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} e^{xA_t + zB_t}\mathbb{1}_{B_t<0}\,dt\right].$$

On définit la mesure d'occupation sur l'axe x, notée $H(z_0, \cdot)$, par

$$H(z_0, S) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} \mathbb{1}_S(A_t)\,dL_s^0(B)\right], \quad S \in \mathcal{B}(\mathbb{R})$$

et on note φ^{z_0} sa transformée de Laplace :

$$\varphi^{z_0}(x) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} e^{xA_t}\,dL_s^0(B)\right], \quad x \in \mathbb{R}.$$

On prouve alors qu'il existe $\eta > 0$ tel que pour $x \in (-\eta, 0)$, $y \leq 0$ et $z \geq 0$, l'équation fonctionnelle suivante est satisfaite : pour tout $z_0 = (a_0, b_0) \in \mathbb{R}^2$, on a

$$\gamma_-(x, z)\varphi_-^{z_0}(x, z) + \gamma_+(x, y)\varphi_+^{z_0}(x, y) + \gamma(x, y, z)\varphi^{z_0}(x) = -e^{xa_0 + yb_0\mathbb{1}_{b_0>0} + zb_0\mathbb{1}_{b_0<0}} \tag{1.3.12}$$

où les fonctions γ_+, γ_- et γ sont données par :

$$\begin{cases} \gamma_+(x, y) = \frac{1}{2}(x, y)\cdot\Sigma^+(x, y) + (x, y)\cdot\mu^+ = \frac{1}{2}(\Sigma_{11}^+x^2 + 2\Sigma_{12}^+xy + \Sigma_{22}^+y^2) + \mu_1^+x + \mu_2^+y, \\ \gamma_-(x, z) = \frac{1}{2}(x, z)\cdot\Sigma^-(x, z) + (x, z)\cdot\mu^- = \frac{1}{2}(\Sigma_{11}^-x^2 + 2\Sigma_{12}^-xz + \Sigma_{22}^-z^2) + \mu_1^-x + \mu_2^-z, \\ \gamma(x, y, z) = q_1x + \frac{1}{2}\big(y(1 + q_2) + z(q_2 - 1)\big). \end{cases}$$

Définition des paramètres. Comme pour les mouvements browniens réfléchis dans le quadrant, certains angles particuliers délimitent différentes zones d'asymptotiques. Avant de présenter les asymptotiques, on introduit quelques notations. Tout d'abord, les deux noyaux $\gamma_+(x, y)$ et $\gamma_-(x, z)$ dans l'équation fonctionnelle (1.3.12) forment deux équations d'ellipses réelles. On note x_{max}^+ et x_{min}^+ (resp. x_{max}^- et x_{min}^-) les abscisses maximales et minimales de l'ellipse d'équation $\gamma_+(x, y) = 0$ (resp. $\gamma_-(x, z) = 0$), voir (4.1.14) et Figure 1.12. On définit alors les points col $(x(\alpha), y(\alpha))$ et $(x(\alpha), z(\alpha))$, selon si $\alpha \in (0, \pi)$ ou $\alpha \in (\pi, 2\pi)$:

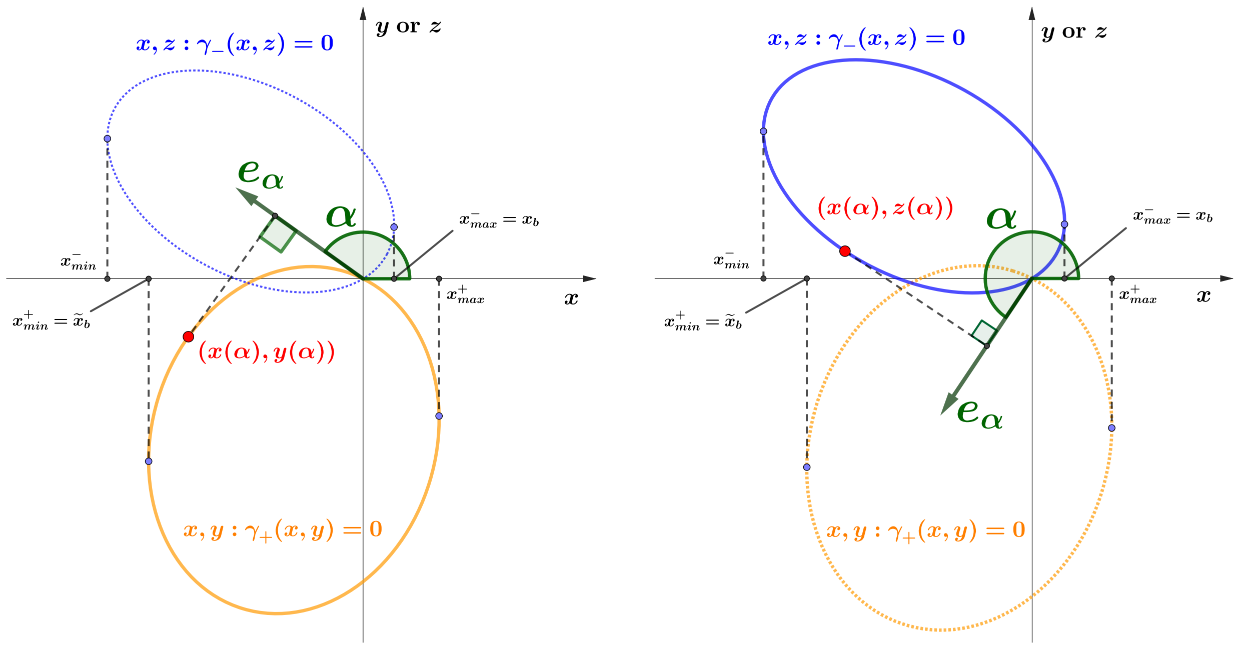
$$(x(\alpha), y(\alpha)) = \underset{\gamma_+(x,y)=0}{\text{argmax}}\,(\cos(\alpha)x + \sin(\alpha)y)\,, \quad \text{si} \quad \alpha \in (0, \pi) \tag{1.3.13}$$

$$(x(\alpha), z(\alpha)) = \underset{\gamma_-(x,z)=0}{\text{argmax}}\,(\cos(\alpha)x + \sin(\alpha)z)\,, \quad \text{si} \quad \alpha \in (\pi, 2\pi) \tag{1.3.14}$$

(voir Figure 1.12).

Selon les positions relatives de x_{max}^+ et x_{max}^- (resp. x_{min}^+ et x_{min}^-), on définit les angles $\alpha_b, \widetilde{\alpha}_b \in (0, 2\pi)$ tels que $x(\alpha_b) = \min(x_{max}^+, x_{max}^-)$ et $x(\widetilde{\alpha}_b) = \max(x_{min}^+, x_{min}^-)$ (cf Section 4.1.4.2, Chapitre 4). On décrit en Figure 1.13 l'ensemble des cas possibles avec les dispositions correspondantes de α_b et $\widetilde{\alpha}_b$. On définit alors $\mathcal{M} \subset [0, 2\pi]$ comme les arcs rouges dans la Figure 1.13, à savoir :

$$\mathcal{M} = \begin{cases} [\alpha_b, \widetilde{\alpha}_b] \cup [\pi, 2\pi] & \text{dans le cas} \quad \text{A} \;:\; x_{max}^+ > x_{max}^- \;\text{ et }\; x_{min}^+ < x_{min}^- \\ [0, \widetilde{\alpha}_b] \cup [\pi, \alpha_b] & \text{dans le cas} \quad \text{B} \;:\; x_{max}^+ < x_{max}^- \;\text{ et }\; x_{min}^+ < x_{min}^- \\ [0, \pi] \cup [\widetilde{\alpha}_b, \alpha_b] & \text{dans le cas} \quad \text{C} \;:\; x_{max}^+ < x_{max}^- \;\text{ et }\; x_{min}^+ > x_{min}^- \\ [\alpha_b, \pi] \cup [\widetilde{\alpha}_b, 2\pi] & \text{dans le cas} \quad \text{D} \;:\; x_{max}^+ > x_{max}^- \;\text{ et }\; x_{min}^+ > x_{min}^-. \end{cases}$$



(a) Si $\alpha \in (0, \pi)$, la construction se fait sur l'ellipse $\{(x, y) : \gamma_+(x, y) = 0\}$.

(b) Si $\alpha \in (\pi, 2\pi)$, la construction se fait sur l'ellipse $\{(x, z) : \gamma_-(x, z) = 0\}$.

FIGURE 1.12 : Construction géométrique de $(x(\alpha), y(\alpha))$ et $(x(\alpha), z(\alpha))$. Le vecteur e_α est défini par $e_\alpha = (\cos(\alpha), \sin(\alpha))$.

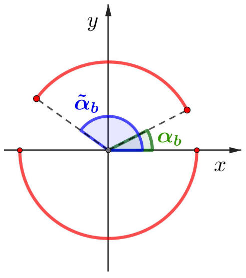
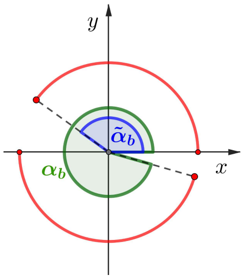
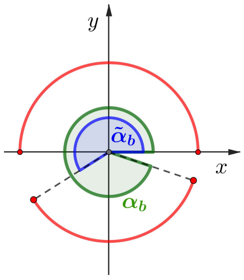
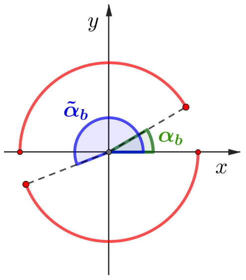
Cas A	Cas B	Cas C	Cas D
$x^+_{max} > x^-_{max}$ $x^+_{min} < x^-_{min}$	$x^+_{max} < x^-_{max}$ $x^+_{min} < x^-_{min}$	$x^+_{max} < x^-_{max}$ $x^+_{min} > x^-_{min}$	$x^+_{max} > x^-_{max}$ $x^+_{min} > x^-_{min}$
			

FIGURE 1.13 : Angles α_b et $\widetilde{\alpha}_b$ en fonction des paramètres. Les arcs $\mathcal{M}$ en rouge forment la frontière de Martin minimale du processus.

Asymptotiques des fonctions de Green. On présente à présent les principales asymptotiques des fonctions de Green dans les différentes régions angulaires délimitées par les directions 0, π, α_b et $\widetilde{\alpha}_b$. Les constantes $C(\alpha)$, $C_{br}(\alpha)$ ainsi que les fonctions $f_0(z_0), f_\pi(z_0)$ et $h_\alpha(z_0)$ apparaissant dans ces asymptotiques sont définies respectivement en (4.1.28) dans le Théorème 4.10 du Chapitre 4, et immédiatement après la présentation des asymptotiques.

- Si $\alpha_0 \in \mathring{\mathcal{M}}$, la fonction de Green admet une asymptotique de type point col :

  $$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} C(\alpha_0)h_{\alpha_0}(z_0)r^{-1/2}e^{-r\left(\cos(\alpha)\,x(\alpha)+\sin(\alpha)\times \substack{y(\alpha) \text{ si } \alpha_0\in(0,\pi) \\ \text{ou} \\ z(\alpha) \text{ si } \alpha_0\in(\pi,2\pi)}\right)}$$

  où $C(\alpha_0) > 0$ et où $h_\alpha(z_0) > 0$ est une fonction harmonique positive en z_0.

- Si $\alpha_0 \in (0, \alpha_b)$ dans les cas A et D ou si $\alpha_0 \in (\alpha_b, 2\pi)$ dans les cas B et C, on a :

  $$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0,\alpha\notin\mathcal{M}}}{\sim} C_{br}(\alpha)f_0(z_0)r^{-3/2}e^{-r\left(\cos(\alpha)x(\alpha_b)+\sin(\alpha)\times \substack{y(\alpha_b) \text{ si } \alpha\in(0,\pi) \\ \text{ou} \\ z(\alpha_b) \text{ si } \alpha\in(\pi,2\pi)}\right)}. \tag{1.3.15}$$

  Si $\alpha_0 \in (\widetilde{\alpha}_b, \pi)$ dans les cas A et B ou si $\alpha_0 \in (\pi, \widetilde{\alpha}_b)$ dans les cas C et D, on obtient l'asymptotique analogue en remplaçant $f_0(z_0)$ par une fonction $f_\pi(z_0)$, et en remplaçant $x(\alpha_b), y(\alpha_b)$ et $z(\alpha_b)$ par $x(\widetilde{\alpha}_b), y(\widetilde{\alpha}_b)$ et $z(\widetilde{\alpha}_b)$.

- Si $\alpha_0 = 0$ ou π, on obtient l'asymptotique (1.3.6) le long de $[0, 2\pi]\backslash\mathcal{M}$ et une asymptotique similaire à (1.3.15) le long de $\mathcal{M}$ (que l'on ne réécrit pas pour des raisons de lisibilité), dont la constante en facteur est la fonction $f_0(z_0)$ issue de (1.3.15) pour le cas $\alpha_0 = 0$ ou la fonction analogue $f_\pi(z_0)$ pour $\alpha_0 = \pi$.

- Si $\alpha_0 = \alpha_b$, alors plusieurs cas apparaissent selon la vitesse de convergence $\alpha \to \alpha_b$ comparée à $r \to +\infty$ (cf disjonction de cas du Théorème 4.9 Chapitre 4). Présentons l'asymptotique significative : sous de bonnes conditions sur les vitesses de $\alpha \to \alpha_b$ et $r \to +\infty$ (voir cas (v) Théorème 4.9), il existe une constante C telle que

  $$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_b}}{\sim} \left(C(\alpha_b)h_{\alpha_b}(z_0) + Cf_0(z_0)\right)r^{-1/2}e^{-r\left(\cos(\alpha)x(\alpha)+\sin(\alpha)\times \substack{y(\alpha) \text{ si } \alpha_b\in(0,\pi) \\ \text{ou} \\ z(\alpha) \text{ si } \alpha_b\in(\pi,2\pi)}\right)}. \tag{1.3.16}$$

  On observe le même comportement en $\widetilde{\alpha}_b$. C'est précisément à ce point qu'apparaît la non-minimalité de la frontière de Martin : on obtient en effet l'ensemble des combinaisons linéaires de h_{α_b} et de f_0 dans les "constantes" dans (1.3.16).

Fonctions harmoniques et frontière de Martin. Les fonctions h_α, f_0 et f_π sont harmoniques, minimales, et sont données par les formules suivantes.

- Si $\alpha \in \mathcal{M} \cap (0, \pi)$, alors

  $$h_\alpha(z_0) = \begin{cases} e^{a_0x(\alpha)+b_0y(\alpha)} - \dfrac{\gamma(x(\alpha), y(\alpha), Z^+(x(\alpha)))}{\gamma(x(\alpha), Y^-(x(\alpha)), Z^+(x(\alpha)))}e^{a_0x(\alpha)+b_0Y^-(x(\alpha))} & \text{si } b_0 \geq 0 \\ \left(1 - \dfrac{\gamma(x(\alpha), y(\alpha), Z^+(x(\alpha)))}{\gamma(x(\alpha), Y^-(x(\alpha)), Z^+(x(\alpha)))}\right)e^{a_0x(\alpha)+b_0Z^+(x(\alpha))} & \text{si } b_0 < 0. \end{cases} \tag{1.3.17}$$

- Si $\alpha \in \mathcal{M} \cap (\pi, 2\pi)$, alors

$$h_\alpha(z_0) = \begin{cases} \left(1 - \dfrac{\gamma(x(\alpha), Y^-(x(\alpha)), z(\alpha))}{\gamma(x(\alpha), Y^-(x(\alpha)), Z^+(x(\alpha)))}\right) e^{a_0 x(\alpha) + b_0 Y^-(x(\alpha))} & \text{si } b_0 \geq 0 \\ e^{a_0 x(\alpha) + b_0 z(\alpha)} - \dfrac{\gamma(x(\alpha), Y^-(x(\alpha)), z(\alpha))}{\gamma(x(\alpha), Y^-(x(\alpha)), Z^+(x(\alpha)))} e^{a_0 x(\alpha) + b_0 Z^+(x(\alpha))} & \text{si } b_0 < 0. \end{cases} \tag{1.3.18}$$

- Si $x_b = x^+_{max} < x^-_{max}$ (cas B et C Figure 1.13), alors

$$f_0(a_0, b_0) = \begin{cases} \left(b_0 - \dfrac{\Sigma^+_{22}}{(\Sigma^+_{22} + \Sigma^-_{22})\gamma(x_b, Y^-(x_b), Z^+(x_b))}\right) e^{x_b a_0 + b_0 Y^\pm(x_b)} & \text{si } b_0 \geq 0 \\ \dfrac{-\Sigma^+_{22}}{(\Sigma^+_{22} + \Sigma^-_{22})\gamma(x_b, Y^-(x_b), Z^+(x_b))} e^{x_b a_0 + b_0 Z^+(x_b)} & \text{si } b_0 < 0. \end{cases} \tag{1.3.19}$$

- Si $x_b = x^-_{max} < x^+_{max}$ (cas A et D Figure 1.13), alors

$$f_0(a_0, b_0) = \begin{cases} \dfrac{-\Sigma^-_{22}}{(\Sigma^+_{22} + \Sigma^-_{22})\gamma(x_b, Y^-(x_b), Z^+(x_b))} e^{x_b a_0 + b_0 Y^-(x_b)} & \text{si } b_0 \geq 0 \\ \left(-b_0 - \dfrac{\Sigma^-_{22}}{(\Sigma^+_{22} + \Sigma^-_{22})\gamma(x_b, Y^-(x_b), Z^+(x_b))}\right) e^{x_b a_0 + b_0 Z^\pm(x_b)} & \text{si } b_0 < 0. \end{cases} \tag{1.3.20}$$

Dans les expressions précédentes, $Y^\pm$ et $Z^\pm$ sont les branches caractérisées par $\gamma_+(x, Y^\pm(x)) = 0$ et $\gamma_-(x, Z^\pm(x))$ comme présenté en Section 1.2.3.3 (voir également Figure 4.2b, Chapitre 4). L'expression explicite de f_π est symétrique par rapport à f_0.

La frontière de Martin se compose de deux arcs de cercle correspondant aux directions de $\mathcal{M}$, ainsi que de deux "antennes" dans les directions α_b et $\widetilde{\alpha}_b$ issues des asymptotiques (1.3.16) et des asymptotiques analogues en $\widetilde{\alpha}_b$. Plus précisément, la frontière de Martin Γ du processus est donnée par :

$$\Gamma = \left[\{e^{i\alpha}\}_{\alpha \in \mathcal{M}} \cup \{ue^{i\alpha}\}_{u \in [1,2],\, \alpha = \alpha_b, \widetilde{\alpha}_b}\right] / \mathcal{R} \ \sim \ \mathbb{S}^1 \tag{1.3.21}$$

où la relation d'équivalence $\mathcal{R}$ identifie les points e^{i0} et $2e^{i\alpha_b}$, ainsi que $e^{i\pi}$ et $2e^{i\widetilde{\alpha}_b}$ (voir Figure 1.14). Enfin, la frontière de Martin minimale Γ_{min} est donnée par

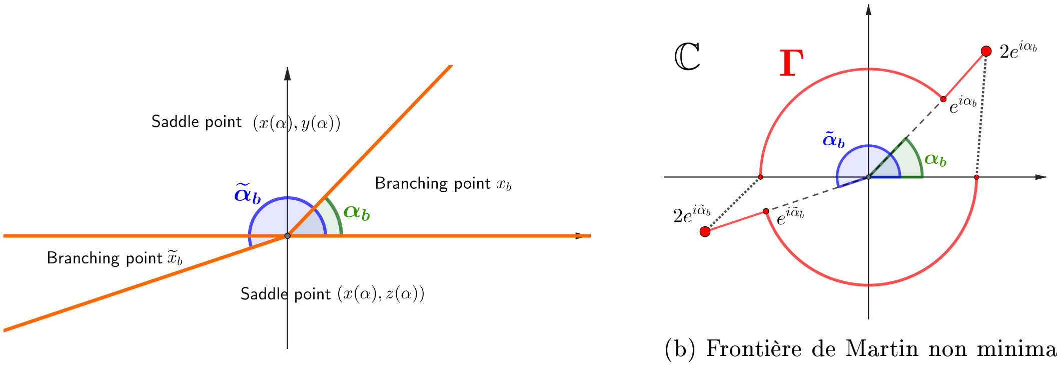
$$\Gamma_{min} = \{e^{i\alpha}\}_{\alpha \in \mathcal{M}} \tag{1.3.22}$$

(voir Figure 1.13) et les fonctions harmoniques correspondantes sont les fonctions h_α et f_0, f_π.

Remarque 1.26 (EDP satifaite par les fonctions harmoniques)*.* L'équation aux dérivées partielles associée à l'harmonicité d'une fonction h pour le processus est donnée par :

$$\begin{cases} \left(\frac{1}{2}\nabla \cdot \Sigma^+ \nabla + \mu^+ \cdot \nabla\right) h(z) = 0 & \text{pour } z \in \mathbb{R} \times (0, +\infty), \\ \left(\frac{1}{2}\nabla \cdot \Sigma^- \nabla + \mu^- \cdot \nabla\right) h(z) = 0 & \text{pour } z \in \mathbb{R} \times (-\infty, 0), \\ (q_1, 1 + q_2)\nabla h(x, 0^+) = (-q_1, 1 - q_2)\nabla h(x, 0^-) & \text{pour } x \in \mathbb{R}, \\ h(x, 0^+) = h(x, 0^-) \text{ and } (1, 0) \cdot \nabla h(x, 0^+) = (1, 0) \cdot \nabla h(x, 0^-) & \text{pour } x \in \mathbb{R}. \end{cases} \tag{1.3.23}$$

On peut par ailleurs vérifier directement que les fonctions obtenues f_0, f_π, and $h_\alpha, \alpha \in \mathcal{M} \backslash \{0, \pi\}$ satisfont ces équations.



(a) Zones correspondantes à différents types d'asymptotiques.

(b) Frontière de Martin non minimale, les pointillés fins représentent un recollement topologique.

Figure 1.14 : Types d'asymptotiques et frontière de Martin dans le cas D.

Chapter 2

Asymptotics for the Green's functions of a transient reflected Brownian motion in a wedge

Ce chapitre est issu de l'article [53] co-écrit avec Irina Kourkova et Sandro Franceschi, publié dans le journal *Queueing Systems* dans une revue spécialisée sur le mouvement Brownien réfléchi.

Résumé

Dans ce chapitre, nous considérons un mouvement Brownien transient réfléchi obliquement dans un quadrant bidimensionnel. Un développement asymptotique précis des fonctions de Green est obtenu dans toutes les directions.

À cette fin, nous déterminons tout d'abord une équation fonctionnelle de noyau reliant les transformées de Laplace des fonctions de Green. Nous prolongeons ensuite analytiquement ces transformées de Laplace et étudions leurs singularités. Les asymptotiques sont obtenues en appliquant la méthode du point col à la transformée de Laplace inverse sur la surface de Riemann engendrée par le noyau.

Abstract

In this chapter, we consider a transient Brownian motion reflected obliquely in a two-dimensional wedge. A precise asymptotic expansion of the Green's functions is found in all directions.

To this end, we first determine a kernel functional equation connecting the Laplace transforms of the Green's functions. We then extend the Laplace transforms analytically and study its singularities. We obtain the asymptotics applying the saddle point method to the inverse Laplace transform on the Riemann surface generated by the kernel.

Contents

2.1 Introduction

Context

In this chapter we consider a transient obliquely reflected Brownian motion in a cone of angle $\beta \in (0,\pi)$ with two different reflection laws on two boundary rays of the cone (see Section 1.1.1.3). We denote by $\widetilde{g}(\rho\cos(\omega),\rho\sin(\omega))$ the Green's function of this process in polar coordinates. This chapter determines the asymptotics of $\widetilde{g}(\rho\cos(\omega),\rho\sin(\omega))$ as $\rho \to \infty$ and $\omega \to \omega_0$ for any given angle $\omega_0 \in [0,\beta]$, see Section 1.3.1 for the presentation of the results. It extends results of [37] in two aspects. First, asymptotic results are obtained in any convex two dimensional cone with two different reflection laws on its boundaries. While in the half plane of [37] the Laplace transform of the Green's function is easily made explicit, this is not the case for a cone as above. Laplace transforms of the Green functions in this case are expressed in [49] in terms of some integrals as solutions of Riemann boundary problems, which hardly suit further analysis. Second, Theorem 2.2 provides Green function's asymptotics in any direction of the cone and not only along straight rays as in [37], namely when the angle ω tends to a given angle ω_0 and not just equals it. The asymptotics depends on the rate of convergence of $\omega \to \omega_0$ and allows to determine the full Martin boundary of the process.

In [52] the asymptotics of the stationary distribution for recurrent Brownian motion in a cone is found along all regular directions $\omega_0 \in (0,\beta)$, while some special directions ω_0 are left for future work. The asymptotics are obtained by studying the singularities and applying the saddle point method to the inverse Laplace transform of the stationary distribution. This chapter applies the approach of Section 1.2.3 to Green's functions and provides further developments: the new techniques allow to treat all special directions where asymptotics depends more deeply on the rate of convergence of ω to ω_0. This is the case when $\omega_0 = 0$ or β, see Theorem 2.3, and also when the *saddle point meet a pole* of the boundary Laplace transform, see Theorem 2.4.

Main results

We now precisely present the main results of this chapter. We consider an obliquely reflected standard Brownian motion in a cone of angle $\beta \in (0,\pi)$ starting from $\widetilde{z}_0$, of reflection angles $\delta \in (0,\pi)$ and

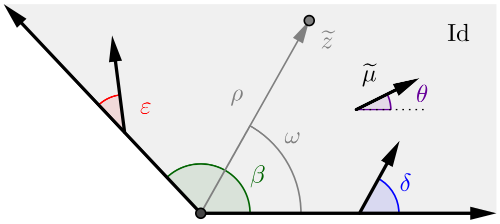


Figure 2.1: The cone of angle β, the reflection angles δ and ε and the drift $\widetilde{\mu}$ with its direction θ. In grey the point $\widetilde{z}$ of polar coordinates ρ and ω.

$\varepsilon \in (0,\pi)$ and of drift $\widetilde{\mu}$ of angle $\theta \in (0,\beta)$ with the horizontal axis, see Figure 2.1. We assume that

$$\delta + \varepsilon < \beta + \pi.$$

This well known condition ensures that the process is a semi-martingale reflected Brownian motion [120, 121]. The process is transient since we assumed that $\theta \in (0,\beta)$ which means that the drift belongs to the cone. If we assume that p_t is the transition probability of this process, the Green's function is defined for $\widetilde{z}$ inside the cone by

$$\widetilde{g}(\widetilde{z}) = \int_0^\infty \widetilde{p}_t(\widetilde{z}_0, \widetilde{z}) \mathrm{d}t.$$

For $\omega \in (0,\beta)$ and $\rho > 0$ we will denote $\widetilde{z} = (\rho\cos\omega, \rho\sin\omega)$ the polar coordinates in the cone. Note that the tilde symbol $\widetilde{\ }$ stands for quantities linked to the standard reflected Brownian motion in the β-cone. The same notations without the tilde symbol will stand for the corresponding process in the quadrant $\mathbb{R}_+^2$, see Remark 2.6 bellow.

Before presenting our results in more detail, we need to make the following remark.

Remark 2.1 (Notation). Along all this chapter, we will use the symbol $\sim$ to express an asymptotic expansion of a function. If for some functions f and g_k we state that $f(x) \sim \sum_{k=1}^n g_k(x)$ when $x \to x_0$, it means that $g_k(x) = o(g_{k-1}(x))$ and that $f(x) - \sum_{k=1}^n g_k(x) = o(g_n(x))$ when $x \to x_0$.

We now state the main result of the chapter. We define the angles

$$\omega^* := \theta - 2\delta \quad \text{and} \quad \omega^{**} := \theta + 2\epsilon.$$

We can remark that $\omega^* < \theta < \omega^{**}$.

Theorem 2.2 (Asymptotics in the general case)**.** *We consider a standard reflected Brownian motion in a wedge of opening β, of reflection angles δ and ε and a drift $\widetilde{\mu}$ of angle θ, see Figure 2.1. Then, the Green's function $\widetilde{g}(\rho\cos\omega, \rho\sin\omega)$ of this process has the following asymptotics when $\omega \to \omega_0 \in (0,\beta)$ and $\rho \to \infty$, for all $n \in \mathbb{N}$:*

- *If $\omega^* < \omega_0 < \omega^{**}$ then*

$$\widetilde{g}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to\omega_0}}{\sim} e^{-2\rho|\widetilde{\mu}|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{1}{\sqrt{\rho}} \sum_{k=0}^n \frac{\widetilde{c}_k(\omega)}{\rho^k} \tag{2.1.1}$$

- *If $\omega_0 < \omega^*$ then*

$$\widetilde{g}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to\omega_0}}{\sim} c^* e^{-2\rho|\widetilde{\mu}|\sin^2(\omega+\delta-\theta)} + e^{-2\rho|\widetilde{\mu}|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{1}{\sqrt{\rho}} \sum_{k=0}^n \frac{\widetilde{c}_k(\omega)}{\rho^k} \tag{2.1.2}$$

- *If* $\omega^{**} < \omega_0$ *then*

$$\widetilde{g}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to\omega_0}}{\sim} c^{**}e^{-2\rho|\widetilde{\mu}|\sin^2(\omega-\epsilon-\theta)} + e^{-2\rho|\widetilde{\mu}|\sin^2\left(\frac{\omega-\theta}{2}\right)}\frac{1}{\sqrt{\rho}}\sum_{k=0}^{n}\frac{\widetilde{c_k}(\omega)}{\rho^k} \tag{2.1.3}$$

where c^* *and* c^{**} *are positive constants and* $c_k(\omega)$ *are constants depending on* ω *such that* $\widetilde{c_k}(\omega) \underset{\omega\to\omega_0}{\longrightarrow} \widetilde{c_k}(\omega_0)$.

There are four cases which are illustrated by Figure 1.9.

Our second result states the asymptotics near the edges when $\omega \to 0$ or $\omega \to \beta$.

Theorem 2.3 (Asymptotics along the edges)**.** *We now assume that* $\omega_0 = 0$ *and let* $\rho \to \infty$ *and* $\omega \to \omega_0 = 0$. *In these case, we have* $\widetilde{c_0}(\omega) \underset{\omega\to 0}{\sim} c'\omega$ *and* $\widetilde{c_1}(\omega) \underset{\omega\to 0}{\sim} c''$ *for some non-negative constants* c' *and* c'' *which are non null at least when* $\omega^* < 0$. *Then, the Green's function* $\widetilde{g}(\rho\cos\omega, \rho\sin\omega)$ *has the following asymptotics:*

- *When* $\omega^* < 0$ *the asymptotics given by* (2.1.1) *remains valid. In particular, we have*

$$\widetilde{g}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to 0}}{\sim} e^{-2\rho|\widetilde{\mu}|\sin^2\left(\frac{\omega-\theta}{2}\right)}\frac{1}{\sqrt{\rho}}\left(c'\omega + \frac{c''}{\rho}\right)$$

- *When* $\omega^* > 0$ *the asymptotics given by* (2.1.2) *remains valid. In particular, we have*

$$\widetilde{g}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to 0}}{\sim} c^*e^{-2\rho|\widetilde{\mu}|\sin^2(\omega+\delta-\theta)}.$$

Therefore, when $\omega^* < 0$, *there is a competition between the two first terms of the sum* $\sum_{k=0}^{n}\frac{\widetilde{c_k}(\omega)}{\rho^k}$ *to know which one is dominant between* $c'\omega$ *and* $\frac{c''}{\rho}$. *More precisely:*

- *If* $\rho\sin\omega \underset{\substack{\rho\to\infty \\ \omega\to 0}}{\longrightarrow} \infty$ *then the first term is dominant.*
- *If* $\rho\sin\omega \underset{\substack{\rho\to\infty \\ \omega\to 0}}{\longrightarrow} c > 0$ *then both terms contribute, they have the same order of magnitude.*
- *If* $\rho\sin\omega \underset{\substack{\rho\to\infty \\ \omega\to 0}}{\longrightarrow} 0$ *then the second term is dominant.*

A symmetric result holds when we take $\omega_0 = \beta$. *The asymptotics given by* (2.1.1) *remains valid when* $\beta < \omega^{**}$ *and* (2.1.3) *remain valid when* $\omega^{**} < \beta$ *and there is a competition between the two first terms of the sum to know which one is dominant which depends of the limit of* $\rho\sin(\beta - \omega)$.

We will explain later in Propositions 2.46 and 2.47 that ω^* and ω^{**} correspond in some sense to the poles of the Laplace transforms of the Green's functions and that ω correspond to the saddle point obtained when we will inverse the Laplace transform. Our third result states the asymptotics when the saddle point *meet* the poles which means when $\omega \to \omega^*$ or $\omega \to \omega^{**}$.

Theorem 2.4 (Asymptotics when the saddle point *meet* a pole)**.** *We now assume that* $\omega_0 = \omega^* = \theta - 2\delta$ *and let* $\omega \to \omega^*$ *and* $\rho \to \infty$. *Then, the Green's function* $\widetilde{g}(\rho\cos\omega, \rho\sin\omega)$ *has the following asymptotics:*

- *When* $\rho(\omega - \omega^*)^2 \to 0$ *then asymptotics are given by* (2.1.2) *but the constant* c^* *of the first term has to be replaced by* $\frac{1}{2}c^*$.
- *When* $\rho(\omega - \omega^*)^2 \to c > 0$ *for some constant* c *then:*

  - *If* $\omega < \omega^*$ *the asymptotics are still given by* (2.1.2) *but the constant* c^* *of the first term has to be replaced by* $\frac{1}{2}c^*(1+\Phi(\sqrt{c}A))$ *for some constant* A*.*
  - *If* $\omega > \omega^*$ *the asymptotics are still given by* (2.1.2) *but the constant* c^* *of the first term has to be replaced by* $\frac{1}{2}c^*(1-\Phi(\sqrt{c}A))$ *for some constant* A*.*

  In the previous items we denoted $\Phi(z) := \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt$*.*

- *When* $\rho(\omega-\omega^*)^2 \to \infty$ *then:*
  - *If* $\omega < \omega^*$ *the asymptotics are given by* (2.1.2)
  - *If* $\omega > \omega^*$ *the asymptotics are given by* (2.1.1) *and we have* $\widetilde{c_0}(\omega) \underset{\omega\to\omega^*}{\sim} \frac{c}{\omega-\omega^*}$ *for some constant* c*.*

A symmetric result holds when we assume that $\omega_0 = \omega^{**} = \theta + 2\epsilon$*.*

The following remark concerns the Martin boundary.

Remark 2.5 (Martin boundary). The Martin boundary associated to this process can be computed from the asymptotics of the Green's function obtained in the previous theorems. The corresponding harmonic functions can also be obtained thanks to the constants of the dominant terms of the asymptotics. See Section 6 of [37] which briefly reviews some elements of the theory in a similar context.

Remark 2.6 (Equivalence between cones and quadrant). As mentioned in Section 1.1.1.3, our strategy of proof is to first establish our results in the quadrant for a general covariance matrix and then to extend the results to all convex cones with covariance matrix identity by a linear transformation. In Section 2.11, this equivalence is established by means of a simple linear transformation defined in (2.11.2). Therefore, all the results established for the RBMs in quadrant with general covariance matrices can be transposed directly to RBMs in cones.

In the three previous theorems we consider a Brownian motion which has covariance matrix identity. But all the results stated above may easily be extended to all covariance matrices thanks to the simple linear transformation mentioned in the previous remark. The next remark explains how to proceed, in line with what is stated in Section 2.11.

Remark 2.7 (Generalisation to any covariance matrix in any convex cone). We consider $\widehat{Z}_t$ an obliquely reflected Brownian motion in a cone of angle $\widehat{\beta}_0 \in (0,\pi)$ starting from $\widehat{z}_0$, of reflection angles $\widehat{\delta}$ and $\widehat{\varepsilon}$, of drift $\widehat{\mu}$ of angle $\widehat{\theta}$ and of covariance matrix $\widehat{\Sigma}$. We introduce the angle $\widehat{\beta}_1 := \arccos\left(-\frac{\widehat{\sigma}_{12}}{\sqrt{\widehat{\sigma}_{11}\widehat{\sigma}_{22}}}\right) \in (0,\pi)$ and the linear transformation

$$\widehat{T} := \begin{pmatrix} \frac{1}{\sin\widehat{\beta}_1} & \cot\widehat{\beta}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\widehat{\sigma}_{11}}} & 0 \\ 0 & \frac{1}{\sqrt{\widehat{\sigma}_{22}}} \end{pmatrix}$$

Then, the process $\widetilde{Z}_t := \widehat{T}\widehat{Z}_t$ is an obliquely reflected *standard* Brownian motion in a cone of angle $\beta \in (0,\pi)$ starting from $\widetilde{z}_0 := \widehat{T}\widehat{z}_0$, of reflection angles δ and ε and of drift $\widetilde{\mu} := \widehat{T}\widehat{\mu}$ of angle θ. The angle parameters are in $(0,\pi)$ and are determined by

$$\tan\beta = \frac{\sin\widehat{\beta}_1}{\frac{1}{\tan\widehat{\beta}_0}\sqrt{\frac{\widehat{\sigma}_{22}}{\widehat{\sigma}_{11}}} + \cos\widehat{\beta}_1}\ , \qquad \tan\theta = \frac{\sin\widehat{\beta}_1}{\frac{1}{\tan\widehat{\theta}}\sqrt{\frac{\widehat{\sigma}_{22}}{\widehat{\sigma}_{11}}} + \cos\widehat{\beta}_1}\ ,$$

$$\tan\delta = \frac{\sin\widehat{\beta}_1}{\frac{1}{\tan\widehat{\delta}}\sqrt{\frac{\widehat{\sigma}_{22}}{\widehat{\sigma}_{11}}} + \cos\widehat{\beta}_1}\ , \qquad \tan(\beta-\varepsilon) = \frac{\sin\widehat{\beta}_1}{\frac{1}{\tan(\widehat{\beta}_0-\widehat{\varepsilon})}\sqrt{\frac{\widehat{\sigma}_{22}}{\widehat{\sigma}_{11}}} + \cos\widehat{\beta}_1}.$$

Thanks to this linear transformation, we obtain the following relation between the Green's function of $\widehat{Z}_t$ denoted by $\widehat{g}(\widehat{z})$ for $\widehat{z}$ inside the cone of angle $\widehat{\beta}_0$ and the Green's function of $\widetilde{Z}_t$ denoted by $\widetilde{g}(\widetilde{z})$ for $\widetilde{z}$ inside the cone of angle β:

$$\widehat{g}(\widehat{z}) = \frac{1}{\sqrt{\det \widehat{\Sigma}}} \widetilde{g}(\widehat{T}\widehat{z}).$$

Therefore, the previous formula allows us to extend our results from $\widetilde{g}$ to $\widehat{g}$.

Plan and strategy of proof

In this chapter, the results will be first established in a quadrant for any covariance matrix and then transferred to any cone in the last section.

The first step in solving our problem is to determine a functional equation relating the Laplace transforms of the Green's functions in the quadrant and on the edges, see Section 2.2. Next, we continue these Laplace transforms and study their singularities, see Section 2.3. Then, we use the inversion Laplace transform formula combined with the functional equation to express the Green's functions as a sum of simple integrals, see Section 2.4. Doetsch's book [33] is one of the leading references on Laplace transforms. To determine the asymptotics, we first use complex analysis to obtain Tauberian results which link the poles of the Laplace transforms to the asymptotics of the Green's functions. Then, we use a double refinement of the classical saddle-point method: the uniform method of the steepest descent. One of the reference books on this classical approach is those of Fedoryuk [42]. Appendix 2.A, which gives a generalized version of the classical Morse Lemma by introducing a parameter dependency, will be useful to refine this saddle-point method. Section 2.5 studies the saddle point, Section 2.6 explains how we shift the integration contour and thus determines the contribution of the encountered poles to the asymptotics. Section 2.7 shows that some part of the new integration contour are negligible. Section 2.8 establishes the contribution of the saddle point to the asymptotics and states the main result. Section 2.9 studies the asymptotics along axes and Section 2.10 the asymptotics in the technical case where the saddle point meet a pole. Appendix 2.A states a technical result useful to this section. Finally, Section 2.11 explains how to transfer to any convex cone the asymptotic results obtained in the quadrant in the previous sections and thus concludes the proof of Theorems 2.2, 2.3 and 2.4.

2.2 Convergence of Laplace transforms and functional equation

Transient reflected Brownian motion in a cone

Let $(Z_t)_{t\geq 0} = (z_0 + \mu t + B_t + RL_t)_{t\geq 0}$ be a (continuous) semimartingale reflected Brownian motion (SRBM) in $\mathbb{R}^2_+$ on a filtered probability space where $\mu = (\mu_1, \mu_2)^\top \in \mathbb{R}^2$ is the drift, Σ the covariance matrix associated to the Brownian motion B, $R = (r_{ij})_{1\leqslant i,j\leqslant 2} \in \mathbb{R}^{2\times 2}$ the reflection matrix, and $(L_t)_{t\geq 0} = ((L^1_t, L^2_t)^\top)_{t\geq 0}$ the local times on the edges associated to the process. We will assume that $\det(\Sigma) > 0$, i.e. that Σ is positive-definite. See Figure 2.2 to visualize the parameters of this process. We recall the following classical result concerning the existence of such a process, see for example [115, 120].

Proposition 2.8 (Existence and uniqueness of SRBM). *There exists an SRBM with parameters (μ, Σ, R) if and only if Σ is a covariance matrix and R is completely-$\mathcal{S}$, i.e.*

$$r_{11} > 0,\ r_{22} > 0,\ \text{ and } [\det(R) > 0 \quad \text{or} \quad r_{21},\ r_{12} > 0]. \tag{2.2.1}$$

In this case, the SRBM is unique in law and defines a Feller continuous strong Markov process.

Condition (2.2.1) will therefore be required throughout the chapter. The recurrence and transience conditions of those processes are well known, see [68, 119]. In our case, the SRBM will be systematically transient because of the following assumption of positive drift, which will be held throughout the rest of the chapter.

Assumption 2.9 (Positivity of the drift)**.** *We assume that* $\mu_1 > 0$ *and* $\mu_2 > 0$*.*

Note that this assumption is equivalent to that made in the introduction: $\theta \in (0, \beta)$.

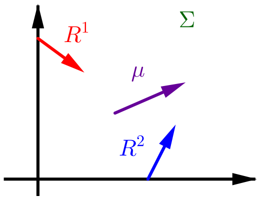


Figure 2.2: SRBM parameters in the quadrant: drift μ, reflection vectors R^1 and R^2 and covariance matrix Σ.

Green's function

We are working in the transient case and we will focus on the Green's functions.

Definition 2.10 (Green's measures and densities)**.** *The Green's measure* G *inside the quadrant is defined by*

$$G(z_0, A) := \mathbb{E}_{z_0}\left[\int_0^\infty \mathbb{1}_A(Z_t)dt\right] = \int_A g(z)dz$$

for $z_0 \in \mathbb{R}_+^2$ *and* $A \subset \mathbb{R}^2$ *and admits a density* g *with respect to the Lebesgue measure. The density* g *is called the Green's function.*

For $i \in \{1, 2\}$*, we define* H_i *the Green's measures on the edges of the quadrant which also has densities* h_i *with respect to the Lebesgue measure, namely*

$$H_i(z_0, A) := \mathbb{E}_{z_0}\left[\int_0^\infty \mathbb{1}_A(Z_t)dL_t^i\right] = \int_A h_i(z)dz.$$

The measure H_1 *has its support on the vertical axis and* H_2 *has its support on the horizontal axis.*

Throughout the chapter one should be kept in mind that in the notations g and h_i we omit the dependence on the starting point z_0.

Proof. In the recurrent case, Harrison and Williams proved in [65] that the invariant measure has a density according to the Lebesgue measure. The proof done in that article extends to the transient case and justify the existence of a density with respect to the Lebesgue measure for the Green's measures. Indeed, the proof of Lemma (9) of section 7 in [65] shows that for a Borel set A of Lebesgue measure 0, we have

$$\mathbb{E}\left[\int_0^{+\infty} \mathbb{1}_A(Z_t)dt\right] = 0.$$

This is even an equivalence, but we don't need it here. Since the proof does not requires the recurrence property, this gives the desired result by the Radon Nikodym theorem. The same argument applies to the densities of H_i for $i = 1, 2$, see theorem (1), section 8 in [65]. $\square$

Remark 2.11 (Partial differential equation). Let us denote $\mathcal{L} = \frac{1}{2}\nabla \cdot \Sigma\nabla + \mu \cdot \nabla$ the generator of the SRBM inside the quadrant and $\mathcal{L}^* = \frac{1}{2}\nabla \cdot \Sigma\nabla - \mu \cdot \nabla$ its dual operator. Then, the Green's function g satisfies

$$\mathcal{L}^* g = -\delta_{z_0}$$

in the sense of distributions $\mathcal{D}'((\mathbb{R}_+^*)^2)$.

Let us define the matrix $R^* = 2\Sigma - R\ \mathrm{diag}(R)^{-1}\mathrm{diag}(\Sigma)$. We denote R_1^* and R_2^* the two columns of R^*. Then, the following boundary conditions holds

$$\begin{cases} \partial_{R_1^*} g(z) - 2\mu_1 g(z) = 0 \text{ for } z \in \{0\} \times \mathbb{R}_+ \\ \partial_{R_2^*} g(z) - 2\mu_2 g(z) = 0 \text{ for } z \in \mathbb{R}_+ \times \{0\} \end{cases}$$

where $\partial_{R_i^*} = R_i^* \cdot \nabla$.

Sketch of proof of the remark. The partial differential equation of the Green's function and its boundary conditions are derived from the forward equation of the transition kernel establish in [62], see Equation (8.3). However, we provides here a direct elementary proof of the fact that $\mathcal{L}^* g = -\delta_{z_0}$. Let $\varphi \in C_c^\infty((\mathbb{R}_+^*)^2)$. We apply Ito's formula and we take the expectation,

$$\mathbb{E}[\varphi(Z_t)] = \varphi(z_0) + \mathbb{E}\left[\int_0^t \mathcal{L}\varphi(Z_s) ds\right].$$

One may remark that there are no boundary terms since φ cancel on a neighborhood of the boundaries. Since we are in the transient case and since φ is bounded, the left term converges to 0 while t goes to infinity by the dominated convergence theorem. Since successive derivatives of φ are bounded, $\mathcal{L}\varphi(a,b)$ is bounded by an exponential function up to a multiplication constant. Thanks to convergence domain of the Laplace transform (see Proposition 2.14 bellow), we obtain by dominated convergence that $\varphi(z_0) = -\mathbb{E}\left[\int_0^{+\infty} \mathcal{L}\varphi(Z_s)ds\right] = -\int_{\mathbb{R}_+^2} \mathcal{L}\varphi(z) g(z) dz$ which implies that $\mathcal{L}^* g = -\delta_{z_0}$. □

Furthermore, it is preferable to have continuity of the Green's function to talk about their asymptotic behaviour. This is the subject of the following comment.

Remark 2.12 (Smoothness of the Green's functions). By the strictly elliptic regularity theorem, we may deduce from $\mathcal{L}^* g = -\delta_{z_0}$ that the density g has a $\mathcal{C}^\infty$ version on $(\mathbb{R}_+^*)^2 \backslash \{z_0\}$. We won't go into more detail here about the proof of this result. In the remainder of this article, we will assume that this property is true and that g is continuous on $(\mathbb{R}_+^2)^* \setminus \{z_0\}$.

Laplace transform and functional equation

Definition 2.13 (Laplace transform of the Green's functions)**.** *For $(x,y) \in \mathbb{C}^2$ we define the Laplace transforms of the Green's measures by*

$$\varphi(x,y) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,y)\cdot Z_t} dt\right] = \int_{\mathbb{R}_+^2} e^{(x,y)\cdot z} g(z) dz$$

and

$$\varphi_1(y) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,y)\cdot Z_t} dL_t^1\right] = \int_{\mathbb{R}_+} e^{yb} h_1(b) db, \quad \varphi_2(x) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,y)\cdot Z_t} dL_t^2\right] = \int_{\mathbb{R}_+} e^{xa} h_2(a) da.$$

Let us remark that φ_1 does not depend on x and φ_2 does not depend on y. One has to remember the dependence on the starting point z_0 even though we omit it in the notations.

Since Green's measures are not probability measures, the convergence of their Laplace transforms is not guaranteed. For example $\varphi(0)$ is not finite. Convergence domains had already been studied in [49] but we need stronger results. The following proposition establishes the convergence when the real part of x and y is negative.

Proposition 2.14 (Convergence of the Laplace transform)**.** *Assuming that $\mu_1 > 0$ and $\mu_2 > 0$,*

- *$\varphi_1(y)$ converges (at least) on $y \in \{y \in \mathbb{C}, \Re(y) < 0\}$*

- $\varphi_2(x)$ *converges (at least) on* $x \in \{x \in \mathbb{C}, \Re(x) < 0\}$
- $\varphi(x, y)$ *converges (at least) on* $(x, y) \in \{(x, y) \in \mathbb{C}^2, \Re(x) < 0 \text{ and } \Re(y) < 0\}$.

Before proving this proposition, we state the functional equation that will be central in this chapter. First, we need to define for $(x, y) \in \mathbb{C}^2$ the following polynomials

$$\begin{cases} \gamma(x, y) = \frac{1}{2}(x, y) \cdot \Sigma(x, y) + (x, y) \cdot \mu = \frac{1}{2}(\sigma_{11}x + 2\sigma_{12}xy + \sigma_{22}y^2) + \mu_1 x + \mu_2 y \\ \gamma_1(x, y) = R^1 \cdot (x, y) = r_{11}x + r_{21}y \\ \gamma_2(x, y) = R^2 \cdot (x, y) = r_{12}x + r_{22}y \end{cases}$$

where R^1, R^2 are the two columns of the reflection matrix R. The polynomial γ is called the kernel.

Proposition 2.15 (Functional equation)**.** *If* $\Re(x) < 0$ *and* $\Re(y) < 0$*, then*

$$-\gamma(x, y)\varphi(x, y) = \gamma_1(x, y)\varphi_1(y) + \gamma_2(x, y)\varphi_2(x) + e^{(x,y)\cdot z_0}. \tag{2.2.2}$$

The proofs of these two proposition are deeply linked. So we'll be gathering their proofs.

Proof of Propositions 2.14 and 2.15. The main idea of the proof is to take the expectation of Itô's formula applied to the SRBM and to use a sign argument to justify the limit when $t \to +\infty$. The beginning of the proof is inspired of the Proposition 5 of [49].

Let $(x, y) \in (\mathbb{R}_-^*)^2$, Itô's formula applied to $f(z) := e^{(x,y)\cdot z}$ gives

$$\begin{aligned} f(Z_t) - f(z_0) &= \int_0^t \nabla f(Z_s).dB_s + \int_0^t \mathcal{L}f(Z_s)ds + \sum_{i=1}^{2} \int_0^t R_i \cdot \nabla f(Z_s)dL_s^i & (2.2.3) \\ &= \int_0^t \nabla f(Z_s).dB_s + \gamma(x, y) \int_0^t e^{(x,y)\cdot Z_s}ds + \sum_{i=1}^{2} \gamma_i(x, y) \int_0^t e^{(x,y)\cdot Z_s}dL_s^i & (2.2.4) \end{aligned}$$

where $\mathcal{L} = \frac{1}{2}\nabla \cdot \Sigma\nabla + \mu \cdot \nabla$ is the generator of the Brownian motion. Since $(x, y) \in (\mathbb{R}_-^*)^2$, the integral $\int_0^t \nabla f(Z_s).dB_s$ is a martingale (its quadratic variation is bounded by $C.t$ for a constant $C > 0$) and its expectation cancels out. Therefore,

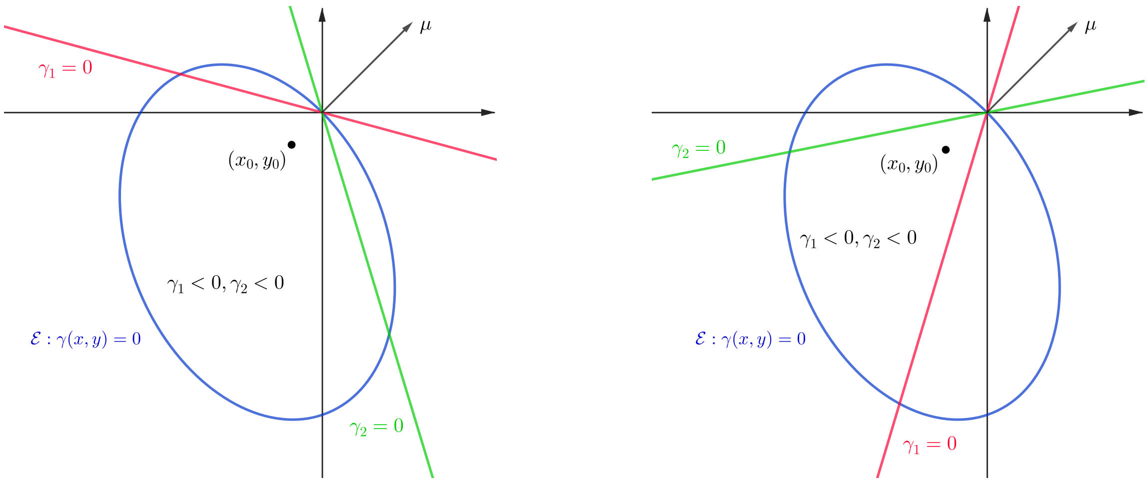
$$\begin{aligned} &\mathbb{E}_{z_0}\left[e^{(x,y)\cdot Z_t}\right] - e^{(x,y)\cdot z_0} - \gamma(x, y)\mathbb{E}_{z_0}\left[\int_0^t e^{(x,y)\cdot Z_s}ds\right] \\ &\qquad = \mathbb{E}_{z_0}\left[\gamma_1(x, y) \int_0^t e^{(x,y)\cdot Z_s}dL_s^1 + \gamma_2(x, y) \int_0^t e^{(x,y)\cdot Z_s}dL_s^2\right]. \qquad (2.2.5) \end{aligned}$$

The expectations in the left-hand side of the previous equation are finite because $(x, y) \in (\mathbb{R}_-^*)^2$, the first one is bounded by 1 and the second one by t. This implies that the expectation of the right-hand side is also finite.

The aim now is to take the limit of (2.2.5) when t goes to infinity to show the finiteness of the Laplace transforms and the functional equation. First, since $(x, y) \in (\mathbb{R}_-^*)^2$ and $\|Z_t\| \underset{t\to\infty}{\longrightarrow} +\infty$ a.s., the expectation $\mathbb{E}_x\left[e^{(x,y)\cdot Z_t}\right]$ converges toward 0 when $t \to \infty$ by the dominated convergence theorem. Secondly, by the monotone convergence theorem the expectation $\mathbb{E}_{z_0}\left[\int_0^t e^{(x,y)\cdot Z_s}ds\right]$ converges in $[0, \infty]$ to $\varphi(x, y) = \mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,y)\cdot Z_s}ds\right]$.

Let us assume for a moment that it is possible to choose $(x_0, y_0) \in (\mathbb{R}_-^*)^2$ such that $\gamma(x_0, y_0) < 0$, $\gamma_1(x_0, y_0) < 0$ and $\gamma_2(x_0, y_0) < 0$. We use a proof by contradiction assuming that we have $\mathbb{E}_{z_0}\left[\int_0^\infty e^{(x_0,y_0)\cdot Z_s}ds\right] = +\infty$. Since $\gamma(x_0, y_0) < 0$, it implies that the left-hand side of (2.2.5) will be positive for t large enough. But, since $\gamma_1(x_0,y_0)<0$ and $\gamma_2(x_0,y_0)<0$, the right-hand side of (2.2.5) is always negative. We obtain a contradiction and we deduce that $\varphi(x_0,y_0)=\mathbb{E}_{z_0}\left[\int_0^\infty e^{(x_0,y_0)\cdot Z_s}ds\right]$ is finite. Hence the limit of the right-hand side of (2.2.5) is also finite and converges by the monotone convergence theorem to $\gamma_1(x_0,y_0)\varphi_1(y_0)+\gamma_2(x_0,y_0)\varphi_2(x_0)$. We deduce that $\varphi_1(y_0)$ and $\varphi_2(x_0)$ are also finite and that the functional equation (2.2.2) is satisfied in (x_0,y_0). This implies that for all x and y in $\mathbb{C}$ such that $\Re x<x_0$ and $\Re y<y_0$ the Laplace transforms $\varphi(x,y)$, $\varphi_1(y)$ and $\varphi_2(x)<\infty$ are finite and the functional equation (2.2.2) is satisfied by taking the limit of (2.2.5) when $t\to\infty$.

All that remains is to show that we can always choose x_0 and y_0 *as close to* 0 *as we like*, such that $(x_0,y_0)\in(\mathbb{R}_-^*)^2$, $\gamma(x_0,y_0)<0$, $\gamma_1(x_0,y_0)<0$ and $\gamma_2(x_0,y_0)<0$ and the proof of Propositions 2.14 and 2.15 will be complete. Let us denote $\mathcal{E}$ the ellipse of equation $\gamma(x,y)=0$. One may observe that the interior of the ellipse $\mathcal{E}$ defined by $\gamma(x,y)<0$ contains a neighbourhood of 0 intersecting $(\mathbb{R}_-^*)^2$ by Assumption 2.9 on the positivity of the drift. Indeed, the drift is an external normal to the ellipse at $(0,0)$. We consider two cases coming from the existence condition of the process (2.2.1). First case, $r_{11}>0$, $r_{22}>0$, $r_{12}>0$ and $r_{21}>0$, see Figure 2.3a. In this case, one may see directly see that $\gamma_1(x,y)<0$ and $\gamma_2(x,y)<0$ on $(\mathbb{R}_-^*)^2$. It is therefore easy to pick (x_0,y_0) close enough to $(0,0)$ which satisfies the required conditions. Second case, $r_{11}>0$, $r_{22}>0$ and $\det(R)>0$, see Figure 2.3b. In this case, the cone defined by $\gamma_1<0$ and $\gamma_2<0$ has a non-empty intersection with $(\mathbb{R}_-^*)^2$. Hence, one can still choose (x_0,y_0) as close as we want to $(0,0)$ inside the desired cone and the ellipse $\mathcal{E}$.



(a) Case $r_{11}>0$, $r_{22}>0$, $r_{12}>0$ and $r_{21}>0$.

(b) Case $r_{11}>0$, $r_{22}>0$ and $\det R>0$.

Figure 2.3: For $(x,y)\in\mathbb{R}^2$, illustration of the domain where $\gamma_1<0$ and $\gamma_2<0$.

□

The following lemma follows from the functional equation and states that the boundary green's densities h_1 and h_2 are equal, up to some constant, to the bivariate green's function g on the axes.

Proposition 2.16 (Green's densities on the boundaries)**.** *The Green's density g is related to the boundary Green's densities h_i by the formulas*

$$r_{11}h_1(b)=\frac{\sigma_{11}}{2}g(0,b)\quad \textit{and}\quad r_{22}h_2(a)=\frac{\sigma_{22}}{2}g(a,0).$$

Proof. The initial value formula of a Laplace transform gives

$$x\varphi(x,y)\underset{x\to-\infty}{\longrightarrow}-\int_0^\infty e^{yb}g(0,b)db.$$

Therefore, by dividing the functional equation (2.2.2) by x and taking the limit when x tends to infinity, we obtain

$$\frac{1}{2}\sigma_{11}\int_0^\infty e^{yb}g(0,b)db = r_{11}\varphi_1(y) = r_{11}\int_0^\infty e^{yb}h_1(b)db$$

which implies the result. □

2.3 Continuation and properties of $\varphi_1(x)$ and $\varphi_2(y)$

The first step of this analytic approach is to study the kernel.

Lemma 2.17 (Kernel study)**.** *(i) Equation $\gamma(x,y)=0$ determines an algebraic function $Y(x)$ [resp. $X(y)$] with two branches*

$$Y^{\pm}(x) = \frac{1}{\sigma_{22}}\Big(-\sigma_{12}x-\mu_2 \pm \sqrt{(\sigma_{12}^2-\sigma_{11}\sigma_{22})x^2+2(\mu_2\sigma_{12}-\mu_1\sigma_{22})x+\mu_2^2}\Big)$$

The function $Y(x)$ [resp. $X(y)$] has two branching points x_{min} and x_{max} [resp. y_{min} and y_{max}] given by

$$x_{min} = \frac{\mu_2\sigma_{12}-\mu_1\sigma_{22}-\sqrt{D_1}}{\det(\Sigma)}, \quad x_{max} = \frac{\mu_2\sigma_{12}-\mu_1\sigma_{22}+\sqrt{D_1}}{\det(\Sigma)},$$

$$y_{min} = \frac{\mu_1\sigma_{12}-\mu_2\sigma_{11}-\sqrt{D_2}}{\det(\Sigma)}, \quad y_{max} = \frac{\mu_1\sigma_{12}+\mu_2\sigma_{11}-\sqrt{D_2}}{\det(\Sigma)},$$

where $D_1=(\mu_2\sigma_{12}-\mu_1\sigma_{22})^2+\mu_2^2\det(\Sigma)$ and $D_2=(\mu_1\sigma_{12}-\mu_2\sigma_{11})^2+\mu_1^2\det(\Sigma)$. Both of them are real and $x_{min}<0<x_{max}$ [resp. $y_{min}<y_{max}$]. The branches of $Y(x)$ [resp. $X(y)$] take real values if and only if $x\in[x^{min},x^{max}]$ [resp. $y\in[y_{min},y_{max}]$]. Furthermore $Y^-(0)=-\frac{2\mu_2}{\sigma_{22}}<0$, $Y^-(x_{max})<0$, $Y^+(0)=0$, $Y^+(x_{max})<0$. See Figure 2.4.

(ii) For any $u\in\mathbb{R}$

$$\mathrm{Re}Y^{\pm}(u+iv) = \frac{1}{\sigma_{22}}\Big(-\sigma_{12}u-\mu_2\pm\frac{1}{\sqrt{2}}\sqrt{(u-x_{min})(x_{max}-u)+v^2+|(u+iv-x_{min})(x_{max}-u-iv)|}\Big).$$

(iii) Let $\delta=\infty$ if $\sigma_{12}\geq 0$ and $\delta=-\mu_2/\sigma_{12}-x_{max}>0$ if $\sigma_{12}<0$. Then for some $\epsilon>0$ small enough

$$\mathrm{Re}Y^-(u+iv)<0 \quad \textit{for } u\in]-\epsilon,\ x_{max}+\delta[, \quad v\in\mathbb{R}.$$

Proof. Points (i) and (ii) follow from elementary considerations. The fact that $Y^+(x_{max})<0$ implies the inequality $-\sigma_{12}x_{max}-\mu_2<0$, so that $\delta>0$. Furthermore by (ii) $\mathrm{Re}\,Y^-(u+iv)\leq\mathrm{Re}Y^-(u)$ which is strictly negative for $u\in]-\epsilon,x_{max}+\delta[$ by the analysis made in (i). □

Lemma 2.18 (Continuation of the Laplace transform)**.** *Function $\varphi_2(x)$ can be meromorphically continued to the (cut) domain*

$$\{x=u+iv \mid u<x_{max}+\delta, v\in\mathbb{R}\}\setminus[x_{max},x_{max}+\delta] \tag{2.3.1}$$

by the formula :

$$\varphi_2(x) = \frac{-\gamma_1(x,Y^-(x))\varphi_1(Y^-(x))-\exp\big(a_0x+b_0Y^-(x)\big)}{\gamma_2(x,Y^-(x))}. \tag{2.3.2}$$

A symmetric continuation formula holds for φ_1.

Proof. By Lemma 2.17 (iii) for any $x=u+iv$ with $u\in]-\epsilon,0[$ the following equation is valid.

$$\gamma(x,Y^-(x))\varphi(x,Y^-(x)) = \gamma_1(x,Y^-(x))\varphi_1(Y^-(x))+\gamma_2(x,Y^-(x))\varphi_2(x)+\exp(a_0x+b_0Y^-(x)).$$

Since $\gamma(x,Y^-(x))=0$, the statement follows. □

Proposition 2.19 (Poles of the Laplace transform, necessary condition)**.** *(i) $x = 0$ is not a pole of $\varphi_2(x)$, so that $\varphi_2(0) = \mathbb{E}[L^2_\infty] < +\infty$. The local time spent by the process on the horizontal axis is finite.*

(ii) If x^* is a pole of $\varphi_2(x)$ in the domain (2.3.1), then $(x^*, Y^-(x^*))$ is a unique non-zero solution of the system of two equations

$$\gamma(x,y) = 0, \qquad \gamma_2(x,y) = r_{12}x + r_{22}y = 0. \tag{2.3.3}$$

Moreover, x^* is real and belongs to $]0, x_{max}[$. Furthermore, this solution exist only if

$$x_{max}r_{12} + Y^\pm(x_{max})r_{22} > 0.$$

(iii) If y^{**} is a pole of $\varphi_1(y)$, then $(X^-(y^{**}), y^{**})$ is a unique non-zero solution of the system of two equations

$$\gamma(x,y) = 0, \qquad \gamma_1(x,y) = r_{11}x + r_{21}y = 0. \tag{2.3.4}$$

Moreover, y^{**} is real and belongs to $]0, y_{max}[$. Furthermore, this solution exist only if

$$y_{max}r_{21} + X^\pm(y_{max})r_{11} > 0.$$

When these solutions exist we have

$$x^* = 2\frac{\mu_2\frac{r_{12}}{r_{22}} - \mu_1}{\sigma_{11} - 2\sigma_{12}\frac{r_{12}}{r_{22}} + \sigma_{22}\left(\frac{r_{12}}{r_{22}}\right)^2} \quad \textit{and} \quad y^{**} = 2\frac{\mu_1\frac{r_{21}}{r_{11}} - \mu_2}{\sigma_{11}\left(\frac{r_{21}}{r_{11}}\right)^2 - 2\sigma_{12}\frac{r_{21}}{r_{11}} + \sigma_{22}}. \tag{2.3.5}$$

Then, we define

$$y^* := Y^+(x^*) \quad \text{and} \quad x^{**} := X^+(y^*).$$

See Figure 2.4 to visualize all these points.

Proof. (i) The observation that $\gamma_2(0, Y^-(0)) = r_{22} \times Y^-(0) \neq 0$ implies the first statement.

(ii) If x^* is a pole of $\varphi_2(x)$, then $(x^*, Y^-(x^*))$ should be a solution the system (2.3.3) above by by the continuation formula (2.3.2) and the continuity of φ_1 [resp. φ_2] on $\{\Re y \leq 0\}$ [resp. $\{\Re x \leq 0\}$]. This system has one solution $(0,0)$ and the second one (x°, y°), which is necessarily real. Then $x^\circ \in [x_{min}, x_{max}]$ and y° is either $Y^-(x^\circ)$ or $Y^+(x^\circ)$. But x° can be a pole of $\varphi_2(x)$, if only it is within $]0, x_{max}]$ and $y^\circ = Y^-(x^\circ)$. This last condition implies $\frac{r_{12}}{r_{22}} > \frac{-Y^\pm(x_{\max})}{x_{max}}$. ☐

Proposition 2.20 (Poles of the Laplace transforms, sufficient condition)**.** *The pole x^* (resp. y^{**}) of φ_2 (resp. φ_1) exists if (and only if) $x_{max}r_{12} + Y^\pm(x_{max})r_{22} > 0$ (resp. $y_{max}r_{21} + X^\pm(y_{max})r_{11} > 0$).*

Proof. The conditions of the previous proposition are necessary. The next two lemmas prove the sufficiency. In those, we denote the dependence of Laplace transforms with the initial condition z_0 by $\varphi_1^{z_0}, \varphi_2^{z_0}$ instead of φ_1, φ_2. The proof is done for x^*, but is of course symmetrical for y^{**}. ☐

Lemma 2.21 (Existence of the pole for a starting point)**.** *If $x_{max}r_{12} + Y^\pm(x_{max})r_{22} > 0$, there exists $z_0 \in \mathbb{R}^2_+$ such that x^* is a pole of $\varphi_2^{z_0}$.*

Proof. The denominator of the continuation formula (2.3.2) vanishes since we assume that $x_{max}r_{12} + Y^\pm(x_{max})r_{22} > 0$. We are looking for a z_0 such that the numerator doesn't vanish at x^*, which will imply that z_0 is a pole of φ_2. If $\gamma_1(x^*, Y^-(x^*)) \geq 0$, this is obvious thanks to the exponential term and a sign argue. We suppose now that $-C := \gamma_1(x^*, Y^-(x^*)) < 0$. We make a proof by contradiction assuming that

$$\forall z_0 = (a_0, b_0) \in \mathbb{R}^2_+, \quad -C\varphi_1^{(a_0,b_0)}(Y^-(x^*)) + e^{a_0x^* + b_0Y^-(x^*)} = 0. \tag{2.3.6}$$

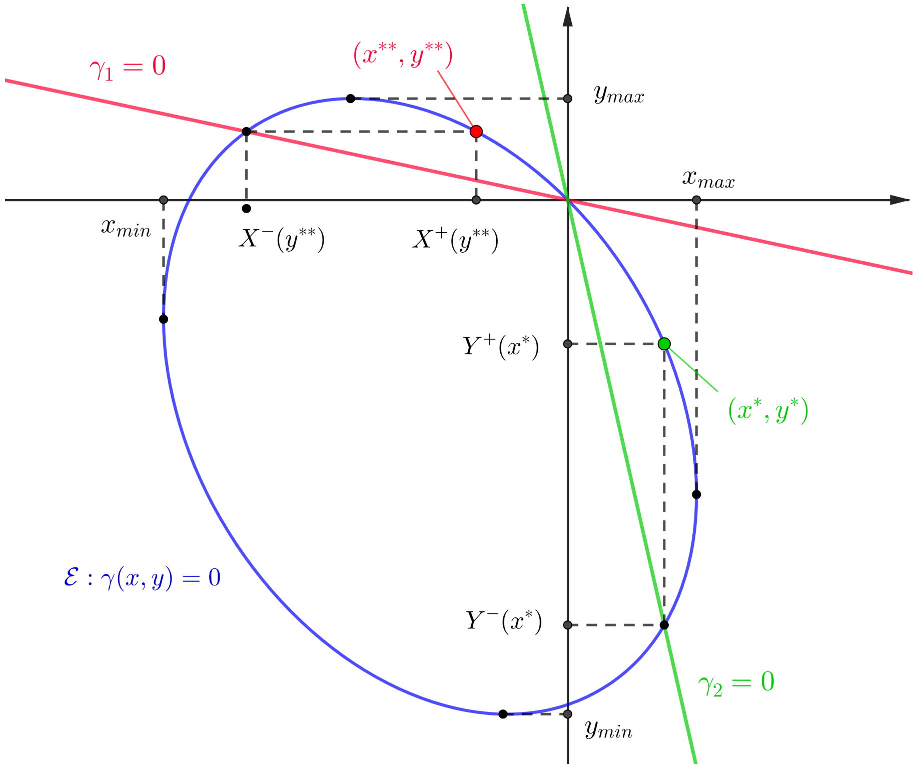


Figure 2.4: In the real plane (x, y), graphic representation of poles x^* and y^{**} when both exist.

Let T be the stopping time defined by the first hitting time of the axis $\{x = 0\}$, i.e. $T = \inf\{t \geq 0, Z_t^1 = 0\}$ with $Z = (Z^1, Z^2)$. (It is possible that $T = +\infty$). Firstly, since the the Stieltjes measure dL^1 is supported by $\{Z^1 = 0\}$ and since Z is a strong Markov process, for a starting point $z_0 = (a_0, b_0)$ we have:

$$\varphi_1^{(a_0,b_0)}(Y^-(x^*)) = \mathbb{E}_{(a_0,b_0)}\left[\int_T^{+\infty} e^{Z_t^2.Y^-(x^*)} dL_t^1 \mathbb{1}_{T<+\infty}\right] \tag{2.3.7}$$

$$= \mathbb{E}_{(a_0,b_0)}\left[\mathbb{E}_{Z_T}\left[\int_0^{+\infty} e^{Z_t^2.Y^-(x^*)} dL_t^1\right] \mathbb{1}_{T<+\infty}\right] \tag{2.3.8}$$

$$= \mathbb{E}_{(a_0,b_0)}\left[\varphi_1^{(0,Z_T^2)}(Y^-(x^*)) \mathbb{1}_{T<+\infty}\right]. \tag{2.3.9}$$

Conditioning by the value of Z_T^2, using (2.3.6) and $Y^-(x^*) \leq 0$, we get :

$$\varphi_1^{(a_0,b_0)}(Y^-(x^*)) = \int_0^{+\infty} \varphi_1^{(0,b)}(Y^-(x^*)) \mathbb{P}_{(a_0,b_0)}(T < +\infty, Z_T^2 = db) \tag{2.3.10}$$

$$= \int_0^{+\infty} \frac{1}{C} e^{0.x^* + bY^-(x^*)} \mathbb{P}_{(a_0,b_0)}(T < +\infty, Z_T^2 = db) \leq \frac{1}{C} \mathbb{P}_{(a_0,b_0)}(T < +\infty) \leq \frac{1}{C}. \tag{2.3.11}$$

But, (a_0, b_0) can be chosen such that $e^{a_0 x^* + b_0 Y^-(x^*)}$ is as huge as wanted because $x^* > 0$. This is in contradiction with (2.3.6). □

Lemma 2.22 (Existence of a pole for all starting points)**.** *If x^* is a pole of $\varphi_2^{z_0}$ for some $z_0 \in \mathbb{R}_+^2$, then x^* is a pole of $\varphi_2^{z_0'}$ for every $z_0' \in \mathbb{R}_+^2$.*

The proof of this lemma is postponed bellow Proposition 2.24 since it needs this proposition to be established.

Lemma 2.23 (Nature of the branching point of φ_2)**.** *Let $x \to x_{max}$ with $x < x_{max}$, we have*

- *If $\gamma_2(x_{max}, Y^-(x_{max})) = 0$, i.e. $x^* = x_{max}$, then*

  $$\varphi_2(x) = \frac{C}{\sqrt{x_{max} - x}} + O(1)$$

  for a constant $C > 0$.

- *If $\gamma_2(x_{max}, Y^-(x_{max})) \neq 0$, then*

  $$\varphi_2(x) = C_1 + C_2\sqrt{x_{max} - x} + O(x_{max} - x)$$

  for constants $C_1 \in \mathbb{R}$ and $C_2 > 0$.

Proof. Thanks to Lemma 2.17, Y^- can be written as $Y^-(x) = Y^-(x_{max}) - c\sqrt{x_{max} - x} + O(x_{max} - x)$ where $c > 0$. Let's carry out an elementary asymptotic expansion of the quotient of the continuation formula (2.18). First of all,

$$\begin{aligned}\frac{1}{\gamma_2(x, Y^-(x))} &= \frac{1}{\gamma_2(x_{max}, Y^-(x_{max})) - r_{22}c\sqrt{x_{max} - x} + O(x_{max} - x)} \\ &= \begin{cases} \frac{-1}{r_{22}c\sqrt{x_{max}-x}} & \text{if } \gamma_2(x_{max}, Y^-(x_{max}) = 0, \\ \frac{1}{\gamma_2(x_{max}, Y^-(x_{max}))}\left(1 + \frac{r_{22}c\sqrt{x_{max}-x}}{\gamma_2(x_{max}, Y^-(x_{max}))} + O(x_{max} - x)\right) & \text{if } \gamma_2(x_{max}, Y^-(x_{max})) \neq 0. \end{cases}\end{aligned}$$

Secondly, for the numerator,

$$\begin{aligned}\gamma_1(x, Y^-(x))\varphi_1(Y^-(x)) + e^{a_0x + b_0Y^-(x)} = \\ \left(\gamma_1(x_{max}, Y^-(x_{max})) - r_{21}c\sqrt{x_{max} - x} + O(x_{max} - x)\right) \\ \times \left(\varphi_1(Y^-(x_{max})) - c\varphi_1'(Y^-(x_{max}))\sqrt{x_{max} - x} + O(x_{max} - x)\right) \\ + e^{a_0x_{max} + b_0Y^-(x_{max})}(1 - cb_0\sqrt{x_{max} - x} + O(x_{max} - x))\end{aligned} \tag{2.3.12}$$

Combining the two asymptotic expansions, we obtain the desired formula with

$$C = \frac{\gamma_1(x_{max}, Y^-(x_{max}))\varphi_1(Y^-(x_{max})) + e^{a_0x_{max} + b_0Y^-(x_{max})}}{r_{22}c}$$

and

$$\begin{aligned}C_2 = \frac{1}{\gamma_2(x_{max}, Y^-(x_{max}))}\Big[r_{21}c\varphi_1(Y^-(x_{max})) + c\gamma_1(x_{max}, Y^-(x_{max}))\varphi_1'(Y^-(x_{max})) + cb_0e^{a_0x_{max} + b_0Y^-(x_{max})} \\ - \frac{r_{22}c}{\gamma_2(x_{max}, Y^-(x_{max}))}\left(\gamma_1(x_{max}, Y^-(x_{max}))\varphi_1(Y^-(x_{max})) + e^{a_0x_{max} + b_0Y^-(x_{max})}\right)\Big]\end{aligned}$$

□

The following proposition states the asymptotics of the Green's functions h_1 and h_2 on the boundaries. We note that we obtain the same asymptotics as in Theorem 2.3 and 2.41 with $\alpha \to 0$, which is consistent with the link made between h_1, h_2 and g in Proposition 2.16.

Proposition 2.24 (Asymptotics of the Green's functions on the boundary h_1 and h_2)**.** *In this lemma we denote by c a constant which can be different from one line to another.*

1. *Suppose that we have a pole $x^* \in]0, x_{max}[$ for φ_2. Then, the Green's function h_2 has the following asymptotics*
$$h_2(u) \underset{u\to\infty}{\sim} ce^{-x^*u}.$$
2. *Suppose that $x^* = x_{max}$, then*
$$h_2(u) \underset{u\to\infty}{\sim} cu^{-1/2}e^{-x_{max}u}.$$
3. *Suppose that there is no pole in $]0, x_{max}[$ and that $x^* \neq x_{max}$, then,*
$$h_2(u) \underset{u\to\infty}{\sim} cu^{-3/2}e^{-x_{max}u}.$$

A symmetric result holds for h_1.

Proof. The result directly follows from classical Tauberian inversion lemmas which link the asymptotics of a function at infinity to the first singularity of its Laplace transform (which is here given in Lemma 2.23). We refer here Theorem 37.1 of Doetsch's book [33] and more precisely we apply the special case stated in Lemma C.2 of [24]. To apply this lemma we have to verify the analyticity and the convergence to 0 at infinity of φ_2 in a domain $\mathcal{G}_\delta(x_{max}) := \{z \in \mathbb{C} : z \neq x_{max}, |\arg(z - x_{max})| > \delta\}$ for some $\delta \in (0, \pi/2)$. But this follows directly from the continuation procedure of Lemma 2.18 :the exponential part of the continuation formula (2.3.2) tends to 0 in a domain $\mathcal{G}_\delta(x_{max})$ for some $\delta \in (0, \pi/2)$ by using (ii) of lemma 2.17. Note that the convergence to 0 also follows from Lemma 2.D.1. Then, Lemma 2.23 gives the nature at the branching point x_{max} which is the smallest singularity except in the case where there is a pole in $]0, x_{max}[$, then the pole x^* is the smallest singularity. □

Remark 2.25. We can remark in the proof of Lemma 2.23 that $O(1)$ and $O(x_{max} - x)$ of this lemma are locally uniform according to z_0. Which means that $\sup_{z_0' \in V} \left| \varphi_2^{(z_0')}(x) - \frac{C^{(z_0')}}{\sqrt{x_{max} - x}} \right| = O(1)$ as $x \to x^*$ when $\gamma_2(x_{max}, Y^-(x_{max})) = 0$ for a sufficiently small neighborhood V of z_0 (and the same holds for $O(x_{max} - x)$ in the other case). This imply that the results of Proposition 2.24 hold locally uniformly in z_0. Indeed it is enough to adapt the Tauberian lemmas of [33] used in the proof of Proposition 2.24 in a slightly more technical but quite similar way. Note that the constants c of this proposition depend continuously on z_0.

Proof of Lemma 2.22. Let $z_0 = (a_0, b_0)$ be a starting point such that x^* is a pole of $\varphi_2^{z_0}$. Then, the continuation formula (2.3.2) implies that $-\gamma_1(x^*, Y^-(x^*))\varphi_1^{z_0}(Y^-(x^*)) - \exp\left(a_0x^* + b_0Y^-(x^*)\right) \neq 0$. By continuity with respect to the starting point (which follows from the integral formula given in [49] or from [89]), there exists a neighbourhood V of z_0 such that $-\gamma_1(x^*, Y^-(x^*))\varphi_1^{z_0'}(Y^-(x^*)) - \exp\big(a_0'x^* + b_0'Y^-(x^*)\big) \neq 0$ for all $z_0' = (a_0', b_0') \in V$. Therefore, by the continuation formula, x^* is a pole of $\varphi_2^{z_0'}$ for all $z_0' \in V$. From Proposition 2.24 and by continuity of the constant of this proposition according to z_0' we deduce the following. If x^* is a pole of $\varphi_2^{z_0'}$, there exists a constant c such that for all $z_0' \in V$ we have $h_2^{(z_0')}(u) = ce^{-x^*u}(1 + o(1))$ (notice that $o(1)$ is uniform in z_0' in the sense of Remark 2.25 and that c is continuous in z_0'). For $z_0'' \in \mathbb{R}^2$ we introduce the stopping time
$$T_V := \inf\{t > 0 : Z_t^{z_0''} \in V\}$$
where $Z_t^{z_0''}$ denotes the process starting from z_0''. By the strong Markov property applied to T_V we have for some constant C and when $u \to \infty$,
$$h_2^{z_0''}(u) \geqslant \mathbb{P}_{z_0''}(T_V < \infty) \inf_{z_0' \in V} h_2^{z_0'}(u) = Ce^{-x^*u}(1 + o(1)).$$
We deduce by Proposition 2.24 that z_0'' is necessarily a pole. □

We conclude this section with the following lemma which will be needed in Section 2.6.

Lemma 2.26 (Boundedness of the Laplace transform)**.** *Let* $\eta \in]0,\delta[$*, we have*

$$\sup_{\substack{u\in[X^{\pm}(y_{\max})-\eta,x_{max}+\eta]\\ |v|>\epsilon}} |\varphi_2(u+iv)| < \infty.$$

Proof. Clearly, for any $x=u+iv$ with $u<0$, $|\varphi_2(u+iv)| \leq \varphi_2(u)$. Then for any $\epsilon>0$,

$$\sup_{u\in[X^{\pm}(y_{\max})-\eta,-\epsilon]} |\varphi_2(u+iv)| < \infty. \tag{2.3.13}$$

For any $x=u+iv$ with $u\in[-\epsilon,x_{max}+\eta]$ Lemma 2.18 applies and gives the representation (2.3.2). Let us consider all its terms. By Lemma 2.17 (ii), for any fixed $u\in\mathbb{R}$, function $\mathrm{Re}Y^-(u+iv)$ is strictly decreasing as $|v|$ goes from 0 to infinity. Moreover for any $u\in[-\epsilon,x_{max}+\delta]$

$$\mathrm{Re}Y^-(u+iv) \leq -\frac{1}{\sqrt{2}\sigma_{22}}|v|.$$

Then,

$$|\varphi_1(Y^-(u+iv))| \leq \varphi_1(\mathrm{Re}\,Y^-(u+iv)) \leq \varphi_1\left(\frac{-1}{\sqrt{2}\sigma_{22}}|v|\right) \leq \varphi_1(0). \tag{2.3.14}$$

By Lemma 2.19 (i) $\varphi_1(0)<\infty$. It follows that

$$\sup_{u\in[-\epsilon,x_{max}+\delta]} \varphi_1(Y^-(u+iv)) < \infty. \tag{2.3.15}$$

By Lemma 2.17 (i) there exists a constant $d_1>0$ such that

$$|\gamma_1(u+iv,Y^-(u+iv))| \leq d_1|v|, \quad \forall u\in[-\epsilon,x_{max}+\eta],\ |v|\geq\epsilon. \tag{2.3.16}$$

Note that $|\gamma_2(u+iv,Y^-(u+iv))| \geq |r_{12}u+r_{22}\mathrm{Re}Y^-(u+iv)|$. Then by Lemma 2.17 (ii) and also by Lemma 2.19 (ii) there exists a constant $d_2>0$ such that

$$|\gamma_2(u+iv,Y^-(u+iv))| \geq d_2|v|, \quad \forall u\in[-\epsilon,x_{max}+\eta],\ |v|\geq\epsilon. \tag{2.3.17}$$

Finally by Lemma 2.17 (ii)

$$|\exp(a_0(u+iv)+b_0Y^-(u+iv))| = \exp(a_0u+b_0\mathrm{Re}\,Y^-(u+iv)) \leq \exp\left(\left(a_0-b_0\frac{\sigma_{12}}{\sigma_{22}}\right)u-\frac{b_0}{\sqrt{2}\sigma_{22}}|v|\right) \tag{2.3.18}$$

for any $u\in[-\epsilon,x_{max}+\eta]$ and v with $|v|>\epsilon$. Then the estimate (2.3.13), the representation (2.3.2) combined with the estimates (2.3.15), (2.3.16), (2.3.17) and (2.3.18) lead to the statement of the lemma. □

2.4 Inverse Laplace transform: from a double integral to simple integrals

By the Laplace transform inversion formula ([33, Theorem 24.3 and 24.4] and [16]), for any $\epsilon>0$ small enough,

$$g(a,b) = \frac{1}{(2\pi i)^2}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi(x,y)\exp(-ax-by)dxdy.$$

in the sense of principal value convergence.

Lemma 2.27 (Inverse Laplace transform as a sum of simple integrals)**.** *Let* $z_0 = (a_0, b_0)$ *be the starting point of the process. For any* $(a, b) \in \mathbb{R}_+^2$ *where either* $a > a_0, b > 0$ *or* $b > b_0, a > 0$ *the following representation holds :*

$$g(a,b) = I_1(a,b) + I_2(a,b) + I_3(a,b)$$

where

$$I_1(a,b) = \frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi_2(x)\gamma_2(x,Y^+(x))\exp(-ax-bY^+(x))\frac{dx}{\gamma'_y(x,Y^+(x))},$$

$$I_2(a,b) = \frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi_1(y)\gamma_1(X^+(y),y)\exp(-aX^+(y)-by)\frac{dy}{\gamma'_x(X^+(y),y)},$$

$$I_3(a,b) = \frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \exp(a_0x+b_0Y^+(x))\exp(-ax-bY^+(x))\frac{dx}{\gamma'_y(x,Y^+(x))} \quad \textit{if } b > b_0,$$

$$I_3(a,b) = \frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \exp(a_0X^+(y)+b_0y)\exp(-aX^+(y)-by)\frac{dy}{\gamma'_x(X^+(y),y)} \quad \textit{if } a > a_0.$$

The two different formulas for I_3 will be useful in Section 2.9 to study the asymptotics along the axes.

Proof. For any $\epsilon > 0$ small enough $\gamma(-\epsilon, -\epsilon) < 0$. Then

$$\mathrm{Re}\gamma(-\epsilon+iv_1, -\epsilon+iv_2) < 0 \quad \forall v_1, v_2 \in \mathbb{R} \tag{2.4.1}$$

since Σ is a covariance matrix. Then by (2.2.2)

$$g(a,b) = \frac{-1}{(2\pi i)^2}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{\gamma_1(x,y)\varphi_1(y)+\gamma_2(x,y)\varphi_2(x)+\exp(a_0x+b_0y),}{\gamma(x,y)}\exp(-ax-by)dxdy$$

Now, let us consider for example the second term. It can be written as

$$\frac{-1}{(2\pi i)^2}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\varphi_2(x)\exp(-ax)\Big(\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\frac{\gamma_2(x,y)}{\gamma(x,y)}\exp(-by)dy\Big)dx.$$

Note that the convergence in the sense of the principal value of this integral can be guaranteed by integrating by parts. Now, it just remains to show that

$$\frac{-1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\frac{\gamma_2(x,y)}{\gamma(x,y)}\exp(-by)dy = \frac{\gamma_2(x,Y^+(x))}{\gamma'_y(x,Y^+(x))}\exp\big(-bY^+(x)\big) \tag{2.4.2}$$

Let $x = -\epsilon$. The equation $\gamma(-\epsilon, y) = 0$ has two solutions $Y^+(-\epsilon) > 0$ and $Y^-(-\epsilon) < 0$. (In fact, for $\epsilon > 0$ small enough $Y^+(-\epsilon)$ is close to $Y^+(0) = 0$ staying positive and $Y^-(-\epsilon)$ is close to $Y^-(0) = -2\mu_2/\sigma_{22} < 0$). Let $x = -\epsilon + iv$. Functions $Y^+(-\epsilon+iv)$ and $Y^-(-\epsilon+iv)$ are continuous in v. By (2.4.1) their real parts do not equal $-\epsilon$ for no $v \in \mathbb{R}$. Thus $\mathrm{Re}Y^+(-\epsilon+iv) > -\epsilon$ and $\mathrm{Re}Y^-(-\epsilon+iv) < -\epsilon$ for all $v \in \mathbb{R}$. Let us construct the contour $[-\epsilon-iR, -\epsilon+iR] \cup \{t+iR, \mid t \in [-\epsilon, 0]\} \cup \{Re^{it} \mid t \in]-\pi/2+\pi/2[\} \cup \{t-iR, \mid t \in [-\epsilon, 0]\}$, see Figure 2.5.

For any fixed $x = -\epsilon + iv$, the integral over this contour taken in the counter-clockwise direction of the function $\frac{\gamma_2(x,y)}{\gamma(x,y)}\exp(-by)$ equals the residue of this function multiplied by $2\pi i$, which is exactly the result announced in (2.4.2). It suffices to show that the integral over $\{t+iR \mid t \in [-\epsilon, 0]\} \cup \{Re^{it} \mid t \in]-\pi/2+\pi/2[\} \cup \{t-iR \mid t \in [-\epsilon, 0]\}$ converges to zero as $R \to \infty$. The integral over the half of the circle $\{Re^{it} \mid t \in]-\pi/2+\pi/2[\}$ equals

$$\int_{-\pi/2}^{\pi/2}\frac{\gamma_2(x,Re^{it})}{\gamma(x,Re^{it})}\exp(-bRe^{it})iRe^{it}dt.$$

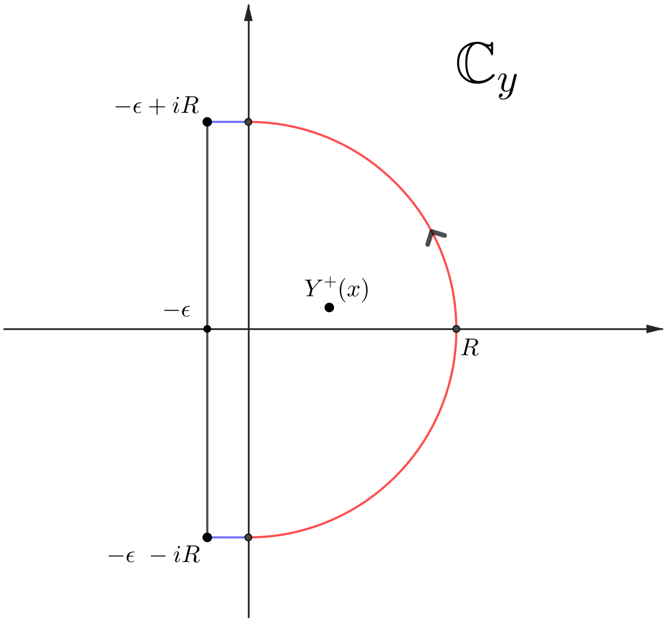


Figure 2.5: Integral contour in the complex plane $\mathbb{C}_y$, with the pole $Y^+(x)$.

We have $\sup_{R>R_0} \sup_{t\in]-\pi/2,\pi/2[} \left|\frac{\gamma_2(x,Re^{it})}{\gamma(x,Re^{it})} iRe^{it}\right| < \infty$ for $R_0 = R_0(x) > 0$ large enough, while $|\exp(-bRe^{it})| = \exp(-bR\cos t) \to 0$ as $R \to \infty$ for any $t \in]-\pi/2, \pi/2[$ since $b > 0$. Hence, the integral over the half of the circle converges to zero as $R \to \infty$ by the dominated convergence theorem. Let us look at the integral over segment $\{t + iR \mid t \in [-\epsilon, 0]\}$. For any fixed $x = -\epsilon + iv$, there exists a constant $C(x) > 0$ such that for any R large enough

$$\sup_{u\in[-\epsilon,0]} \left|\frac{\gamma_2(x,u+iR)}{\gamma(x,u+iR)}\right| \leq \frac{C(x)}{R}.$$

Therefore

$$\left|\int_{-\epsilon}^{0} \frac{\gamma_2(x,u+iR)}{\gamma(x,u+iR)} \exp(-b(u+iR))du\right| \leq \epsilon \exp(b\epsilon)\frac{C(x)}{R} \underset{R\to\infty}{\longrightarrow} 0.$$

The representation of $I_1(a,b)$ follows.

The reasoning is the same for the third term. The integral over the half of the circle equals

$$\int_{-\pi/2}^{\pi/2} \frac{\exp(-(b-b_0)Re^{it})}{\gamma(x,Re^{it})} iRe^{it}dt.$$

We have $\sup_{R>R_0} \sup_{t\in]-\pi/2,\pi/2[} \left|\frac{1}{\gamma(x,Re^{it})} iRe^{it}\right| < \infty$ while $|\exp(-(b-b_0)Re^{it})| = \exp(-(b-b_0)R\cos t) \to 0$ as $R \to \infty$ for any $t \in]-\pi/2, \pi/2[$ since $b - b_0 > 0$. The integral over the half of the circle converges to zero as $R \to \infty$ by the dominated convergence theorem once again. For any fixed $x = -\epsilon + iv$, there exists a constant $C(x) > 0$ such that for any R large enough

$$\sup_{u\in[-\epsilon,0]} \left|\frac{1}{\gamma(x,u+iR)}\right| \leq \frac{C(x)}{R^2}.$$

Therefore

$$\left|\int_{-\epsilon}^{0} \frac{\exp(-(b-b_0)(u+iR))}{\gamma(x,u+iR)}du\right| \leq \epsilon \exp((b-b_0)\epsilon)\frac{C(x)}{R^2} \to 0, \quad R \to \infty.$$

The representations for $I_2(a,b)$ and $I_3(a,b)$ with $a > a_0$ are obtained in the same way. $\square$

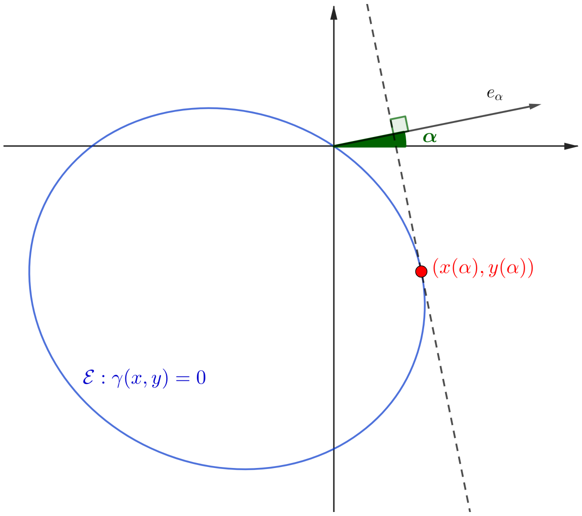


Figure 2.6: Graphic representation of the saddle point. We denote $e_\alpha = (\cos(\alpha), \sin(\alpha))$.

Remark. Let us introduce some notations a, b, c, $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ by

$$\gamma(x,y) = a(x)y^2 + b(x)y + c(x) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y). \tag{2.4.3}$$

Then functions in the integrand can be represented as

$$\gamma'_y(x, Y^+(x)) = a(x)(Y^+(x) - Y^-(x)) = 2a(x)Y^+(x) + b(x) = \sqrt{b^2(x) - 4a(x)c(x)} \tag{2.4.4}$$

$$\gamma'_x(X^+(y), y) = \widetilde{a}(y)(X^+(y) - X^-(y)) = 2\widetilde{a}(y)X^+(y) + \widetilde{b}(y) = \sqrt{\widetilde{b}^2(y) - 4\widetilde{a}(y)\widetilde{c}(y)} \tag{2.4.5}$$

All these forms will be used in the following.

2.5 Saddle point and contour of the steepest descent

Our aim is to study the integrals I_1, I_2 and I_3 of Lemma 2.27 using the saddle point method. One of the reference books about this approach is those of Fedoryuk [42].

Saddle point

For $\alpha \in [0, 2\pi[$ we define

$$(x(\alpha), y(\alpha)) := \text{argmax}_{(x,y):\gamma(x,y)=0}(x\cos\alpha + y\sin\alpha). \tag{2.5.1}$$

We are going to see that this point turns out to be the saddle point of the functions inside the exponentials of the integrals I_1, I_2 and I_3. See Figure 2.6 for a geometric interpretation of this point.

The map $\alpha : [0, 2\pi[\to \{(x,y) : \gamma(x,y) = 0\}$ is a diffeomorphism. Functions $x(\alpha), y(\alpha)$ are in the class $C^\infty([0, 2\pi])$. For any $\alpha \in [0, \pi/2]$ function $\cos(\alpha)x + \sin(\alpha)Y^+(x)$ reaches its maximum at the unique point on $[X^\pm(y_{max}), x_{max}]$ called $x(\alpha)$. This function is strictly increasing on $[X^\pm(y_{max}), x(\alpha)]$ and strictly decreasing on $[x(\alpha), x_{max}]$. Function $\cos(\alpha)X^+(y) + \sin(\alpha)y$ reaches its maximum on $[Y^\pm(x_{max}), y_{max}]$ at the unique point $y(\alpha)$. It is strictly increasing on $[Y^\pm(x_{max}), y(\alpha)]$ and strictly decreasing on $[y(\alpha), y_{max}]$.

Thus $x(0) = x_{max}$, $y(0) = Y^\pm(x_{max})$, $x(\pi/2) = X^\pm(y_{max})$, $y(\pi/2) = y_{max}$. Finally, $x(\alpha) = 0$ and $y(\alpha) = 0$ if $(\cos(\alpha), \sin(\alpha)) = \left(\frac{\mu_1}{\sqrt{\mu_1^2+\mu_2^2}}, \frac{\mu_2}{\sqrt{\mu_1^2+\mu_2^2}}\right)$. We denote this direction corresponding to the drift by α_μ.

Let's define the functions

$$F(x,\alpha) = -\cos(\alpha)x - \sin(\alpha)Y^+(x) + \cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha), \tag{2.5.2}$$

$$G(y,\alpha) = -\cos(\alpha)X^+(y) - \sin(\alpha)y + \cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha).$$

The function F appears to be (up to a constant) the function inside exponential of the integral I_1, and the function G appears to be (up to a constant) the function inside the exponential of the integral I_2, see Lemma 2.27. We have

$$F(x(\alpha),\alpha) = 0 \quad \forall\alpha \in [0,\pi/2]$$

and

$$F'_x(x(\alpha),\alpha) = 0 \quad \forall\alpha \in]0,\pi/2], \text{ but not at } \alpha = 0.$$

In the same way $G(y(\alpha),\alpha) = 0$ for any $\alpha \in [0,\pi/2]$ and $G'_y(y(\alpha),\alpha) = 0$ for any $\alpha \in [0,\pi/2[$ but not at $\alpha = \pi/2$. Then $(Y^+(x(\alpha)))' = -\text{ctan}(\alpha)$ and $(X^+(y(\alpha)))' = -\tan(\alpha)$.

Using the identities $\gamma(x, Y^+(x)) \equiv 0$ and $\gamma(X^+(y), y) \equiv 0$, we get :

$$(Y^+(x))' \Big|_{x=x(\alpha)} = -\frac{\gamma'_x(x(\alpha),y(\alpha))}{\gamma'_y(x(\alpha),y(\alpha))} = -\frac{\cos(\alpha)}{\sin(\alpha)}, \quad \alpha \in]0,\pi/2] \tag{2.5.3}$$

$$(X^+(y))' \Big|_{y=y(\alpha)} = -\frac{\gamma'_y(x(\alpha),y(\alpha))}{\gamma'_x(x(\alpha),y(\alpha))} = -\frac{\sin(\alpha)}{\cos(\alpha)}, \quad \alpha \in [0,\pi/2[$$

$$(Y^+(x))'' \Big|_{x=x(\alpha)} = -\frac{\sigma_{11} + 2\sigma_{12}(-\text{ctan}\,(\alpha)) + \sigma_{22}(-\text{ctan}\,(\alpha))^2}{\gamma'_y(x(\alpha),y(\alpha))}$$

$$(X^+(y))'' \Big|_{x=x(\alpha)} = -\frac{\sigma_{11}(-\tan(\alpha))^2 + 2\sigma_{12}(-\tan\,(\alpha)) + \sigma_{22}}{\gamma'_x(x(\alpha),y(\alpha))}$$

$$F''_x(x(\alpha),\alpha) = \frac{\sigma_{11}\sin^2(\alpha) + 2\sigma_{12}\sin(\alpha)\cos(\alpha) + \sigma_{22}\cos^2(\alpha)}{\gamma'_y(x(\alpha),y(\alpha))\sin\alpha} > 0 \quad \alpha \in]0,\pi/2], \tag{2.5.4}$$

$$G''_y(y(\alpha),\alpha) = \frac{\sigma_{11}\sin^2(\alpha) + 2\sigma_{12}\sin(\alpha)\cos(\alpha) + \sigma_{22}\cos^2(\alpha)}{\gamma'_x(x(\alpha),y(\alpha))\cos(\alpha)} > 0 \quad \alpha \in [0,\pi/2[,$$

the strict inequality coming from (2.4.4), (2.4.5) and the positive-definite form of Σ.

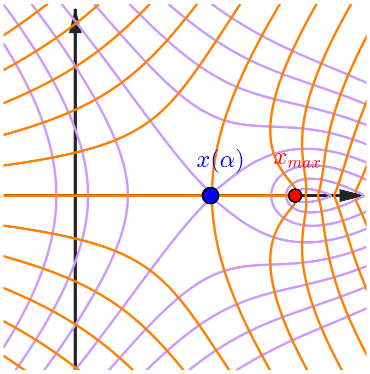


Figure 2.7: Level sets of $\Re(F)$ in purple and of $\Im(F)$ in orange. The saddle point $x(\alpha)$ is represented in blue and the branch point x_{max} is in red.

The values of $x(\alpha)$ and $y(\alpha)$ are given by the following formulas.

$$x(\alpha) = \frac{(\mu_2\sigma_{12} - \mu_1\sigma_{22})}{\det(\Sigma)} + \frac{1}{\det(\Sigma)}(\sigma_{22} - \tan(\alpha)\sigma_{12})\sqrt{\frac{\mu_2^2\sigma_{11} - 2\mu_1\mu_2\sigma_{12} + \mu_1^2\sigma_{22}}{\sigma_{11}\tan^2(\alpha) - 2\sigma_{12}\tan(\alpha) + \sigma_{22}}} \tag{2.5.5}$$

$$y(\alpha) = \frac{(\mu_1\sigma_{12} - \mu_2\sigma_{11})}{\det(\Sigma)} + \frac{1}{\det(\Sigma)}\left(\sigma_{11} - \frac{1}{\tan(\alpha)}\sigma_{12}\right)\sqrt{\frac{\mu_1^2\sigma_{22} - 2\mu_1\mu_2\sigma_{12} + \mu_2^2\sigma_{11}}{\frac{\sigma_{22}}{\tan^2(\alpha)} - 2\frac{\sigma_{12}}{\tan(\alpha)} + \sigma_{11}}} \tag{2.5.6}$$

Indeed, using the same calculations as in section 4.2 of [52], the equation $0 = \frac{d}{dx}\left[\gamma(x, Y^+(x))\right]|_{x=x(\alpha)}$ combined with the first equation of (2.5.3) gives a linear relationship between $x(\alpha)$ and $y(\alpha)$. Injecting this condition in the polynomial equation $\gamma(x(\alpha), y(\alpha)) = 0$, we get two possible values for $x(\alpha)$ and $y(\alpha)$. The choice of the sign depends then of α and we get this expression.

Contour of the steepest descent

Before continuing, the reader should read Appendix 2.A which states a parameter dependent Morse lemma. Let $\alpha_0 \in]0, \pi/2]$. We apply Lemma 2.A.1 to F defined in (2.5.2). Let us fix any $\epsilon \in]0, K[$ and consider any α such that $|\alpha - \alpha_0| < \eta$, where constants K and η are taken from the definition of $\Omega(0, \alpha_0)$ in Lemma 2.A.1. Then, for any α we can construct the contour of the steepest descent

$$\Gamma_{x,\alpha} = \{x(it, \alpha) \mid t \in [-\epsilon, \epsilon]\}.$$

Clearly

$$F(x(it, \alpha), \alpha) = -t^2.$$

We denote by $x_\alpha^+ = x(i\epsilon, \alpha)$ and $x_\alpha^- = x(-i\epsilon, \alpha)$ its ends. Then

$$F(x_\alpha^+, \alpha) = -\epsilon^2, \quad F(x_\alpha^-, \alpha) = -\epsilon^2. \tag{2.5.7}$$

Since $F_x''(x(\alpha), \alpha) \neq 0$, the contours in a neighborhood of $x(\alpha)$ where the function F is real are orthogonal, see Figure 2.7. One of them is the real axis. The other is the contour of the steepest descent, which is the orthogonal to the real axis. It follows that $\mathrm{Im} x_{\alpha_0}^+ > 0$ and $\mathrm{Im} x_{\alpha_0}^- < 0$. By continuity of $x(i\epsilon, \alpha)$ on α for any $\eta > 0$ small enough, there exists $\nu > 0$ such that

$$\mathrm{Im}\, x_\alpha^+ > \nu, \quad \mathrm{Im}\, x_\alpha^- < -\nu \quad \forall \alpha : |\alpha - \alpha_0| < \eta. \tag{2.5.8}$$

In the same way, for any $\alpha \in [0, \pi/2[$, we may define by the generalized Morse lemma the function $y(\omega, \alpha)$ w.r.t. $G(y, \alpha)$. Let $\alpha_0 \in [0, \pi/2[$. We can construct the contour of the steepest descent

$$\Gamma_{y,\alpha} = \{y(it, \alpha) \mid t \in [-\epsilon, \epsilon]\}$$

with end points $y_\alpha^+ = y(i\epsilon, \alpha)$ and $y_\alpha^- = y(-i\epsilon, \alpha)$ and the property analogous to (2.5.8).

We note that for any $\alpha =]0, \pi/2[$

$$\Gamma_{x,\alpha} = \overleftarrow{\underrightarrow{X^+(\Gamma_{y,\alpha})}}, \quad \Gamma_{y,\alpha} = \overleftarrow{\underrightarrow{Y^+(\Gamma_{x,\alpha})}}. \tag{2.5.9}$$

The arrows mean that the direction has to be changed because of the facts that $(X^+(y))'\Big|_{y=y(\alpha)} < 0$ and $(Y^+(x))'\Big|_{x=x(\alpha)} < 0$.

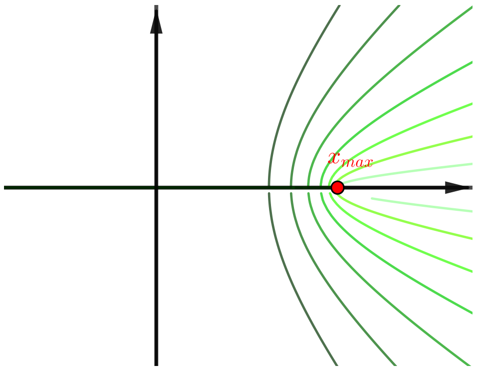


Figure 2.8: Steepest descent contour for $\Re(F)$ according to α, the closer alpha is to zero, the lighter green the corresponding contour becomes. When $\alpha \to 0$ this contour tends the half line $[x_{max}, \infty)$.

Case where $\alpha_0 = 0$

In this case $\Gamma_{y,0}$ is now well defined, but not $\Gamma_{x,0}$ (since $F''(x(0), 0) = \infty$), see Figure 2.8. We may define then

$$\Gamma_{x,0} = \overleftarrow{\underrightarrow{X^+(\Gamma_{y,0})}}$$

with end points $x_0^+ = X^+(y_0^+) = x_{max} + \epsilon^2$ and $x_0^- = X^+(y_0^-) = x_{max} + \epsilon^2$. In fact, for $\alpha = 0$, we have $G(y, 0) = -X^+(y) + x_{max}$ and $G(y(i\epsilon, 0), 0) = -\epsilon^2$. Thus $\Gamma_{x,0}$ runs the real segment from $x_{max} + \epsilon^2$ to x_{max} and back $x_{\max} + \epsilon^2$. Figure 2.8 illustrates why this phenomenon happens when $\alpha = 0$. Again by continuity on α we may find $\eta > 0$ and $\nu > 0$ small enough, such that

$$Rex_\alpha^+ - x_{max} > \nu, \quad Rex_\alpha^- - x_{max} > \nu, \quad \forall \alpha \in [0, \eta]. \tag{2.5.10}$$

If $\alpha_0 = \pi/2$, $\Gamma_{x,\pi/2}$ is well defined, but not $\Gamma_{y,\pi/2}$. We put then

$$\Gamma_{y,\pi/2} = \overleftarrow{\underrightarrow{Y^+(\Gamma_{x,\pi/2})}}$$

with end points $y_\alpha^+ = Y^+(x_\alpha^+)$ and $y_\alpha^- = Y^+(x_\alpha^-)$.

2.6 Shift of the integration contours and contribution of the poles

We are going to define the integration contours of I_1, I_2 and I_3 thanks to the contours of the steepest descent studied in the previous section. First, let us define

$$S_{x,\alpha}^+ = \{x_\alpha^+ + it \mid t \geq 0\ \}, \quad S_{x,\alpha}^- = \{x_\alpha^- - it \mid t \geq 0\},$$

$$S_{y,\alpha}^+ = \{y_\alpha^+ + it \mid t \geq 0\}, \quad S_{y,\alpha}^- = \{y_\alpha^- - it \mid t \geq 0\}.$$

Now, let us construct the integration contours $T_{x,\alpha} = S_{x,\alpha}^- + \Gamma_{x,\alpha} + S_{x,\alpha}^+$ and $T_{y,\alpha} = S_{y,\alpha}^- + \Gamma_{y,\alpha} + S_{y,\alpha}^+$ for any $\alpha \in [0, \pi/2]$. See Figure 2.9 which illustrates these integration contours.

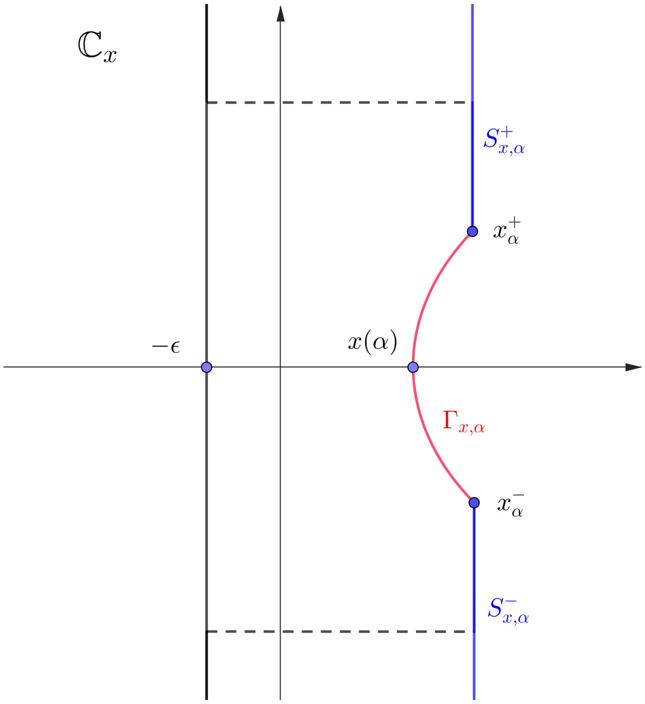


Figure 2.9: In the complex plane $\mathbb{C}_x$, shift of the integration contour passing through the saddle point along the steepest line.

Case where the saddle point meet the pole

The only exception to define these contours will be for $\alpha \in [0, \pi/2]$ such that $x(\alpha) = x^* \in]0, x_{max}[$ is a pole of $\varphi_2(x)$ and $y(\alpha) = y^{**} \in]0, y^{max}[$ is a pole of $\varphi_1(y)$. We call these directions α^* and α^{**}, so that $x(\alpha^*) = x^*$, $y(\alpha^*) = Y^+(x^*) = y^*$, $y(\alpha^{**}) = y^{**}$, $x(\alpha^{**}) = X^+(y^{**}) = x^{**}$. When the poles x^* and y^{**} exists, we recall that by the Lemma 2.19 :

$$x^* = 2\frac{\mu_2\frac{r_{12}}{r_{22}} - \mu_1}{\sigma_{11} - 2\sigma_{12}\frac{r_{12}}{r_{22}} + \sigma_{22}\left(\frac{r_{12}}{r_{22}}\right)^2} \quad \text{and} \quad y^{**} = 2\frac{\mu_1\frac{r_{21}}{r_{11}} - \mu_2}{\sigma_{11}\left(\frac{r_{21}}{r_{11}}\right)^2 - 2\sigma_{12}\frac{r_{21}}{r_{11}} + \sigma_{22}}. \tag{2.6.1}$$

We also recall that by definition

$$y^* := Y^+(x^*) \quad \text{and} \quad x^{**} := X^+(y^{**}). \tag{2.6.2}$$

We remark that we have $y^* = -\frac{r_{12}}{r_{22}}x^*$ (resp. $x^{**} = -\frac{r_{21}}{r_{11}}y^{**}$) if and only if x^* (resp y^{**}) is not a pole of φ_2 (resp. φ_1) because of the condition on x^* and y^{**} to be poles.

If the pole x^* exists, then $\alpha^* \in]0, \alpha_\mu[$, and if y^* exists, then $\alpha^{**} \in]\alpha_\mu, \pi/2[$. We denote for convenience $\alpha^* = -\infty$ if x^* does not exist and $\alpha^{**} = +\infty$ if y^* does not exist.

If $\alpha = \alpha^* \in]0, \alpha_\mu[$, we modify in the definition of $T_{x,\alpha}$ the contour $\Gamma_{x,\alpha}$ by $\widetilde{\Gamma}_{x,\alpha}$, which is the half of the circle centred at $x(\alpha^*)$ going from $x^+_{\alpha^*}$ to $x^-_{\alpha^*}$ in the counter-clockwise direction. The same modification is made for $\alpha = \alpha^{**} \in]\alpha_\mu, \pi/2[$.

The next lemma perform the shift of the integration contour and take into account the contribution of the crossed poles. Recall that I_1, I_2 and I_3 are defined in Lemma 2.27.

Lemma 2.28 (Contribution of the poles to the asymptotics). *Let $\alpha \in [0, \pi/2]$. Then for any $a, b > 0$*

$$I_1(a,b) = \frac{\left(-\text{res}_{x=x^*}\varphi_2(x)\right)\gamma_2(x^*, y^*)}{\gamma'_y(x^*, y^*)} \exp(-ax^* - by^*) \times \mathbf{1}_{\alpha<\alpha^*}$$

$$+\frac{1}{2\pi i}\int_{T_{x,\alpha}} \frac{\varphi_2(x)\gamma_2(x,Y^+(x))}{\gamma_y'(x,Y^+(x))}\exp(-ax-bY^+(x))dx,$$

$$I_2(a,b)=\frac{(-\mathrm{res}_{y=y^{**}}\varphi_1(y))\gamma_1(x^{**},y^{**})}{\gamma_x'(x^{**},y^{**})}\exp(-ax^{**}-by^{**})\times \mathbf{1}_{\alpha>\alpha^{**}}$$

$$+\frac{1}{2\pi i}\int_{T_{y,\alpha}} \frac{\varphi_1(y)\gamma_1(X^+(y),y)}{\gamma_x'(X^+(y),y)}\exp(-aX^+(y)-by)dy,$$

$$I_3(a,b)=\frac{1}{2\pi i}\int_{T_{x,\alpha}} \exp((a_0-a)x+(b_0-b)Y^+(x))\frac{dx}{\gamma_y'(x,Y^+(x))}\quad \textit{if } b>b_0$$

$$I_3(a,b)=\frac{1}{2\pi i}\int_{T_{x,\alpha}} \exp((a_0-a)X^+(y)+(b_0-b)y)\frac{dy}{\gamma_x'(X^+(y),y)}\quad \textit{if } a>a_0.$$

One may remark that we have $\gamma_2(x^*,y^*)\mathrm{res}_{x^*}\varphi_2<0$ and $\gamma_1(x^{**},y^{**})\mathrm{res}_{y^{**}}\varphi_1<0$.

Proof. We start from the result of Lemma 2.27 and we use Cauchy theorem to shift the integration contour. We take into account the poles by the residue theorem noting that $x^*<x(\alpha)$ if and only if $\alpha<\alpha^*$ and that $y^{**}<y(\alpha)$ if and only if $\alpha^{**}<\alpha$. In order to get the representation of I_1 by shifting the contour, we want to show that the integrals on the dotted lines of Figure 2.9 tends to 0 when these lines goes to infinity. To do so, it suffices to show that for any $\eta>0$ small enough ,

$$\sup_{u\in[X^+(y_{max})-\eta,x^{max}+\eta]}\left|\frac{\varphi_2(u+iv)\gamma_2(u+iv,Y^+(u+iv))}{\gamma_y'(u+iv,Y^+(u+iv))}\exp(-a(u+iv)-bY^+(u+iv))\right|\to 0,\quad \text{as } v\to\infty.$$

In reality, it would be sufficient to study the supremum on $[-\epsilon,x^{max}+\eta]$. By Lemma 2.26 for any $\epsilon>0$,

$$\sup_{u\in[X^+(y_{max})-\eta,x^{max}+\eta],|v|\geq\epsilon}|\varphi_2(u+iv)|<\infty.$$

Let us observe that by (2.4.4)

$$\gamma_y'(x,Y^+(x))=\sqrt{b^2(x)-4a(x)c(x)}=\sqrt{(\sigma_{12}^2-\sigma_{11}\sigma_{22})x^2+2(\mu_2\sigma_{12}-\mu_1\sigma_{22})x+\mu_2^2}. \tag{2.6.3}$$

This function equals zero only at real points x_{min} and x_{max} and grows linearly in the absolute value as $|\Im x|\to\infty$. By Lemma 2.17 (i) function $|\gamma_2(x,Y^+(x))|$ grows linearly as well as $|\Im x|\to\infty$. Then for any $\epsilon>0$

$$\sup_{\substack{u\in[X^+(y_{max})-\eta,x^{max}+\eta],\\ |v|\geq\epsilon}}\left|\frac{\gamma_2(u+iv,Y^+(u+iv))}{\gamma_y'(u+iv,Y^+(u+iv))}\exp(-a(u+iv))\right|<\infty,$$

Finally,

$$\sup_{u\in[X^+(y_{max})-\eta,x^{max}+\eta]}|\exp(-bY^+(u+iv))|=\sup_{u\in[X^+(y_{max})-\eta,x^{max}+\eta]}\exp(-b\mathrm{Re}Y^+(u+iv))\to 0,$$

as $|v|\to\infty$ due to Lemma 2.17 (ii) and the fact that $b>0$. The other representations are obtained in the same way. In the representations of $I_3(a,b)$ we use the facts that $a-a_0>0$ and $b-b_0>0$. □

2.7 Exponentially negligible part of the asymptotics

Let's recall the integration contours $T_{x,\alpha} = S^-_{x,\alpha} + \Gamma_{x,\alpha} + S^+_{x,\alpha}$ and $T_{y,\alpha} = S^-_{y,\alpha} + \Gamma_{y,\alpha} + S^+_{y,\alpha}$ for any $\alpha \in [0, \pi/2]$. This section establishes a domination of the integrals on the contours $S^{\pm}_{x,\alpha}$ and $S^{\pm}_{y,\alpha}$. This domination is useful in the following sections to show that these integrals are negligible. We will see that the asymptotics of integrals I_1, I_2 and I_3 of contour $T_{x,\alpha}$ and $T_{y,\alpha}$ are given by the integrals on the lines of steepest descent $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$.

Lemma 2.29 (Negligibility of the integrals on $S^{\pm}_{x,\alpha}$ and $S^{\pm}_{y,\alpha}$)**.** *For any couple $(a, b) \in \mathbb{R}^2_+$ we may define $\alpha(a, b)$ the angle in $[0, \pi/2]$ such that $\cos(\alpha) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha) = \frac{b}{\sqrt{a^2+b^2}}$.*

- *Let $\alpha_0 \in]0, \pi/2]$. Then for any η small enough and any $r_0 > 0$ there exists a constant $D > 0$ such that for any couple (a, b) such that $\sqrt{a^2 + b^2} > r_0$ and $|\alpha(a, b) - \alpha_0| < \eta$ we have*

$$\Big| \int_{S^+_{x,\alpha}} \frac{\varphi_2(x)\gamma_2(x, Y^+(x))}{\gamma'_y(x, Y^+(x))} \exp\big(- ax - bY^+(x)\big)dx \Big| \leq \frac{D}{b} \exp\Big(- ax(\alpha) - by(\alpha) - \epsilon^2 \sqrt{a^2 + b^2}\Big) \tag{2.7.1}$$

  and if furthermore $b > b_0$ we have

$$\Big| \int_{S^+_{x,\alpha}} \exp((a_0-a)x+(b_0-b)Y^+(x)) \frac{dx}{\gamma'_y(x, Y^+(x))} \Big| \leq \frac{D}{b - b_0} \exp\Big(-ax(\alpha)-by(\alpha)-\epsilon^2 \sqrt{a^2 + (b - b_0)^2}\Big). \tag{2.7.2}$$

- *Let $\alpha_0 \in [0, \pi/2[$. Then for any η small enough and any $r_0 > 0$ there exists a constant $D > 0$ such that for any couple (a, b) such that $\sqrt{a^2 + b^2} > r_0$, $|\alpha(a, b) - \alpha_0| \leq \eta$ we have*

$$\Big| \int_{S^+_{y,\alpha}} \frac{\varphi_1(y)\gamma_1(X^+(y), y)}{\gamma'_x(X^+(y), y)} \exp\big(- aX^+(y) - by\big)dy \Big| \leq \frac{D}{a} \exp\Big(- ax(\alpha) - by(\alpha) - \epsilon^2 \sqrt{a^2 + b^2}\Big). \tag{2.7.3}$$

  and if furthermore $a > a_0$ we have

$$\Big| \int_{S^+_{y,\alpha}} \exp((a_0-a)X^+(y)+(b_0-b)y) \frac{dy}{\gamma'_x(X^+(y), y)} \Big| \leq \frac{D}{a - a_0} \exp\Big(-ax(\alpha)-by(\alpha)-\epsilon^2 \sqrt{(a - a_0)^2 + b^2}\Big). \tag{2.7.4}$$

The same estimations hold true for $S^-_{x,\alpha}$ and $S^-_{y,\alpha}$.

Proof. With definition (2.5.2) and notation (2.5.7), the estimate (2.7.1) can be written as

$$\Big| \int_{v>0} \frac{\varphi_2(x^+_\alpha + iv)\gamma_2(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))}{\gamma'_y(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))} \exp\big(- \sqrt{a^2 + b^2}\big(F(x^+_\alpha + iv, \alpha) - F(x^+_\alpha, \alpha)\big)dx \Big| \leq \frac{D}{b} \tag{2.7.5}$$

with $\alpha = \alpha(a, b)$. Let us prove it.

Let first $\alpha_0 \in]0, \pi/2]$. If $\alpha_0 \neq \pi/2$, let us fix $\eta > 0$ so small that $\alpha_0 - \eta > 0$, and $\alpha_0 + \eta \leq \pi/2$. If $\alpha_0 = \pi/2$, let us fix any small $\eta > 0$ and consider only $\alpha \in [\pi/2 - \eta, \pi/2]$.

By Lemma 2.26 and remark (2.5.8)

$$\sup_{v \geq 0, |\alpha - \alpha_0| \leq \eta} |\varphi_2(x^+_\alpha + iv)| < \infty. \tag{2.7.6}$$

By the observation (2.4.4) $\gamma'_y(x, Y^+(x)) = 0$ only if $x = x_{min}, x_{max}$. Then by (2.5.8) we have

$$\inf_{v \geq 0, |\alpha - \alpha_0| \leq \eta} |\gamma'_y(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))| > 0. \tag{2.7.7}$$

Furthermore again by (2.6.3) and Lemma 2.17 (ii) we have

$$\sup_{v\geq 0, |\alpha-\alpha_0|\leq\eta}\left|\frac{\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))}{\gamma_y'(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))}\right| < \infty. \tag{2.7.8}$$

Finally

$$|\exp\big(-\sqrt{a^2+b^2}(F(x_\alpha^+ + iv,\alpha) - F(x_\alpha^+,\alpha))\big)| = \exp\big(-b(\mathrm{Re}Y^+(x_\alpha^+ + iv) - \mathrm{Re}Y^+(x_\alpha^+))\big). \tag{2.7.9}$$

By Lemma 2.17 (ii) function $\mathrm{Re}Y^+(x_\alpha^+ + iv) - \mathrm{Re}Y^+(x_\alpha^+)$ equals 0 at $v = 0$ and strictly increasing as v goes from zero to infinity. Moreover, it grows linearly as $v\to\infty$: there exists a constant $c > 0$ such that for any α such that $|\alpha-\alpha_0|\leq\eta$ and any v large enough

$$\mathrm{Re}Y^+(x_\alpha^+ + iv) - \mathrm{Re}Y^+(x_\alpha^+) \geq cv. \tag{2.7.10}$$

It follows from (2.7.6), (2.7.8), (2.7.9) and (2.7.10) that the left hand side of (2.7.5) is bounded by

$$C\int_0^\infty \exp(-bcv)dv = C\times(cb)^{-1}$$

with some constant $C > 0$ and all couples (a, b) with $|\alpha(a, b) - \alpha_0| \leq \eta$.

As for the integral (2.7.2), let us make the change of variables $B = b-b_0 > 0$. Next, we proceed exactly as for (2.7.1). The only different detail is the elementary estimation $\sup_{|\alpha-\alpha_0|\leq\eta, v>0}|\exp(a_0(x_\alpha^+ + iv))| < \infty$. We obtain then the bound $\frac{D'}{B}\exp(-ax(\alpha) - By(\alpha) - \epsilon\sqrt{a^2+B^2})$ with some $D' > 0$. Then with $D = D'\exp(b_0 y(\alpha))$ the estimation (2.7.2) follows.

The proofs for (2.7.3) and (2.7.4) are symmetric. □

The previous lemma will be useful in Section 2.8 to establish the asymptotics when $\alpha_0 \in]0, \pi/2[$. In the next lemma we are going to show the negligibility of the integrals in the two missing cases when $\alpha_0 = 0$ or $\pi/2$ which will be useful in Section 2.9.

Remark 2.30 (Pole and branching point). In the next lemma and in Section 2.9 and 2.10, we exclude the case $\gamma_2(x_{max}, Y^\pm(x_{max})) = 0$ [resp. $\gamma_1(X^\pm(y_{max}), y_{max}) = 0$] such that the branching point and the pole of $\varphi_2(x)$ coincides. This case correspond to $x^* = x_{max}$ [resp. $y^* = y_{max}$], i.e. $\alpha^* = 0$ [resp. $\alpha^{**} = \pi/2$]. Note that we already obtained the asymptotics of h_1 and h_2 in these specific cases in Proposition 2.24.

Lemma 2.31 (Negligibility of the integrals on $S_{x,\alpha}^\pm$ and $S_{y,\alpha}^\pm$, case where $\alpha_0 = 0$ or $\pi/2$)**.** *For any $\eta > 0$ small enough and any $r_0 > 0$ there exists a constant $D > 0$ such that for any couple (a, b) such that $\sqrt{a^2+b^2} > r_0$ and $0 < \alpha(a, b) < \eta$ we have*

$$\Big|\int_{S_{x,\alpha}^+}\frac{\varphi_2(x)\gamma_2(x, Y^+(x))}{\gamma_y'(x, Y^+(x))}\exp\big(-ax - bY^+(x)\big)dx\Big| \leq D\exp\Big(-ax(\alpha) - by(\alpha) - \epsilon^2\sqrt{a^2+b^2}\Big) \tag{2.7.11}$$

and if furthermore $b > b_0$ we have

$$\Big|\int_{S_{x,\alpha}^+}\exp((a_0-a)x + (b_0-b)Y^+(x))\frac{dx}{\gamma_y'(x, Y^+(x))}\Big| \leq D\exp\Big(-ax(\alpha) - by(\alpha) - \epsilon^2\sqrt{a^2+(b-b_0)^2}\Big). \tag{2.7.12}$$

The same estimations hold true for $S_{x,\alpha}^-$. For any couple (a, b) such that $\sqrt{a^2+b^2} > r_0$ and $0 < \pi/2 - \alpha(a, b) < \eta$, a symmetric result holds for the integrals on $S_{y,\alpha}^+$ and $S_{y,\alpha}^-$.

Proof. Let now $\alpha_0 = 0$ so that $x(\alpha_0) = x_{max}$. Our aim is to prove (2.7.11) which is reduced to the estimate

$$\Big|\int_{v>0} \frac{\varphi_2(x_\alpha^+ + iv)\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))}{\gamma_y'(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))} \exp\big(-aiv - b(Y^+(x_\alpha^+ + iv) - Y^+(x_\alpha^+))\big)dv\Big| \le D \tag{2.7.13}$$

Let us fix any $\eta > 0$ small enough and consider $\alpha \in]0, \eta]$. By (2.4.4) the denominator $\gamma_y'(x, Y^+(x))$ has zero at $x = x_{max}$ but not at other points in a neighborhood of x_{max}. Then by (2.5.10) we have

$$\inf_{0\le\alpha\le\eta} |\gamma_y'(x_\alpha^+, Y^+(x_\alpha^+))| > 0. \tag{2.7.14}$$

The function $\varphi_2(x)$ has a branching point at $x_{\max}$. But it follows from the representation (2.3.2) that it is bounded in a neighborhood of x_{max} cut along the real segment due to Remark 2.30. Hence, this integral has no singularity at $v = 0$ for none $\alpha \in]0, \eta]$ so that

$$\sup_{0\le\alpha\le\eta} \frac{\varphi_2(x_\alpha^+)\gamma_2(x_\alpha^+, Y^+(x_\alpha^+))}{\gamma_y'(x_\alpha^+, Y^+(x_\alpha^+))} < \infty \tag{2.7.15}$$

Let us consider the asymptotics of the integrand in (2.7.13) as $v \to \infty$. It is clear that $Y^+(x_\alpha^+ + iv)$ grows linearly as $v \to \infty$ and so do functions γ_2 and γ_y' of this argument. Function $\varphi_2(x_\alpha + iv)$ is defined by the formula of the analytic continuation

$$\varphi_2(x_\alpha + iv) = -\frac{\gamma_1(x_\alpha^+ + iv, Y^-(x_\alpha^+ + iv))\varphi_1(Y^-(x_\alpha^+ + iv)) + \exp\big(a_0(x_\alpha^+ + iv) + b_0 Y^-(x_\alpha^+ + iv)\big)}{\gamma_2(x_\alpha^+ + iv, Y^-(x_\alpha^+ + iv))} \tag{2.7.16}$$

We know that $Y^-(x_\alpha^+ + iv)$ varies linearly as well as $v \to \infty$, and moreover $Re Y^-(x_\alpha + iv) \le -c_1 - c_2 v$ for all $v \ge 0$ and $\alpha \in]0, \eta]$ with some $c_1, c_2 > 0$. Then by Lemma 2.D.1 in Appendix 2.D,

$$|\varphi_2(x_\alpha^+ + iv)| \le C v^{\lambda - 1} \tag{2.7.17}$$

for any $\alpha \in]0, \eta]$ and $v > V_0$ with some $C > 0$, $V_0 > 0$ and $\lambda < 1$. Hence the integrand

$$\frac{\varphi_2(x_\alpha^+ + iv)\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))}{\gamma_y'(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))}$$

is about $O(v^{\lambda-1})$ as $v \to \infty$. The positivity of $\mathrm{Re}Y^+(x_\alpha^+ + iv) - \mathrm{Re}Y^+(x_\alpha^+)$ for any $v \ge 0$ and the inequality (2.7.10) in the exponent stay valid for any $\alpha \in]0, \eta]$, so that the exponential term is bounded in the absolute value by $\exp(-cbv)$ with some $c > 0$. But for η small enough, the assumption $\alpha(a, b) \in]0, \eta]$ implies the arbitrary smallness of b. In the limiting case $b = 0$ the integral in the l.h.s of (2.7.13) is not absolutely convergent. In order to prove the required estimate (2.7.13), we proceed by integration by parts, this integral equals

$$\frac{\varphi_2(x_\alpha^+ + iv)\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))}{\gamma_y'(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))(-ai - b(Y^+(x_\alpha^+ + iv))_v')} \exp\Big(-aiv - b(Y^+(x_\alpha^+ + iv) - Y^+(x_\alpha))\Big)\Big|_{v=0}^{v=\infty} \tag{2.7.18}$$

$$-\int_0^\infty \Big(\frac{\varphi_2(x_\alpha^+ + iv)\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))}{\gamma_y'(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))(-ai - b(Y^+(x_\alpha^+ + iv))_v')}\Big)_v' \exp\big(-aiv - b(Y^+(x_\alpha^+ + iv) - Y^+(x_\alpha^+))\big)dv. \tag{2.7.19}$$

Note that although in this case $x_{\alpha_0} = x_{max}$ which is a branching point for $Y^+(x)$, we have the first and second derivative bounded

$$\sup_{\alpha\in[0,\eta]} \Big|Y(x_\alpha^+ + iv)'\Big|_{v=0}\Big| < \infty, \quad \sup_{\alpha\in[0,\eta]} \Big|Y(x_\alpha^+ + iv)''\Big|_{v=0}\Big| < \infty \tag{2.7.20}$$

by remark (2.5.10). Furthermore $Y^{\pm}(x_{\alpha}^{+}+iv)'$ is of the constant order and $Y^{\pm}(x_{\alpha}^{+}+iv)''$ is not greater than $O(1/v)$ as $v\to\infty$.

The term (2.7.18) at $v=0$ is bounded in the absolute value by some constant due to (2.7.15) and (2.7.20). It converges to zero as $v\to\infty$ by all said above for any $\alpha\in[0,\infty]$, $a,b\geq 0$. To evaluate (2.7.19), we compute the derivative in its integrand and show that it is of the order $O(v^{\lambda-2})$ as $v\to\infty$. We skip technical details of this computation but outline fact that $\varphi_2(x_{\alpha}^{+}+iv)'_v$ is computed via the representation (2.7.16) and $|\varphi_1(Y^{-}(x_{\alpha}^{+}+iv))'_v|$ is evaluated again by Lemma 2.D.1, namely it is of the order not greater than $O(v^{\lambda-2})$ as $v\to\infty$. Thus the integral (2.7.19) is absolutely convergent for any $a,b\geq 0$ and can be bounded by some constant as well. This finishes the proof of (2.7.11). The proof of (2.7.12) is symetric. ☐

Note that the proof of Lemma 2.29 uses essentially the result of Lemma 2.26 which bounds the Laplace transforms. The proof of Lemma 2.31, for its part, uses a stronger result stated in Appendix 2.D which gives a more precise estimate of the Laplace transform near infinity.

Following the lines of the proof we could establish a better estimate, namely the one that the integral is bounded by some universal constant divided by a, but we do not need it for our purposes.

Remark 2.32 (Negligibility). When $\alpha(a,b)\to\alpha_0\in]0,\pi/2[$, Equations (2.7.1), (2.7.2), (2.7.3), (2.7.4) of Lemma 2.29 give quite satisfactory estimates to prove the negligibility of the integrals on the contours $S_{x,\alpha}^{\pm}$ and $S_{y,\alpha}^{\pm}$ with respect to integrals on contours $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$, see Lemma 2.33 below. In fact

$$\frac{\exp(-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+b^2})}{b}=o\Big(\frac{\exp(-ax(\alpha)-by(\alpha))}{\sqrt[4]{a^2+b^2}}\Big),$$

$$\frac{\exp(-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+b^2})}{a}=o\Big(\frac{\exp(-ax(\alpha)-by(\alpha))}{\sqrt[4]{a^2+b^2}}\Big).$$

When $\alpha(a,b)\to 0$ or $\pi/2$, Equations (2.7.11) and (2.7.12) of Lemma 2.31 give satisfactory estimates to prove the negligibility which will be useful in Section 2.9 to compute the asymptotics along the axes.

2.8 Essential part of the asymptotics and main theorem

This section is dedicated to the asymptotics of $g(a,b)=I_1+I_2+I_3$ when $\alpha(a,b)\to\alpha_0\in]0,\pi/2[$. The next lemma determines the asymptotics of the integrals on the lines of steepest descent $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$ of the shifted contours.

For any couple $(a,b)\in\mathbb{R}_+^2$ we define $\alpha(a,b)$ the angle in $[0,\pi/2]$ such that $\cos(\alpha)=\frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha)=\frac{b}{\sqrt{a^2+b^2}}$ and we define $r\in\mathbb{R}_+$ such that $r=\sqrt{a^2+b^2}$.

Lemma 2.33 (Contribution of the saddle point to the asymptotics)**.** *Let $\alpha_0\in]0,\pi/2[$. Let $\alpha(a,b)\to\alpha_0=$ and $r=\sqrt{a^2+b^2}\to\infty$. Then for any $n\geq 0$ we have*

$$\frac{1}{2\pi i}\int_{\Gamma_{x,\alpha}}\frac{\varphi_2(x)\gamma_2(x,Y^{+}(x))}{\gamma'_y(x,Y^{+}(x))}\exp(-ax-bY^{+}(x))dx+\frac{1}{2\pi i}\int_{\Gamma_{y,\alpha}}\frac{\varphi_1(y)\gamma_1(X^{+}(y),y)}{\gamma'_x(X^{+}(y),y)}\exp(-aX^{+}(y)-by)dy$$

$$+\frac{1}{2\pi i}\int_{\Gamma_{x,\alpha}}\exp((a_0-a)X^{+}(y)+(b_0-b)y)\frac{dy}{\gamma'_x(X^{+}(y),y)}$$

$$\sim\exp(-ax(\alpha(a,b))-by(\alpha(a,b)))\sum_{k=0}^{n}\frac{c_k(\alpha(a,b))}{\sqrt[4]{a^2+b^2}(a^2+b^2)^{k/2}} \tag{2.8.1}$$

with some constants $c_0(\alpha),c_1(\alpha),\dots,c_n(\alpha)$ continuous at α_0. Namely

$$c_0(\alpha)=\frac{\gamma_1(x(\alpha),y(\alpha))\varphi_1(y(\alpha))+\gamma_2(x(\alpha),y(\alpha))\varphi_2(x(\alpha))+\exp(a_0x(\alpha)+b_0y(\alpha))}{\sqrt{2\pi(\sigma_{11}\sin^2(\alpha)+2\sigma_{12}\sin(\alpha)\cos(\alpha)+\sigma_{22}\cos^2(\alpha))}}\times C(\alpha), \tag{2.8.2}$$

where

$$C(\alpha)=\sqrt{\frac{\sin(\alpha)}{\gamma_y'(x(\alpha),y(\alpha))}}=\sqrt{\frac{\cos(\alpha)}{\gamma_x'(x(\alpha),y(\alpha)}}.$$

Proof. Consider the first integral. Let us make the change of variables $x=x(it,\alpha)$, see Section 2.5 and Appendix 2.A. Then it becomes

$$\frac{\exp(-ax(\alpha)-by(\alpha))}{2\pi}\int_{-\epsilon}^{\epsilon}f(it,\alpha)\exp(-\sqrt{a^2+b^2}t^2)dt$$

where

$$f(it,\alpha)=\frac{\varphi_2(x(it,\alpha))\gamma_2(x(it,\alpha),Y^+(x(it,\alpha)))}{\gamma_y'(x(it,\alpha),Y^+(x(it,\alpha)))}x_\omega'(it,\alpha).$$

Let us take $\Omega(\alpha_0)$ from Lemma 2.A.1 with K and η defined in this lemma. For any $\alpha\in[\alpha_0-\eta,\alpha_0+\eta]$ and $t\in[-\epsilon,\epsilon]$ we have

$$\Big|f(it,\alpha)-\sum_{l=0}^{2n}f^{(l)}(0,\alpha)\frac{(it)^l}{l!}\Big|\le C|t|^{2n+1}$$

with the constant

$$C=\sup_{\substack{|\omega|=K,\\ |\alpha-\alpha_0|\le\eta}}\Big|\frac{f(\omega,\alpha)-\sum_{l=0}^{2n}f^{(l)}(0,\alpha)\frac{\omega^l}{l!}}{\omega^{2n+1}}\Big|$$

by the maximum modulus principle and the fact that $f(\omega,\alpha)$ is in class C^∞ in $\Omega(\alpha_0)$. The integral

$$\int_{-\epsilon}^{\epsilon}t^l\exp(-\sqrt{a^2+b^2}t^2)dt$$

equals 0 if l is odd. By the change of variables $s=\sqrt[4]{a^2+b^2}t$ it equals

$$\frac{(l-1)(l-3)...(1)}{2^{l/2}}\frac{\sqrt{\pi}}{(\sqrt[4]{a^2+b^2})^{l+1}}+O\Big(\frac{\exp(-\sqrt{a^2+b^2}\epsilon)}{(\sqrt[4]{a^2+b^2})^{l+1}}\Big),\quad \sqrt{a^2+b^2}\to\infty$$

if l is even. The constant comes from the fact that $\int_{-\infty}^{+\infty}t^le^{-s^2}ds=\frac{(l-1)(l-3)...(1)}{2^{l/2}}\sqrt{\pi}$. By the same reason

$$\int_{-\epsilon}^{\epsilon}|t|^{2n+1}\exp(-\sqrt{a^2+b^2}t^2)dt=O\Big(\frac{1}{(\sqrt[4]{a^2+b^2})^{2n+2}}\Big),\quad \sqrt{a^2+b^2}\to\infty.$$

The representation (2.8.1) for the first integral follows with the constants

$$c_l^1(\alpha)=\frac{(l-1)(l-3)...(1)}{2^{l/2}}\frac{\sqrt{\pi}}{2\pi}\frac{(-1)^lf^{(2l)}(0,\alpha)}{(2l)!}.$$

In particular

$$c_0^1(\alpha)=\frac{1}{2\sqrt{\pi}}\times\frac{\gamma_2(x(\alpha),y(\alpha))\varphi_2(x(\alpha))}{\gamma_y'(x(\alpha),y(\alpha))}\times x_\omega'(0,\alpha).$$

Using the expressions (2.A.1) and (2.5.4), we get

$$c_0^1(\alpha)=\frac{\gamma_2(x(\alpha),y(\alpha))\varphi_2(x(\alpha))}{\sqrt{2\pi(\sigma_{11}\sin^2(\alpha)+2\sigma_{12}\sin(\alpha)\cos(\alpha)+\sigma_{22}\cos^2(\alpha))}}\times\sqrt{\frac{\sin(\alpha)}{\gamma_y'(x(\alpha),y(\alpha))}}.$$

In the same way, using the variable y instead of x, we get the asymptotic expansions of the second and the third integral with constants $c_0^2(\alpha),\dots,c_n^2(\alpha),c_0^3(\alpha),\dots,c_n^3(\alpha)$. Namely,

$$c_0^2(\alpha)+c_0^3(\alpha)=\frac{\gamma_1(x(\alpha),y(\alpha))\varphi_1(y(\alpha))+\exp(a_0x(\alpha)+b_0y(\alpha))}{\sqrt{2\pi(\sigma_{11}\sin^2(\alpha)+2\sigma_{12}\sin(\alpha)\cos(\alpha)+\sigma_{22}\cos^2(\alpha))}}\times\sqrt{\frac{\cos(\alpha)}{\gamma_x'(x(\alpha),y(\alpha))}}.$$

By (2.5.3) $\sin(\alpha)\gamma'_x(x(\alpha),y(\alpha)) = \cos(\alpha)\gamma'_y(x(\alpha),y(\alpha))$. This implies the representation (2.8.1) and concludes the proof with $c_k(\alpha) = \sum_{i=1}^{3} c_k^i(\alpha)$. ☐

Now, we summarise all our reasoning to show the main result. We will justify later that the constants $c_0(\alpha)$ are not zero.

Theorem 2.34 (Asymptotics in the quadrant, general case)**.** *We consider a reflected Brownian motion in the quadrant of parameters (Σ,μ,R) satisfying conditions of Proposition 2.8 and Assumption 2.9. Then, the Green's density function $g(r\cos(\alpha), r\sin(\alpha))$ of this process has the following asymptotics when $\alpha \to \alpha_0 \in (0,\pi/2)$ and $r\to\infty$, for all $n\in\mathbb{N}$ we have:*

- *If $\alpha^* < \alpha_0 < \alpha^{**}$ then*

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to\alpha_0}}{\sim} e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}\sum_{k=0}^{n}\frac{c_k(\alpha)}{r^k} \tag{2.8.3}$$

- *If $\alpha_0 < \alpha^*$ then*

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to\alpha_0}}{\sim} c^* e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)} + e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}\sum_{k=0}^{n}\frac{c_k(\alpha)}{r^k} \tag{2.8.4}$$

- *If $\alpha^{**} < \alpha_0$ then*

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to\alpha_0}}{\sim} c^{**} e^{-r(\cos(\alpha)x^{**}+\sin(\alpha)y^{**})} + e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}\sum_{k=0}^{n}\frac{c_k(\alpha)}{r^k} \tag{2.8.5}$$

where explicit expression of the saddle point coordinate $x(\alpha)$ and $y(\alpha)$ are given by (2.5.5) and (2.5.6), the coordinates of the poles x^*, y^*, y^{**}, x^{**} are given by (2.6.1) and (2.6.2), the constants are given by

$$c^* = \frac{(-res_{x=x^*}\varphi_2(x))\gamma_2(x^*,y^*)}{\gamma'_y(x^*,y^*)} > 0 \quad \text{and} \quad c^{**} = \frac{(-res_{y=y^{**}}\varphi_1(y))\gamma_1(x^{**},y^{**})}{\gamma'_y(x^{**},y^{**})} > 0$$

and c_k are constants depending on α and such that $c_k(\alpha) \underset{\alpha\to\alpha_0}{\longrightarrow} c_k(\alpha_0)$ where $c_0(\alpha)$ is given by (2.8.2). We have $c_0(\alpha) > 0$ at least when $\alpha^* < \alpha_0 < \alpha^{**}$ where it gives the dominant term of the asymptotics (2.8.3).

Proof. The theorem follows directly from several lemmas put together. By Lemma 2.27 which inverse the Laplace transform, $g(a,b)$ can be expressed as of the sum of three simple integrals $I_1+I_2+I_3$. Those integrals has been rewritten in Lemma 2.28, thanks to the residue theorem, as the sum of residues and integrals whose contour locally follows the steepest descent line through the saddle point. This has been done in Section 2.6 using Morse's Lemma, see Lemma 2.A.1. Residues are present if $0 < x^* < x(\alpha)$ or $0 < y^{**} < y(\alpha)$. Besides, we proved in Lemma 2.29 the negligibility of the integrals of the lines $S^{\pm}_{x,\alpha}$ and $S^{\pm}_{y,\alpha}$ compared to the integrals on the steepest descent lines. The main asymptotics is then given by the poles plus the asymptotics of the steepest descent integrals. A disjunction of cases concerning the pole's contributions gives the three cases of the theorem (we recall that $\alpha^* < \alpha^{**}$). In the second case, when $\alpha_0 < \alpha^*$, φ_2 has a pole and then $c^* \neq 0$ because we have $\frac{r_{12}}{r_{22}} > \frac{-Y^{\pm}(x_{max})}{x_{max}}$ which imply $\gamma_2(x^*,y^*) \neq 0$. The same holds for c^{**}. Finally, Lemma 2.33 gives the desired asymptotic expansion of the integrals on the lines of the steepest descent. The fact that $c_0(\alpha_0) \neq 0$ when $\alpha^* < \alpha_0 < \alpha^{**}$ is postponed to Lemma 2.35 and Lemma 2.36. ☐

For the relevance of the asymptotics, constants $c_0(\alpha)$ shall not be zero at least when $\alpha^* < \alpha_0 < \alpha^{**}$, that is when the poles are not involved in the asymptotic. We divide the proof in two lemmas.

Most of the quantities studied so far depend on the starting point of the process, even if this dependence is not explicit in the notations. In the following, we add a power z_0 (or (a_0, b_0)) in the notation of the objects which correspond to a process whose starting point is $z_0 = (a_0, b_0)$. For example, we will note $h_1^{z_0}$ or $\varphi_1^{z_0}$ when we want to emphasise the dependency on the starting point.

Lemma 2.35 (Non nullity of the constant $c(\alpha)$ for at least a starting point)**.** *If $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha^*, \alpha^{**}\}$, there exist some starting point $z_0 \in \mathbb{R}_+^2$ such that $c_0^{z_0}(\alpha) \neq 0$.*

Proof. Let $z_0 = (a_0, b_0)$ the starting point of the process. We proceed by contradiction assuming that $c_0^{(a_0,b_0)}(\alpha) = 0$ for all $a_0, b_0 \geq 0$. Since $x(\alpha) \leq 0$ or $y(\alpha) \leq 0$, we suppose without loss of generality that $y(\alpha) \leq 0$. We have then, by (2.8.2) and the continuation formula:

$$c_1\varphi_1^{(a_0,b_0)}(y(\alpha)) - c_2\varphi_1^{(a_0,b_0)}(Y^-(x(\alpha))) = \gamma_2(x(\alpha), y(\alpha))e^{a_0 x(\alpha)+b_0 Y^-(x(\alpha))} - \gamma_2(x(\alpha), Y^-(x(\alpha)))e^{a_0 x(\alpha)+b_0 y(\alpha)} \tag{2.8.6}$$

with $c_1 = \gamma_1(x(\alpha), Y^-(x(\alpha)))\gamma_2(x(\alpha), y(\alpha))$ and $c_2 = \gamma_1(x(\alpha), Y^-(x(\alpha)))\gamma_2(x(\alpha), Y^-(x(\alpha)))$. Remark that $\gamma_2(x(\alpha), Y^-(x(\alpha))) \neq 0$ since we assume $\alpha \neq \alpha^*$. The right term of (2.8.6) is unbounded on the set of all (a_0, b_0) belonging to $\mathbb{R}_+^2$ since $Y^-(x(\alpha)) < y(\alpha) = Y^+(x(\alpha))$. Then, it is sufficient to show that the supremum of the left term is bounded according to (a_0, b_0). We denote $h_1^{(a_0,b_0)}$ the density of H_1 according to the Lebesgue measure corresponding to the starting point (a_0, b_0). We have

$$c_1\varphi_1^{(a_0,b_0)}(y(\alpha)) - c_2\varphi_1^{(a_0,b_0)}(Y^-(x(\alpha))) = \int_0^\infty \left(c_1 e^{y(\alpha)z} - c_2 e^{Y^-(x(\alpha))z}\right) h_1^{(a_0,b_0)}(z)dz =: I. \tag{2.8.7}$$

Similarly to the proof of Lemma 2.21, we introduce T the first hitting time of the axis $\{x = 0\}$. By the strong Markov property, we obtain on the same way:

$$\begin{aligned} I &= \mathbb{E}_{(a_0,b_0)}\left[\mathbb{1}_{T<+\infty}\mathbb{E}_{(0,Z_T^2)}\left[\int_0^{+\infty} \mathbb{1}_{\{0\}\times\mathbb{R}_+}(Z_t)\left(c_1 e^{y(\alpha)Z_t^2} - c_2 e^{Y^-(x(\alpha))Z_t^2}\right) dL_t^1\right]\right] &(2.8.8)\\ &= \int_0^{+\infty}\int_0^{+\infty} \left(c_1 e^{y(\alpha)z} - c_2 e^{Y^-(x(\alpha))z}\right) h_1^{(0,y)}(z)dz\mathbb{P}(T < +\infty, Z_T^2 = dy) &(2.8.9)\\ &= \int_0^{+\infty} \left(c_1\varphi_1^{(0,y)}(y_\alpha) - c_2\varphi_1^{(0,y)}(Y^-(x_\alpha))\right)\mathbb{P}(T < +\infty, Z_T^2 = dy). &(2.8.10)\end{aligned}$$

Using the identity (2.8.6) in (2.8.10) (where we see the relevance of going to the y-axis), we get the bound

$$|I| \leq |\gamma_2(x(\alpha), y(\alpha))| + |\gamma_2(x(\alpha), Y^-(x(\alpha)))| \tag{2.8.11}$$

since $y(\alpha) \leq 0$. The right term of (2.8.6) is therefore bounded in (a_0, b_0), which is absurd according to what we said earlier in the proof. □

Lemma 2.36 (Non nullity of the constant $c(\alpha)$ for all starting points)**.** *For all $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that $\alpha^* < \alpha < \alpha^{**}$ and $z_0 \in \mathbb{R}_+^2$, we have $c_0^{z_0}(\alpha) \neq 0$.*

Proof. Let's denote $z_0 = (a_0, b_0)$ the point obtained in Lemma 2.35 such that $c_0^{z_0}(\alpha) \neq 0$. We use again the continuity of the Laplace transform in z_0 (see the proof of Lemma 2.22) to remark that $c^{z_0'}(\alpha) \neq 0$ for all z_0' in an open neighborhood V of z_0. Let $z_0'' \in \mathbb{R}_+^2$ be the starting point of the process $Z^{(z_0'')}$ and let $T = \inf\{t \geq 0, Z_t^{(z_0'')} \in V\}$ be the hitting time of V. We have $\mathbb{P}_{z_0''}(T < +\infty) = p > 0$. By the strong Markov property,

$$\begin{aligned} g^{z_0''}(r\cos(\alpha), r\sin(\alpha)) &\geq p \inf_{z_0' \in V} g^{z_0'}(r\cos(\alpha), r\sin(\alpha)) &(2.8.12)\\ &\geq p \inf_{z_0' \in V}\left[c_0^{z_0'}(\alpha)(1 + o_{r\to\infty}(1))\right] e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}. &(2.8.13)\end{aligned}$$

Furthermore, V can be chosen bounded and such that $\inf_{z_0'\in V} c_0^{z_0'}(\alpha) > 0$. The issue is that the term $o_{r\to\infty}(1)$ may depend on z_0'. We then have to refer to the proof of Lemma 2.29. We remark that the only thing depending on the initial condition is the constant D of Lemma 2.29, which is based on Lemma 2.26. If the supremum on $z_0' \in V$ of the quantity of Lemma 2.26 is finite, then the result is true. This fact is verified easily from the proof of this lemma, because V is bounded and $\varphi_1^{z_0'}(0)$ is continuous in z_0'. □

2.9 Asymptotics along the axes : $\alpha \to 0$ or $\alpha \to \frac{\pi}{2}$

In this section we study the asymptotics of the Green's function g along the axes. We recall the assumption $\alpha^* \neq 0$ and $\alpha^{**} \neq \pi/2$ made in Remark 2.30.

Let us recall that for any couple $(a,b) \in \mathbb{R}_+^2$ we define $r = \sqrt{a^2+b^2}$ and $\alpha(a,b)$ the angle in $[0,\pi/2]$ such that $\cos(\alpha) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha) = \frac{b}{\sqrt{a^2+b^2}}$.

Lemma 2.37 (Contribution of the saddle point to the asymptotics when $\alpha \to 0$ or $\pi/2$)**.**

(i) Let $a \to \infty$, $b > 0$ and $\alpha(a,b) \to 0$. Then the asymptotics of (2.8.1) remains valid with $c_0(\alpha) \to 0$ as $\alpha \to 0$. Moreover, we have $c_0(\alpha) \sim c'\alpha$ and $c_1(\alpha) \sim c''$ as $\alpha \to 0$ where c' and c'' are non null constants at least when $\alpha^* < 0$.

(ii) When $b \to \infty$, $a > 0$ and $\alpha(a,b) \to \pi/2$ the same result holds.

Remark 2.38 (Competition between the two first term of the asymptotics)*.* The previous lemma states that when $\alpha \to 0$ and $r \to \infty$, there is a competition between the first two terms of the sum of the asymptotic development given in (2.8.1), namely the first term $\frac{c_0(\alpha)}{\sqrt{r}} \sim \frac{c'\alpha}{\sqrt{r}} \sim \frac{c'b}{r\sqrt{r}}$ and the second term $\frac{c_1(\alpha)}{r\sqrt{r}} \sim \frac{c''}{r\sqrt{r}}$ may have the same order of magnitude. If $b \to 0$, the second term is dominant. If $b \to c$ where c is a positive constant they both contribute to the asymptotic. If $b \to \infty$ (and also $b = o(a)$ since $\alpha \to 0$), the first term is dominant.

Proof. We first prove (i). For any α close to 0, $\Gamma_{x,\alpha}$ lies in a neighborhood of $x(\alpha)$. Using the continuation formula of $\varphi_2(x)$ (2.3.2), the definition of F (2.5.2), and the fact that $\Gamma_{x,\alpha} = \underrightarrow{\overleftarrow{X^+(\Gamma_{y,\alpha})}}$ (2.5.9), the first integral of (2.8.1) becomes

$$\frac{e^{-ax(\alpha)-by(\alpha)}}{2i\pi}\int_{\underrightarrow{\overleftarrow{X^+(\Gamma_{y,\alpha})}}} \frac{\gamma_2(x,Y^+(x))\Big(-\gamma_1(x,Y^-(x))\varphi_1(Y^-(x)) - e^{a_0x+b_0Y^-(x)}\Big)}{\gamma_2(x,Y^-(x))\gamma_y'(x,Y^+(x))}$$
$$\times \exp\left(\sqrt{a^2+b^2}F(x,\alpha)\right)dx.$$

Let us make the change of variables $x = X^+(y)$. Taking into account the fact that $Y^+(X^+(y)) = y$, the relation $\gamma_x'(X^+(y),y)(X^+(y))' + \gamma_y'(X^+(y),y) \equiv 0$ and the direction of $\underrightarrow{\overleftarrow{X^+(\Gamma_{y,\alpha})}}$, the first integral becomes

$$\frac{e^{-ax(\alpha)-by(\alpha)}}{2i\pi}\int_{\Gamma_{y,\alpha}} \frac{\gamma_2(X^+(y),y)\Big(-\gamma_1(X^+(y),Y^-(X^+(y)))\varphi_1(Y^-(X^+(y))) - e^{a_0X^+(y)+b_0Y^-(X^+(y))}\Big)}{\gamma_2(X^+(y),Y^-(X^+(y)))\gamma_x'(X^+(y),y)}$$
$$\times \exp\left(\sqrt{a^2+b^2}G(y,\alpha)\right)dy. \tag{2.9.1}$$

Let us sum it up with the second and the third integral, for which we use the representation valid for $a > a_0$. Then we have to find the asymptotics of the integral

$$\frac{e^{-ax(\alpha)-by(\alpha)}}{2i\pi}\int_{\Gamma_{y,\alpha}} \frac{\gamma_2(X^+(y),Y^-(X^+(y)))H(X^+(y),y) - \gamma_2(X^+(y),y)H(X^+(y),Y^-(X^+(y)))}{\gamma_2(X^+(y),Y^-(X^+(y)))\gamma_x'(X^+(y),y)}$$

$$\times \exp\left(\sqrt{a^2+b^2}G(y,\alpha)\right)dy$$

where

$$H(X^+(y),y) = \gamma_1(X^+(y),y)\varphi_1(y) + \exp(a_0X^+(y)+b_0y).$$

Finally, let us note, that with notations (2.4.3)

$$Y^-(X^+(y)) = \frac{c(X^+(y))}{a(X^+(y))\times Y^+(X^+(y))} = \frac{\sigma_{11}(X^+(y))^2+2\mu_1X^+(y)}{\sigma_{22}y}.$$

Function $X^+(y)$ is holomorphic in a neighborhood of $Y^{\pm}(x_{max})$. By (2.4.5) we have $\gamma_x'(X^+(y),y) = \sqrt{\widetilde{b}^2(y)-4\widetilde{a}(y)\widetilde{c}(y)}$ which is holomorphic in a neighborhood of $Y^{\pm}(x_{max})$ and different from zero. Finally $\gamma_2(x_{\max},Y^{\pm}(x_{max})) \neq 0$ by our assumption of Remark 2.30. It follows that the integrand in (2.9.1) is a holomorphic function in a neighborhood of $Y^{\pm}(x_{max})$. Then the whole saddle point procedure of Lemma 2.33 applied to $G(y,\alpha)$ with $\alpha = 0$ and replacing the function $f(it,\alpha)$ by

$$\begin{aligned}f(it,\alpha) = [\gamma_2(X^+(y(it,\alpha)),&Y^-(X^+(y(it,\alpha))))H(X^+(y(it,\alpha)),y(it,\alpha)) - \\ &\gamma_2(X^+(y(it,\alpha)),y(it,\alpha))H(X^+(y(it,\alpha)),Y^-(X^+(y(it,\alpha))))] \\ &\times\frac{y_\omega'(it,\alpha)}{\gamma_2(X^+(y(it,\alpha)),Y^-(X^+(y(it,\alpha))))\gamma_x'(X^+(y(it,\alpha)),y(it,\alpha))}\end{aligned}$$

where $y(it,\alpha)$ is the path given by the parameter Morse Lemma (Lemma 2.A.1). We get the asymptotic development (2.8.1) as $\alpha \to 0$. We then have a competition $c_0(\alpha) + \frac{c_1(\alpha)}{r} + O\left(\frac{1}{r^2}\right)$ with $c_0(\alpha) = \frac{1}{2\sqrt{\pi}}f(0,\alpha)$ and $c_1(\alpha) = -\frac{1}{4\sqrt{\pi}}\frac{f_\omega''(0,\alpha)}{4!}$. When $\alpha \to 0$ we have $c_0(\alpha) \sim c'\alpha$ and $c_1(\alpha) \sim c''$ for a non null constants c' and c'', see Lemma 2.39 and Remark 2.40 bellow.

The proof of (ii) is exactly the same except that we use the other representation of $I_3(a,b)$. ☐

Lemma 2.39 (Non nullity of c'). *When $\alpha \to 0$ we have $c_0(\alpha) \sim c'\alpha$ and the constant c' is non null at least when $\alpha^* < 0$.*

Proof. It is clear that $c_0(0) = 0$ because $c_0(\alpha)$ coincides with (2.8.2) by uniqueness of asymptotic development and this expression tends to 0 as α goes to 0 due to $C(\alpha)$. Let us now deal with the behaviour of $c_0(\alpha)$ when $\alpha \to 0$ remembering that $c_0(\alpha) = \frac{1}{2\sqrt{\pi}}f(0,\alpha)$ where we use the notations of the proof of Lemma 2.37. First, we have thanks to Lemma 2.17

$$\begin{aligned}y(\alpha) - Y^-(X^+(y(\alpha))) &= Y^+(X^+(y(\alpha)) - Y^-(X^+(y(\alpha))) \\ &= \frac{2}{\sigma_{22}}\sqrt{(\sigma_{11}\sigma_{22}-\sigma_{12}^2)(x_{max}-X^+(y(\alpha))(X^+(y(\alpha))-x_{min})}.\end{aligned}$$

We also have $(X^+(y))'\Big|_{y=y(0)} = 0$ and $(X^+(y))''\Big|_{y=y(0)} = -\frac{\sigma_{22}}{\gamma_x'(x_{max},Y^{\pm}(x_{max}))}$, so that

$$x_{max} - X^+(y(\alpha)) = \frac{\sigma_{22}}{2\gamma_x'(x_{max},Y^{\pm}(x_{max}))}\alpha^2(1+o(1)), \text{ as } \alpha \to 0.$$

Finally

$$y(\alpha) - Y^-(X^+(y(\alpha))) \sim \sqrt{\frac{2(\sigma_{11}\sigma_{22}-\sigma_{12}^2)(x_{max}-x_{min})}{2\sigma_{22}\gamma_x'(x_{max},Y^{\pm}(x_{max}))}} \times \alpha \sim \Pi \times \alpha,$$

where Π is defined as the constant in front of α.

Since $\gamma_2(x,y) = r_{12}x + r_{22}y$ and $\gamma_2(x_{max},Y^-(x_{max}))\gamma_x'(x_{max},Y^-(x_{max})) \neq 0$, we obtain that when $\alpha \to 0$

$$\begin{aligned}c_0(\alpha) &= \frac{-r_{22}H(x_{max},Y^{\pm}(x^{max}))\times(\Pi\alpha) + \gamma_2(x_{max},Y^{\pm}(x_{max}))H_y'(x_{max},Y^{\pm}(x_{max}))\times(\Pi\alpha)+o(\alpha)}{\gamma_2(x_{max},Y^-(x_{max}))\gamma_x'(x_{max},Y^-(x_{max}))+o(1)} \\ &= \alpha(c'+o(1))\end{aligned}$$

where c' is the corresponding constant.

Let us prove that $c' \neq 0$. We have to show that

$$-r_{22}H(x_{max}, Y^{\pm}(x^{max})) + \gamma_2(x_{max}, Y^{\pm}(x_{max}))H'_y(x_{max}, Y^{\pm}(x_{max})) \neq 0$$

i.e. that

$$-r_{22}\Big(\gamma_1(x_{max}, Y^{\pm}(x^{max}))\varphi_1(Y^{\pm}(x_{max})) + e^{a_0 x_{max} + b_0 Y^{\pm}(x_{max})}\Big) + \gamma_2(x_{max}, Y^{\pm}(x_{max}))\times$$
$$\Big(r_{21}\varphi_1(Y^{\pm}(x_{max})) + \gamma_1(x_{max}, Y^{\pm}(x_{max}))\varphi'_1(Y^{\pm}(x_{max})) + b_0 e^{a_0 x_{max} + b_0 Y^{\pm}(x_{max})}\Big) \neq 0$$

The equation can be rewritten as

$$c_1\varphi_1(Y^{\pm}(x_{max})) + c_2\varphi'_1(Y^{\pm}(x_{max})) \neq (c_3 + c_4 b_0)e^{a_0 x_{max} + b_0 Y^{\pm}(x_{max})} \tag{2.9.2}$$

with c_1, c_2, c_3, c_4 constants not depending on the initial conditions. Note that $c_3 = -r_{22} \neq 0$ by (2.2.1) and $c_4 = \gamma_2(x_{max}, Y^{\pm}(x_{max})) \neq 0$ by the assumption done in Remark 2.30. Furthermore, with the same method employed in the proof of Lemmas 2.21 and 2.35, the left term of (2.9.2) is bounded in (a_0, b_0). Since $x_{max} > 0$ and $Y^{\pm}(x_{max}) < 0$, the right term of (2.9.2) is not bounded in (a_0, b_0). Hence, (2.9.2) holds for at least one (a_0, b_0). By a similar argument developed in the proof Lemmas 2.22 and 2.36 (using the fact that $\alpha^* < 0$), we show that $c' \neq 0$ only for one starting point (a_0, b_0) imply that $c' \neq 0$ for all starting points. Finally, (2.9.2) holds for every initial condition which concludes the proof of the fact that $c_0(\alpha) \sim c'\alpha$ for a non null constant c'. □

Remark 2.40 (Non nullity of c''). We admit here that $c'' \neq 0$. A proof inspired by what has been done in the previous lemma to show that $c' \neq 0$ would work. The same techniques has also been employed in Lemmas 2.21 and 2.22 to characterize the poles by showing the non nullity of a constant and in Lemmas 2.35 and 2.36 to show the non nullity of $c_0(\alpha)$. Here, it would be too technical to be detailed and would not provide any additional elements of understanding compared with the previous applications of this method.

We now have everything we need to prove our second main result which states the full asymptotic expansion of the Green's function g along the edges.

Theorem 2.41 (Asymptotics along the edges for the quadrant). *We now assume that $\alpha_0 = 0$ and let $r \to \infty$ and $\alpha \to \alpha_0 = 0$. In this case, we have $c_0(\alpha) \underset{\alpha\to 0}{\sim} c'\alpha$ and $c_1(\alpha) \underset{\alpha\to 0}{\sim} c''$ for some constants c' and c'' which are non null at least when $\alpha^* < 0$. Then, the Green's function $g(r\cos(\alpha), r\sin(\alpha))$ has the following asymptotics:*

- *When $\alpha^* < 0$ the asymptotics given by (2.8.3) remains valid. In particular, we have*

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to 0}}{\sim} e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}\left(c'\alpha + \frac{c''}{r}\right).$$

- *When $\alpha^* > 0$ the asymptotics given by (2.8.4) remains valid. In particular, we have*

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to 0}}{\sim} c^* e^{-r(\cos(\alpha)x^* + \sin(\alpha)y^*)}.$$

Therefore, when $\alpha^* < 0$, there is a competition between the two first terms of the sum $\sum_{k=0}^{n}\frac{c_k(\alpha)}{r^k}$ to know which one is dominant between $c'\alpha$ and $\frac{c''}{r}$. More precisely:

- *If $r\sin\alpha \underset{\substack{r\to\infty \\ \alpha\to 0}}{\longrightarrow} \infty$ then the first term is dominant.*
- *If $r\sin\alpha \underset{\substack{r\to\infty \\ \alpha\to 0}}{\longrightarrow} c > 0$ then both terms contribute, they have the same order of magnitude.*

- *If* $r \sin \alpha \underset{\substack{r\to\infty \\ \alpha\to 0}}{\longrightarrow} 0$ *then the second term is dominant.*

A symmetric result holds when we take $\alpha_0 = \frac{\pi}{2}$*. The asymptotics given by* (2.8.3) *remains valid when* $\frac{\pi}{2} < \alpha^{**}$ *and* (2.8.5) *remain valid when* $\alpha^{**} < \frac{\pi}{2}$ *and there is a competition between the two first terms of the sum to know which one is dominant which depends of the limit of* $r\cos(\alpha)$*.*

Proof. The theorem follows directly from several lemmas put together. First, in Lemma 2.27 we inverse the Laplace transform and we express the Green's function g as the sum of three integrals. Then, in Lemma 2.28 we shift the integration contour of the integrals to reveal the contribution of the poles to the asymptotics applying the residue theorem. In Lemma 2.31 we show the negligibility of some integrals which imply that the asymptotic expansion is given by the integrals on the steepest descent contour. Finally, in Lemma 2.37 states the asymptotics expansion of these integrals given by the saddle point method. ☐

2.10 Asymptotics when a pole meets the saddle point : $\alpha \to \alpha^*$ or $\alpha \to \alpha^{**}$

In this section we study the asymptotics of the Green's function $g(r\cos\alpha, r\sin\alpha)$ when $\alpha \to \alpha_0$ in the special case where $\alpha_0 = \alpha^*$ or $\alpha_0 = \alpha^{**}$, that is when *the pole meet the saddle point.*

Let us introduce the following notation for shortness :

$$R(\alpha) = x'_\omega(0,\alpha) = \sqrt{\frac{2}{F''_{xx}(x(\alpha),\alpha)}} = \sqrt{\frac{2}{-\sin(\alpha)(Y^+(x))''\,|_{x(\alpha)}}} \tag{2.10.1}$$

$$= \sqrt{\frac{2\sin(\alpha)\gamma'_y(x(\alpha),y(\alpha))}{\sigma_{11}\sin^2(\alpha) + 2\sigma_{12}\sin(\alpha)\cos(\alpha) + \sigma_{22}\cos^2(\alpha)}}.$$

We recall that for $(a,b) \in \mathbb{R}_+^2$ we define $r = \sqrt{a^2+b^2}$ and $\alpha(a,b)$ the angle in $[0,\pi/2]$ such that $\cos(\alpha) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha) = \frac{b}{\sqrt{a^2+b^2}}$.

Lemma 2.42 (Asymptotics of the integral on steepest descent line when $\alpha \to \alpha^*$)**.** *Let* $\alpha(a,b) \to \alpha^*$ *as* $r = \sqrt{a^2+b^2} \to \infty$*. Then*

$$I := \frac{1}{2\pi i} \int\limits_{\Gamma_{x,\alpha}} \frac{\gamma_2(x,Y^+(x))\varphi_2(x)}{\gamma'_y(x,Y^+(x))} \exp\left(\sqrt{a^2+b^2}F(x,\alpha(a,b))\right)dx$$

has the following asymptotics.

(i) Let $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2 \to 0$*, then*

$$I \sim \frac{-1}{2}\frac{\gamma_2(x^*,y^*)\mathrm{res}_{x=x^*}\varphi_2}{\gamma'_y(x^*,y^*)} \qquad \text{if } \alpha(a,b) > \alpha^*,$$

$$I \sim \frac{1}{2}\frac{\gamma_2(x^*,y^*)\mathrm{res}_{x=x^*}\varphi_2}{\gamma'_y(x^*,y^*)} \qquad \text{if } \alpha(a,b) < \alpha^*.$$

(ii) Let $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2 \to c > 0$*. Let*

$$A(\alpha^*) = \frac{-x'_\alpha(\alpha^*)}{R(\alpha^*)}. \tag{2.10.2}$$

Then

$$I \sim \frac{-1}{2}\exp(cA^2(\alpha^*))\Big(1-\Phi\Big(\sqrt{c}A(\alpha^*)\Big)\Big) \times \frac{\gamma_2(x^*,y^*)\mathrm{res}_{x=x^*}\varphi_2}{\gamma'_y(x^*,y^*)} \qquad \text{if } \alpha(a,b) > \alpha^*,$$

$$I \sim \frac{1}{2}\exp(cA^2(\alpha^*))\Big(1-\Phi\Big(\sqrt{c}A(\alpha^*)\Big)\Big) \times \frac{\gamma_2(x^*,y^*)\mathrm{res}_{x=x^*}\varphi_2}{\gamma'_y(x^*,y^*)} \qquad \text{if } \alpha(a,b) < \alpha^*$$

where

$$\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt. \tag{2.10.3}$$

(iii) Let $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2 \to \infty$*. Then*

$$I \sim \frac{\gamma_2(x^*,y^*)R(\alpha^*)}{2\sqrt{\pi}\gamma'_y(x^*,y^*)} \times \frac{\mathrm{res}_{x=x^*}\varphi_2}{(x(\alpha(a,b))-x(\alpha^*))} \times \frac{1}{\sqrt[4]{a^2+b^2}}.$$

Proof. Starting the proof as in Lemma 2.33, we obtain again that

$$I \sim \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} f(it,\alpha(a,b))\exp(-\sqrt{a^2+b^2}t^2)dt \tag{2.10.4}$$

where

$$f(it,\alpha(a,b)) = \frac{\gamma_2(x(it,\alpha),Y^+(x(it,\alpha)))\varphi_2(x(it,\alpha))}{\gamma'_y(x(it,\alpha),Y^+(x(it,\alpha)))} \times x'_\omega(it,\alpha).$$

The function φ_2 is a sum of a holomorphic function and of the term $\frac{\mathrm{res}_{x^*}\varphi_2}{x-x^*}$ which after the change of variables takes the form $\frac{\mathrm{res}_{x^*}\varphi_2}{x(it,\omega)-x^*}$. This term should be worked out.

We have $x(0,\alpha^*) = x(\alpha^*)$. By the theorem of implicit function there exists a function $\omega(\alpha)$ in the class $\mathcal{C}^\infty$ such that

$$x(\omega(\alpha),\alpha) \equiv x^* \quad \forall\alpha : |\alpha-\alpha^*| \le \widetilde{\eta} \qquad \omega(\alpha^*) = 0$$

with some $\widetilde{\eta}$ small enough. Furthermore, differentiating this equality we get

$$\omega'(\alpha) = \frac{-x'_\alpha(\omega(\alpha),\alpha)}{x'_\omega(\omega(\alpha),\alpha)},$$

so that

$$\omega'(\alpha^*) = -\frac{x'_\alpha(\alpha^*)}{x'_\omega(0,\alpha^*)}.$$

The formula

$$\omega(\alpha) = \frac{(x(\alpha^*)-x(\alpha))}{x'_\omega(0,\alpha^*)}(1+o(1)) = \frac{(x(\alpha^*)-x(\alpha))}{R(\alpha^*)}(1+o(1)) \quad \text{as } \alpha \to \alpha^*. \tag{2.10.5}$$

provides the asymptotics of $\omega(\alpha)$ as $\alpha \to \alpha^*$. Note also that the main part of $\omega(\alpha)$ is real.

Let us introduce function

$$\Psi(\omega,\alpha) = \begin{cases} \frac{\omega-\omega(\alpha)}{x(\omega,\alpha)-x(\omega(\alpha),\alpha)} & \text{if } \omega \neq \omega(\alpha) \\ \frac{1}{x'_\omega(\omega(\alpha),\alpha)} & \text{if } \omega = \omega(\alpha) \end{cases}$$

This function is holomorphic in ω for any fixed α and continuous as a function of three real variables. We can write the integral (2.10.4) as

$$\frac{1}{2\pi i}\int_{-\epsilon}^{\epsilon} f(it,\alpha)(it-\omega(\alpha))\frac{\exp(-\sqrt{a^2+b^2}t^2)}{t+i\omega(\alpha)}dt.$$

There exists a constant $C > 0$ such that

$$\Big| f(it,\alpha)(it-\omega(\alpha))-f(0,\alpha)(0-\omega(\alpha))\Big| \leq C|t|.\ \forall (it,\alpha) \in \widetilde{\Omega}(0,\alpha^*) = \{(\omega,\alpha) : |\omega| \leq K, |\alpha-\alpha_0| \leq \min(\eta,\widetilde{\eta})\}. \tag{2.10.6}$$

It suffices to take C the maximum of the modulus of $(f(\omega,\alpha)(\omega-\omega(\alpha)) - f(0,\alpha)(0-\omega(\alpha)))\omega^{-1}$ on $\{(\omega,\alpha) : |\omega| = K, |\alpha - \alpha_0| \leq \min(\eta_0,\widetilde{\eta})\}$.

Moreover since $\Im\omega(\alpha) = o(\Re\omega(\alpha))$ as $\alpha \to \alpha^*$, then by Lemma (2.C.1) (i) for any α close to α^* the inequality

$$\frac{|t|}{|t+i\omega(\gamma)|} \leq 2$$

is valid for all $t \in \mathbb{R}$. The integral

$$\int_{\mathbb{R}} 2\exp(-\sqrt{a^2+b^2}t^2)dt = O(\frac{1}{\sqrt[4]{a^2+b^2}})$$

is of smaller order than any asymptotic announced in the statement of the lemma. Hence, it suffices to show that the integral

$$\frac{1}{2\pi i}\int\limits_{-\epsilon}^{\epsilon} f(0,\alpha)(0-\omega(\alpha))\frac{\exp(-\sqrt{a^2+b^2}t^2)}{t+i\omega(\alpha)}dt$$

has the expected asymptotic. Note that by (2.10.5)

$$\varphi(x(\alpha))(-\omega(\alpha))x'_{\omega}(0,\alpha) \to \operatorname{res}_{x^*}\varphi \text{ as } \alpha \to \alpha^*,$$

so that

$$f(0,\alpha)(-\omega(\alpha)) \to \frac{\gamma_2(x^*,y^*)}{\gamma'_y(x^*,y^*)} \times \operatorname{res}_{x^*}\varphi.$$

It remains to study

$$\frac{1}{2\pi i}\int\limits_{-\epsilon}^{\epsilon} \frac{\exp(-\sqrt{a^2+b^2}t^2)}{t+i\omega(\alpha)}dt.$$

For any $t \in \mathbb{R} \setminus [-\epsilon,\epsilon]$ the denominator in the integral is bounded from below

$$|t+i\omega(\alpha)| \geq ||t|-\omega(\alpha)| \geq \epsilon - \omega(\alpha) \geq \epsilon/2$$

for any α close enough to α^* while

$$\int_{\mathbb{R}} \exp(-\sqrt{a^2+b^2}t^2)dt = O(\frac{1}{\sqrt[4]{a^2+b^2}})$$

is of smaller order than the one expected in the lemma. Finally it suffices to prove that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\exp(-\sqrt{a^2+b^2}t^2)}{t+i\omega(\alpha(a,b))}dt. \tag{2.10.7}$$

has the right asymptotic. By the change of variables, equation (2.10.7) is equals to

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\exp(-s^2)}{s+i\omega(\alpha)\sqrt[4]{a^2+b^2}}ds. \tag{2.10.8}$$

Let now $\alpha > \alpha^*$ [resp. $\alpha < \alpha^*$]. Then $x(\alpha) < x(\alpha^*)$ [resp. $x(\alpha) > x(\alpha^*)$] and by (2.10.5) $\Re\omega(\alpha) > 0$ [resp. $\Re\omega(\alpha) < 0$]. By Lemma 2.C.1 (iii) this integral equals

$$\frac{-1}{2}\exp(\sqrt{a^2+b^2}\omega^2(\alpha))\Big(1-\Phi(\sqrt[4]{a^2+b^2}\omega(\alpha))\Big) \quad \text{if } \alpha > \alpha^*$$

$$\frac{1}{2}\exp(\sqrt{a^2+b^2}\omega^2(\alpha))\Big(1-\Phi(-\sqrt[4]{a^2+b^2}\omega(\alpha))\Big) \quad \text{if } \alpha<\alpha^*.$$

If $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2 \to c \geq 0$ then by (2.10.5) $\sqrt{a^2+b^2}\omega(\alpha(a,b))^2 \to cA^2(\alpha^*)$, the results of (i) and (ii) are immediate. Let now $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2 \to \infty$. Then by Lemma 2.C.1 (ii) the asymptotics of this integral is

$$\frac{\sqrt{\pi}}{2\pi i \times (i\omega(\alpha(a,b)))\sqrt[4]{a^2+b^2}}$$

where the asymptotics of $\omega(\alpha(a,b))$ is found in (2.10.5). The result follows. □

For further use one may remark that

$$\begin{aligned}-\cos(\alpha)x^*-\sin(\alpha)y^* &= -\cos(\alpha)x(\alpha)-\sin(\alpha)Y^+(x(\alpha))+R^{-2}(\alpha^*)(x(\alpha)-x^*)^2(1+o(1))\\ &= -\cos(\alpha)x(\alpha)-\sin(\alpha)Y^+(x(\alpha))+A^2(\alpha^*)(\alpha-\alpha^*)^2(1+o(1)), \quad \alpha\to\alpha^* \qquad (2.10.9)\end{aligned}$$

with notations $R(\alpha)$ and $A(\alpha)$ above (2.10.1) et (2.10.2).

In fact, by Taylor expansion at $x(\alpha)$ and definition of the saddle point (the first derivative is zero):

$$-\cos(\alpha)x^*-\sin(\alpha)y^* = -\cos(\alpha)x(\alpha)-\sin(\alpha)Y^+(x(\alpha))-\frac{1}{2}\sin(\alpha)(Y^+(x))''\mid_{x=x(\alpha)}(x(\alpha)-x^*)^2(1+o(1)), \quad \alpha\to\alpha^*.$$

Let us remind that

$$-\frac{1}{2}\sin(\alpha)(Y^+(x))''\mid_{x=x(\alpha)} = (R(\alpha))^{-2} = R(\alpha^*)^{-2}(1+o(1)), \quad \alpha\to\alpha^*.$$

The following lemma is useful to determine the asymptotics of the value of I_1 found in Lemma 2.28.

Lemma 2.43 (Combined contribution of the pole and saddle point to the asymptotics when $\alpha\to\alpha^*$**).** *Let $r=\sqrt{a^2+b^2}\to\infty$ and $\alpha(a,b)\to\alpha^*$. The sum*

$$I := \frac{(-\text{res}_{x=x^*}\varphi_2(x))\gamma_2(x^*,y^*)}{\gamma_y'(x^*,y^*)}\exp(-ax^*-by^*)\times\mathbf{1}_{\alpha<\alpha^*}+\int_{\Gamma_{x,\alpha}}\frac{\gamma_2(x,Y^+(x))\varphi_2(x)}{\gamma_y'(x,Y^+(x))}\exp(-ax-bY^+(x))dx \qquad (2.10.10)$$

has the following asymptotic.

(i) If $\alpha>\alpha^*$ and $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2\to\infty$. Then

$$I \sim \frac{\exp(-ax(\alpha(a,b))-by(\alpha(a,b)))}{\sqrt[4]{a^2+b^2}}\frac{C(\alpha^*)\gamma_2(x^*,y^*)}{\sqrt{2\pi}\sqrt{\sigma_{11}\sin^2(\alpha^*)+2\sigma_{12}\sin(\alpha^*)\cos(\alpha^*)+\sigma_{22}\cos^2(\alpha^*)}}\times\frac{\text{res}_{x^*}\varphi_2}{x(\alpha(a,b))-x^*},$$

where

$$C(\alpha^*) = \sqrt{\frac{\sin(\alpha^*)}{\gamma_y'(x^*,y^*)}} = \sqrt{\frac{\cos(\alpha^*)}{\gamma_x'(x^*,y^*)}}.$$

(ii) If $\alpha>\alpha^*$ and $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2\to c>0$, then

$$I \sim \frac{-1}{2}\exp(-ax^*-by^*)\Big(1-\Phi\Big(\sqrt{c}A(\alpha^*)\Big)\Big)\times\frac{\gamma_2(x^*,y^*)\text{res}_{x=x^*}\varphi}{\gamma_y'(x^*,y^*)}.$$

where $A(\alpha^*)$ and Φ are defined in (2.10.2), (2.10.1) and (2.10.3).

(iii) If $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2\to 0$, then

$$I \sim \frac{-1}{2}\exp(-ax^*-by^*)\times\frac{\gamma_2(x^*,y^*)\text{res}_{x=x^*}\varphi}{\gamma_y'(x^*,y^*)}.$$

(iv) If $\alpha < \alpha^*$ *and* $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2 \to c > 0$, *then*

$$I \sim \frac{-1}{2}\exp(-ax^*-by^*)\Big(1+\Phi\big(\sqrt{c}A(\alpha^*)\big)\Big) \times \frac{\gamma_2(x^*,y^*)\mathrm{res}_{x=x^*}\varphi}{\gamma_y'(x^*,y^*)}.$$

(v) If $\alpha < \alpha^*$ *and* $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2 \to \infty$, *then*

$$I \sim \exp(-ax^*-by^*) \times \frac{-\gamma_2(x^*,y^*)\mathrm{res}_{x=x^*}\varphi}{\gamma_y'(x^*,y^*)}.$$

Proof. Let us note that

$$\int_{\Gamma_{x,\alpha}} \frac{\gamma_2(x,Y^+(x))\varphi_2(x)}{\gamma_y'(x,Y^+(x))}\exp(-ax-bY^+(x))dx \tag{2.10.11}$$
$$= \exp(-ax(\alpha)-by(\alpha))\int\limits_{\Gamma_{x,\alpha}} \frac{\gamma_2(x,Y^+(x))\varphi_2(x)}{\gamma_y'(x,Y^+(x))}\exp\big(-\sqrt{a^2+b^2}F(x,\alpha(a,b))\big)dx$$

(i) The result just follows from the representation (2.10.11) and Lemma 2.42 (iii) with $R(\alpha^*)$ defined in (2.10.1).

(ii) By (2.10.9) the representation (2.10.11) can be also written as

$$\exp(-ax^*-by^*-\sqrt{a^2+b^2}A^2(\alpha^*)(\alpha(a,b)-\alpha^*)^2(1+o(1)))\int\limits_{\Gamma_{x,\alpha}} \frac{\gamma_2(x,Y^+(x))\varphi_2(x)}{\gamma_y'(x,Y^+(x))}\exp(-\sqrt{a^2+b^2}F(x,\alpha(a,b)))dx \tag{2.10.12}$$

The the result follows from Lemma 2.42 (ii).

(iii) We have to consider three subcases.

Let first $\alpha(a,b) > \alpha^*$. Then the announced result follows from Lemma 2.42 (i) and the representation (2.10.12) where $\sqrt{a^2+b^2}A^2(\alpha^*)(\alpha(a,b)-\alpha^*) \to 0$.

If $\alpha = \alpha^*$, then the contour $\Gamma_{x,\alpha}$ is special and a direct computation leads to the result. We refer to Lemma 19 of [37] which deals with a similar case.

If $\alpha(a,b) < \alpha^*$, then by Lemma 2.42 (i) the asymptotics of the second term of (2.10.10) is the same as in the case $\alpha(a,b) > \alpha^*$ but with the opposite sign. It should be summed with the first term. The sum of their constants $1/2-1$ provides the result.

(iv) By the representation (2.10.12) and Lemma 2.42 (ii) the asymptotic of the second term of (2.10.10) is the same as in the case (ii) but with opposite sign. It should be summed with the first term. The sum of their constants $\frac{1}{2}(1-\Phi(\sqrt{c}A(\alpha^*)))-1$ leads to the result.

(v) By Lemma (2.42) (iii) and the representation (2.10.12) the second term of (2.10.10) has the asymptotics

$$\exp\Big(-ax^*-by^*-\sqrt{a^2+b^2}A^2(\alpha^*)(\alpha(a,b)-\alpha^*)^2(1+o(1))\Big)$$
$$\times\frac{\gamma_2(x^*,y^*)R(\alpha^*)}{2\sqrt{\pi}\gamma_y'(x^*,y^*)} \times \frac{\mathrm{res}_{x=x^*}\varphi}{(x(\alpha(a,b))-x(\alpha^*))} \times \frac{1}{\sqrt[4]{a^2+b^2}}.$$

Since $\frac{\exp(-\sqrt{a^2+b^2}A^2(\alpha^*)(\alpha(a,b)-\alpha^*)^2)}{(\alpha(a,b)-\alpha^*)\sqrt[4]{a^2+b^2}}$ converges to 0 in this case, the order of the second term in (2.10.10) is clearly smaller than the one of the first term which dominates the asymptotics.

□

Remark 2.44 (Consistency of the results). The results of (i) and (v) are perfectly "continuous" with asymptotics along directions upper and lower than α^*. Namely, if in (i) we substitute $\varphi(x(\alpha))$ instead of $\frac{res_{x^*}\varphi}{x(\alpha(a,b))-x^*}$, we obtain the asymptotics for angles higher than α^*. The result (v) stays valid for angles lower than α^*.

We now summarize the previous results to obtain our last main result.

Theorem 2.45 (Asymptotics in the quadrant when the saddle point *meet* a pole)**.** *We now assume that $\alpha_0 = \alpha^*$ and let $\alpha \to \alpha^*$ and $r \to \infty$. Then, the Green's density function $g(r\cos\alpha, r\sin\alpha)$ has the following asymptotics:*

- *When $r(\alpha-\alpha^*)^2 \to 0$ then the principal term of the asymptotics is given by (2.8.4) but the constant c^* of the first term has to be replaced by $\frac{1}{2}c^*$.*
- *When $r(\alpha-\alpha^*)^2 \to c > 0$ for some constant c then:*
    - *If $\alpha < \alpha^*$ the principal term of the asymptotics is still given by (2.8.4) but the constant c^* of the first term has to be replaced by $\frac{1}{2}c^*(1+\Phi(\sqrt{c}A))$ for some constant A.*
    - *If $\alpha > \alpha^*$ the principal term of the asymptotics is still given by (2.8.4) but the constant c^* of the first term has to be replaced by $\frac{1}{2}c^*(1-\Phi(\sqrt{c}A))$ for some constant A.*

  In the previous items we denoted $\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt$.
- *When $r(\alpha-\alpha^*)^2 \to \infty$ then:*
    - *If $\alpha < \alpha^*$ the principal term of the asymptotics is given by (2.8.4).*
    - *If $\alpha > \alpha^*$ the principal term of the asymptotics is given by (2.8.3) and we have $c_0(\alpha) \underset{\alpha\to\alpha^*}{\sim} \frac{c}{\alpha-\alpha^*}$ for some constant c.*

A symmetric result holds when we assume that $\alpha_0 = \alpha^{**}$.

Proof. The theorem follows directly from several lemmas put together. The Green's function g is still given by the sum $I_1+I_2+I_3$, see Lemma 2.27. We apply again Lemma 2.28 to take into account the contribution of the poles and Lemma 2.29 which show the negligibility of some integrals in the final asymptotics. Furthermore, by the proof of Lemma 2.33, $I_2+I_3 = O\left(\frac{e^{-r\cos(\alpha^*)x(\alpha^*)-r\sin(\alpha^*)y(\alpha^*))}}{\sqrt{r}}\right)$ when $r\to\infty$ and $\alpha\to\alpha^*$ (recall that $\alpha^* < \alpha^{**}$). With Lemma 2.43, we see in each cases that I_2+I_3 is negligible compared to I_1 when $r\to\infty$ and $\alpha\to\alpha^*$. Indeed, in the case $\alpha>\alpha^*$ and $r(\alpha-\alpha^*)^2\to\infty$, the domination of I_1 is due to the term $\frac{1}{x(\alpha)-x^*}$. For the other cases, the domination of I_1 is due to the factor $\frac{1}{\sqrt{r}}$ in the asymptotics of I_2+I_3.

The proof is similar for $\alpha_0 = \alpha^{**}$. □

2.11 Asymptotics in a cone

From the quadrant to the cone

Let us describe the linear transformation which maps the reflected Brownian motion in the quarter plane (of covariance matrix Σ and reflecting vectors R^1 and R^2) to a reflected Brownian motion in a wedge with identity covariance matrix. We take

$$\beta = \arccos\left(-\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}\right) \in (0,\pi) \tag{2.11.1}$$

and we define

$$T = \begin{pmatrix} \dfrac{1}{\sin\beta} & \cot\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{\sqrt{\sigma_{11}}} & 0 \\ 0 & \dfrac{1}{\sqrt{\sigma_{22}}} \end{pmatrix} \tag{2.11.2}$$

which satisfy $T\Sigma T^\top = \mathrm{Id}$. Then, if Z_t is a reflected Brownian motion in the quadrant of parameters (Σ, μ, R), the process $\widetilde{Z_t} = TZ_t$ is a reflected Brownian motion in the cone of angle β and of parameters $(T\Sigma T^\top, T\mu, TR) = (\mathrm{Id}, \widetilde{\mu}, TR)$. The new reflection matrix TR correspond to reflections of angles δ and ϵ defined in $(0, \pi)$ by

$$\tan\delta = \frac{\sin\beta}{\frac{r_{12}}{r_{22}}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos\beta} \qquad \text{and} \qquad \tan\varepsilon = \frac{\sin\beta}{\frac{r_{21}}{r_{11}}\sqrt{\frac{\sigma_{11}}{\sigma_{22}}} + \cos\beta}. \tag{2.11.3}$$

The new drift has an angle $\theta = \arg\widetilde{\mu}$ with the horizontal axis, it satisfies

$$\tan\theta = \frac{\sin\beta}{\frac{\mu_1}{\mu_2}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos\beta}. \tag{2.11.4}$$

The assumption $\mu_1 > 0$ and $\mu_2 > 0$ is equivalent to $\theta \in (0, \beta)$.

Green's functions in the cone

Let us denote g^{z_0} the density of $G(z_0, \cdot)$. For $z \in \mathbb{R}_+^2$ we have

$$g^{z_0}(z) = \int_0^\infty p_t(z_0, z)\mathrm{d}t.$$

Let us recall that we denote $\widetilde{G}(\widetilde{z_0}, \widetilde{A})$ the Green measure of $\widetilde{Z_t}$ and $\widetilde{g}^{\widetilde{z_0}}(\widetilde{z})$ its density. It is straightforward to see that for $A \in \mathbb{R}_+^2$ we have $G(z_0, A) = \widetilde{G}(Tz_0, TA)$ and then

$$g^{z_0}(z) = |\det T|\widetilde{g}^{Tz_0}(Tz) = \frac{1}{\sqrt{\det\Sigma}}\widetilde{g}^{\widetilde{z_0}}(\widetilde{z}) \tag{2.11.5}$$

where $\widetilde{z_0} = Tz_0$ and $\widetilde{z} = Tz$.

Polar coordinates

For any $z = (a, b) \in \mathbb{R}_+^2$ we may define the polar coordinate in the quadrant $(r, \alpha) \in \mathbb{R}_+ \times [0, \frac{\pi}{2}]$ by

$$z = (a, b) = (r\cos\alpha, r\sin\alpha). \tag{2.11.6}$$

We now define the polar coordinates in the β-cone (ρ, ω) by

$$\widetilde{z} = (\rho\cos\omega, \rho\sin\omega). \tag{2.11.7}$$

For $\widetilde{z} = Tz$ we obtain by a direct computation that

$$(r\cos\alpha, r\sin\alpha) = (\rho\sqrt{\sigma_{11}}\cos(\beta - \omega), \rho\sqrt{\sigma_{22}}\sin\omega). \tag{2.11.8}$$

and that

$$\tan\omega = \frac{\sin\beta}{\frac{1}{\tan\alpha}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos\beta}. \tag{2.11.9}$$

We deduce that

$$\widetilde{g}^{\widetilde{z_0}}(\rho\cos\omega, \rho\sin\omega) = \sqrt{\det\Sigma}\ g^{z_0}(\rho\sqrt{\sigma_{11}}\cos(\beta - \omega), \rho\sqrt{\sigma_{22}}\sin\omega). \tag{2.11.10}$$

Saddle point

The ellipse $\mathcal{E} = \{(x,y) \in \mathbb{R}^2 : \gamma(x,y) = 0\}$ can be easily parametrized by the following

$$\mathcal{E} = \{(\widetilde{x}(t), \widetilde{y}(t)) : t \in [0, 2\pi]\},$$

where

$$\begin{cases} \widetilde{x}(t) = \frac{x_{max}+x_{min}}{2} + \frac{x_{max}-x_{min}}{2}\cos(t), \\ \widetilde{y}(t) = \frac{y_{max}+y_{min}}{2} + \frac{y_{max}-y_{min}}{2}\cos(t-\beta). \end{cases} \tag{2.11.11}$$

see Proposition 5 of [52]. Noticing that

$$-\cos\theta = \frac{x_{max}+x_{min}}{x_{max}-x_{min}}, \quad \text{and} \quad -\cos(\beta-\theta) = \frac{y_{max}+y_{min}}{y_{max}-y_{min}}$$

and that

$$2|\widetilde{\mu}| = \sqrt{\sigma_{11}}(x_{max}-x_{min})\sin\beta = \sqrt{\sigma_{22}}(y_{max}-y_{min})\sin\beta$$

we obtain

$$\begin{cases} \widetilde{x}(t) = \frac{|\widetilde{\mu}|}{\sqrt{\sigma_{11}}\sin\beta}(\cos t - \cos\theta) = \frac{2|\widetilde{\mu}|}{\sqrt{\sigma_{11}}\sin\beta}\sin(\frac{\theta-t}{2})\sin(\frac{t+\theta}{2}) \\ \widetilde{y}(t) = \frac{|\widetilde{\mu}|}{\sqrt{\sigma_{22}}\sin\beta}(\cos(t-\beta) - \cos(\theta-\beta)) = \frac{2|\widetilde{\mu}|}{\sqrt{\sigma_{22}}\sin\beta}\sin(\frac{\theta-t}{2})\sin(\frac{t+\theta-2\beta}{2}). \end{cases} \tag{2.11.12}$$

The following result gives an expression of the saddle point in term of the polar coordinate in the cone.

Proposition 2.46 (Saddle point in polar coordinate)**.** *For* $\alpha \in (0, \frac{\pi}{2})$ *and* $\omega \in (0, \beta)$ *previously defined and linked by* (2.11.9) *we have*

$$(x(\alpha), y(\alpha)) = (\widetilde{x}(\omega), \widetilde{y}(\omega)) \tag{2.11.13}$$

where $(x(\alpha), y(\alpha))$ *is the saddle point defined in* (2.5.1)*.*

Proof. Let $\alpha \in (0, \frac{\pi}{2})$, we are looking for the point $(x(\alpha), y(\alpha))$ which maximizes the quantity $x\cos\alpha + y\sin\alpha$ for (x,y) in the ellipse $\mathcal{E} = \{(x,y) \in \mathbb{R}^2 : \gamma(x,y) = 0\}$. We are looking for a $t \in (0,\beta)$ cancelling the derivative of $\widetilde{x}(t)\cos\alpha + \widetilde{y}(t)\sin\alpha$ w.r.t t. By (2.11.12) we obtain that $\widetilde{x}'(t)\cos\alpha + \widetilde{y}'(t)\sin\alpha = 0$ if and only if

$$-\frac{1}{\sqrt{\sigma_{11}}}\sin t\cos\alpha - \frac{1}{\sqrt{\sigma_{22}}}\sin(t-\beta)\sin\alpha = 0.$$

Writing $\sin(t-\beta) = \sin t\cos\beta - \cos t\sin\beta$ it directly leads to $\tan t = \frac{\sin\beta}{\frac{1}{\tan\alpha}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}}+\cos\beta}$. Then by (2.11.9) we obtain $\tan t = \tan\omega$ and we deduce that $t = \omega$ maximizes $\widetilde{x}(t)\cos\alpha + \widetilde{y}(t)\sin\alpha$ and therefore $(x(\alpha), y(\alpha)) = (\widetilde{x}(\omega), \widetilde{y}(\omega))$. □

Poles

Let us recall that x^* is the pole of $\varphi_2(x)$ when it has one, and y^{**} is the pole of $\varphi_1(y)$ when it has one, see Proposition 2.20. We defined α^* and α^{**} such that $x(\alpha^*) = x^*$ and $y(\alpha^{**}) = y^{**}$. Now, we may define the corresponding ω^* and ω^{**} linked by linked by formula (2.11.9) and such that

$$x^* = \widetilde{x}(\omega^*) = \widetilde{x}(-\omega^*) \quad \text{and} \quad y^{**} = \widetilde{y}(\omega^{**}) = \widetilde{y}(2\beta - \omega^{**}). \tag{2.11.14}$$

Proposition 2.47 (Poles in polar coordinate)**.** *We have*

$$\omega^* = \theta - 2\delta \quad \textit{and} \quad \omega^{**} = \theta + 2\epsilon. \tag{2.11.15}$$

We have, $\alpha < \alpha^*$ *if and only if* $\omega < \omega^*$*, and* $\alpha > \alpha^{**}$ *if and only if* $\omega > \omega^{**}$*. Then,* x^* *is the pole of* $\varphi_2(x)$ *if and only if* $\theta - 2\delta > 0$*, and* y^{**} *is a pole of* $\varphi_1(y)$ *if and only if* $\theta + 2\epsilon < \beta$*.*

Proof. When the pole of φ_2 exists we have $\gamma_2(x^*, Y^-(x^*)) = 0$. Let us recall that in (2.6.2) we defined $y^* := Y^+(x^*) = \widetilde{y}(\omega^*)$. Therefore, we have $Y^-(x^*) = \widetilde{y}(-\omega^*)$. We are looking for the solutions of equation

$$\gamma_2(\widetilde{x}(t), \widetilde{y}(t)) = 0, \tag{2.11.16}$$

which is the intersection of the ellipse $\mathcal{E}$ and the line $\gamma_2 = 0$. There are two solutions, the first one is elementary and is given by $t = \theta$, that is $(\widetilde{x}(t), \widetilde{y}(t)) = (0, 0)$. The second one is by definition $(\widetilde{x}(-\omega^*), \widetilde{y}(-\omega^*)) = (x^*, Y^-(x^*))$. By (2.11.12), the equation (2.11.16) gives

$$r_{12}\frac{1}{\sqrt{\sigma_{11}}}\sin\left(\frac{-\omega^* + \theta}{2}\right) + r_{22}\frac{1}{\sqrt{\sigma_{22}}}\sin\left(\frac{-\omega^* + \theta}{2} - \beta\right) = 0$$

With some basic trigonometry, we obtain that

$$\tan\frac{-\omega^* + \theta}{2} = \frac{\sin\beta}{\frac{r_{12}}{r_{22}}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos\beta} = \tan(\delta).$$

We deduce that $\omega^* = \theta - 2\delta$. A symmetric computation leads to $\omega^{**} = \theta + 2\epsilon$. The necessary and sufficient condition for the existence of the poles comes from Proposition 2.20 and some trigonometry. The inequalities on α transfer on ω by equation (2.11.9). ☐

Asymptotics in the cone

We now compute the exponential decay rate in term of the polar coordinate in the cone.

Proposition 2.48 (Exponential decay rate)**.** *For α and ω previously defined and linked by (2.11.9) we have*

$$r\cos(\alpha)x(\alpha) + r\sin(\alpha)y(\alpha) = 2\rho|\widetilde{\mu}|\sin^2\left(\frac{\omega - \theta}{2}\right) \tag{2.11.17}$$

$$r\cos(\alpha)x(\alpha^*) + r\sin(\alpha)y(\alpha^*) = 2\rho|\widetilde{\mu}|\sin^2\left(\frac{2\omega - \omega^* - \theta}{2}\right) \tag{2.11.18}$$

Proof. By Equations (2.11.8) and (2.11.13) and some basic trigonometry properties we obtain the desired result. ☐

Proofs of Theorems 2.2, 2.3 and 2.4. Equation (2.11.5) and Propositions 2.46, 2.47, 2.48 combined to Theorem 2.34 (resp. Theorems 2.41 and 2.45), leads to Theorem 2.2 (resp. Theorems 2.3 and 2.4). ☐

2.A Parameter-dependent Morse Lemma

The following lemma is a parameter-dependent Morse lemma. It is an intuitive result but we could not find it in the known literature.

Lemma 2.A.1. *Assume that $\alpha_0 \in \mathbb{R}$ is a constant, $\alpha \mapsto x(\alpha)$ is a function which is $\mathcal{C}^\infty$ near α_0, and $(x, \alpha) \mapsto F(x, \alpha)$ is a function which is analytic as a function of the first variable x and $\mathcal{C}^\infty$ as a function of the second variable α near $(x(\alpha_0), \alpha_0)$. Furthermore we assume that for all α near α_0 we have*

$$F(x(\alpha), \alpha) = 0, \quad F'_x(x(\alpha), \alpha) = 0, \quad F''(x(\alpha), \alpha) > 0.$$

There exists a neighborhood of $(0, \alpha_0)$ in $\mathbb{C} \times \mathbb{R}$

$$\Omega(0, \alpha_0) = \{(\omega, \alpha) \in \mathbb{C} \times \mathbb{R} : |\omega| \leq K, |\alpha - \alpha_0| \leq \eta\}$$

with some $K, \eta > 0$ and a function $x(\omega, \alpha)$ defined in $\Omega(0, \alpha_0)$ such that

$$F(x(\omega, \alpha), \alpha) = \omega^2, \quad \forall \omega : |\omega| \leq K$$

$$x(0, \alpha) = x(\alpha) \quad \forall \alpha : |\alpha - \alpha_0| \leq \eta.$$

Furthermore $x(\omega, \alpha)$ is in the class $\mathcal{C}^\infty$ as function of three real variables $\Re\omega, \Im\omega, \alpha$ and holomorphic of ω for any fixed α. Finally

$$x'_\omega(0, \alpha) = \sqrt{\frac{2}{F''_x(x(\alpha), \alpha))}}. \tag{2.A.1}$$

Proof. This is an adaptation of Morse lemma to the dependence of the parameter a α. Consider $T(z, \alpha) = F(z + x(\alpha), \alpha)$. Then $T(0, \alpha) = 0$, $T'_z(0, \alpha) = 0$ and $T''_z(0, \alpha) = F''_x(x(\alpha), \alpha) > 0$ for any α close to α_0. Then the following representation holds

$$T(z, \alpha) = z^2 F''_x(x(\alpha), \alpha)/2 + z^3 h(z, \alpha) \tag{2.A.2}$$

which allows to define

$$S(z, \alpha) = z\sqrt{F''_x(x(\alpha), \alpha)/2 + zh(z, \gamma)}$$

with one of two branches of the square root. Let us choose the one that takes the value $+F''_x(x(\alpha), \alpha)/2$ at $z = 0$. Due to elementary properties of the function F and the fact that $x(\alpha)$ is in class $\mathcal{C}^\infty$, function $h(z, \alpha)$ in the representation of T above is in class $\mathcal{C}^\infty$ in a neighborhood of $\mathcal{O}(0, \alpha_0) \subset \mathbb{C} \times \mathbb{R}$ as a function of three real variables and also holomorphic in z for any fixed α. Furthermore

$$S'_z(0, \alpha_0) = F''_x(x(\alpha_0), \alpha_0)/2 \neq 0. \tag{2.A.3}$$

Then by the theorem of implicit function (the real one to establish the announced properties in $\mathbb{R}^3$ and the complex one to show the holomorphicity), there exists a function $z(\omega, \alpha)$ in a neighborhood of $(0, \alpha_0)$ which is in the class $\mathcal{C}^\infty$ in three variables and holomorphic in ω such that

$$S(z(\omega, \alpha), \alpha) \equiv \omega, \qquad z(0, \alpha_0) = 0. \tag{2.A.4}$$

This means that $T(z(\omega, \alpha), \alpha) \equiv \omega^2$ for any couple (ω, α) in this neighborhood. In particular, function $z(0, \alpha)$ solves the equation $S(z, \alpha) \equiv 0$ in the variable z. Since $S'_z(0, \alpha_0) \neq 0$, a function in the class $\mathcal{C}^\infty$ of real variable α satisfying this equation and vanishing at α_0 is unique by the theorem of implicit function. But we know already that $S(0, \alpha) = 0$ for any α close to α_0. Hence, $z(0, \alpha) \equiv 0$ for any α close to α_0.

Let now

$$x(\omega, \alpha) = z(\omega, \alpha) + x(\alpha).$$

where $x(\alpha)$ is in the class $\mathcal{C}^\infty$. It satisfies all expected properties. Furthermore $F(x(\omega, \alpha), \alpha) \equiv \omega^2$. Differentiating this identity twice, we obtain (2.A.1). □

2.B Non-degenerate RBM : Green's functions near zero and Laplace transforms near infinity

We introduce the parameter

$$\lambda = \frac{\delta + \epsilon - \pi}{\beta}$$

where β is the angle of the cone, and ϵ and δ the angles of reflection which can be expressed in term of the covariance matrix Σ and the reflection matrix R, see Section 2.11. This parameter λ is well known in the SRBM literature and is usually denoted α but to avoid any confusion of notation we have called it lambda in this chapter. It is well known that existence conditions of the SRBM stated in (2.2.1) are equivalent to

$$\lambda < 1.$$

Lemma 2.B.1 (Laplace transforms behaviour near infinity and Green's functions near zero)**.** *For some constants C_1 and C_2, the Laplace transforms φ_1 and φ_2 satisfy*

$$\varphi_1(y) \sim C_1 y^{\lambda-1} \text{ when } |y| \to \infty \quad \text{and} \quad \varphi_2(x) \sim C_2 x^{\lambda-1} \text{ when } |x| \to \infty \tag{2.B.1}$$

and their derivatives satisfy

$$\varphi_1'(y) \sim C_1(\lambda-1) y^{\lambda-2} \text{ when } |y| \to \infty \quad \text{and} \quad \varphi_2'(x) \sim C_2(\lambda-1) x^{\lambda-2} \text{ when } |x| \to \infty. \tag{2.B.2}$$

Furthermore, the Green's functions on the boundaries h_1 and h_2 satisfy

$$h_1(v) \sim C_1\Gamma(-\lambda+1) v^{-\lambda} \text{ when } |v| \to 0 \quad \text{and} \quad h_2(u) \sim C_2\Gamma(-\lambda+1) u^{-\lambda} \text{ when } |u| \to 0. \tag{2.B.3}$$

We have noted Γ the gamma function.

We give the sketch of the proof of the previous Lemma which rely on the resolution of Boundary Value Problem studied in [49] which cannot be fully detailed here due to technical aspects. This lemma is not crucial for establishing the results of this chapter. It is only used to simplify the proof of Lemma 2.31 which is useful only in the special case where we are looking for the asymptotics along the axes.

Sketch of proof. The article [49] states in Theorem 11 an explicit expression for the Laplace transform φ_1. This result is obtained by solving a Carleman Boundary Value Problem coming from the functional equation (2.2.2). The solution is the product of the solution of the corresponding homogeneous problem and an integral, namely

$$\varphi_1(y) = X(W(y)) \left(\frac{1}{2\pi} \int_{\mathcal{R}^-} \frac{g(t)}{X^+(t)} \frac{dt}{W(y) - W(t)} + C \right),$$

where we took the notations of Theorem 11 in [49] and its proof. Since $\frac{g(t)}{X^+(s)}$ converges to 0 when t tends to infinity, the integral $\frac{1}{2\pi} \int_0^1 \frac{g(t)}{X^+(t)} \frac{dt}{W(y)-W(t)}$ converges to a constant when $y \to \infty$ by classical complex analysis results, see (5.2.17) of [41]. The function $X(W(y))$ is the solution to the corresponding homogeneous BVP which is studied in detail in the recurrent case in [56]. Proposition 19 of [56] shows that $X(W(y))) \sim y^{\lambda-1}$ when y tends to infinity which concludes the proof of (2.B.1).

Integral Hardy–Littlewood Tauberian theorems (see for example Karamata's theorem and Ikehara's theorem [116, §7.4 & 7.5] and [33, Thm 33.3 & 33.7]) state that, with some hypothesis, for a function f and its Laplace transform $\mathcal{L}(f)$, for $\lambda \geqslant -1$, $f(t) \sim Ct^{-\lambda}$ when $t \to 0$ is equivalent to $\mathcal{L}(f)(x) \sim C\Gamma(-\lambda+1)x^{\lambda-1}$ when $x \to \infty$. Equation (2.B.3) follows from Tauberian theorem and from (2.B.1).

The proof of (2.B.2) follows from (2.B.3), from Tauberian theorem and from the properties of the derivative Laplace transform, namely $\mathcal{L}(tf(t)) = \frac{d}{dx}\mathcal{L}(f)(x)$. ☐

2.C Technical Results

This following lemma is useful in Section 2.10 for finding out how the asymptotics behaves as the saddle point approaches the pole.

Lemma 2.C.1. *(i) If $C > 0$ is such that $C^2 \geq 1 + \frac{B^2}{A^2}$, then*

$$\frac{|s|}{|s + i(A+iB)|} \leq C \quad \forall s \in \mathbb{R}.$$

(ii) Let $|A| \to \infty$ and $B = o(A)$ as $|A| \to \infty$. Then

$$\int_{-\infty}^{\infty} \frac{\exp(-s^2)}{s + i(A+iB)} ds \sim \frac{\sqrt{\pi}}{i(A+iB)}.$$

(iii) Let

$$\Pi(w) = \int_{-\infty}^{\infty} \frac{\exp(-s^2)}{s+iw} ds$$

with $\Re w \neq 0$. This function is holomorphic in each half plane $\{w : \Re w > 0\}$ and $\{w : \Re w < 0\}$ and can be made explicit:

$$\Pi(w) = \pi i \exp(w^2)(1 - \Phi(-w)) \quad \forall w : \ \Re w < 0$$

$$\Pi(w) = -\pi i \exp(w^2)(1 - \Phi(w)) \quad \forall z : \ \Re w > 0$$

where $\Phi(w) = \frac{2}{\sqrt{\pi}} \int_0^w \exp(-s^2) ds$.

Proof. (i) Elementary computation.

(ii) We have $\int_{-\infty}^{\infty} \frac{\exp(-s^2)}{i(A+iB)} ds = \frac{\sqrt{\pi}}{i(A+iB)}$. It suffices to show that

$$\int_{\mathbb{R}} \frac{|s|}{|s+i(A+iB)|} \exp(-s^2) ds$$

converges to 0 for any A with absolute value large enough to have $\frac{|A|}{|B|} \geq 1$. Then by (i) $\frac{|s|}{|s+i(A+iB)|} \leq 2$ for any $s \in \mathbb{R}$. Since the integral $\int_{\mathbb{R}} 2\exp(-s^2) ds$ converges, the dominated convergence theorem applies and we get the asymptotic.

(iii) Let us define for any $z > 0$ and $w > 0$

$$\Pi(z, w) = \int_{-\infty}^{\infty} \frac{\exp(-zs^2)}{s+iw} ds$$

Then

$$\begin{aligned}
\Pi'_z(z, w) &= \int_{-\infty}^{\infty} \frac{-s^2 \exp(-zs^2)}{s+iw} ds = \int_{-\infty}^{\infty} \frac{((iw)^2 - s^2 - (iw)^2) \exp(-zs^2)}{s+iw} ds \\
&= \int_{-\infty}^{\infty} (iw - s) \exp(-zs^2) ds + w^2 \int_{-\infty}^{\infty} \frac{\exp(-zs^2)}{s+iw} ds \\
&= iw\sqrt{\frac{\pi}{z}} + w^2 \Pi(w, z).
\end{aligned}$$

Solving this differential equation we get that $\Pi(w, z) = c(w, z) \exp(w^2 z)$ where $c'_z(w, z) = iw\sqrt{\frac{\pi}{z}} \exp(-w^2 z)$. Taking into account the fact that $\Pi(+\infty, w) = 0$, we obtain

$$\Pi(z, w) = -iw\sqrt{\pi} \exp(w^2 z) \int_z^{\infty} t^{-1/2} \exp(-w^2 t) dt = -iw\sqrt{\pi} \exp(w^2 z) \int_{w\sqrt{z}}^{\infty} \exp(-s^2) ds$$

$$= -iw\pi \exp(w^2 z)\Big(1 - \Phi(w\sqrt{z})\Big).$$

Let now $z = 1$. Then

$$\Pi(1, w) = -\pi i \exp(w^2)(1 - \Phi(w))$$

for any real positive w. Using the holomorphicity of $\Phi(w)$ in $\{w \in \mathbb{C} : \Re w > 0\}$ we prove the statement (iii). Finally we note that for any w with $\Re w < 0$: $\Pi(-w) = -\Pi(w)$.

□

2.D Green's functions near zero and Laplace transforms near infinity

We introduce the parameter

$$\lambda = \frac{\delta+\epsilon-\pi}{\beta}$$

where β is the angle of the cone, and ϵ and δ the angles of reflection which can be expressed in term of the covariance matrix Σ and the reflection matrix R, see Section 2.11. This parameter λ is well known in the SRBM literature and is usually denoted α but to avoid any confusion of notation we have called it lambda in this article. It is well known that existence conditions of the SRBM stated in (2.2.1) are equivalent to

$$\lambda < 1.$$

Lemma 2.D.1 (Laplace transforms behaviour near infinity and Green's functions near zero)**.** *For some constants C_1 and C_2, the Laplace transforms φ_1 and φ_2 satisfy*

$$\varphi_1(y) \sim C_1 y^{\lambda-1} \text{ when } |y|\to\infty \quad \text{and} \quad \varphi_2(x) \sim C_2 x^{\lambda-1} \text{ when } |x|\to\infty \tag{2.D.1}$$

and their derivatives satisfy

$$\varphi_1'(y) \sim C_1(\lambda-1) y^{\lambda-2} \text{ when } |y|\to\infty \quad \text{and} \quad \varphi_2'(x) \sim C_2(\lambda-1) x^{\lambda-2} \text{ when } |x|\to\infty. \tag{2.D.2}$$

Furthermore, the Green's functions on the boundaries h_1 and h_2 satisfy

$$h_1(v) \sim C_1\Gamma(-\lambda+1)v^{-\lambda} \text{ when } |v|\to 0 \quad \text{and} \quad h_2(u) \sim C_2\Gamma(-\lambda+1)u^{-\lambda} \text{ when } |u|\to 0. \tag{2.D.3}$$

We have noted Γ the gamma function.

We give the sketch of the proof of the previous Lemma which rely on the resolution of Boundary Value Problem studied in [49] which cannot be fully detailed here due to technical aspects. This lemma is not crucial for establishing the results of this chapter. It is only used to simplify the proof of Lemma 2.31 which is useful only in the special case where we are looking for the asymptotics along the axes.

Sketch of proof. The article [49] states in Theorem 11 an explicit expression for the Laplace transform φ_1. This result is obtained by solving a Carleman Boundary Value Problem coming from the functional equation (2.2.2). The solution is the product of the solution of the corresponding homogeneous problem and an integral, namely

$$\varphi_1(y) = X(W(y))\left(\frac{1}{2\pi}\int_{\mathcal{R}^-}\frac{g(t)}{X^+(t)}\frac{dt}{W(y)-W(t)} + C\right),$$

where we took the notations of Theorem 11 in [49] and its proof. Since $\frac{g(t)}{X^+(s)}$ converges to 0 when t tends to infinity, the integral $\frac{1}{2\pi}\int_0^1 \frac{g(t)}{X^+(t)}\frac{dt}{W(y)-W(t)}$ converges to a constant when $y\to\infty$ by classical complex analysis results, see (5.2.17) of [41]. The function $X(W(y))$ is the solution to the corresponding homogeneous BVP which is studied in detail in the recurrent case in [56]. Proposition 19 of [56] shows that $X(W(y))) \sim y^{\lambda-1}$ when y tends to infinity which concludes the proof of (2.D.1).

Integral Hardy–Littlewood Tauberian theorems (see for example Karamata's theorem and Ikehara's theorem [116, §7.4 & 7.5] and [33, Thm 33.3 & 33.7]) state that, with some hypothesis, for a function f and its Laplace transform $\mathcal{L}(f)$, for $\lambda \geqslant -1$, $f(t)\sim Ct^{-\lambda}$ when $t\to 0$ is equivalent to $\mathcal{L}(f)(x) \sim C\Gamma(-\lambda+1)x^{\lambda-1}$ when $x\to\infty$. Equation (2.D.3) follows from Tauberian theorem and from (2.D.1).

The proof of (2.D.2) follows from (2.D.3), from Tauberian theorem and from the properties of the derivative Laplace transform, namely $\mathcal{L}(tf(t)) = \frac{d}{dx}\mathcal{L}(f)(x)$. □

Chapter 3

Martin boundary of a degenerate Reflected Brownian Motion in a wedge

Ce chapitre est issu de l'article [101] publié dans le journal *Stochastic Systems*.

Résumé

Nous considérons un mouvement Brownien dégénéré dans le quart de plan, avec drift dirigé vers l'extérieur, et réfléchi obliquement sur le bord. Dans ce chapitre, nous calculons explicitement les transformées de Laplace des fonctions de Green associées au processus. Ces transformées de Laplace sont exprimées comme une somme infinie de produits par itération d'une équation fonctionnelle, étroitement liée à la méthode de compensation. Nous établissons également les asymptotiques des fonctions de Green le long de tous les chemins possibles et déterminons la frontière de Martin (minimale). Enfin, nous fournissons des formules explicites pour toutes les fonctions harmoniques positives correspondantes.

Abstract

We consider an outward degenerate drifted Brownian motion in the quarter plane with oblique reflections on the boundaries. In this chapter, we explicitly compute the Laplace transforms of the Green's functions associated with the process. These Laplace transforms are expressed as an infinite sum of products by iterating a functional equation, which is deeply linked to the compensation method. We also derive the asymptotics of the Green's functions along all possible paths and determine the (minimal) Martin boundary. Finally, we provide explicit formulae for all the corresponding positive harmonic functions.

Contents

3.1 Introduction and main results

Context

We consider a class of degenerate transient SRBMs in the quadrant (see Section 1.1.1.3 and precise definition below) with conditions (3.1.1)-(3.1.3). This chapter aims to solve both of the following problems:

(P_1) Obtaining the Laplace transforms of the Green's functions,

(P_2) Computing the asymptotics of the Green's functions along all trajectories of the SRBM.

Solutions to (P_1) in the half-plane can be expressed directly in terms of a rational function of two variables $(x, Y(x))$, where $Y(x)$ is a branch of a certain two-valued algebraic function, as detailed in [37]. However, solving (P_1) in a general cone presents a significantly greater challenge. Specifically, for non-degenerate SRBM in the quarter plane with three domains, the Laplace transforms are obtained as singular integral representations via boundary-value problems like in Franceschi and al. [49, 56] (see Section 1.2.4.1). Although these expressions are explicit, they are not particularly amenable to in-depth analysis. For this class of processes, we express the Laplace transforms of the Green's functions in terms of infinite series in product form using the compensation method, see Section 1.2.4.2.

To solve (P_2), we adapt the approach described in Section 1.2.3. A key difference with the non-degenerate case studied in Chapter 2 is that the kernel equation defines a parabola in $\mathbb{R}^2$ instead of an ellispe. This procedure provides asymptotic developments of Green's functions with arbitrarily many terms, but with unknown multiplicative constants. These constants may be derived – albeit somewhat indirectly – from the solutions to (P_1). The significance of these constants – viewed as functions of the starting point of the process – extends beyond asymptotic precision: they also yield all positive harmonic functions for the DRBM via the Martin boundary theory, see Appendice A.

Main results

The degenerate reflected Brownian, assumptions.

We consider a degenerate Brownian motion $(Z_t)_{t\geq 0}$ in a quadrant, with oblique reflection at the boundaries. By *degenerate* we mean that the covariance matrix is of rank 1. This obliquely reflected process was studied in [71] and its rigorous definition is provided in Section 3.2. The parameters of the degenerate reflected Brownian motion are given by:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & -\sigma_1\sigma_2 \\ -\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, R = \begin{pmatrix} 1 & r_2 \\ r_1 & 1 \end{pmatrix} = \begin{pmatrix} R_1 & R_2 \end{pmatrix} \tag{3.1.1}$$

where Σ is the degenerate covariance matrix ($\det(\Sigma) = 0$), μ is the drift and columns of R represent the reflection directions from the axes. The direction $v = (v_1, v_2)^T = (\sigma_1, -\sigma_2)$ is antidiagonal, i.e. $v_1 v_2 < 0$ (see Figure 3.1). When the process does not hit the boundaries, it behaves like a one-dimensional Brownian motion along the direction v (plus the drift). Our main assumptions in this chapter are as follows:

$$\mu_1 > 0, \quad \mu_2 > 0, \tag{3.1.2}$$

$$r_1 > -\frac{\sigma_2}{\sigma_1}, \quad r_2 > -\frac{\sigma_1}{\sigma_2}. \tag{3.1.3}$$

Assumption (3.1.2) ensures that the process is transient, whereas (3.1.3) specifies that the reflection vectors $R_1 = (1, r_1)^T$ (on $\{x = 0\}$) and $R_2 = (1, r_2)^T$ (on $\{y = 0\}$) point outward from the direction v of the Brownian motion (see Figure 3.1).

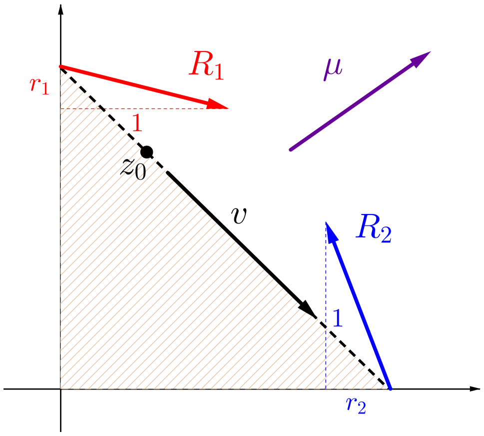


Figure 3.1: Reflections R_1, R_2 on the edges, the drift μ, and the direction v of the degenerate Brownian motion. The process starting from z_0 never reaches the hatched region.

In Sections 1–8 we state and prove our results under the additional assumption

$$\sigma_1 = \sigma_2 = 1, \quad \mu_1 + \mu_2 = 1. \tag{3.1.4}$$

Results for the general case, i.e. without Assumption (3.1.4), are stated and proved in Section 3.9. In fact, they are easily deduced from the results under (3.1.4) by means of a simple space-time transformation.

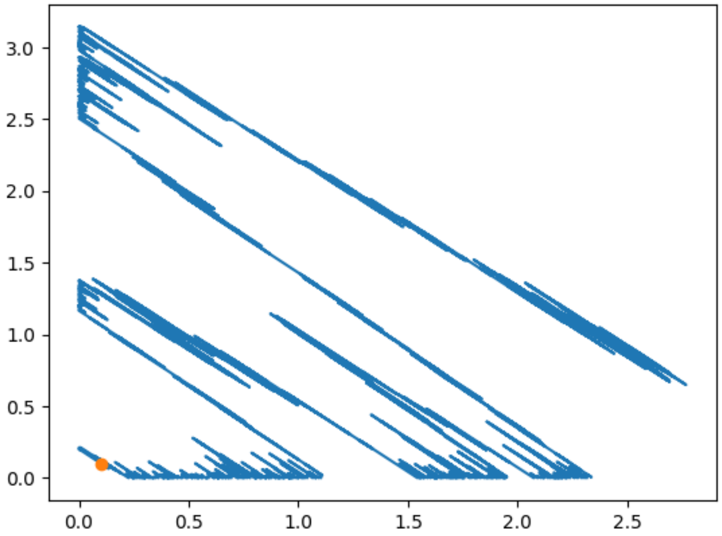


Figure 3.2: Example of a typical path (over a finite time horizon) of the drifted degenerate Brownian motion. The initial point is marked in orange.

Green's functions

We show that for any starting point $z_0 \in \mathbb{R}_+^2$, there exists a density $g^{z_0}(\cdot)$ of the Green's measure $G(z_0, \cdot)$ on the quadrant defined as

$$G(z_0, A) := \int_0^{+\infty} \mathbb{P}_{z_0}(Z_t \in A) dt = \int_A g^{z_0}(z) dz. \tag{3.1.5}$$

Functions $g^{z_0}(\cdot)$ are called the Green's functions. We also define the Green's measures on the sides of the wedge

$$H_i(z_0, A) := \mathbb{E}_{z_0}\left[\int_0^\infty \mathbb{1}_A(Z_t) dL_t^i\right], \quad i = 1, 2 \tag{3.1.6}$$

where $(L_t^1)_{t\geq 0}$ (resp. $(L_t^2)_{t\geq 0}$) is the local time of the process on the axis $\{x = 0\}$ (resp. $\{y = 0\}$). The measure H_1 has its support on the vertical axis and H_2 has its support on the horizontal axis. Laplace transforms $\varphi(x, y)$ of $G(z_0, \cdot)$ and $\varphi_1(y), \varphi_2(x)$ of $H_1(z_0, \cdot), H_2(z_0, \cdot)$ are related by the following functional equation

$$-\gamma(x, y)\varphi(x, y) = \gamma_1(x, y)\varphi_1(y) + \gamma_2(x, y)\varphi_2(x) + e^{(x,y)\cdot z_0}, \quad Re(x) < 0, Re(y) < 0 \tag{3.1.7}$$

where

$$\gamma(x, y) = \frac{1}{2}(x - y)^2 + \mu_1 x + \mu_2 y \tag{3.1.8}$$

and

$$\gamma_1(x, y) = R^1 \cdot (x, y) = x + r_1 y, \qquad \gamma_2(x, y) = R^2 \cdot (x, y) = r_2 x + y. \tag{3.1.9}$$

It can be viewed as a balance equation for Green's measures between the interior and the edges of the quadrant. Let us define

$$\mathcal{P} = \{(x, y) \in \mathbb{R}^2, \ \ \gamma(x, y) = 0\}. \tag{3.1.10}$$

The functional equation (3.3.1) is similar to that in Chapter 2; however, an important difference is that $\mathcal{P}$ is now a parabola rather than an ellipse. This distinction is what allows the compensation method to be effective, leading to explicit expressions for the Laplace transforms and positive harmonic functions.

Explicit expressions for Laplace transforms

The first results of the chapter provide explicit expressions for Laplace transforms φ_1 and φ_2 in terms of infinite series of product forms, given by formulae (3.4.22) and (3.4.21) which we do not specify here. Function φ is derived from φ_1 and φ_2 via the functional equation (3.1.7).

Asymptotics of Green's functions

We now focus on the asymptotics of $g^{z_0}(r\cos(\alpha), r\sin(\alpha))$ as $r \to +\infty$ and $\alpha \to \alpha_0 \in [0, \pi/2]$. For any direction α, we denote by $(x(\alpha), y(\alpha))$ a corresponding point on the parabola given by

$$(x(\alpha), y(\alpha)) = \operatorname{argmax}_{(x,y)\in\mathcal{P}} (\cos(\alpha)x + \sin(\alpha)y), \tag{3.1.11}$$

see Figure 3.3a. It can be computed explicitly as:

$$(x(\alpha), y(\alpha)) = \left(\frac{(\mu_2 - \tan(\alpha)\mu_1)(\mu_2 + \tan(\alpha))(1+\mu_2)}{2(1+\tan(\alpha))^2}, \frac{(\mu_2 - \tan(\alpha)\mu_1)(1+\mu_1(1+\tan(\alpha)))}{2(1+\tan(\alpha))^2} \right). \tag{3.1.12}$$

Let us define two particular directions

$$\alpha^* := \begin{cases} 0 & \text{if} \quad (r_1+1)\mu_2 \leq 2 \\ \arctan\left(\frac{(1+r_1)\mu_2 - 2}{2+(1+r_1)\mu_1}\right) & \text{if} \quad (r_1+1)\mu_2 > 2. \end{cases} \tag{3.1.13}$$

$$\alpha^{**} := \begin{cases} \arctan\left(\frac{(1+r_2)\mu_2 + 2}{(1+r_2)\mu_1 - 2}\right) & \text{if} \quad (r_2+1)\mu_1 > 2 \\ \pi/2 & \text{if} \quad (r_2+1)\mu_1 \leq 2, \end{cases} \tag{3.1.14}$$

see Figure 3.3b for their geometric interpretation. We always have $\alpha^* < \alpha^{**}$ as will be proved in Section 3.5.

In the following theorem we summarize the asymptotics of Green's functions for directions $\alpha_0 \in (0, \pi/2)\backslash\{\alpha^*, \alpha^{**}\}$. The ones for $\alpha_0 \in \{0, \alpha^*, \alpha^{**}, \pi/2\}$ are given later in Theorems 3.33, 3.34 and 3.35.

Theorem 3.1 (Asymptotics in the quadrant, general case)**.** *Assume (3.1.2) to (3.1.4). Then, the Green's density function g^{z_0} of this process has the following asymptotics as $\alpha \to \alpha_0$ and $r \to \infty$.*

- *If $\alpha^* < \alpha_0 < \alpha^{**}$, then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} c_{\alpha_0} h_{\alpha_0}(z_0) \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}. \tag{3.1.15}$$

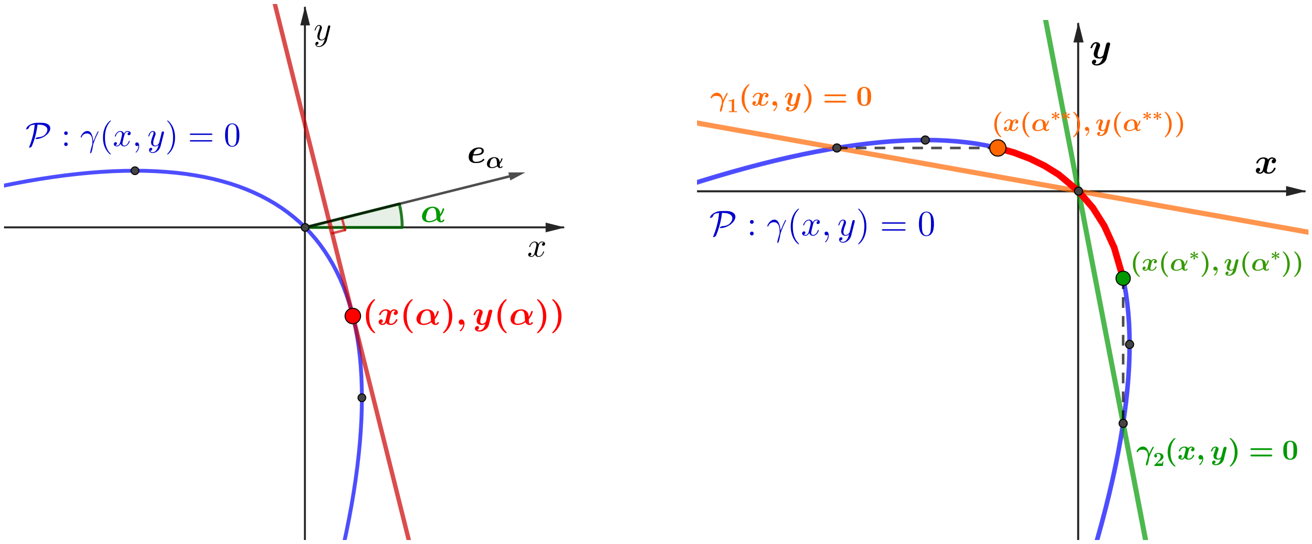
- *If $\alpha_0 < \alpha^*$, then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} c^* h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x(\alpha^*)+\sin(\alpha)y(\alpha^*))}. \tag{3.1.16}$$

- *If $\alpha_0 > \alpha^{**}$, then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} c^{**} h_{\alpha^{**}}(z_0) e^{-r(\cos(\alpha)x(\alpha^{**})+\sin(\alpha)y(\alpha^{**}))}. \tag{3.1.17}$$

where $c_{\alpha_0} = \frac{1}{\sqrt{2\pi(\cos(\alpha_0)+\sin(\alpha_0))}}$, c^* and c^{**} are positive explicit constants only depending on the parameters of the degenerate reflected Brownian motion (see (3.6.1)) and where $h_\alpha(z_0), h_{\alpha^*}(z_0), h_{\alpha^{**}}(z_0)$ are harmonic functions given in Theorem 3.2. Furthermore, $h_\alpha(z_0), h_{\alpha^*}(z_0), h_{\alpha^{**}}(z_0)$ are non-zero.



(a) The point $(x(\alpha), y(\alpha))$ maximises the scalar product $\langle (x, y), e_\alpha \rangle$ where $e_\alpha = (\cos(\alpha), \sin(\alpha))$ and (x, y) belongs to the parabola $\mathcal{P}$.

(b) In the case $0 < \alpha^* < \alpha^{**} < \pi/2$, angles α^* and α^{**} introduced in (3.1.13), (3.1.14) can be defined equivalently using this construction.

Figure 3.3: Geometric interpretation of $(x(\alpha), y(\alpha))$, α^* and α^{**}.

Explicit expressions for positive harmonic functions with the compensation method

Let us recall the following definition: a function $h : \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ is harmonic if and only if for all $t \geq 0$ and $z_0 \in \mathbb{R}_+^2$,

$$\mathbb{E}_{z_0}[h(Z_t)] = h(z_0). \tag{3.1.18}$$

All functions $h_\alpha, \alpha \in [\alpha^*, \alpha^{**}]$ are harmonic. These functions are explicitly stated in Theorem 3.2 below and will be derived in this chapter using the compensation method. The essence of this method is to construct functions that satisfy the partial differential equation along with boundary conditions:

$$\begin{cases} (H_0) & \mathcal{G}h = 0 \qquad \text{on} \quad (0, +\infty)^2, \\ (H_1) & \partial_{R_1} h(0, y) = 0, \quad y \geq 0 \\ (H_2) & \partial_{R_2} h(x, 0) = 0, \quad x \geq 0 \end{cases} \tag{3.1.19}$$

where $\mathcal{G} = \frac{1}{2}\nabla \cdot \Sigma \nabla + \mu \cdot \nabla$. Those function are harmonic as it will be noticed in Section **??**.

For $(a_0, b_0) \in \mathcal{P}$ and $k \in \mathbb{Z} \backslash \{0\}$, we set

$$a_{2k} = -2k^2 + 2(a_0 - b_0 - \mu_2)k + a_0, \quad a_{2k+1} = a_{2k} \tag{3.1.20}$$

$$b_{2k} = -2k^2 + 2(a_0 - b_0 + \mu_1)k + b_0, \quad b_{2k+1} = b_{2k+2} \tag{3.1.21}$$

As illustrated in Figure 3.4, points $(a_p, b_p) \in \mathcal{P}$ are constructed by following the "downstairs" path on the parabola, applying successively automorphisms that leave invariant the first or the second coordinate respectively.

Theorem 3.2 (Explicit expressions for harmonic functions $(h_\alpha)_{\alpha \in [\alpha^*, \alpha^{**}]}$)**.** *Assume (3.1.2) to (3.1.4). Then, the functions $(h_\alpha)_{\alpha \in [\alpha^*, \alpha^{**}]}$ are harmonic and are given by the following formulae*

- *For $\alpha \in (\alpha^*, \alpha^{**})$, taking $(a_0, b_0) = (x(\alpha), y(\alpha))$, we have*

$$h_\alpha : z_0 \longmapsto \sum_{m=-\infty}^{+\infty} \kappa_m(\alpha) e^{z_0 \cdot (a_m, b_m)} \tag{3.1.22}$$

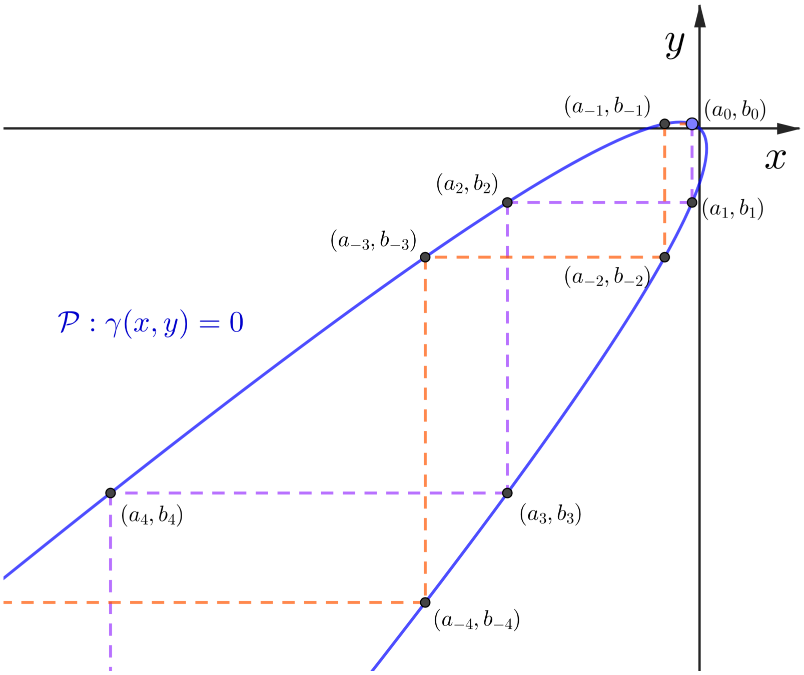


Figure 3.4: Parabola $\mathcal{P}$ and points (a_n, b_n) on the parabola.

where $\kappa_0(\alpha) = 1$ *and*

$$\kappa_m(\alpha) = \begin{cases} (-1)^m \left[\prod_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \frac{\frac{\gamma_1}{\gamma_2}(a_{2k+1}, b_{2k+1})}{\frac{\gamma_1}{\gamma_2}(a_{2k+2}, b_{2k+2})}\right] \frac{\gamma_2(a_0, b_0)}{\gamma_2(a_m, b_m)} & \text{if } m > 0 \\ (-1)^m \left[\prod_{k=0}^{\lfloor \frac{-m}{2} \rfloor - 1} \frac{\frac{\gamma_2}{\gamma_1}(a_{-2k-1}, b_{-2k-1})}{\frac{\gamma_2}{\gamma_1}(a_{-2k-2}, b_{-2k-2})}\right] \frac{\gamma_1(a_0, b_0)}{\gamma_1(a_m, b_m)} & \text{if } m < 0 \end{cases} \tag{3.1.23}$$

(with the convention $\Pi_{k=0}^{-1} = 1$*).*

- *For* $\alpha = \alpha^*$,
  - *If* $\frac{2}{r_2+1} < \mu_2$, *then* $\alpha^* = 0$ *and* $h_0 : z_0 \longmapsto \partial_\alpha \left[h_\alpha(z_0)\right]_{\alpha = 0^+}$.
  - *If* $\frac{2}{r_2+1} > \mu_2$, *then* $\alpha^* > 0$ *and taking* $(a_0, b_0) = (x(\alpha^*), y(\alpha^*))$,

$$h_{\alpha^*} : z_0 \longmapsto e^{z_0 \cdot (a_1, b_1)} + \sum_{m=2}^{+\infty} \widehat{\kappa}_m(\alpha^*) e^{z_0 \cdot (a_m, b_m)} \tag{3.1.24}$$

where

$$\widehat{\kappa}_m(\alpha^*) = (-1)^{m+1} \frac{\gamma_1(a_1, b_1)}{\frac{\gamma_1}{\gamma_2}(a_2, b_2)} \left[\prod_{k=1}^{\lfloor \frac{m}{2} \rfloor - 1} \frac{\frac{\gamma_1}{\gamma_2}(a_{2k+1}, b_{2k+1})}{\frac{\gamma_1}{\gamma_2}(a_{2k+2}, b_{2k+2}}\right] \frac{1}{\gamma_2(a_m, b_m)}.$$

  - *If* $\frac{2}{r_2+1} = \mu_2$, *then* $\alpha^* = 0$ *and taking* $(a_0, b_0) = (x(0), y(0))$,

$$h_0 : z_0 \longmapsto 2e^{z_0 \cdot (a_0, b_0)} + \sum_{m=-\infty}^{-1} \kappa_m(\alpha^*) e^{z_0 \cdot (a_m, b_m)} + \sum_{m=2}^{+\infty} \widetilde{\kappa}_m(\alpha^*) e^{z_0 \cdot (a_m, b_m)} \tag{3.1.25}$$

where

$$\widetilde{\kappa}_m(\alpha^*) = (-1)^{m+1} \frac{\gamma_1(a_1, b_1)}{\frac{\gamma_1}{\gamma_2}(a_2, b_2)} \left[\prod_{k=1}^{\lfloor \frac{m}{2} \rfloor - 1} \frac{\frac{\gamma_1}{\gamma_2}(a_{2k+1}, b_{2k+1})}{\frac{\gamma_1}{\gamma_2}(a_{2k+2}, b_{2k+2})}\right] \frac{1}{\gamma_2(a_m, b_m)}.$$

- *For $\alpha = \alpha^{**}$, symmetrical formulae hold replacing r_1 by r_2, μ_1 by μ_2 and 0 by $\frac{\pi}{2}$.*

Note that if $\alpha < \alpha^*$ or $\alpha > \alpha^{**}$, expression (3.1.22) may define a harmonic function that is not necessarily non-negative everywhere.

The Martin boundary and its minimality are derived from Theorem 3.1 and Theorem 3.2, together with the further technical results in Theorems 3.33, 3.34 and 3.35 concerning the asymptotics of Green functions along the directions $0, \alpha^*, \alpha^{**}$ and $\pi/2$.

Theorem 3.3 (Martin Boundary)**.** *Under (3.1.2) to (3.1.4), the Martin boundary Γ of the degenerate reflected Brownian motion is homeomorphic to $[\alpha^*, \alpha^{**}]$ via the mapping*

$$\alpha \in [\alpha^*, \alpha^{**}] \longmapsto h_\alpha(\cdot)/h_\alpha(0) \in \Gamma. \tag{3.1.26}$$

Furthermore, the Martin boundary is minimal.

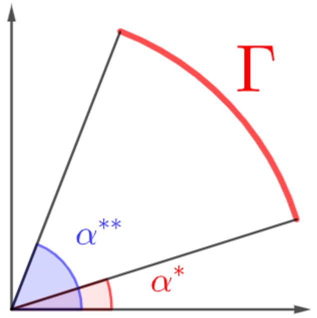


Figure 3.5: Martin boundary Γ when $0 < \alpha^*$ and $\alpha^{**} < \pi/2$.

Remark 3.4 (On Assumptions (3.1.2) and (3.1.3), and possible extensions)*.*

- Regarding assumption (3.1.2), similar results could be established under the more general condition $\mu_1\sigma_2 + \mu_2\sigma_1 > 0$. This condition is equivalent to the orientation of the parabola toward $x \to -\infty$ and $y \to -\infty$. It is also necessary for the convergence of the expressions defining h_α — specifically, equation (3.1.22). Namely, if $\mu_2 < 0$, the Laplace transform φ_2 would have a pole at zero. Due to the technical nature of this paper, we have chosen to restrict our analysis to Assumption (3.1.2). Investigating how the Martin boundary is affected by the presence of such a pole could be an interesting direction for future work.
- If (3.1.3) is not satisfied, the arguments which yield the explicit expressions of the harmonic functions fail. In particular, attempts to construct the functions h_α without this assumption often lead to signed functions which, while possibly harmonic, are not necessarily non-negative. For interested readers, the only step in our argument that fails for general reflection vectors is equation (3.4.16), which may offer a direction for future investigation.

Plan of the chapter

In Section 3.2, we define the degenerate reflected Brownian motion. We then derive the functional equation (3.1.7) in Section 3.3 and meromorphically extend Laplace transforms on the edges up to their singularities. In Section 3.4, we obtain the explicit form of the Laplace transforms iterating the functional equation (3.1.7). Next, in Section 3.5 we carry out preparatory work to derive the asymptotics of Green's functions. These asymptotics are computed in all directions in Sections 3.6 and 3.7 by the saddle-point method. This enables us to prove Theorem 3.1 and Theorem 3.2 by employing the explicit expressions from Section 3.4. In Section 3.8 we establish the asymptotics of the Martin kernel and identify all the harmonic functions. We also prove the minimality of the Martin boundary and conclude the proof of Theorem 3.3. Finally, in Section 3.9 we treat the general case of the model without Assumption (3.1.4) via a linear transformation of space and time.

3.2 Definition of the process

Throughout the following, the filtered space we consider is always the space of continuous functions $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^2)$ with the standard $\sigma-$field and the usual filtration. The following background definition is taken from [115], where the non-degenerate reflected Brownian motion is studied.

Definition 3.5 (Degenerate reflecting Brownian motion)**.** *Let Σ, R and μ be defined as in (3.1.1). A degenerate reflecting Brownian motion (DRBM) associated with the data (Σ, μ, R) is a process $(Z_t)_{t\geq 0}$ and a family of measures $(\mathbb{P}_{z_0})_{z_0\in\mathbb{R}_+^2}$ such that $(Z_t)_{t\geq 0}$ can be written as*

$$Z_t = X_t + RL_t \in \mathbb{R}_+^2, \quad t \geq 0, \tag{3.2.1}$$

where

- *$(X_t - \mu t)_{t\geq 0}$ is an adapted degenerate Brownian motion (with zero drift) of covariance Σ starting from z_0 under $\mathbb{P}_{z_0}$.*
- *L is an adapted 2-dimensional process starting from 0 such that $\mathbb{P}_{z_0}-$almost surely, its components L^1, L^2 are continuous and non-decreasing with $\operatorname{supp}(dL^i) \subset \{t \geq 0, Z_t^i = 0\}$; that is, L^i increases only when $Z_t^i = 0$.*

Note that under $\mathbb{P}_{z_0}$, Z can be written as $(Z_t)_{t\geq 0} = (z_0 + vB_t + \mu t + RL_t)_{t\geq 0}$ where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion and $v = (\sigma_1, -\sigma_2)$ $(= (1, -1)$ under (3.1.4)) is the unique eigenvector (up to a scalar multiplication) associated with the positive eigenvalue of the covariance matrix.

Theorem 3.6 (Existence, uniqueness and Strong Markov property)**.** *Suppose that $|r_1 r_2| < 1$. Then, for any starting point z_0, there exists a DRBM associated with (Σ, μ, R). The processes Z and (Z, L) are pathwise unique (according to the associated degenerate Brownian motion). Furthermore, Z is a semi-martingale, a Feller process (i.e., for any $t \geq 0$, $x \longmapsto \mathbb{E}_x[f(Z_t)]$ is continuous whenever f is bounded and continuous), and a Strong Markov process.*

Proof. Define the matrix $Q = I - R$, whose spectral radius is $\rho(Q) = \sqrt{|r_1 r_2|} < 1$. By Theorem 1 in [63], for any continuous path $x = (x_t)_{t\geq 0} \subset \mathbb{R}^2$ there exists a unique solution $(z_t)_{t\geq 0} = \psi(x)$ of the Skorokod problem

$$z_t = x_t + R(l_t^1, l_t^2)^T, \quad t \geq 0$$

where $(z_t)_{t\geq 0} \subset \mathbb{R}_+^2$ and for $i \in \{1, 2\}$, $(l^i)_{t\geq 0}$ is a continuous, increasing function with $\operatorname{supp}(dl^i) \subset \{t \geq 0, z_t^i = 0\}$. Moreover, ψ is continuous in the topology of uniform convergence on compact sets. This yields the stated result with $Z = \psi(X)$. □

As in the non-degenerate case, there may be existence and uniqueness in law if R is a general $\mathcal{S}$-matrix [115] (without assuming $|r_1 r_2| < 1$), but not pathwise uniqueness [11]. To avoid excessive technicality, we work under assumption $|r_1 r_2| < 1$.

Proposition 3.7 (Transience)**.** *Under conditions (3.1.2) and (3.1.3), the DRBM is a transient Markov process.*

Proof. Consider $w = (\sigma_2, \sigma_1)$, which is orthogonal to the direction of the Brownian motion. It suffices to note that $(Z_t \cdot w)_{t\geq 0}$ is almost surely strictly increasing and tends to $+\infty$ since $Z_t \cdot w \geq \mu \cdot wt$ by (3.1.3). □

We recall the definition of Green's measure $G(z_0,\cdot)$ and $H_i(z_0,\cdot)$ from (3.1.5) and (3.1.6). Assumption (3.1.2) on the drift is crucial for the following proposition.

Proposition 3.8 (Densities and Laplace transforms)**.** *Suppose that assumptions (3.1.2) and (3.1.3) hold. Then, Green's measure $G(z_0,\cdot)$ has a density $g^{z_0}(\cdot)$ with respect to the Lebesgue measure. Functions $g^{z_0}(\cdot)$ are called Green's functions. Furthermore, measures $H_i(z_0,\cdot)$ ($i=1,2$), have densities $f_i^{z_0}(\cdot)$ with respect to the one-dimensional Lebesgue measure.*

Proof. Let A be a compact set of $\mathbb{R}_+^2$ at a positive distance of the edges. Define the stopping times:

$$\sigma=\inf\{t\geq 0,\ Z_t\in A\},\quad \tau=\inf\{t\geq\sigma,\ Z_t\in\partial\mathbb{R}_+^2\}.$$

Considering the back-and-forth trajectories between A and $\partial\mathbb{R}_+^2$ (see [65, Lemma 9 of Section 7]), we can reduce the proof to showing that:

$$\mathbb{E}_{z_0}\left[\int_\sigma^\tau 1_A(Z_s)ds\right]=0.$$

Then, by the Strong Markov property, it suffices to prove the result for a non-reflected degenerate Brownian motion. By Assumption (3.1.2), rotating the plane so that the x-axis aligns with the drift direction reduces the problem to one-dimensional Brownian motion. The proposition then follows from elementary properties of the latter. ☐

Definition 3.9 (Laplace transforms of Green's measures)**.** *We denote the Laplace transforms of $G(z_0,\cdot)$ by*

$$\varphi(x,y):=\mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,y)\cdot Z_t}dt\right]=\int_{\mathbb{R}_+^2}e^{(x,y)\cdot z}g^{z_0}(z)dz$$

and the Laplace transforms of $H_1(z_0,\cdot),H_2(z_0,\cdot)$ by

$$\varphi_1(y):=\mathbb{E}_{z_0}\left[\int_0^\infty e^{(0,y)\cdot Z_t}dL_t^1\right],\quad \varphi_2(x):=\mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,0)\cdot Z_t}dL_t^2\right].$$

For brevity, we omit the dependence on the starting point in the notation for the Laplace transforms. However, when relevant, we will denote this dependence explicitly as $\varphi^{z_0}(x,y),\varphi_1^{z_0}(y)$ and $\varphi_2^{z_0}(x)$.

3.3 Functional equation, kernel and analytic continuation

From now on, we assume (3.1.2) to (3.1.4). As mentioned in the introduction, Laplace transforms $\varphi,\varphi_1,\varphi_2$ are linked by a functional equation.

Proposition 3.10 (Functional equation)**.** *If $Re(x)<0$ and $Re(y)<0$, then $\varphi_1(y)$, $\varphi_2(x)$ and $\varphi(x,y)$ converge and the following equation holds*

$$-\gamma(x,y)\varphi(x,y)=\gamma_1(x,y)\varphi_1(y)+\gamma_2(x,y)\varphi_2(x)+e^{(x,y)\cdot z_0}\tag{3.3.1}$$

where γ,γ_1 and γ_2 are defined in (3.1.8), (3.1.9).

Proof. We apply Itô's formula to the semimartingale $(Z_t)_{t\geq 0}$ and the function $(u,v)\longmapsto e^{xu+yv}$. Then,

$$e^{(x,y)\cdot Z_t}-e^{(x,y)\cdot z_0}=\int_0^t e^{(x,y)\cdot Z_t}(x,y)^T\cdot dB_s+\gamma(x,y)\int_0^t e^{(x,y)\cdot Z_s}ds+\sum_{i=1}^2\gamma_i(x,y)\int_0^t e^{(x,y)\cdot Z_s}dL_s^i.\tag{3.3.2}$$

where $(B_t)_{t\geq 0}=(X_t-\mu t)_{t\geq 0}$ is the non reflected degenerate Brownian motion associated with the process (see Definition 3.5). Next, taking the expectation and letting t to $+\infty$, we derive (3.3.1). See Proposition 2.15, Chapter 2 for a detailed version of the proof in the non-degenerate case. ☐

Considering $\gamma(x,y)$ as a polynomial in x (resp. y) with coefficients depending on y (resp. x), we obtain two complex branches $Y^+(x)$, $Y^-(x)$ (resp. $X^+(y)$, $X^-(y)$) satisfying $\gamma(x,Y^{\pm}(x)) = \gamma(X^{\pm}(y),y) = 0$:

$$Y^{\pm}(x) = x - \mu_2 \pm \sqrt{-2x+\mu_2^2}, \quad X^{\pm}(y) = y - \mu_1 \pm \sqrt{-2y+\mu_1^2}. \tag{3.3.3}$$

We have one branching point $x_{max} = \frac{\mu_2^2}{2} > 0$ $\big($resp. $y_{max} = \frac{\mu_1^2}{2} > 0\big)$ for $Y^{\pm}$ (resp. $X^{\pm}$). The square roots are chosen to be defined as holomorphic functions on $\mathbb{C}\backslash(-\infty,0)$ and take non-negative values on the non-negative reals.

Lemma 3.11. *Let $u,v\in\mathbb{R}$ such that $u+iv\notin[x_{max},+\infty[$. Then, we have:*

$$\Re(Y^{\pm}(u+iv)) = u-\mu_2 \pm \frac{1}{\sqrt{2}}\sqrt{\mu_2^2-2u+\sqrt{(\mu_2^2-2u)^2+4v^2}} \tag{3.3.4}$$

If $u,v\in\mathbb{R}$ satisfy $u+iv\notin[y_{max},+\infty[$, then

$$\Re(X^{\pm}(u+iv)) = u-\mu_1 \pm \frac{1}{\sqrt{2}}\sqrt{\mu_1^2-2u+\sqrt{(\mu_1^2-2u)^2+4v^2}}. \tag{3.3.5}$$

Let $\delta=\min(\mu_1,\mu_2)>0$. Then, $\Re(Y^-(x))<0$ for all x such that $\Re(x)<x_{max}+\delta, x\notin[x_{max},+\infty[$. Similarly, $\Re(X^-(y))<0$ for all y such that $\Re(y)<y_{max}+\delta, y\notin[y_{max},+\infty[$.

Proof. Equations (3.3.4) and (3.3.5) follow directly from the expression (3.1.8) of γ. The last statements come from the inequalities $x_{max}=\frac{\mu_2^2}{2}<\mu_2$ and $y_{max}=\frac{\mu_1^2}{2}<\mu_1$. □

Corollary 3.12 (Continuation of Laplace transforms). *The Laplace transforms φ_1 and φ_2 can be extended as meromorphic functions on $\{y\in\mathbb{C},\Re(y)<y_{max}+\delta\}\backslash[y_{max},y_{max}+\delta]$ and $\{x\in\mathbb{C},\Re(x)<x_{max}+\delta\}\backslash[x_{max},x_{max}+\delta]$ respectively via the formulae:*

$$\varphi_1(y) = \frac{-\gamma_2(X^-(y),y)\varphi_2(X^-(y)) - \exp\big(a_0X^-(y)+b_0y\big)}{\gamma_1(X^-(y),y)} \tag{3.3.6}$$

$$\varphi_2(x) = \frac{-\gamma_1(x,Y^-(x))\varphi_1(Y^-(x)) - \exp\big(a_0x+b_0Y^-(x)\big)}{\gamma_2(x,Y^-(x))}. \tag{3.3.7}$$

Proof. This follows directly from Lemma 3.11 and the functional equation (3.3.1). □

From now on, φ_1 and φ_2 will be considered over their extended domains. Let us define

$$x^* = 2\frac{\mu_2r_2-\mu_1}{(1+r_2)^2}, \qquad y^{**} = 2\frac{\mu_1r_1-\mu_2}{(r_1+1)^2}. \tag{3.3.8}$$

If equation $\gamma_2(x,Y^-(x))=0$ (resp. $\gamma_1(X^-(y),y)=0$) has a solution in the complex plane, then it is unique and is given by $x=x^*$ (resp. $y=y^{**}$). We also define

$$y^* = Y^+(x^*), \quad x^{**} = X^+(y^{**}), \tag{3.3.9}$$

see Figure 3.6.

Proposition 3.13 (Poles of Laplace transform).

(i) $x=0$ (resp. $y=0$) is not a pole of $\varphi_2(x)$ (resp. $\varphi_1(y)$).

(ii) If x (resp. y) is a pole of $\varphi_2(x)$ (resp. $\varphi_1(y)$), then $x=x^*$ (resp. $y=y^{**}$) and $\gamma_1(x^*,Y^-(x^*))=0$ (resp. $\gamma_2(X^-(y^{**}),y^{**})=0$). Furthermore, x^* is a pole of φ_2 (resp. y^{**} is a pole of φ_1) if and only if $(r_2+1)\mu_2>2$ (resp. $(r_1+1)\mu_1>2$).

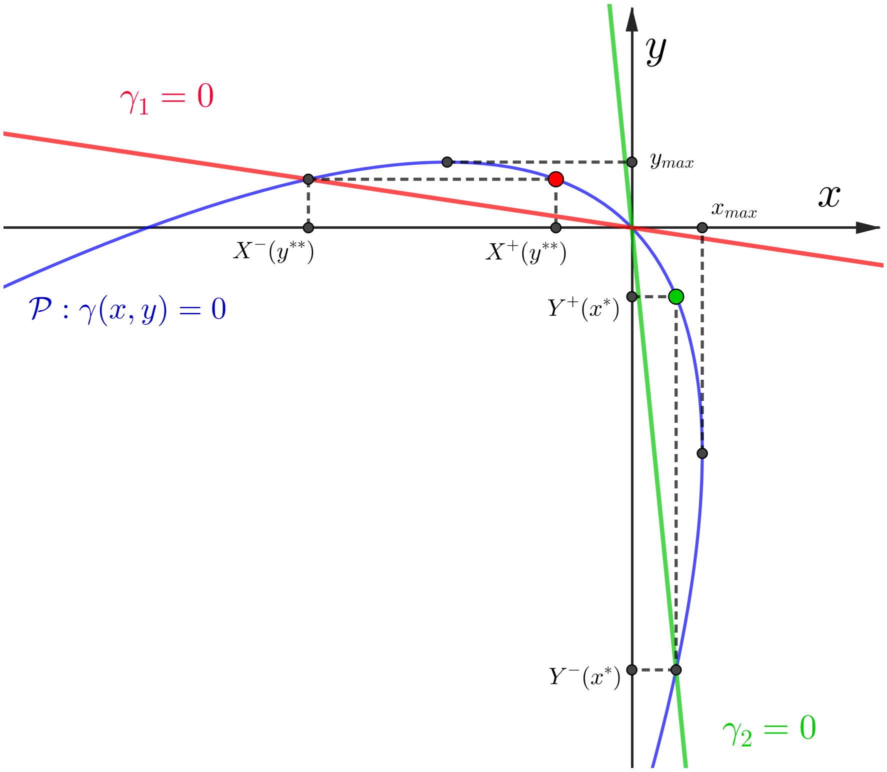


Figure 3.6: In the case of the figure, both φ_1 and φ_2 have a pole.

Proof. The first point follows from the continuation formula (3.3.7) since $\gamma_2(0, Y^-(0)) = -2\mu_2 \neq 0$.

For (ii), if x is a pole of φ_2, it follows from (3.3.7) that $\gamma_2(x, Y^-(x)) = 0$, which implies that $x = x^*$. Moreover, the Laplace transform φ_2 is holomorphic in $\text{Re}(x) < 0$. Thus, x^* being a pole of φ_2, must be positive. Note that equation $\gamma_2(x, Y^-(x)) = 0$ has a positive solution if and only if $\gamma_2(x_{max}, Y^{\pm}(x_{max})) > 0$. This last condition is equivalent to $(r_2 + 1)\mu_2 > 2$.

Let us assume that $(r_2 + 1)\mu_2 > 2$. Then, $x^* > 0$. Since $\gamma_2(x^*, Y^-(x^*)) = 0$, it follows from (3.3.7) that x^* is a pole of φ_2 if the numerator of the right-hand side of (3.3.6) does not vanish at x^*. The last fact holds true, and is actually equivalent to the non-nullity of the function $h_{\alpha^*}(z_0)$ defined in (3.1.24): this equivalence and the non-nullity are postponed to the end of Section 3.6.2.. □

The following proposition provides some estimates for the Laplace transforms. These estimates will be useful in Section 3.5.

Proposition 3.14 (Decay of Laplace transforms on Re $= -\epsilon$)**.** *Let $z_0 = (a_0, b_0) \in \mathbb{R}_+^2$ be an initial condition with $a_0 \neq 0, b_0 \neq 0$ and $\epsilon > 0$. Then there exist constants $c, C > 0$ such that for $l = 1, 2$,*

$$\forall v \in \mathbb{R}, \qquad |\varphi_l^{z_0}(-\epsilon + iv)| \leq Ce^{-c\sqrt{|v|}}. \tag{3.3.10}$$

Proof. The expressions $\gamma_1(-\epsilon+iv, Y^-(-\epsilon+iv))$ and $\gamma_2(-\epsilon+iv, Y^-(-\epsilon+iv))$ grow linearly with respect to v as v tends to $\pm\infty$. Furthermore, expression (3.3.4) provides inequality $\exp(b_0 Y^-(-\epsilon + iv)) \leq C_1 e^{-b_0\sqrt{c_1|v|}}$ for some constants $c_1, C_1 > 0$ and

$$\varphi_1^{z_0}(Y^-(-\epsilon + iv)) \leq \varphi_1^{z_0}(0)e^{(a_0+b_0)Re(Y^-(-\epsilon+iv))} \leq C_2 e^{-c_2\sqrt{|v|}}$$

for some constants $c_2, C_2 > 0$. Finally, equation (3.3.7) implies the conclusion for φ_2. The proof for φ_1 is analogous. □

We also give some further estimates for Laplace transforms that will be useful in Section 3.4.2.

Lemma 3.15 (Decay of Laplace transforms)**.** *Assume that* (3.1.2) *to* (3.1.4) *hold. For any initial point* $z_0 = (a_0, b_0)$ *and* $p \geq 0$,

$$\varphi_2^{z_0}(-p) \leq e^{-p(a_0+b_0)}\varphi_2^{z_0}(0). \tag{3.3.11}$$

The symmetric result holds for $\varphi_1^{z_0}$.

Proof. By (3.1.3), note that the support of the measure $H_2((a_0, b_0), \cdot)$ is $[a_0 + b_0, +\infty)$. Then,

$$\varphi_2^{z_0}(-p) = \int_{a_0+b_0}^{+\infty} e^{-px} f_2^{z_0}(x)dx \leq e^{-p(a_0+b_0)} \int_{a_0+b_0}^{+\infty} f_2^{z_0}(x)dx = e^{-p(a_0+b_0)}\varphi_2^{z_0}(0).$$

□

3.4 The compensation method and the explicit expressions of the Laplace transforms

The heuristic of the compensation method was given in Section 1.2.4.2. We will see in Section 3.6.1 that the harmonic functions we obtain can be written as

$$(x, y) \longmapsto \gamma_1(a_0, b_0)\varphi_1^{(x,y)}(b_0) + \gamma_2(a_0, b_0)\varphi_2^{(x,y)}(a_0) + e^{a_0x+b_0y}.$$

The explicit expressions of φ_1 and φ_2 in Section 3.4.2 then provide the exact formula (1.2.38) suggested by the compensation method. Moreover, the approach of Section 3.6.1 justifies why the harmonic functions given by (1.2.38) are non-negative when (a_0, b_0) is well chosen.

3.4.1 Parabola and automorphisms

Let us recall that $\mathcal{P}$ is the parabola defined by $\mathcal{P} = \{(x, y) \in \mathbb{R}^2, \quad \gamma(x, y) = 0\}$ (see (3.1.10)). Before defining the sequence $((a_n, b_n))_{n\in\mathbb{Z}}$ motivated by Section **??** (see Figure 3.4), we first give a parametrisation of $\mathcal{P}$.

Proposition 3.16 (Parameterisation of $\mathcal{P}$)**.** *The parabola* $\mathcal{P}$ *(see* (3.1.10)*) admits the following parameterisation:*

$$\begin{cases} x(s) &= -\frac{1}{2}s(s - 2\mu_2) \\ y(s) &= -\frac{1}{2}s\,(s + 2\mu_1) \end{cases}, \qquad s \in \mathbb{R}. \tag{3.4.1}$$

This means that $\{(x, y) \in \mathbb{R}^2,\ \gamma(x, y) = 0\} = \{(x(s), y(s)),\ s \in \mathbb{R}\}$.

Proof. The relation $\gamma(x(s), y(s)) = 0$ is easily verified by substituting $x(s), y(s)$ into the expression (3.1.8) of $\gamma(x, y)$. Furthermore, the parameterisation is injective. To show this, assume that

$$\begin{cases} s\,(s - 2\mu_2) = s'\,(s' - 2\mu_2) \\ s\,(s + 2\mu_1) = s'\,(s' + 2\mu_1) \end{cases}.$$

Subtracting the second equation from the first gives $2s(\mu_1 + \mu_2) = 2s'(\mu_1 + \mu_2)$ which implies $s = s'$. Similarly, surjectivity can be verified by elementary considerations. □

To define the "downstairs" as in Figure 3.4, we introduce two transformations on the parabola which leave the first (resp. second) coordinate invariant. This is the aim of the following proposition (which also serves as a definition). This proposition is illustrated by Figure 3.7.

Proposition 3.17 (Automorphisms η, ζ)**.** *For $s \in \mathbb{R}$, we define*

$$\begin{cases} \zeta s = -s + 2\mu_2 \\ \eta s = -s - 2\mu_1. \end{cases} \tag{3.4.2}$$

Then, $x(\zeta s) = x(s)$ and $y(\eta s) = y(s)$ for all $s \in \mathbb{R}$. Therefore, $\varphi_2(x(\zeta s)) = \varphi_2(x(s))$ and $\varphi_1(y(\eta s)) = \varphi_1(y(s))$ in their respective domains of definition. Furthermore, for all $n \in \mathbb{Z}$ and $s \in \mathbb{R}$, we have

$$(\eta\zeta)^n s = s - 2n. \tag{3.4.3}$$

Proof. The formulae $x(-s+2\mu_2) = x(s)$ and $y(-s-2\mu_1) = y(s)$ are easily verified. The expression of $(\eta\zeta)^n$ is a consequence of expressions of η, ζ and of the equation $\mu_1 + \mu_2 = 1$ (see Assumption (3.1.4)). ☐

Note that $\zeta^2 = Id$, $\eta^2 = Id$. By the parameterisation (3.4.1), ζ and η can be regarded as reflections (see (3.4.2)), and their composition as a translation (see (3.4.3)).

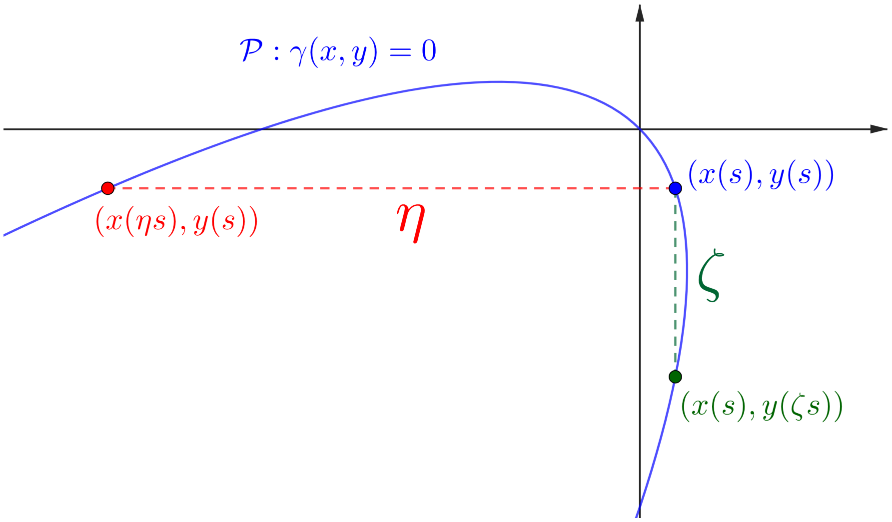


Figure 3.7: Parabola $\mathcal{P}$ and automorphisms η and ζ.

Lemma 3.18 (Explicit form of (a_n, b_n))**.** *Let $s \in \mathbb{R}$ and $(a_0, b_0) = (x(s), y(s))$. For any integer $n \in \mathbb{Z}$, we define*

$$(a_{2n}, b_{2n}) = (x((\eta\zeta)^n s), y((\eta\zeta)^n s)), \quad (a_{2n+1}, b_{2n+1}) = (x(\zeta(\eta\zeta)^n s), y(\zeta(\eta\zeta)^n s))$$

(see Figure 3.4). Then, for any $n \in \mathbb{Z}$ the following expressions hold.

$$a_{2n} = -2n^2 + 2(a_0 - b_0 - \mu_2)n + a_0, \quad a_{2n+1} = a_{2n} \tag{3.4.4}$$

$$b_{2n} = -2n^2 + 2(a_0 - b_0 + \mu_1)n + b_0, \quad b_{2n+1} = b_{2n+2}. \tag{3.4.5}$$

Proof. The invariance of the first and the second coordinate of ζ and η respectively implies equalities $a_{2n+1} = a_{2n}$ and $b_{2n+1} = b_{2n+2}$. The explicit expressions of a_{2n} and b_{2n} are obtained from the explicit expression (3.4.3). ☐

Notation 3.19. *For $s \in \mathbb{R}$, we define*

$$z(s) = (x(s), y(s)) \tag{3.4.6}$$

as the point of the parabola corresponding to the parameter $s \in \mathbb{R}$. *We also define*

$$s_{max} = \mu_2, \quad s_{min} = -\mu_1 \tag{3.4.7}$$

and write $\gamma_i(s)$ *instead of* $\gamma_i(x(s), y(s))$ *for* $i = 1, 2$. *Finally, let* s^* *and* s^{**} *be defined as*

$$s^* = \frac{2}{r_2 + 1}, \quad s^{**} = \frac{-2}{r_1 + 1}. \tag{3.4.8}$$

With these notations,

$$z(s_{max}) = (x_{max}, Y^{\pm}(x_{max})) \quad \text{and} \quad z(s_{min}) = (X^{\pm}(y_{max}), y_{max}).$$

Note that the curve $(z(s))_{s\in[s_{min},s_{max}]}$ is the portion of the parabola from $(x_{max}, Y^+(x_{max}))$ to $(X^+(y_{max}), y_{max})$ going counterclockwise (see Figure 3.6). Furthermore, $x(s^*) = x^*$ and $y(s^{**}) = y^{**}$ with definition (3.3.8). We can now provide explicit expressions for the Laplace transforms φ_1 and φ_2.

3.4.2 Explicit expressions of the Laplace transforms via the compensation approach

Theorem 3.20 (Explicit expressions for Laplace transforms)**.** *Let* $z_0 \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ *be the initial condition. Then, for any* $s \in (\max(s_{min}, s^{**}), \min(s_{max}, s^*))$,

$$\varphi_2^{z_0}(x(s)) = \frac{-1}{\gamma_2(\zeta s)} e^{z_0\cdot z(\zeta s)} + \sum_{n=1}^{+\infty} \left[\prod_{k=0}^{n-1} G(s-2k)\right] \left[\frac{e^{z_0\cdot z(s-2n)}}{\gamma_2(s-2n)} - \frac{e^{z_0\cdot z(\zeta(s-2n))}}{\gamma_2(\zeta(s-2n))}\right] \tag{3.4.9}$$

where

$$G(s) = \frac{\frac{\gamma_1}{\gamma_2}(\zeta s)}{\frac{\gamma_1}{\gamma_2}(s-2)}. \tag{3.4.10}$$

Similarly, for all $s \in (\max(s_{min}, s^{**}), \min(s_{max}, s^*))$

$$\varphi_1^{z_0}(y(s)) = \frac{-1}{\gamma_1(\eta s)} e^{z_0\cdot z(\eta s)} + \sum_{n=1}^{+\infty} \left[\prod_{k=0}^{n-1} \widetilde{G}(s+2k)\right] \left[\frac{e^{z_0\cdot z(s+2n)}}{\gamma_1(s+2n)} - \frac{e^{z_0\cdot z(\eta(s+2n))}}{\gamma_1(\eta(s+2n))}\right] \tag{3.4.11}$$

where

$$\widetilde{G}(s) = \frac{\frac{\gamma_2}{\gamma_1}(\eta s)}{\frac{\gamma_2}{\gamma_1}(s+2)}.$$

Before proving Theorem 3.20, we establish a technical lemma.

Lemma 3.21. *For all* $n \geq 1$ *and* $s \in (\max(s_{min}, s^{**}), \min(s_{max}, s^*))$, *we have* $\gamma_1(s-2n) \neq 0$ *and* $\gamma_2(\zeta(s-2n)) \neq 0$. *Furthermore,* $\gamma_2(s), \gamma_2(\zeta s), \gamma_1(s), \gamma_1(\eta s)$ *are also non-zero.*

Proof. We define two portions of the parabola E^+ and E^- given by

$$E^+ = \{(x, Y^+(x)),\ x \leq X^+(0)\} \quad \text{and} \quad E^- = \{(X^-(y)), y),\ y \leq Y^-(0)\}.$$

By Assumption (3.1.3), the line $\{\gamma_2 = 0\}$ (resp. $\{\gamma_1 = 0\}$) cannot pass through E^- (resp. E^+). Additionally, note that $\eta(E^-) \subset E^+$ and $\zeta(E^+) \subset E^-$. Since $s \in (s_{min}, s_{max})$, $z((\eta\zeta)^n s) = z(s-2n)$ belongs to E^+ for all $n \geq 1$. Thus, $\gamma_2(\zeta(s-2n)) \neq 0$ for any $n \geq 0$. By similar reasoning, $\gamma_1(s-2n) \neq 0$ for any $n \geq 0$. The last statement comes from the fact that $s \in (s^{**}, s^*)$. □

Proof of Theorem 3.20. The main idea of the proof is to get a recursive formula for Laplace transforms. To do this, we rewrite the functional equation (3.3.1) in $z(\zeta s)$ and $z(\eta\zeta s) = z(s-2)$, which holds because $x(\zeta s), y(\zeta s), x(s-2)$ and $y(s-2)$ are negative:

$$\begin{cases} 0 = \gamma_1(\zeta s)\varphi_1(y(\zeta s)) + \gamma_2(\zeta s)\varphi_2(x(\zeta s)) + e^{z_0\cdot z(\zeta s)} \\ 0 = \gamma_1(s-2)\varphi_1(y(s-2)) + \gamma_2(s-2)\varphi_2(x(s-2)) + e^{z_0\cdot z(s-2)}. \end{cases}$$

By the invariance of φ_2 (resp. φ_1) under ζ (resp. η), we have $\varphi_2(x(\zeta s)) = \varphi_2(x(s))$ and $\varphi_1(y(s-2)) = \varphi_1(y(\zeta s))$. Then, by eliminating $\varphi_1(y(\zeta s))$ from the equations (which is possible by Lemma 3.21), we obtain:

$$\begin{aligned} \varphi_2(x(s)) &= \frac{\frac{\gamma_1}{\gamma_2}(\zeta s)}{\frac{\gamma_1}{\gamma_2}(s-2)}\varphi_2(x(s-2)) + \left[\frac{\frac{\gamma_1}{\gamma_2}(\zeta s)e^{z(s-2)\cdot z_0}}{\gamma_1(s-2)} - \frac{e^{z(\zeta s)\cdot z_0}}{\gamma_2(\zeta s)}\right] && (3.4.12) \\ &= G(s)\varphi_2(x(s-2)) + \left[\frac{G(s)}{\gamma_2(s-2)}e^{z_0\cdot z(s-2)} - \frac{e^{z_0\cdot z(\zeta s)}}{\gamma_2(\zeta s)}\right]. && (3.4.13) \end{aligned}$$

Similarly, we get:

$$\varphi_2(x(s-2)) = G(s-2)\varphi_2(x(s-4)) + \left[\frac{G(s-2)}{\gamma_2(s-4)}e^{z_0\cdot z(s-4)} - \frac{e^{z_0\cdot z(\zeta(s-2))}}{\gamma_2(\zeta(s-2))}\right]. \tag{3.4.14}$$

Substituting this into (3.4.13), we get

$$\varphi_2(x(s)) = G(s)G(s-2)\varphi_2(x(s-4)) + G(s)G(s-2)\frac{e^{z_0\cdot z(s-4)}}{\gamma_2(s-4)} - G(s)\frac{e^{z_0\cdot z(\zeta(s-2))}}{\gamma_2(\zeta(s-2))} + \frac{G(s)}{\gamma_2(s-2)}e^{z_0\cdot z(s-2)} - \frac{e^{z_0\cdot z(\zeta s)}}{\gamma_2(\zeta s)}.$$

Then, by induction on N, we obtain the following equality for all $N \geq 1$:

$$\begin{aligned} \varphi_2(x(s)) = \left[\prod_{k=0}^{N} G(s-2k)\right]\varphi_2(x(s-2(N+1))) - \frac{e^{z_0\cdot z(\zeta s)}}{\gamma_2(\zeta s)} + \left[\prod_{k=0}^{N} G(s-2k)\right]\frac{e^{z_0\cdot z(s-2(N+1))}}{\gamma_2(s-2(N+1))} \\ + \sum_{n=1}^{N}\left[\prod_{k=0}^{n-1} G(s-2k)\right]\left[\frac{e^{z_0\cdot z(s-2n)}}{\gamma_2(s-2n)} - \frac{e^{z_0\cdot z(\zeta(s-2n))}}{\gamma_2(\zeta(s-2n))}\right] \end{aligned} \tag{3.4.15}$$

The proof is then reduced to proving the following limit:

$$\left[\prod_{k=0}^{n} G(s-2k)\right]\varphi_2(x(s-2(n+1))) \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.4.16}$$

To justify this, note using formula (3.4.10) and Lemma 3.18 that:

$$G(s-2k) = \frac{(k+a)(k+b)}{(k+c)(k+d)}. \tag{3.4.17}$$

for some constants a, b, c, d defined by:

$$a = \frac{-s}{2} + \frac{r_1}{1+r_1}, \quad b = 1 - \frac{s}{2} + \frac{\mu_2 r_2 - \mu_1}{1+r_2}$$

$$c = \frac{-s}{2} + \frac{1}{1+r_2}, \quad d = 1 - \frac{s}{2} + \frac{\mu_2 - \mu_1 r_1}{1+r_1}.$$

By elementary considerations, the following asymptotic behavior holds:

$$\left[\prod_{k=0}^{n} G(s+2k)\right] \underset{n\to\infty}{\sim} Cn^{a-c+b-d} \tag{3.4.18}$$

where C is a real constant. Moreover,

$$a + b - c - d = 2 - 2\left(\frac{1}{1+r_1} + \frac{1}{1+r_2}\right) \tag{3.4.19}$$

since $\mu_1 + \mu_2 = 1$. Then, the exponential decay in (3.3.11) for φ_2, together with the polynomial rate of expression (3.4.18), yields (3.4.9). Note that inequality (3.3.11) is the only (and crucial) reason why we work under Assumption (3.1.3). Equation (3.4.11) is obtained with symmetric arguments. □

Remark 3.22. The exponent given by (3.4.18) is exactly the parameter -2γ introduced in [35], which determines the algebraic nature of the Laplace transforms for the same degenerate particle model in the recurrent case. Furthermore, the constants $\kappa_m = \kappa_m(\alpha)$ in (3.1.22) satisfy

$$\kappa_m \underset{m\to\pm\infty}{\sim} C_{\pm} m^{-2\gamma-2}. \tag{3.4.20}$$

for some constant $C_{\pm} > 0$ where $-2\gamma - 2 < 0$ by (3.1.3).

In (3.4.9) (resp. (3.4.11)) φ_2 (resp. φ_1) is not given as a function of x (resp. y) but of s. We therefore establish the following corollary.

Corollary 3.23. *The following expressions hold in the domains $Re(x) < x_{max}$ and $Re(y) < y_{max}$, respectively:*

$$\begin{aligned}\varphi_2(x) = {} & \frac{-1}{\gamma_2(x, Y^-(x))} e^{z_0\cdot(x,Y^-(x))} \\ & + \sum_{n=1}^{+\infty}\left[\prod_{k=1}^{n} \frac{\frac{\gamma_1}{\gamma_2}(\psi_{2k-1}(x, Y^+(x))}{\frac{\gamma_1}{\gamma_2}(\psi_{2k}(x, Y^+(x)))}\right]\left[\frac{e^{z_0\cdot\psi_{2n}(x,Y^+(x))}}{\gamma_2(\psi_{2n}(x, Y^+(x))} - \frac{e^{z_0\cdot\psi_{2n+1}(x,Y^+(x))}}{\gamma_2(\psi_{2n+1}(x, Y^+(x))}\right]\end{aligned} \tag{3.4.21}$$

$$\begin{aligned}\varphi_1(y) = {} & \frac{-1}{\gamma_1(X^+(y), y)} e^{z_0\cdot(X^+(y),y)} \\ & + \sum_{n=1}^{+\infty}\left[\prod_{k=1}^{n} \frac{\frac{\gamma_2}{\gamma_1}(\psi_{-2k+1}(X^+(y), y))}{\frac{\gamma_2}{\gamma_1}(\psi_{-2k}(X^+(y), y))}\right]\left[\frac{e^{z_0\cdot\psi_{-2n}(X^+(y),y))}}{\gamma_1(\psi_{-2n}(X^+(y), y))} - \frac{e^{z_0\cdot\psi_{-2n+1}(X^+(y),y)}}{\gamma_1(\psi_{-2n+1}(X^+(y), y))}\right]\end{aligned} \tag{3.4.22}$$

where

$$\psi_{2n}(a, b) = \left(-2n^2 + 2(a - b - \mu_2)n + a, -2n^2 + 2(a - b + \mu_1)n + b\right)$$

and

$$\psi_{2n+1}(a, b) = \left(-2n^2 + 2(a - b - \mu_2)n + a, -2(n+1)^2 + 2(a - b + \mu_1)(n+1) + b\right).$$

Proof. By Lemma 3.18 and equalities $z(s) = (x(s), Y^+(x(s)) = (X^+(y(s)), y(s))$ for $s \in (s_{min}, s_{max})$, equations (3.4.21) and (3.4.22) hold on the curve $\{(x, y) = (x(s), y(s)) : s \in \left((\max(s_{min}, s^{**}), \min(s_{max}, s^*))\right)\}$. By Corollary 3.12, Laplace transforms $\varphi_2(x)$ and $\varphi_1(y)$ are meromorphic on $Re(x) < x_{max}$ and $Re(y) < y_{max}$ respectively. Consequently, the explicit expressions (3.4.21) and (3.4.22) remain valid in these domains. □

3.5 Laplace inverse and saddle-point method

To avoid certain technical complications, we first derive the asymptotic behavior of the Green functions g^{z_0} for $z_0 \neq 0$, and later address the case $z_0 = 0$ with additional arguments.

3.5.1 Inverse Laplace theorem and saddle-point.

Let $z_0 \neq (0,0)$ be a starting point of the process. The inverse Laplace transform formula (see [33, Theorem 24.3 and 24.4] and [16]) yields the following representation for $g^{z_0}(a,b)$: for $\epsilon > 0$ sufficiently small,

$$g^{z_0}(a,b) = \frac{1}{(2\pi i)^2} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi^{z_0}(x,y) \exp(-ax-by) dx dy \tag{3.5.1}$$

where the convergence is in the sense of principal value. This can be justified by the functional equation (3.3.1) and the decay properties of the Laplace transforms established in Proposition 3.14.

Lemma 3.24 (From double to simple integrals)**.** *Denote by* $z_0 = (a_0, b_0)$ *the starting point of the process. Then, for any* $(a,b) \in \mathbb{R}_+^2$ *satisfying* $a > 0$ *or* $b > 0$,

$$g(a,b) = I_1(a,b) + I_2(a,b) + I_3(a,b)$$

where

$$I_1(a,b) = \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi_2(x)\gamma_2(x, Y^+(x)) \exp(-ax - bY^+(x)) \frac{dx}{\partial_y \gamma(x, Y^+(x))},$$

$$I_2(a,b) = \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi_1(y)\gamma_1(X^+(y), y) \exp(-aX^+(y) - by) \frac{dy}{\partial_x \gamma(X^+(y), y)},$$

$$I_3(a,b) = \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \exp(a_0 x + b_0 Y^+(x)) \exp(-ax - bY^+(x)) \frac{dx}{\partial_y \gamma(x, Y^+(x))} \quad \text{if } b > b_0,$$

$$I_3(a,b) = \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \exp(a_0 X^+(y) + b_0 y) \exp(-aX^+(y) - by) \frac{dy}{\partial_x \gamma(X^+(y), y)} \quad \text{if } a > a_0.$$

Proof. By the functional equation (3.3.1), $\varphi(x,y)$ can be decomposed as:

$$\varphi(x,y) = -\frac{\gamma_1(x,y)\varphi_1(y)}{\gamma(x,y)} - \frac{\gamma_2(x,y)\varphi_2(x)}{\gamma(x,y)} - \frac{e^{(x,y)\cdot z_0}}{\gamma(x,y)}. \tag{3.5.2}$$

Substituting this expression into the double integral (3.5.1), $g^{z_0}(a,b)$ is written as the sum of three double integrals. Let us consider the first term, given by

$$\frac{-1}{(2\pi i)^2} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi_2(x) \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{\gamma_2(x,y)}{\gamma(x,y)} e^{-ax-by}\, dy\, dx. \tag{3.5.3}$$

Let C_R be the closed oriented contour defined by

$$C_R = \{-\epsilon + it \mid t \in [-R, R]\} \cup \{-\epsilon + Re^{-i\theta} \mid \theta \in [-\pi/2, \pi/2]\}.$$

By applying the residue theorem along the contour C_R and considering the asymptotics as $R \to +\infty$ (see Lemma 2.27, Chapter 2 for more details), we obtain the identity

$$\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{\gamma_2(x,y)}{\gamma(x,y)} e^{-ax-by}\, dy = \frac{\gamma_2(x, Y^+(x))}{\partial_y \gamma(x, Y^+(x))} e^{-ax-bY^+(x)},$$

so that expression (3.5.3) equals $I_1(a,b)$. The remaining terms are handled analogously.

□

To find the asymptotics of these integrals as $a, b \to +\infty$, we use the saddle-point method. For any $\alpha \in [0, \pi/2]$, let $(x(\alpha), y(\alpha))$ be defined as

$$(x(\alpha), y(\alpha)) = \operatorname{argmax}_{(x,y)\in\mathcal{P}} \left(\cos(\alpha)x + \sin(\alpha)y\right), \tag{3.5.4}$$

see Figure 3.3a. For $\alpha \in [0, \pi/2]$, we define the real number $\mathfrak{s}(\alpha) \in \mathbb{R}$ by

$$\mathfrak{s}(\alpha) = \operatorname{argmax}_{s\in\mathbb{R}}(\cos(\alpha)x(s) + \sin(\alpha)y(s)).$$

Note that $(x(\alpha), y(\alpha)) = (x(\mathfrak{s}(\alpha)), y(\mathfrak{s}(\alpha)))$, using notation (3.4.1). By studying the variations of the function $s \longmapsto x(s)\cos(\alpha) + y(s)\sin(\alpha)$, we prove that

$$\mathfrak{s} \colon \begin{cases} [0, \pi/2] \longrightarrow [s_{min}, s_{max}] \\ \quad \alpha \quad \longmapsto \dfrac{\mu_2 - \tan(\alpha)\mu_1}{1 + \tan(\alpha)} \end{cases} \qquad (\text{with} \quad \mathfrak{s}(\pi/2) = -\mu_1 = s_{min}) \tag{3.5.5}$$

is a C^∞ diffeomorphism, and

$$\mathfrak{s}^{-1} \colon \begin{cases} [s_{min}, s_{max}] \longrightarrow [0, \pi/2] \\ \quad s \quad \longmapsto \arctan\left(\dfrac{\mu_2 - s}{s + \mu_1}\right) \end{cases} \qquad (\text{with} \quad \mathfrak{s}^{-1}(-\mu_1) = \pi/2). \tag{3.5.6}$$

Using the definitions of $\alpha^*, \alpha^{**}, x^*$ and y^{**} given by (3.1.13), (3.1.14) and (3.3.8), if x^* (resp. y^*) is a pole of φ_2 (resp. φ_1), then $x(\alpha^*) = x^*$ (resp. $y(\alpha^{**}) = y^{**}$). Since $s^{**} < 0 < s^*$ (see Notation 3.19), then the monotonicity of (3.5.6) implies that $0 \leq \alpha^* < \alpha_\mu < \alpha^{**} \leq \pi/2$, where $\alpha_\mu = \arctan(\mu_2/\mu_1) \in (0, \pi/2)$ is the angle of the drift. We follow the notation of Chapter 2 and define:

$$F(x, \alpha) = -\cos(\alpha)x - \sin(\alpha)Y^+(x) + \cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha) \tag{3.5.7}$$

$$G(y, \alpha) = -\cos(\alpha)X^+(y) - \sin(\alpha)y + \cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha). \tag{3.5.8}$$

By construction, the equations $\partial_x F(x(\alpha), \alpha) = 0$ and $\partial_y G(y(\alpha), \alpha) = 0$ hold. Then, by differentiating equations (3.5.7) and $\gamma(x, Y^+(x)) = 0$, we get for any $\alpha \in (0, \pi/2]$,

$$(Y^+(x))' \Big|_{x=x(\alpha)} = -\frac{\partial_x\gamma(x(\alpha), y(\alpha))}{\partial_y\gamma(x(\alpha), y(\alpha))} = -\frac{\cos(\alpha)}{\sin(\alpha)}, \quad (Y^+(x))'' \Big|_{x=x(\alpha)} = -\frac{(1 + \operatorname{ctan}(\alpha))^2}{\partial_y\gamma(x(\alpha), y(\alpha))}. \tag{3.5.9}$$

Therefore,

$$\partial^2_{xx}F(x(\alpha), \alpha) = \frac{(\sin(\alpha) + \cos(\alpha))^2}{\partial_y\gamma(x(\alpha), y(\alpha))\sin(\alpha)} > 0, \quad \alpha \in (0, \pi/2]. \tag{3.5.10}$$

Similarly,

$$\partial^2_{yy}G(y(\alpha), \alpha) = \frac{(\sin(\alpha) + \cos(\alpha))^2}{\partial_x\gamma(x(\alpha), y(\alpha))\cos(\alpha)} > 0, \quad \alpha \in [0, \pi/2).$$

3.5.2 Contour of steepest descent

Let $\alpha_0 \in (0, \pi/2]$. The key idea of the saddle-point method is to use the parameterised Morse lemma. Since $\partial^2_{xx}F(x(\alpha), \alpha) > 0$, Lemma 2.A.1 yields some $\epsilon > 0, \eta > 0$ and a family of smooth paths $\Gamma_{x,\alpha} = \{x(it, \alpha) \mid t \in [-\epsilon, \epsilon]\}, |\alpha - \alpha_0| < \eta$ such that

$$\forall t \in [-\epsilon, \epsilon], \qquad F(x(it, \alpha), \alpha) = -t^2. \tag{3.5.11}$$

Define

$$x^+_\alpha = x(i\epsilon, \alpha), \quad x^-_\alpha = x(-i\epsilon, \alpha). \tag{3.5.12}$$

In particular,

$$F(x_\alpha^+,\alpha) = -\epsilon^2, \quad F(x_\alpha^-,\alpha) = -\epsilon^2. \tag{3.5.13}$$

Furthermore, $Im(x_\alpha^+) > 0$ and $Im(x_\alpha^-) < 0$ (see Figure 3.8 and construction in Section 2.5, Chapter 2). The same construction holds for $\Gamma_{y,\alpha} = \{y(it,\alpha) \mid t \in [-\epsilon,\epsilon]\}$ for G and $\alpha_0 \in [0,\pi/2)$. These paths satisfy

$$\Gamma_{x,\alpha} = \overleftarrow{\underrightarrow{X^+(\Gamma_{y,\alpha})}} \quad \text{and} \quad \Gamma_{y,\alpha} = \overleftarrow{\underrightarrow{Y^+(\Gamma_{x,\alpha})}}, \quad 0 < \alpha < \pi/2.$$

The arrows above and below the paths indicate reversed orientations, this notation is taken from [41, Chapter 5.3].

3.5.3 Shift of the integration contours and contribution of the poles

We now apply the saddle-point method. To do this, we shift the integration contours of I_1, I_2 and I_3 to contours passing through the saddle-point and following the steepest descent contours $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$. We define $T_{x,\alpha} = S_{x,\alpha}^- + \Gamma_{x,\alpha} + S_{x,\alpha}^+$ and $T_{y,\alpha} = S_{y,\alpha}^- + \Gamma_{y,\alpha} + S_{y,\alpha}^+$ for $\alpha \in [0,\pi/2]$ where

$$S_{x,\alpha}^+ = \{x_\alpha^+ + it \mid t \geq 0\}, \quad S_{x,\alpha}^- = \{x_\alpha^- - it \mid t \geq 0\},$$

$$S_{y,\alpha}^+ = \{y_\alpha^+ + it \mid t \geq 0\}, \quad S_{y,\alpha}^- = \{y_\alpha^- - it \mid t \geq 0\}.$$

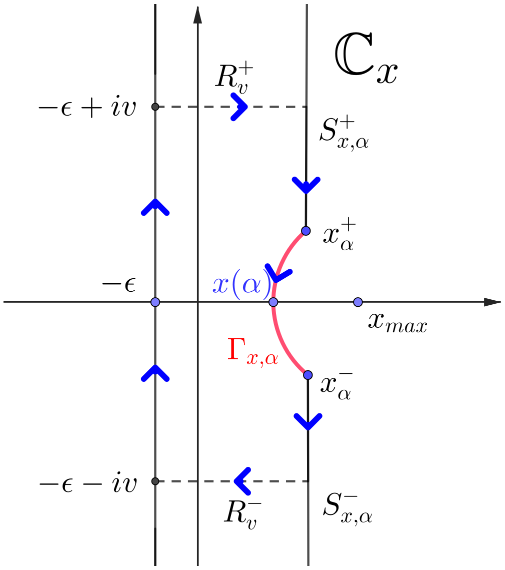


Figure 3.8: Changing path for I_2. Here, $x(\alpha) < x^*$.

Lemma 3.25 (Contour deformation and contribution of the pole)**.** *Let* $\alpha \in [0,\pi/2]\setminus\{\alpha^*,\alpha^{**}\}$ *and* $z_0 \neq (0,0)$ *be the initial condition of the process. Then for any* $a, b > 0$*,*

$$I_1(a,b) = \frac{\big(-\text{res}_{x=x^*}\varphi_2(x)\big)\gamma_2(x^*,y^*)}{\partial_y\gamma(x^*,y^*)}\exp(-ax^*-by^*)\mathbf{1}_{\alpha<\alpha^*} \tag{3.5.14}$$

$$+\frac{1}{2\pi i}\int_{T_{x,\alpha}} \frac{\varphi_2(x)\gamma_2(x,Y^+(x))}{\partial_y\gamma(x,Y^+(x))}\exp(-ax-bY^+(x))dx,$$

$$I_2(a,b) = \frac{\big(-\text{res}_{y=y^{**}}\varphi_1(y)\big)\gamma_1(x^{**},y^{**})}{\partial_x\gamma(x^{**},y^{**})}\exp(-ax^{**}-by^{**})\mathbf{1}_{\alpha>\alpha^{**}} \tag{3.5.15}$$

$$+\frac{1}{2\pi i}\int\limits_{T_{y,\alpha}} \frac{\varphi_1(y)\gamma_1(X^+(y),y)}{\partial_x\gamma(X^+(y),y)}\exp(-aX^+(y)-by)dy,$$

$$I_3(a,b)=\frac{1}{2\pi i}\int\limits_{T_{x,\alpha}} \exp((a_0-a)x+(b_0-b)Y^+(x))\frac{dx}{\partial_y\gamma(x,Y^+(x))} \quad \textit{if } b>b_0, \tag{3.5.16}$$

$$I_3(a,b)=\frac{1}{2\pi i}\int\limits_{T_{y,\alpha}} \exp((a_0-a)X^+(y)+(b_0-b)y)\frac{dy}{\partial_x\gamma(X^+(y),y)} \quad \textit{if } a>a_0. \tag{3.5.17}$$

Proof. The shift of the path is illustrated in Figure 3.8 and is the same as in Lemma 2.28, Chapter 2. The proof of (3.5.14) is a direct consequence of the residue theorem, provided that the integrals over the horizontal contours R_v^+ and R_v^- tend to 0 as v tends to $+\infty$. Then, it remains to prove that for any sufficiently small $\eta>0$,

$$\sup_{u\in[X^+(y_{max})-\eta,x^{max}+\eta]}\left|\frac{\varphi_2(u+iv)\gamma_2(u+iv,Y^+(u+iv))}{\gamma_y'(u+iv,Y^+(u+iv))}\exp(-a(u+iv)-bY^+(u+iv))\right|\to 0, \quad \text{as } v\to\infty.$$

By the functional equation (3.3.1) and continuation formula (3.3.7), the term inside the supremum is equal to

$$\left|\frac{\Big(\gamma_1(u+iv,Y^-(u+iv))\varphi_1(Y^-(u+iv))+e^{a_0(u+iv)+b_0Y^-(u+iv)}\Big)\gamma_2(u+iv,Y^+(u+iv))}{\gamma_2(u+iv,Y^-(u+iv))\gamma_y'(u+iv,Y^+(u+iv))}\right|$$

$$\times|\exp(-a(u+iv)-bY^+(u+iv))|.$$

By (3.3.3), $Re(Y^\pm(u+iv))$ grows like $\pm\sqrt{|v|}$ uniformly in $u\in[X^+(y_{max})-\eta,x^{max}+\eta]$ as $|v|\to+\infty$. Furthermore, $\gamma_2(u+iv,Y^\pm(u+iv))$ grows linearly in v uniformly in $u\in[X^+(y_{max})-\eta,x^{max}+\eta]$ as $v\to+\infty$ by Assumption (3.1.3). The same asymptotics hold for $\gamma_1(u+iv,Y^-(u+iv))$. Moreover, $\partial_y\gamma(u+iv,Y^+(u+iv))=\sqrt{-2(u+iv)+\mu_2^2}$, so this expression grows with rate $\sqrt{v}$, uniformly in $u\in[X^+(y_{max})-\eta,x^{max}+\eta]$. Considering the exponential decay of φ_1 (see Lemma 3.15) we get the conclusion for I_1. Formulae for I_2 and I_3 are obtained similarly. ☐

3.5.4 Negligibility of some integrals

For any pair $(a,b)\in\mathbb{R}_+^2$ let $\alpha(a,b)$ be the angle in $[0,\pi/2]$ such that $\cos(\alpha)=\frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha)=\frac{b}{\sqrt{a^2+b^2}}$. We now aim to evaluate the asymptotics of the integrals over $T_{x,\alpha}^\pm$ and $T_{y,\alpha}^\pm$ in Lemma 3.25 as $\sqrt{a^2+b^2}\to+\infty$ and $\alpha(a,b)\to\alpha_0$ for some $\alpha_0\in[0,\pi/2]$. In the next lemma, we establish exponential bounds for the integrals over the vertical contours $S_{x,\alpha}^\pm$, $S_{y,\alpha}^\pm$. These bounds imply that the main contribution to the above asymptotics comes from the integrals over the steepest descent contours $\Gamma_{x,\alpha}$, $\Gamma_{y,\alpha}$, while those over $S_{x,\alpha}^\pm$ and $S_{y,\alpha}^\pm$ turn out to be negligible.

Lemma 3.26 (Negligibility of the integrals over $S_{x,\alpha}^\pm$ and $S_{y,\alpha}^\pm$)**.** *Suppose $z_0\neq(0,0)$. Let K be a compact neighbourhood of z_0 in the quadrant satisfying $d((0,0),K)>0$. Let $\alpha_0\in[0,\pi/2]$. Then, for sufficiently small $\eta>0$, there exist constants $r_0>0$ and $D_{\alpha_0}>0$ such that for any $z\in K$ and any pair (a,b) satisfying $\sqrt{a^2+b^2}>r_0$ and $|\alpha(a,b)-\alpha_0|<\eta$, the following inequalities hold*

$$\Bigg|\int\limits_{S_{x,\alpha}^\pm}\frac{\varphi_2^z(x)\gamma_2(x,Y^+(x))}{\partial_y\gamma(x,Y^+(x))}\exp\big(-ax-bY^+(x)\big)dx\Bigg|\leq D_{\alpha_0}\exp\Big(-ax(\alpha)-by(\alpha)-\epsilon\sqrt{a^2+b^2}\Big), \tag{3.5.18}$$

$$\Bigg|\int\limits_{S_{y,\alpha}^\pm}\frac{\varphi_1^z(y)\gamma_1(X^+(y),y)}{\partial_x\gamma(X^+(y),y)}\exp\big(-aX^+(y)-by\big)dy\Bigg|\leq D_{\alpha_0}\exp\Big(-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+b^2}\Big). \tag{3.5.19}$$

If $b > b_0$,

$$\Big| \int\limits_{S^{\pm}_{x,\alpha}} \exp((a_0-a)x+(b_0-b)Y^{+}(x))\frac{dx}{\partial_y\gamma(x,Y^{+}(x))}\Big| \leq \frac{D_{\alpha_0}}{b-b_0}\exp\left(-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+(b-b_0)^2}\right). \tag{3.5.20}$$

If $a > a_0$,

$$\Big| \int\limits_{S^{\pm}_{y,\alpha}} \exp((a_0-a)X^{+}(y)+(b_0-b)y)\frac{dy}{\partial_x\gamma(X^{+}(y),y)}\Big| \leq \frac{D_{\alpha_0}}{a-a_0}\exp\left(-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{(a-a_0)^2+b^2}\right). \tag{3.5.21}$$

Proof. We start by showing (3.5.18). Using notations (3.5.7) and (3.5.12), this inequality can be rewritten as

$$\Big| \int\limits_{v>0} \frac{\varphi_2^z(x^+_\alpha+iv)\gamma_2(x^+_\alpha+iv,Y^{+}(x^+_\alpha+iv))}{\partial_y\gamma(x^+_\alpha+iv,Y^{+}(x^+_\alpha+iv))}\exp\left(-aiv-b\left(Y^{+}(x^+_\alpha+iv)-Y^{+}(x^+_\alpha)\right)\right)dx\Big| \leq D_{\alpha_0} \tag{3.5.22}$$

where $\alpha=\alpha(a,b)$.

Suppose first that $\alpha_0>0$. Let $\alpha>0$ and $0<\eta<\alpha_0/2$. Since $\partial_y\gamma(x^+_\alpha+iv,Y^{+}(x^+_\alpha+iv)) = \sqrt{-2(x^+_\alpha+iv)+\mu_2^2}$, this expression does not vanish and grows at rate $\sqrt{|v|}$ as $v\to+\infty$, uniformly in α with $|\alpha-\alpha_0|<\eta$. Similarly, $\gamma_2(x^+_\alpha+iv,Y^{+}(x^+_\alpha+iv))$ grows with speed $|v|$ as $v\to+\infty$, uniformly in α, $|\alpha-\alpha_0|<\eta$. Then we have, for all $v\geq 0$,

$$\sup_{|\alpha-\alpha_0|<\eta}\frac{\gamma_2(x^+_\alpha+iv,Y^{+}(x^+_\alpha+iv))}{\partial_y\gamma(x^+_\alpha+iv,Y^{+}(x^+_\alpha+iv))}\leq C_\eta(1+\sqrt{v})$$

for some constant $C_\eta>0$. If $|\alpha-\alpha_0|<\eta$, then by (3.3.4), there exists a constant $C'_\eta>0$ such that

$$Re\left(\sqrt{a^2+b^2}\left(F(x^+_\alpha+iv,\alpha)-F(x^+_\alpha,\alpha)\right)\right)=b\left(Re(Y^{+}(x^+_\alpha+iv))-Re(Y^{+}(x^+_\alpha))\right)\geq C'_\eta b\sqrt{v} \tag{3.5.23}$$

for any $v\geq 1$. Furthermore, using the continuation formula (3.3.7), the estimates (3.3.11) and the continuity of $\varphi_2^{z_0}(0)$ in z_0 (see (3.4.21)), there exists a constant D such that $|\varphi_2^z(x^+_\alpha+iv)|\leq D$ for all $v\geq 0$, $z\in K$ and $|\alpha-\alpha_0|<\eta$. Then, the left-hand side of (3.5.22) is bounded by

$$DC_\eta C'_\eta\left(2+\int\limits_{v>1}(1+\sqrt{v})e^{-b\sqrt{v}}dv\right)=DC_\eta C'_\eta\left(2+\frac{1}{b^2}+\frac{4}{b^3}\right)\leq D_{\alpha_0}$$

for some constant $D_{\alpha_0}>0$ since $b\to+\infty$ (because $\alpha_0>0$). This inequality implies (3.5.18).

Now suppose that $\alpha_0=0$. We no longer use estimate (3.5.23), as it would produce terms of order $\frac{1}{b}$, and here b may be close to zero. Let $z=(a_1,b_1)\in K$. We write continuation formula (3.3.7) for $\varphi_2^z(x^+_\alpha+iv)$, which splits into two terms:

$$\varphi_2^z(x^+_\alpha+iv)=-\frac{\gamma_1(x^+_\alpha+iv,Y^{-}(x^+_\alpha+iv))\varphi_1^z(Y^{-}(x^+_\alpha+iv))}{\gamma_2(x^+_\alpha+iv,Y^{-}(u+iv))}-\frac{e^{a_1(x^+_\alpha+iv)+b_1Y^{-}(x^+_\alpha+iv)}}{\gamma_2(x^+_\alpha+iv,Y^{-}(u+iv))}, \tag{3.5.24}$$

and we substitute into the right-hand side of (3.5.22). Then, the integral (3.5.22) can be written as the sum of two terms. For the first term, note that there are some constants, $c, C_0>0$ independant on $\alpha\in[0,\eta]$ and $z\in K$, such that

$$\left|\frac{\gamma_2(x^+_\alpha+iv,Y^{+}(x^+_\alpha+iv))\gamma_1(x^+_\alpha+iv,Y^{-}(x^+_\alpha+iv))\varphi_1^z(Y^{-}(x^+_\alpha+iv))}{\gamma_2(x^+_\alpha+iv,Y^{-}(u+iv))\partial_y\gamma(x^+_\alpha+iv,Y^{+}(x^+_\alpha+iv))}\exp\left(-aiv-b\left(Y^{+}(x^+_\alpha+iv)-Y^{+}(x^+_\alpha)\right)\right)\right| \tag{3.5.25}$$

$$\leq C_0(\sqrt{v}+1)e^{(a_1+b_1)Re(Y^-(x_\alpha+iv))}\varphi_1^z(0) \leq C_0(\sqrt{v}+1)e^{(a_1+b_1)c\sqrt{v}}\varphi_1^z(0).$$

for any $v \geq 0$. We recall that function $z \longmapsto \varphi_1^z(0)$ is continuous, and therefore locally bounded. The integral of (3.5.25) over $v > 0$ can then be bounded by a positive constant which is (locally) independent of z and of $0 \leq \alpha \leq \eta$. The second term is given by

$$\int_0^{+\infty} \frac{\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))e^{a_1(x_\alpha^+ + iv)+b_1Y^-(x_\alpha^+ + iv)}}{\gamma_2(x_\alpha^+ + iv, Y^-(u+iv))\partial_y\gamma(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))} \exp\left(-aiv - b\left(Y^+(x_\alpha^+ + iv) - Y^+(x_\alpha^+)\right)\right) dv. \tag{3.5.26}$$

Note that if $b_1 = 0$, the quotient in the integrand is of order $O(1/\sqrt{v})$ as $v \to +\infty$. Moreover, it suffices to bound the integral over $(v_0, +\infty)$ for some $v_0 > 0$, since the integrand is uniformly bounded with respect to $\alpha \in [0, \eta]$ and $z \in K$. By integration by parts, the integral over $(v_0, +\infty)$ equals

$$\frac{\gamma_2(x_\alpha^+ + iv_0, Y^+(x_\alpha^+ + iv_0))e^{a_1(x_\alpha^+ + iv_0)+b_1Y^-(x_\alpha^+ + iv_0)} \exp\Big(- aiv_0 - b(Y^+(x_\alpha^+ + iv_0) - Y^+(x_\alpha))\Big)}{\gamma_2(x_\alpha^+ + iv, Y^-(x_\alpha^+ + iv_0))\partial_y\gamma(x_\alpha^+ + iv_0, Y^+(x_\alpha^+ + iv_0))(-ai - b\frac{d}{dv}(Y^+(x_\alpha^+ + iv))_{v=v_0})} \tag{3.5.27}$$

$$- \int_{v_0}^{\infty} \frac{d}{dv}\left(\frac{\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))e^{a_1(x_\alpha^+ + iv)+b_1Y^-(x_\alpha^+ + iv)}}{\gamma_2(x_\alpha^+ + iv, Y^-(x_\alpha^+ + iv))\partial_y\gamma(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))(-ai - b\frac{d}{dv}(Y^+(x_\alpha^+ + iv))} \right)$$
$$\times \exp\big(- aiv - b(Y^+(x_\alpha^+ + iv) - Y^+(x_\alpha^+))\big)dv.$$

Furthermore, $\frac{d}{dv}(Y^+(x_\alpha^+ + iv)) = i\left(1 - \frac{1}{\sqrt{\mu_2^2 - 2(x_\alpha + iv)}}\right)$ and $Re\left(1 - \frac{1}{\sqrt{\mu_2^2 - 2(x_\alpha + iv)}}\right) \geq 1/2$ for all $v \geq v_0$ with v_0 large enough and $0 < \alpha < \eta$. With some calculations, the integrand of (3.5.27) is of order $O(1/v^{3/2})$ as $v \to +\infty$. Hence, the integral in (3.5.27) is bounded by a positive constant independent of α and of $z \in K$. This establishes the bound in (3.5.18). Inequalities (3.5.19), (3.5.20) and (3.5.21) are obtained similarly. □

3.6 Proof of Theorem 3.1

In Section 3.6.1, we establish the asymptotics stated in Theorem 3.1. In Section 3.6.2, we show that all the constants $h_\alpha(z_0)$ appearing in the asymptotics of Theorem 3.1 are non-zero, which completes the proof of the theorem.

3.6.1 Asymptotics in Theorem 3.1

We now have the tools to derive the asymptotics stated in Theorem 3.1 where $h_{\alpha_0}(z_0)$ is given by (3.1.22), $h_{\alpha^*}(z_0)$ by (3.1.24) (with the symmetric formula for $h_{\alpha^{**}}(z_0)$), and

$$c^* = \frac{\gamma_2(x^*, y^*)}{\partial_y\gamma(x^*, y^*)} \frac{x'(s^*)}{\gamma_2(x'(\zeta s^*), y'(\zeta s^*))}, \quad c^{**} = \frac{\gamma_1(x^{**}, y^{**})}{\partial_x\gamma(x^{**}, y^{**})} \frac{y'(s^{**})}{\gamma_1(x(\eta s^{**}), y'(\eta s^{**}))} \tag{3.6.1}$$

where $x'(s)$ and $y'(s)$ are the derivatives of $x(s)$ and $y(s)$ (see (3.4.1) for the definition of $x(s)$ and $y(s)$, (3.4.8) for s^* and s^{**}, (3.3.8) and (3.3.9) for x^*, x^{**}, y^* and y^{**}, and (3.4.2) for ηs and ζs).

Proof of the asymptotics in Theorem 3.1 when $z_0 \neq (0,0)$. We use the identity $g(a,b) = I_1(a,b) + I_2(a,b) + I_3(a,b)$, using the expressions provided in Lemma 3.25. By the classical saddle-point method (see details in Lemma 2.33, Chapter 2), the sum of the integrals of Lemma 3.25 along $\Gamma_{\alpha,x}$ and $\Gamma_{y,\alpha} = \overleftarrow{\underrightarrow{Y^+(\Gamma_{x,\alpha})}}$ has the following asymptotic expansion

$$\frac{1}{2\pi i}\int_{\Gamma_{x,\alpha}} \frac{\varphi_2(x)\gamma_2(x, Y^+(x))}{\partial_y\gamma(x, Y^+(x))} \exp(-ax - bY^+(x))dx + \frac{1}{2\pi i}\int_{\Gamma_{y,\alpha}} \frac{\varphi_1(y)\gamma_1(X^+(y), y)}{\partial_x\gamma(X^+(y), y)} \exp(-aX^+(y) - by)dy \tag{3.6.2}$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma_{y,\alpha}}\exp((a_0-a)X^+(y)+(b_0-b)y)\frac{dy}{\partial_x\gamma(X^+(y),y)}$$

$$\underset{\substack{r\to\infty\\ \alpha\to\alpha_0}}{=}\quad e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\left(\frac{1}{\sqrt{r}}\sum_{k=0}^{n}\frac{c_k^{z_0}(\alpha)}{r^k}+o\left(\frac{1}{r^n\sqrt{r}}\right)\right).$$

where $(a,b)=(r\cos(\alpha),r\sin(\alpha))$, and

$$c_0^{z_0}(\alpha)=\frac{1}{\sqrt{2\pi(\cos(\alpha)+\sin(\alpha))^2}}\sqrt{\frac{\sin(\alpha)}{\partial_y\gamma(x(\alpha),y(\alpha))}} \tag{3.6.3}$$

$$\times\Big(\gamma_1(x(\alpha),y(\alpha))\varphi_1(y(\alpha))+\gamma_2(x(\alpha),y(\alpha))\varphi_2(x(\alpha))+e^{(x(\alpha),y(\alpha))\cdot z_0}\Big) \tag{3.6.4}$$

$$=\frac{1}{\sqrt{2\pi(\cos(\alpha)+\sin(\alpha))}}h_\alpha(z_0) \tag{3.6.5}$$

by the explicit expressions of $\varphi_1(y(s))$ and $\varphi_2(x(s))$ given in (3.4.9) and (3.4.11), evaluated at $s=\mathfrak{s}(\alpha)$ (see (3.5.5)).

Lemma 3.26 shows that, when $z_0\neq(0,0)$, integrals over $S_x^\pm, S_y^\pm$ are negligible compared to those over paths of steepest descent. Finally, Theorem 3.20 gives the explicit form of residues of Lemma 3.25 providing $h_{\alpha^*}(z_0)$, $h_{\alpha^{**}}(z_0)$. □

For the case $z_0=(0,0)$, we establish two preliminary lemmas. The first one is a consequence of the general Martin boundary theory.

Lemma 3.27. *For $\alpha\in[\alpha^*,\alpha^{**}]$, $z\longmapsto h_\alpha(z)$ is harmonic on $\mathbb{R}_+^2\backslash\{(0,0)\}$.*

Proof. For $z_0=(a_0,b_0)\neq(0,0)$, we may consider the process evolving in $\mathbb{R}_+^2\cap\{(x,y),\ x+y\geq a_0+b_0\}$. Since h_α is the limit of the quotient of Green's kernels, [78] implies its harmonicity over all these domains, and thus over $\mathbb{R}_+^2\backslash\{(0,0)\}$. □

Lemma 3.28. *Let Θ be the contour defined by $\Theta:=\{z\in\mathbb{R}_+^2\ :\ |z|=1\}$ and $T_\Theta:=\inf\{t\geq 0, Z_t\in\Theta\}$ the stopping time at Θ. Then, for all $z_0\in\mathbb{R}_+^2$ satisfying $|z_0|<1$,*

$$h_{\alpha_0}(z_0)=\int_\Theta h_{\alpha_0}(z)\mathbb{P}_{z_0}(Z_{T_\Theta}=dz). \tag{3.6.6}$$

Proof. Suppose first that $z_0\neq(0,0)$. The process $(h_\alpha(Z_t))_{t\geq 0}$ is a martingale: indeed, for $t,s\geq 0$,

$$\mathbb{E}_{z_0}\left[h_\alpha(Z_{t+s})|\mathcal{F}_t\right]=\mathbb{E}_{Z_t}\left[h_\alpha(Z_s)\right]=h_\alpha(Z_t)$$

by the strong Markov property and the harmonicity of h_α (see Lemma 3.27). Furthermore, under $\mathbb{P}_{z_0}$, the process $(h_\alpha(Z_{t\wedge T_\Theta}))_{t\geq 0}$ is bounded above by $\sup_{|z|\leq 1}h_\alpha(z)<\infty$ since h_α is continuous. Then, by the optional stopping theorem for bounded martingales, we obtain $h_{\alpha_0}(z_0)=\mathbb{E}_{z_0}\left[h(Z_{T_\Theta})\right]$ which is precisely the desired equality.

Now suppose $z_0=(0,0)$ and consider a sequence $(z_n)_{n\geq 1}$ in the quarter plane converging to $(0,0)$ such that $0<|z_n|<1$. By continuity of h_α, $h_\alpha(z_n)$ converges to $h_\alpha(z_0)$ as n goes to $+\infty$. Since equation (3.6.6) holds for all nonzero initial conditions, it suffices to show that

$$\int_\Theta h_{\alpha_0}(z)\mathbb{P}_{z_n}(Z_{T_\Theta}=dz)\underset{n\to+\infty}{\longrightarrow}\int_\Theta h_{\alpha_0}(z)\mathbb{P}_{(0,0)}(Z_{T_\Theta}=dz).$$

By continuity and boundedness of h_α on $\{z\in\mathbb{R}_+^2,|z|\leq 1\}$), it is enough to show that $\mathcal{L}_{z_n}Z_{T_\Theta}\underset{n\to+\infty}{\longrightarrow}\mathcal{L}_{z_0}Z_{T_\Theta}$ weakly where $\mathcal{L}_zZ_{T_\Theta}$ denotes the law of Z_{T_Θ} with initial condition $Z_0=z$. This follows from Assumption (3.1.2), combined with [63, Theorem 1], which ensures the continuity of the mapping from the non-reflected to the reflected path under the topology of uniform convergence on compacts. □

We can now prove Theorem 3.1 in the case $z_0 = (0,0)$.

Proof of the asymptotics in Theorem 3.1 for $z_0 = (0,0)$. By continuity of the process and by the Strong Markov property, if (a,b) lies at a distance > 1 from $(0,0)$, then

$$g^{(0,0)}(a,b) = \int_\Theta g^z(a,b)\mathbb{P}(Z_{T_\Theta} = dz). \tag{3.6.7}$$

Since the constant C from the saddle-point method depends continuously on z_0 (see Lemma 2.33, Chapter 2) and since the constants D_{α_0} in Lemma 3.26 are locally uniform in z_0, then for any compact set K in the quadrant $\mathbb{R}_+^2$ with $d((0,0),K) > 0$, we have

$$\sup_{z\in K}\left| g^z(r\cos(\alpha), r\sin(\alpha)) - e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}\sum_{k=0}^{n}\frac{c_k^z(\alpha)}{r^k}\right| \underset{\substack{r\to\infty\\ \alpha\to\alpha_0}}{=} o\left(\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{r^n\sqrt{r}}\right). \tag{3.6.8}$$

By this expansion,the asymptotics of (3.6.7) yield

$$g^{(0,0)}(a,b) \underset{\substack{r\to\infty\\ \alpha\to\alpha_0}}{\sim} \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}\int_\Theta h_{\alpha_0}(z)\mathbb{P}(Z_{T_\Theta} = dz). \tag{3.6.9}$$

Lemma 3.28 combined with (3.6.9) gives the result. □

3.6.2 Positivity of $h_\alpha(z_0)$.

To make our asymptotics consistent, we prove here the positivity of the constants $h_\alpha(z_0)$.

Lemma 3.29. *Let $\alpha \in (\alpha^*, \alpha^{**})$. Then, for every $R > 0$, there exists z_0 such that $|z_0| \geq R$ and $h_\alpha(z_0) > 0$. If $\alpha^* > 0$ (resp. $\alpha^{**} < \pi/2$), then the same result holds for h_{α^*} (resp. $h_{\alpha^{**}}$).*

Proof. By the explicit formulas (3.1.22) for h_α, (3.1.24) for h_{α^*}, and its equivalent for $h_{\alpha^{**}}$, the following asymptotics hold as $r \to +\infty$ for $\alpha \in (\alpha^*, \alpha^{**})$, and for $\alpha = \alpha^*$ (resp. $\alpha = \alpha^{**}$) if $\alpha^* > 0$ (resp. $\alpha^{**} < \pi/2$):

$$h_\alpha(r\cos(\alpha), r\sin(\alpha)) \underset{r\to\infty}{\sim} e^{r(x(\alpha)\cos(\alpha)+y(\alpha)\sin(\alpha))}. \tag{3.6.10}$$

The conclusion follows with $z_0 = (r\cos(\alpha), r\sin(\alpha))$ for r large enough. □

The following Lemma is inspired by Lemmas 2.35 and 2.36, Chapter 2and establishes the positivity of constants $h_\alpha(z_0)$ in the framework of Theorem 3.1.

Lemma 3.30 (Positivity of $h_\alpha(z)$). *Let $\alpha \in (\alpha^*, \alpha^{**})$, or $\alpha = \alpha^*$ (resp. $\alpha = \alpha^{**}$) if $\alpha^* > 0$ (resp. if $\alpha^{**} < \pi/2$). Let $z \in \mathbb{R}_+^2$. Then, $h_\alpha(z) \neq 0$.*

Proof. Let $\alpha \in (\alpha^*, \alpha^{**})$ and let z_0 such that both coordinates are larger than those of z, and such that $h_\alpha(z_0) > 0$, see Lemma 3.29. Let V be a compact neighbourhood of z_0, and denote by $T_V := \inf\{t \geq 0 : Z_t \in V\}$ the hitting time of V. By the hypothesis on z, $\mathbb{P}_z(T_V < +\infty) > 0$. By the strong Markov property,

$$g^z(r\cos(\alpha), r\sin(\alpha)) \geq \mathbb{P}_z(T_V < +\infty)\inf_{z_0'\in V} g^{z_0'}(r\cos(\alpha), r\sin(\alpha)) \tag{3.6.11}$$

$$\geq \mathbb{P}_z(T_V < +\infty)\inf_{z_0'\in V}(h_\alpha(z_0') + \varepsilon_{z_0',\alpha}(r))\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}. \tag{3.6.12}$$

for r large enough where (3.6.8) provides the asymptotics

$$\sup_{z_0'\in V}|\varepsilon_{z_0',\alpha}(r)| \underset{r\to\infty}{\longrightarrow} 0.$$

Furthermore, by continuity of h_α, the set V can be chosen to satisfy $\inf_{z_0' \in V} h_\alpha(z_0') > 0$. On the other hand, we also have

$$g^z(r\cos(\alpha), r\sin(\alpha)) = \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}(h_\alpha(z) + \varepsilon_{z_0',\alpha}(r))$$

where $\varepsilon_{z_0',\alpha}(r) \longrightarrow 0$ as $r \to +\infty$. Therefore, comparing the two expressions, we conclude that $h_\alpha(z) > 0$. If $\alpha = \alpha^*$ or $\alpha = \alpha^{**}$, the proof is analogous. □

End of the proof of Proposition 3.13. The remaining part of the proof is equivalent to showing that $h_{\alpha^*}(z_0) > 0$: indeed, $h_{\alpha^*}(z_0)$ is equal to $\text{res}_{x=x^*}\varphi_2(x)$ up to a non-null multiplicative constant (see (3.1.24) and (3.4.9)). The positivity is established in the previous lemma. □

3.7 Asymptotics of Green's kernel in the particular directions $0, \alpha^*, \alpha^{**}$ and $\pi/2$.

In Section 3.7.1, we study the asymptotics of Green's functions in the direction $\alpha = 0$ under the assumption that $\gamma_2(x_{max}, Y^\pm(x_{max})) \neq 0$. In Section 3.7.2, we provide these asymptotics in the direction α^* if $\alpha^* > 0$. Then, in Section 3.7.3 we analyse the limiting case where $\alpha^* = 0$ and $\gamma_2(x_{max}, Y^\pm(x_{max})) = 0$. The analysis of the directions $\alpha = \pi/2$, $\alpha = \alpha^{**}$ if $\alpha^* < \pi/2$ and $\alpha^{**} = \pi/2, \gamma_1(X^\pm(y_{max}, y_{max})) = 0$ is symmetrical. We then derive the proof of Theorem 3.2 in Section 3.7.4.

3.7.1 Case $\alpha \to 0$ if $\gamma_2(x_{max}, Y^\pm(x_{max})) \neq 0$.

Before deriving the asymptotics, let us relate Green's densities g^{z_0} to $f_1^{z_0}, f_2^{z_0}$.

Proposition 3.31 (Link between densities). *Let $a, b \geq 0$. Suppose $z_0 \neq (a, 0)$ and $z_0 \neq (0, b)$. Then, we have*

$$f_1^{z_0}(b) = \frac{1}{2}g^{z_0}(0, b) \quad \textit{and} \quad f_2^{z_0}(a) = \frac{1}{2}g^{z_0}(a, 0).$$

Proof. By the functional equation (3.3.1), if $x, y < 0$ then

$$-\frac{\gamma(x,y)}{x}\varphi(x,y) = \frac{\gamma_1(x,y)}{x}\varphi_1(y) + \frac{\gamma_2(x,y)}{x}\varphi_2(x) + \frac{e^{(x,y)\cdot z_0}}{x}. \tag{3.7.1}$$

Furthermore, by elementary properties of Laplace transforms,

$$x\varphi(x,y) \underset{x\to-\infty}{\longrightarrow} -\int_0^{+\infty} e^{by} g(0,b)db.$$

Then, letting $x \to -\infty$ in (3.7.1), we get $\int_0^{+\infty} e^{by}g(0,b)db = \varphi_1(y) = \int_0^{+\infty} e^{by}f_1(b)db$. The injectivity of the Laplace transform concludes the proof. The case of f_2 is symmetrical. □

Lemma 3.32 (Asymptotics at $\alpha = 0$). *Suppose that φ_2 does not have a pole. Let $\kappa = \left(\frac{1+\mu_2}{2}\Gamma(1/2)\right)^{-1}$ where Γ denotes the usual Gamma function. Then,*

$$f_2^{z_0}(x) \sim_{x\to+\infty} \frac{\kappa\partial_\alpha[h_\alpha(z_0)]_{\alpha=0}}{x^{3/2}}e^{-x_{max}x}.$$

Proof. Note that $Y^-(x) = Y^-(x_{max}) - \sqrt{2(x_{max}-x)} + o(\sqrt{x - x_{max}})$ by (3.3.3). Then, using the continuation formula (3.3.7), φ_2 is continuous at x_{max} and

$$\begin{aligned}\frac{\varphi_2(x)-\varphi_2(x_{max})}{\sqrt{2(x_{max}-x)}} &= r_1\varphi_1(Y^-(x_{max})) + \gamma_1(x_{max}, Y^\pm(x_{max}))\varphi_1'(Y^\pm(x_{max})) + b_0 e^{z_0\cdot(x_{max}, Y^\pm(x_{max}))} \\ &+ \frac{\gamma_1(x_{max}, Y^\pm(x_{max}))\varphi_1(Y^-(x_{max})) + e^{z_0\cdot(x_{max},Y^\pm(x_{max}))}}{\gamma_2(x_{max}, Y^\pm(x_{max}))} + o_{x\to x_{max}}(1) \\ &= r_1\varphi_1(Y^-(x_{max})) + \gamma_1(x_{max}, Y^\pm(x_{max}))\varphi_1'(Y^\pm(x_{max})) + b_0 e^{z_0\cdot(x_{max}, Y^\pm(x_{max}))} \\ &+ \varphi_2(x_{max}) + o_{x\to x_{max}}(1) \\ &=: A + o_{x\to x_{max}}(1)\end{aligned}$$

where $z_0 = (a_0, b_0)$. Then, by the Tauberian theorem given by [24, Lemma C.2], we obtain

$$f_2^{z_0}(x) \underset{x\to+\infty}{\sim} \frac{A}{\Gamma(1/2)x^{3/2}} e^{-x_{max}x}.$$

It then remains to show that $\frac{A}{\Gamma(1/2)} = \kappa\partial_\alpha[h_\alpha(z_0)]_{\alpha=0}$. From equation (3.5.6), we have:

$$\begin{aligned}\partial_\alpha[h_\alpha(z_0)]_{\alpha=0^+} &= (\mathfrak{s}^{-1})'(s_{max})\partial_s\Big[\gamma_1(x(s),y(s))\varphi_1(y(s)) + \gamma_2(x(s),y(s))\varphi_2(x(s)) + e^{a_0x(s)+b_0y(s)}\Big]_{s=s_{max}} \\ &= \left(\frac{-(\mu_1+\mu_2)}{(\mu_1+\mu_2)^2}\right)\big(r_1y'(s_{max})\varphi_1(y(s_{max})) + \gamma_1(x(s_{max}),y(s_{max}))y'(s_{max})\varphi_1'(y(s)) \\ &+ y'(s_{max})\varphi_2(x(s_{max})) + y'(s_{max})b_0e^{a_0x(s_{max})+b_0y(s_{max})}\big) \\ &= \frac{1+\mu_2}{2}A\end{aligned}$$

since $y'(s_{max}) = y'(\mu_2) = -\frac{1+\mu_2}{2}$ and $x'(s_{max}) = 0$. The conclusion follows. □

We can now establish the asymptotics of Green's functions as $\alpha \to 0$. We use the notation $c_0(\alpha)$ and $c_1(\alpha)$ for the constants in the first and second terms, respectively, in the asymptotic expansion of Green's fonctions (cf (3.6.8)).

Theorem 3.33 (Asymptotics with $\alpha\to 0$)**.** *Suppose $\gamma_2(x_{max}, Y^\pm(x_{max})) \neq 0$. Then,*

$$h_\alpha(z_0) \underset{\alpha\to 0}{\sim} \alpha\partial_\alpha[h_\alpha(z_0)]_{\alpha=0} \tag{3.7.2}$$

and thus $c_0^{z_0}(\alpha) \underset{\alpha\to 0}{\sim} \frac{1}{\sqrt{2\pi}}\alpha\partial_\alpha[h_\alpha(z_0)]_{\alpha=0}$. Furthermore, if $\alpha^*=0$,

$$c_1^{z_0}(\alpha) \underset{\alpha\to 0}{\longrightarrow} 2\kappa\partial_\alpha[h_\alpha(z_0)]_{\alpha=0}. \tag{3.7.3}$$

Moreover,

- *If $\alpha^* = 0$ (i.e. φ_2 has no pole), then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to 0}}{\sim} \partial_\alpha[h_\alpha(z_0)]_{\alpha=0}\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}\left(\frac{\alpha}{\sqrt{2\pi}} + \frac{2\kappa}{r}\right).$$

- *If $\alpha^* > 0$ (i.e. φ_2 has a pole), then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to 0}}{\sim} c^* h_{\alpha^*}(z_0)e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)}$$

  where $h_{\alpha^*}(z_0)$ and c^* are given by (3.1.24) and (3.6.1), respectively.

Moreover, the constants $\partial_\alpha[h_\alpha(z_0)]_{\alpha=0}$ *and* $h_{\alpha^*}(z_0)$ *are nonzero in the corresponding asymptotics.*

Proof. First, (3.7.2) follows from the regularity of $h_\alpha(z_0)$ in α and from the convergence $h_\alpha(z_0) \longrightarrow 0$ as $\alpha \to 0$ (see (3.1.22)). Now, we analyze the asymptotics of the sum of the three integrals in (3.6.2) along the saddle-point curves as $\alpha \to 0$. The integrands in the second and third terms are holomorphic in a neighbourhood of the saddle point $Y^+(x_{max})$. The integrand of the first term, namely φ_2, has a branching point at x_{max}. For this reason, we perform the change of variables $\Gamma_{x,\alpha} = \overleftarrow{\underrightarrow{X^+(\Gamma_{y,\alpha})}}$:

$$\int_{\Gamma_{x,\alpha(a,b)}} \frac{\varphi_2(x)\gamma_2(x,Y^+(x))}{\partial_y\gamma(x,Y^+(x))}\exp(-ax-bY^+(x))dx = \int_{\Gamma_{y,\alpha(a,b)}} \frac{\gamma_2(X^+(y),y)\varphi_2(X^+(y))}{\partial_x\gamma(X^+(y),y)}\exp(-aX^+(y)-by)dy. \tag{3.7.4}$$

Additionally, from (3.3.7):

$$\varphi_2(X^+(y)) = \frac{-\gamma_1(X^+(y),Y^-(X^+(y)))\varphi_1(Y^-(X^+(y))) - \exp\left(a_0X^+(y)+b_0Y^-(X^+(y))\right)}{\gamma_2(X^+(y),Y^-(X^+(y)))}. \tag{3.7.5}$$

Note that $X^+(y)$ is holomorphic in a neighbourhood of $Y^\pm(x_{max})$. The crucial point is that $Y^-(X^+(y))$ is also holomorphic there. Indeed, it can be expressed as

$$Y^-(X^+(y)) = \frac{X^+(y)^2 + 2\mu_1X^+(y)}{y}. \tag{3.7.6}$$

To see this, note that $Y^-(x)$ and $Y^+(x)$ are the two roots of $y \longmapsto \frac{1}{2}(x-y)^2 + \mu_1x + \mu_2y$. Then, by Vieta's equations and since $Y^+(X^+(y)) = y$, (3.7.6) follows immediately. Since $\gamma_2(x_{max}, Y^\pm(x_{max}) \neq 0$, it follows from (3.7.5) that $\varphi_2(X^+(y))$ is holomorphic at $Y^\pm(x_{max})$, so the saddle-point method applies to the right-hand side of (3.7.4). Then, asymptotics of Green's functions become

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} c^*h_{\alpha^*}(z_0)e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)}\mathbb{1}_{\alpha^*>0} \tag{3.7.7}$$

$$+ e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}\left(c_0(\alpha) + \frac{c_1(\alpha)}{r}\right)$$

for all $\alpha_0 \in [0, \epsilon]$ where $\epsilon > 0$ is sufficiently small. It remains to show that if $\alpha^* = 0$, then (3.7.3) holds. With $\alpha = 0$, equation (3.7.7) becomes

$$g(r,0) \underset{r\to\infty}{\sim} \frac{e^{-rx_{max}}}{r^{3/2}}c_1(0). \tag{3.7.8}$$

By Proposition 3.31, $g(r,0) = 2f_2(r)$. Finally, Lemma 3.32 applies and completes the asymptotic analysis. The non-vanishing of $\partial_\alpha[h_\alpha(z_0)]_{\alpha=0^+}$ is analogous to the case $\alpha \in (\alpha^*, \alpha^{**})$, see Section 3.6.2. The non-vanishing of $h_{\alpha^*}(z_0)$ if $\alpha^* > 0$ is already proved in Lemma 3.30. ☐

3.7.2 Case $\alpha \to \alpha^*$ when $\alpha^* > 0$

Theorem 3.34 (Asymptotics with $\alpha \to \alpha^*$)**.** *Suppose* $\alpha^* > 0$ *(i.e.,* φ_2 *has a pole). Then:*

- *If* $r(\alpha - \alpha^*)^2 \to 0$*, then:*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha^*}}{\sim} \frac{1}{2}h_{\alpha^*}(z_0)e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)}. \tag{3.7.9}$$

where $h_{\alpha^*}(z_0)$ *is given by* (3.1.24).

- *If* $r(\alpha-\alpha^*)^2 \to K > 0$ *for some constant* K*, then for* $\alpha < \alpha^*$ *(resp.* $\alpha > \alpha^*$*),*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to\alpha^*}}{\sim} c_K h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)} \tag{3.7.10}$$

$$\left(resp. \quad g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to\alpha^*}}{\sim} \widetilde{c}_K h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)} \right).$$

where constants $c_K > 0$*,* $\widetilde{c}_K > 0$ *are independant of initial condition* z_0*.*

- *If* $r(\alpha-\alpha^*)^2 \to \infty$*, then:*
  - *If* $\alpha < \alpha^*$*, then*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to\alpha^*}}{\sim} h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)}. \tag{3.7.11}$$

  - *If* $\alpha > \alpha^*$*, then:*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to\alpha^*}}{\sim} h_{\alpha^*}(z_0) e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha)} \frac{1}{\sqrt{r}} \frac{C}{\alpha-\alpha^*} \tag{3.7.12}$$

  where C *is a positive constant independent of initial condition* z_0*.*

Furthermore, $h_{\alpha^*}(z_0) > 0$*. Constants* $c_K, \widetilde{c}_K$ *and* C*, are made explicit in Section 2.10, Chapter 2.*

The proof is analogous to Section 2.10, Chapter 2 , which compares the asymptotic contribution of the pole term and the saddle-point term in the expressions of Lemma 3.25 for $g^{z_0} = I_1 + I_2 + I_3$. The non-vanishing of $h_{\alpha^*}(z_0)$ was already proved in Lemma 3.30.

3.7.3 Last particular case: $\gamma_2(x_{max}, Y^{\pm}(x_{max})) = 0$

Theorem 3.35. *Suppose* $\gamma_2(x_{max}, Y^{\pm}(x_{max})) = 0$ *(so* $\alpha^* = 0$*). Then,*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty\\ \alpha\to 0}}{\sim} e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha)} \frac{h_0(z_0)}{\sqrt{r}}. \tag{3.7.13}$$

where $h_0(z_0)$ *is given by (3.1.25). Furthermore,* $h_0(z_0) \neq 0$*.*

Proof. The proof follows the same approach as that of Theorem 3.33 but here $c_0(\alpha) \longrightarrow h_0(z_0) \neq 0$. Thus, we only consider the first term in (3.6.2) with the representation (3.7.4) for $I_1(a,b)$. First, note from (3.3.3) that $\frac{d}{dy}\left[X^+(y)\right]_{y=Y^{\pm}(x_{max})} = 0$. Then, by (3.7.6):

$$\frac{d}{dy}\left[\gamma_2(X^+(y), Y^-(X^+(y)))\right]_{y=Y^{\pm}(x_{max})} = -\left(\frac{(\mu_2/2)^2 + 2\mu_1\mu_2^2/2}{(\mu_2^2/2-\mu_2)^2}\right) = -1. \tag{3.7.14}$$

Hence, $\gamma_2(X^+(y), Y^-(X^+(y))) = (y - Y^{\pm}(x_{max}))(-1 + o_{y\to Y^{\pm}(x_{max})}(1))$. Furthermore, $\gamma_2(X^+(y), y) = (y - Y^{\pm}(x_{max}))(1 + o_{y\to Y^{\pm}(x_{max})}(1))$ by similar calculations.

Using the same arguments as in the proof of Theorem 3.33, the function

$$\frac{\gamma_2(X^+(y), y)\varphi_2(X^+(y))}{\partial_x\gamma(X^+(y), y)} \tag{3.7.15}$$

is holomorphic in a neighbourhood of $Y^{\pm}(x_{max})$, expect possibly in $Y^{\pm}(x_{max})$ where $\varphi_2(X^+(y))$ has a simple pole by (3.7.5) and (3.7.14). Since $\gamma_2(X^+(y), y)$ has a zero of the same order at this point, then the quantity (3.7.15) turns out to be holomorphic at $Y^{\pm}(x_{max})$ as well. Moreover, by (3.4.21), the following asymptotic expansion holds as $y \to Y^{\pm}(x_{max})$

$$\frac{\gamma_2(X^+(y),y)\varphi_2(X^+(y))}{\partial_x\gamma(X^+(y),y)} = (-1+o_{y\to Y^{\pm}(x_{max})}(1))\Big(- e^{z_0\cdot(X^+(y),Y^-(X^+(y)))} \tag{3.7.16}$$

$$+ \frac{\gamma_1(\psi_1(X^+(y),y))}{\frac{\gamma_1}{\gamma_2}(\psi_2(X^+(y),y))} \sum_{n=1}^{+\infty} \left[\prod_{k=2}^{n} \frac{\frac{\gamma_1}{\gamma_2}(\psi_{2k-1}(X^+(y),y)}{\frac{\gamma_1}{\gamma_2}(\psi_{2k}(X^+(y),y))}\right] \left[\frac{e^{z_0\cdot\psi_{2n}(X^+(y),y))}}{\gamma_2(\psi_{2n}(X^+(y),y))} - \frac{e^{z_0\cdot\psi_{2n+1}(X^+(y),y)}}{\gamma_2(\psi_{2n+1}(X^+(y),y))}\right]\Big).$$

This implies (3.1.25). The proof of the non-vanishing of $h_0(z_0)$ is analogous to Lemma 3.30. ☐

3.7.4 Proof of Theorem 3.2

This is a direct consequence of Theorem 3.1, 3.33, 3.34, and 3.35.

3.8 Harmonic functions and Martin boundary

In this Section, we prove Theorem 3.3. In particular, we show in Section 3.8.1 that the Martin boundary is homeomorphic to $[\alpha^*, \alpha^{**}]$ and in Section 3.8.2 that the Martin boundary is minimal.

3.8.1 Context of Martin the boundary

In this section, we consider the construction of the Martin boundary as presented in [102, Section 7.1] for elliptic processes and we adapt this approach to reflected degenerate processes. This method allows us to consistently link the harmonic functions $(h_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$ found in Theorem 3.2 and the Martin boundary. Note that another general construction of the Martin compactification is presented in [78].

Definition 3.36 (Martin kernel). *For $z_0, z_1 \in \mathbb{R}^2_+$, we define the Martin kernel*

$$k(z_0,z_1) = \begin{cases} \frac{g^{z_0}(z_1)}{g^{(0,0)}(z_1)} & \text{if} \quad z_1 \neq (0,0) \\ 0 & \text{if} \quad z_1 = (0,0) \quad \text{or} \quad z_0 = z_1 \end{cases} \tag{3.8.1}$$

and the Martin metric

$$\rho(z_1,z_2) = \int_{\mathbb{R}^2_+} \frac{|k(x,z_1)-k(x,z_2)|}{1+|k(x,z_1)-k(x,z_2)|} e^{-|x|^2} dx. \tag{3.8.2}$$

By usual considerations [102], ρ is a metric equivalent to the Euclidean one on $\mathbb{R}^2_+$. A sequence $(y_n)_{n\geq 0}$ of $\mathbb{R}^2_+$ is called a Martin sequence if $(k(\cdot,y_n))_{n\geq 0}$ converges pointwise. Two Martin sequences are said to be equivalent if their limit functions are equal. We then define M as the quotient of the set of all Martin sequences by this equivalence relation. Each $\xi \in M$ is then naturally associated with function denoted by $k(\cdot,\xi)$. The metric ρ extends naturally to M with the same formula, so that the map

$$\iota\colon \begin{cases} \mathbb{R}^2_+ \longrightarrow (M,\rho) \\ \quad z \longmapsto k(\cdot,z) \end{cases} \tag{3.8.3}$$

is injective and continuous. We define the Martin Boundary Γ as $\Gamma = M \backslash \iota(\mathbb{R}^2_+)$.

Lemma 3.37. *Let $(h_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$ be defined in Theorem 3.2. Then, the map*

$$\Phi : \alpha \in [\alpha^*,\alpha^{**}] \longmapsto h_\alpha(\cdot)/h_\alpha(0) \in \Gamma$$

is a homeomorphism.

Before proving this lemma, we recall some properties of the family $(h_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$.

Remark 3.38. Note that for $z\in\mathbb{R}_+^2$

- If $\alpha^*>0$, then by (3.1.22),
$$h_\alpha(z)\underset{\substack{\alpha\to\alpha^*\\ \alpha>\alpha^*}}{\longrightarrow}+\infty.$$
- If $\alpha^*=0$ and $\gamma_2(x_{max},Y^\pm(x_{max})=0$, then by (3.1.25),
$$h_\alpha(z)\underset{\substack{\alpha\to 0\\ \alpha>0}}{\longrightarrow}h_0(z)>0.$$
- If $\alpha^*=0$ and $\gamma_2(x_{max},Y^\pm(x_{max})\neq 0$, then by (3.1.24),
$$h_\alpha(z)\underset{\substack{\alpha\to 0\\ \alpha>0}}{\longrightarrow}0.$$

Proof of Lemma 3.37. By Theorems 3.1, 3.2, 3.33, 3.34, and 3.35, Φ is surjective. To prove the continuity of Φ, note that a sequence $(\xi_n)_{n\geq 0}$ converges to some $\xi\in M$ if $k(\cdot,\xi_n)$ converges pointwise toward $k(\cdot,\xi)$ almost everywhere. Therefore, the proof of the continuity of Φ is reduced to showing that, for any $z\in\mathbb{R}_+^2$, the map $\alpha\longmapsto\Phi(\alpha)(z)$ is continuous. Let $z\in\mathbb{R}_+^2$.

- By (3.1.22), the map $\alpha\mapsto\Phi(\alpha)(z)$ is continuous on (α^*,α^{**}).
- If $\alpha^*=0$ and $\gamma_2(x_{max},Y^\pm(x_{max}))\neq 0$, then we have:
$$\Phi(\alpha)(z)=\frac{h_\alpha(z)}{h_\alpha(0)}=\frac{h_\alpha(z)}{\alpha}\frac{\alpha}{h_\alpha(0)}\underset{\alpha\to 0}{\longrightarrow}\frac{[\partial_\alpha h_\alpha(z)]_{\alpha=0}}{[\partial_\alpha h_\alpha(0)]_{\alpha=0}}=\Phi(0)(z)$$
so $\alpha\mapsto\Phi(\alpha)(z)$ is continuous at $\alpha^*=0$.
- If $\alpha^*>0$, then $\frac{h_\alpha(z)}{h_\alpha(0)}$ can be written as
$$\frac{\sum_{m=-\infty}^{0}\kappa_m(\alpha)e^{z\cdot(a_m(\mathfrak{s}(\alpha)),b_m(\mathfrak{s}(\alpha)))}+\frac{1}{\gamma_2(\zeta\mathfrak{s}(\alpha))}\sum_{m=1}^{+\infty}\kappa_m(\alpha)e^{z\cdot(a_m(\mathfrak{s}(\alpha)),b_m(\mathfrak{s}(\alpha)))}}{\sum_{m=-\infty}^{0}\kappa_m(\alpha)+\frac{1}{\gamma_2(\zeta\mathfrak{s}(\alpha))}\sum_{m=1}^{+\infty}\kappa_m(\alpha)}$$
where $(a_m(\mathfrak{s}(\alpha)),b_m(\mathfrak{s}(\alpha)))$, are defined by (3.1.20) and (3.1.21), with $(a_0(\mathfrak{s}(\alpha)),b_0(\mathfrak{s}(\alpha)))=(x(\alpha),y(\alpha))$, and $\kappa_m(\alpha)$ is given by (3.1.23). Since $\gamma_2(\zeta(\mathfrak{s}(\alpha)))\underset{\alpha\to\alpha^*}{\longrightarrow}0$ (see Notations 3.19 and Proposition 3.13), the expected continuity in α^* follows from standard continuity theorems on series.
- The remaining case $\alpha^*=0$ and $\gamma_2(x_{max},Y^\pm(x_{max}))=0$ is analogous.
- The proof of the continuity of Φ at α^{**} is symmetric.

Next, let us show that Φ is injective. By the explicit expressions in Theorem 3.2, the following asymptotics hold as $r\to+\infty$. For $\alpha^*<\alpha<\alpha^{**}$ and $0\leq\theta\leq\pi/2$, we have:
$$h_\alpha(r\cos(\theta),r\sin(\theta))\underset{r\to\infty}{\sim}e^{r(x(\alpha)\cos(\theta)+y(\alpha)\sin(\theta))}.\tag{3.8.4}$$

If $\alpha^*=0$, then for any $0<\theta\leq\pi/2$,
$$h_0(r\cos(\theta),r\sin(\theta))\underset{r\to\infty}{\sim}r\sin(\theta)e^{r(x(0)\cos(\theta)+y(0)\sin(\theta))}.\tag{3.8.5}$$

If $\alpha^*>0$ and $0\leq\theta\leq\pi/2$, then
$$h_{\alpha^*}(r\cos(\theta),r\sin(\theta))\underset{r\to\infty}{\sim}e^{r(x(\alpha^*)\cos(\theta)+y(\alpha^*)\sin(\theta))}.\tag{3.8.6}$$

The corresponding symmetric asymptotic behavior holds for $h_{\alpha^{**}}$. If $\alpha, \alpha' \in [\alpha^*, \alpha^{**}]$ are distinct, then by (3.5.4) and the preceding formulae,

$$\frac{h_\alpha(r\cos(\alpha), r\sin(\alpha))}{h_{\alpha'}(r\cos(\alpha), r\sin(\alpha))} \underset{r\to\infty}{\longrightarrow} +\infty.$$

Hence, $h_\alpha \neq Ch_{\alpha'}$ for any constant C, and Φ is injective.

Since Φ is continuous, and since $[\alpha^*, \alpha^{**}]$ is compact, $\Phi(F)$ is closed in Γ for any closed subset F of $[\alpha^*, \alpha^{**}]$. Therefore, Φ is a homeomorphism. ☐

Corollary 3.39. *The following properties hold:*

(i) If $\eta, \xi \in M$ satisfy $k(\cdot, \xi) = k(\cdot, \eta)$, then $\eta = \xi$.

(ii) The metric space (M, ρ) is compact.

(iii) $\iota(\mathbb{R}^2_+)$ is dense in M with respect to ρ.

(iv) If a sequence $(y_n)_{n\geq 0} \subset \mathbb{R}^2_+$ converges to $\eta \in \Gamma$ with respect to ρ, then $k(\cdot, y_n)$ converges pointwise to $k(\cdot, \eta)$.

Proof. Properties (i), (iii), and (iv) follow directly from our construction. We now prove (ii). Let $(y_n)_{n\geq 0}$ be a sequence in M. Then:

- Either $(y_n)_{n\geq 0}$ has infinitely many points in Γ, in which case it has a convergent subsequence since Γ is compact (see Lemma 3.37).
- Or $(y_n)_{n\geq 0}$ has a bounded subsequence, in which case the conclusion follows as $\rho|_{\mathbb{R}^2_+\times\mathbb{R}^2_+}$ is equivalent to the Euclidean metric.
- Or $(y_n)_{n\geq 0}$ has a subsequence that tends to infinity. Since $[0, \pi/2]$ is compact, $(y_n)_{n\geq 0}$ has a subsequence that tends to infinity in some direction $\alpha \in [0, \pi/2]$. By Theorems 3.1, 3.33, 3.34, and 3.35 this subsequence converges (with respect to the metric ρ) to $\mathbb{1}_{\alpha<\alpha^*} h_{\alpha^*} + \mathbb{1}_{\alpha^*\leq\alpha\leq\alpha^{**}} h_\alpha + \mathbb{1}_{\alpha<\alpha^{**}} h_{\alpha^{**}}$.

☐

Remark 3.40. By Corollary 3.39, M is the Martin compactification in the sense of [78, 102], and the Martin Boundary Γ is homeomorphic to $[\alpha^*, \alpha^{**}]$.

In particular, by [78, Theorem 4] the following representation theorem holds.

Theorem 3.41 (Integral representation)**.** *If h is a non-negative harmonic function, then there exists a Radon measure μ_h on $[\alpha^*, \alpha^{**}]$ satisfying*

$$\forall z \in \mathbb{R}^2_+, \quad h(z) = \int_{[\alpha^*, \alpha^{**}]} h_\alpha(z) d\mu_h(\alpha). \tag{3.8.7}$$

Furthermore, every function defined by (3.8.7) is harmonic.

3.8.2 Minimality of functions $(h_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$ and Martin boundary

In this section, we prove that the Martin boundary is minimal.

Definition 3.42 (Minimal harmonic function)**.** *A non-negative harmonic function h is said to be minimal if, for every pair of non-negative harmonic functions f_1 and f_2 satisfying $f_1 + f_2 = h$, both f_1 and f_2 are proportional to h.*

Proposition 3.43 (Γ is minimal)**.** *The Martin boundary is minimal in the sense that if* $\eta \in \Gamma$, *then* $k(\cdot, \eta)$ *is minimal. In particular, the measure* μ_h *in representation* (3.8.7) *is unique.*

To prove this we state the following lemma.

Lemma 3.44. *Let* $\alpha^* < \alpha_1 < \alpha^{**}$ *and* $\epsilon > 0$*. Then, there exist constants* $\eta > 0$ *and* $r_0 > 0$ *such that*

$$h_\alpha(r\cos(\alpha_1), r\sin(\alpha_1)) \geq \frac{1}{2} e^{r(x(\alpha_1)\cos(\alpha_1)+y(\alpha_1)\sin(\alpha_1)-\epsilon)} \tag{3.8.8}$$

for all $r \geq r_0$ *and* $\alpha \in [\alpha_1 - \eta, \alpha_1 + \eta]$.

Proof. This follows from Theorem 3.2, where explicit formulae of h_α are given. □

Proof of Proposition 3.43. Let $\alpha_0 \in [\alpha^*, \alpha^{**}]$. We aim to prove that if $h_{\alpha_0} = \int_{[\alpha^*,\alpha^{**}]} h_\alpha d\mu(\alpha)$ for some Radon measure μ, then μ is the Dirac measure at α_0. This directly implies the minimality of Γ using Definition 3.42 and Theorem 3.41. It suffices to show that the support of μ is exactly $\{\alpha_0\}$. Suppose first that $\alpha^* < \alpha_0 < \alpha^{**}$. Let us prove that $\mu((\alpha^*, \alpha^{**})\backslash\{\alpha_0\}) = 0$. Let $\alpha_1 \in (\alpha^*, \alpha^{**})\backslash\{\alpha_0\}$. First, by (3.5.4), we can choose $\epsilon > 0$ such that

$$x(\alpha_1)\cos(\alpha_1) + y(\alpha_1)\sin(\alpha_1) - \epsilon > x(\alpha_0)\cos(\alpha_1) + y(\alpha_0)\sin(\alpha_1). \tag{3.8.9}$$

Secondly, by Lemma 3.44, there exists some $\eta > 0$ such that

$$h_\alpha(r\cos(\alpha), r\sin(\alpha)) \geq \frac{1}{2} e^{r(x(\alpha_1)\cos(\alpha_1)+y(\alpha_1)\sin(\alpha_1)-\epsilon)}$$

for $\alpha \in [\alpha_1 - \eta, \alpha_1 + \eta]$ and $r \geq r_0$ large enough. Then,

$$h_{\alpha_0}(r\cos(\alpha_1), r\sin(\alpha_1)) \geq \int_{\alpha\in[\alpha_1-\eta,\alpha_1+\eta]} h_\alpha(re_{\alpha_1})\mu(d\alpha) \underset{r\geq r_0}{\geq} \frac{\mu([\alpha_1-\eta,\alpha_1+\eta])}{2} e^{r(x(\alpha_1)\cos(\alpha_1)+y(\alpha_1)\sin(\alpha_1)-\epsilon)}.$$

Considering $\theta = \alpha_1$ and $\alpha = \alpha_0$ in (3.8.4), we obtain, for $r \geq r_1$ large enough:

$$e^{r(x(\alpha_1)\cos(\alpha_0)+y(\alpha_1)\sin(\alpha_0)-\epsilon)} \geq \frac{\mu([\alpha_1-\eta,\alpha_1+\eta])}{2} e^{r(x(\alpha_1)\cos(\alpha_1)+y(\alpha_1)\sin(\alpha_1)-\epsilon)}.$$

By (3.8.9), the asymptotics of the previous inequality as $r \to +\infty$ yield $\mu([\alpha_1 - \eta, \alpha_1 + \eta]) = 0$. Therefore, μ can be written as $\mu = A\delta_{\alpha_0} + B\delta_{\alpha^*} + C\delta_{\alpha^{**}}$ for some non-negative constants A, B, and C, i.e., $(1 - A)h_{\alpha_0} = Bh_{\alpha^*} + Ch_{\alpha^{**}}$. Now, considering the asymptotics (3.8.4), (3.8.5), and (3.8.6), we immediately get $B = C = 0$ and $A = 1$. Hence, μ is the Dirac measure at α_0 and h_{α_0} is minimal. The cases $\alpha_0 = \alpha^*$ and $\alpha_0 = \alpha^{**}$ are treated similarly. □

Proof of Theorem 3.3. This is a direct consequence of Lemma 3.37, Remark 3.40 and Propostion 3.43. □

3.9 From Assumption (3.1.4) to the general case

We stated and proved Theorems 3.1, 3.2 and 3.3 under Assumption (3.1.4). In this section, we generalise these theorems without assuming (3.1.4). To achieve this, we apply transformations to the $x-$axis, $y-$axis, and time t, in order to reduce the problem to a process $\widetilde{Z}$ which satisfies (3.1.4).

Proposition 3.45 (Space-time dilatation)**.** *Let $(Z_t)_{t\geq 0}$ be a degenerate reflected Brownian motion with parameters*

$$\Sigma = \begin{pmatrix} \sigma_1^2 & -\sigma_1\sigma_2 \\ -\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, R = \begin{pmatrix} 1 & r_2 \\ r_1 & 1 \end{pmatrix}.$$

Then the process

$$(\widetilde{Z}_t)_{t\geq 0} := \left(\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right) \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & \frac{1}{\sigma_2} \end{pmatrix} Z\left(\frac{t}{\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right)^2} \right) \right)_{t\geq 0}$$

is a degenerate reflected Brownian motion with parameters

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \widetilde{\mu} := \frac{1}{\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right)} \begin{pmatrix} \frac{\mu_1}{\sigma_1} \\ \frac{\mu_2}{\sigma_2} \end{pmatrix}, \widetilde{R} := \begin{pmatrix} 1 & r_2\frac{\sigma_2}{\sigma_1} \\ r_1\frac{\sigma_1}{\sigma_2} & 1 \end{pmatrix}.$$

Furthermore, $\widetilde{Z}$ satisfies (3.1.2) to (3.1.4).

Proof. This is a direct consequence of Definition 3.5 applying the corresponding transformation to (3.2.1). □

Theorem 3.46 (Harmonic functions and Martin boundary: general case)**.** *Suppose that (3.1.2) and (3.1.3). Let α^*, α^{**} and $(\widetilde{h}_\alpha)_{\alpha\in[\alpha^*,\alpha^{**}]}$ be the angles and the harmonic functions in Theorem 3.2 for the degenerate reflected Brownian motion of parameters $\left(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \widetilde{\mu}, \widetilde{R}\right)$ (using the notation of Proposition 3.45). Then, the family of minimal harmonic functions for the initial process is given by*

$$(x_0, y_0) \longmapsto \widetilde{h}_\alpha \left(\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right) \left(\frac{x_0}{\sigma_1}, \frac{y_0}{\sigma_2}\right) \right), \qquad \alpha \in [\alpha^*, \alpha^{**}]. \tag{3.9.1}$$

Furthermore, the Martin boundary remains homeomorphic to $[\alpha^*, \alpha^{**}]$ and is minimal.

Proof. Let $\psi : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ be the map defined by $\psi(z) = \left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right) \begin{pmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & \frac{1}{\sigma_2} \end{pmatrix} z$. We denote by $G(z_0, \cdot)$ (resp $\widetilde{G}(\widetilde{z}_0, \cdot)$) the Green's measure associated with $(Z_t)_{t\geq 0}$ (resp. with $(\widetilde{Z}_t)_{t\geq 0}$) and $g^{z_0}(z)$ (resp. $\widetilde{g}^{\widetilde{z}_0}(\widetilde{z})$) the corresponding Green's functions, where $\widetilde{z}_0 = \psi(z_0)$. Note that

$$\begin{aligned} G(z_0, A) &= \int_0^{+\infty} \mathbb{P}_{z_0}(Z_t \in A) dt \\ &= \int_0^{+\infty} \mathbb{P}_{z_0} \left(Z\left(\frac{u}{\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right)^2} \right) \in A \right) \frac{du}{\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right)^2} \\ &= \int_0^{+\infty} \mathbb{P}_{\widetilde{z}_0} \left(\widetilde{Z}_u \in \psi(A) \right) \frac{du}{\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right)^2}. \end{aligned}$$

Furthermore,

$$\mathbb{P}_{z_0} \left(\widetilde{Z}_u \in \psi(A) \right) = \int_{\psi(A)} \mathbb{P}_{\widetilde{z}_0} \left(\widetilde{Z}_u = u \right) du = \int_A \mathbb{P}_{\widetilde{z}_0} \left(\widetilde{Z}_u = \psi(v) \right) |Jac(\psi)| dv.$$

Therefore, the following holds for all $z_0, a \in \mathbb{R}_+^2$:

$$g^{z_0}(a) = \frac{1}{\sigma_1\sigma_2} \widetilde{g}^{\widetilde{z}_0}(\psi(a)). \tag{3.9.2}$$

Then,

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) = \frac{1}{\sigma_1\sigma_2}\widetilde{g}^{\psi(z_0)}\left(\widetilde{r}\cos(\widetilde{\alpha}), \widetilde{r}\sin(\widetilde{\alpha})\right) \tag{3.9.3}$$

with $\widetilde{\alpha} = \arctan\left(\frac{\sigma_2}{\sigma_1}\tan(\alpha)\right)$. The conclusion follows from relation (3.9.3). $\square$

Chapter 4

On the Green's functions and Martin boundary structure of a planar diffusion in a discontinuous layered medium

Ce chapitre est issu de l'article [54] co-écrit avec Irina Kourkova et Sandro Franceschi, et est soumis au journal *Electronic Journal of Probability.*

Résumé

Nous considérons un processus de diffusion bidimensionnel dans un plan à deux couches, régi par des matrices de covariance distinctes dans les demi-plans supérieur et inférieur, ainsi que par deux vecteurs de dérive dirigés à l'opposé de l'axe des x. Nous analysons d'abord le cas où le générateur du processus est sous forme divergence, c'est-à-dire lorsque le flux est continu à travers l'interface. Nous étendons ensuite l'étude à une classe plus large de processus dont le comportement à l'interface constitue un analogue bidimensionnel oblique du Skew Brownian motion.

Nous fournissons une analyse théorique détaillée de ce processus transient. Nos principaux résultats sont les suivants : (i) nous calculons explicitement les transformées de Laplace des fonctions de Green ; (ii) nous établissons les asymptotiques exactes des fonctions de Green le long de toutes les trajectoires possibles dans le plan ; (iii) nous déterminons toutes les fonctions harmoniques positives, en identifiant les frontières de Martin complète et minimale, qui s'avèrent être distinctes. La non-minimalité de la frontière de Martin constitue un phénomène remarquable pour des diffusions.

Afin d'obtenir une description analytique du processus, nous développons une version à trois variables de la méthode dite du noyau, en obtenant et en exploitant une équation fonctionnelle faisant intervenir les transformées de Laplace inconnues des fonctions de Green et deux noyaux connus $\gamma_+(x, y)$ et $\gamma_-(x, z)$. L'introduction de variables auxiliaires indépendantes y et z, associées à chacun des demi-plans, constitue une idée clé.

Abstract

We consider a two-dimensional diffusion process in a two-layered plane, governed by distinct covariance matrices in the upper and lower half-planes and by two drift vectors pointed away from the x-axis. We first analyze the case where the generator of the process is in divergence form, that is, when the

flux is continuous across the interface. Then we extend the study to a broader class of processes whose behavior at the interface forms an oblique two-dimensional analogue of the skew Brownian motion.

We provide a detailed theoretical analysis of this transient process. Our main results are as follows: (i) we derive explicit Laplace transforms of the Green's functions; (ii) we compute exact asymptotics of the Green's functions along all possible trajectories in the plane; (iii) We determine all positive harmonic functions, identifying the full and minimal Martin boundaries, which turn out to be distinct. The nonminimality of the Martin boundary is a noteworthy phenomenon for diffusions.

To obtain an analytical description of the process, we fully develop a three-variable version of the so-called kernel method by deriving and exploiting a functional equation involving unknown Laplace transforms of Green's functions and two known kernels $\gamma_+(x,y)$ and $\gamma_-(x,z)$. The introduction of independent auxiliary variables y and z, associated with each half-plane, is a key idea.

Contents

4.1 Introduction and main results

4.1.1 Context, goals and strategy

The study of diffusion processes in multi-dimensional discontinuous or layered media is a challenging problem. Diffusion processes with discontinuous coefficients arise naturally in various applied contexts, such as geophysics, ecology, and astrophysics, where the medium may be composed of layers with different diffusion and advection properties (see [84, 85] and references therein). It arises also naturally in stochastic control models [118]. In such models, discontinuities of the diffusion matrix act as permeable interfaces, leading to complex boundary behaviors (see Section 1.1.2 for a brief presentation of skew processes). In the present chapter, we consider the process defined in Section 1.1.2.2, as a generalisation of the processes in layered media studied by Lejay [85].

Our main focus is on the case where the generator of the process is in divergence form, that is, when the flux is continuous across the interface. In the last section of this chapter, we also consider a broader class of processes whose behavior can be viewed as a natural two-dimensional generalization of a Skew Bang-Bang Brownian Motion (SBBBM), with oblique interaction at the interface.

From a structural viewpoint, our model also connects with discrete analogues such as random walks or queueing processes evolving under different dynamics in distinct regions. The semi-martingale reflecting Brownian motion (SRBM) in a cone and its discrete counterparts are now well understood in dimension two and beyond (see, e.g., [41, 121]), but models involving different generators in several cones sharing a common boundary have been much less studied. Among discrete examples, the joining-the-shortest-queue (JSQ) model (studied by Kurkova and Suhov [81], Kobayashi et al. [77]) and the doubly two-sided quasi-birth-and-death (DQBD) process (studied by Miyazawa [94, 96, 97]) display related features, although symmetries sometimes reduce them to single-kernel functional equations. In contrast, our diffusion model genuinely involves two distinct kernels corresponding to the two half-planes. Some time-space continuous models for the multi-level single server queue are also studied in [76, 95].

Our aim is to provide a detailed theoretical analysis of the transient process presented above. The main results of the present paper are as follows:

(i) We make explicit the Laplace transforms of the Green's functions.

(ii) We derive the exact asymptotics of the Green's functions along all possible trajectories in $\mathbb{R}^2$.

(iii) As a consequence, we obtain explicit expressions for all positive harmonic functions and determine the full Martin boundary. We also identify the minimal Martin boundary, which turns out to be strictly smaller than the full one.

The nonminimalty of the full Martin boundary reveals a genuinely new phenomenon, which contrasts with many known results for diffusions with regular coefficients, whether free, killed or confined in cones or in other domains, see [37], Chapter 2 and DeBlassie's papers [26, 27, 28]. This phenomenon can be also expected for diffusions in two-dimensional cones with special assumptions on drifts making the process escape to infinity along the boundaries, see [72, 79] for the proof in the case of the analogous discrete random walks. However, the minimal Martin boundary in the last example is reduced to only two points, while for our process it is homeomorphic to a union of arcs of a circle.

The main tool of our analysis is a version of the kernel method presented in Section 1.2.3 with two kernels, $\gamma^+(x,y)$ and $\gamma^-(x,z)$, which share the same first variable and have distinct second variables. The crucial idea of our method is to introduce the independent variables y and z, which correspond to the two half-planes. This approach was first fruitfully applied for other techniques in Ignatyuk et al. [73] and Miyazawa [97] (see Section 1.2.3.1).

In this chapter, we extend the kernel method to three variables by fully developing it through the derivation and exploitation of a functional equation involving the Laplace transforms of Green's functions, see (4.1.11) below. To the best of our knowledge, this is the first example of a three-variable implementation of the method, leading to a complete analytical characterization of the model.

Detailed outline of the chapter. In Section 4.1.2, we introduce the main process. In Section 4.1.3, we define the Green's functions and state the three-variable kernel functional equation for their Laplace transforms. In Section 4.1.4 we state our main results. In Section 4.1.4.1, we first present result (i), providing explicit formulas for the Laplace transforms of the Green's functions in Theorem 4.5. Then, in Section 4.1.4.2, we state result (ii), which provides the exact asymptotics of the Green's functions g^{z_0} along all trajectories in the plane $\mathbb{R}^2$. In other words, we consider the limit of $g^{z_0}(r\cos\alpha, r\sin\alpha)$ as $r\to\infty$ and $\alpha\to\alpha_0$, where $\alpha_0\in[0,2\pi)$ and $z_0\in\mathbb{R}^2$ denotes the initial state. We highlight two key explicit angles for the asymptotic study,

$$\alpha_b \text{ and } \widetilde{\alpha}_b\in[0,2\pi]$$

which are related to branching points coming from the branches $Y^{\pm}(x)$ and $Z^{\pm}(x)$ related to $\gamma_+(x,y)$ and $\gamma_-(x,z)$ as in Definitions (4.1.20) and (4.1.21). These angles define the domain $\mathcal{M}$ which takes the following form depending on the configuration (see Figure 4.5 below) :

$$\mathcal{M}=[\alpha_b,\widetilde{\alpha}_b]\cup[\pi,2\pi] \text{ or } [0,\widetilde{\alpha}_b]\cup[\pi,\alpha_b] \text{ or } [0,\pi]\cup[\widetilde{\alpha}_b,\alpha_b] \text{ or } [\alpha_b,\pi]\cup[\widetilde{\alpha}_b,2\pi].$$

The different types of directions are studied in Theorems 4.6, 4.7, 4.8, 4.9 and 4.10, see Figure 4.6. In Section 4.1.4.3, we state result (iii) in Theorem 4.13, concerning the harmonic functions, the full Martin boundary, and the minimal Martin boundary which is homeomorphic to $\mathcal{M}$. In Section 4.2, we study the process in greater detail, establish its existence and uniqueness in the case where the generator is in divergence form and prove the main functional equation. In Section 4.3, we prove Theorem 4.5 using the functional equation from which we derive Theorem 4.6 applying Tauberian theorems; in Section 4.4, we prove Theorems 4.7–4.10, the proofs use variants of the saddle-point method; and in the short Section 4.5, we prove Theorem 4.13 which follows from the asymptotic results. Finally, in Section 4.6, we extend our analysis to a two-dimensional oblique skew diffusion, where the flux is not necessarily continuous: substantially new results under this assumption are found in Theorems 4.46, 4.47 and 4.48.

4.1.2 Planar skew bang-bang diffusion with piecewise constant coefficients

We now define a two-dimensional generalization of the SBBBM, which acts obliquely at the interface between two half-planes. The diffusion coefficients and the drift vector are piecewise constant, taking different values in the upper and lower half-planes. Let Σ^+,Σ^- be covariance matrices and μ^+,μ^- vectors in $\mathbb{R}^2$. We define the matrices $\Sigma(y)$ and $\sigma(y)$ and the vector $\mu(y)$ such that

$$\Sigma(y)=\sigma(y)\sigma^\top(y)=\Sigma^-\mathbb{1}_{y<0}+\Sigma^+\mathbb{1}_{y\geq 0}\quad\text{and}\quad \mu(y)=\mu^-\mathbb{1}_{y<0}+\mu^+\mathbb{1}_{y\geq 0}.$$

Let $q=(q_1,q_2)\in\mathbb{R}\times(-1,1)$. This vector represents a singular drift on the axis $\{y=0\}$. The parameter $q_2=2\beta-1$ corresponds to the classical skewness intensity at the boundary, when the process hits the axis $y=0$, it restarts on the $y>0$ side with probability $\beta=(1+q_2)/2$, and on the $y<0$ side with probability $1-\beta=(1-q_2)/2$. The parameter q_1 fixes the singular drift component along the axis and thus introduces the oblique direction of the skew behaviour at the interface.

Definition 4.1 (Planar skew bang-bang diffusion with piecewise constant coefficients)**.** *We define a planar skew bang-bang diffusion associated with coefficients* $(\Sigma^+,\Sigma^-,\mu^+,\mu^-)$ *and with the skew vector* q*, as a continuous adapted process* $(Z_t)_{t\geq 0}=(A_t,B_t)_{t\geq 0}$ *on* $\mathbb{R}^2$*, such that* Z *can be expressed almost surely as*

$$Z_t=(A_t,B_t)=(a_0,b_0)+\int_0^t\sigma(B_s)\,dW_s+\int_0^t\mu(B_s)\,ds+qL_t^0(B),\quad t\geq 0,\tag{4.1.1}$$

where:

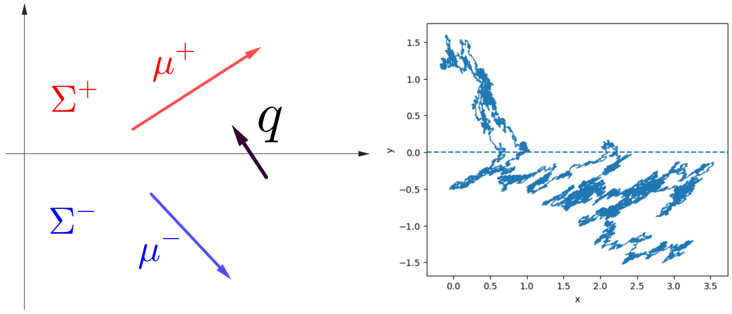


Figure 4.1: Parameters and path of the process. Locally, the process Z behaves like a planar Brownian motion with covariance Σ^+ (resp. Σ^-) and drift μ^+ (resp. μ^-) in the upper (resp. lower) half-plane. When the process hits the axis $\{y = 0\}$, it is subject to a singular drift in the direction of q.

(i) $z_0 = (a_0, b_0) \in \mathbb{R}^2$ *is the starting point;*

(ii) $(W_t)_{t\geq 0}$ *is an adapted two-dimensional standard Brownian motion;*

(iii) $(L_t^0(B))_{t\geq 0}$ *is the symmetric local time at* 0 *of* B *defined by*

$$L_t^0(B) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[-\varepsilon,\varepsilon]}(B_s)\, ds. \tag{4.1.2}$$

Let q_0 be the vector defined by

$$q_0 = \left(\frac{\Sigma_{12}^+ - \Sigma_{12}^-}{\Sigma_{22}^+ + \Sigma_{22}^-}, \frac{\Sigma_{22}^+ - \Sigma_{22}^-}{\Sigma_{22}^+ + \Sigma_{22}^-} \right). \tag{4.1.3}$$

In all sections of the chapter, except Section 4.6, we will assume that

$$q = q_0.$$

In this case, the generator, understood in the sense of Strook [113, Theorem II.3.1], can be written in divergence form; see (4.1.4) below. This form is relevant in physical applications [85]. The transmission conditions then ensure the continuity of the probability flux across the boundary, see the partial differential equation (4.1.40). The following proposition is proved in Section 4.2.1.

Proposition 4.2 (Existence, uniqueness and divergence form)**.** *There exists a unique solution* Z_t *(both pathwise and in law) to the stochastic differential equation* (4.1.1) *associated with the skew vector* $q = q_0$*. This process defines a strong Markov process and its generator can be expressed as follows:*

$$\mathcal{L} = \frac{1}{2}\nabla \cdot (\Sigma\nabla) + \mu \cdot \nabla. \tag{4.1.4}$$

For the remainder of this chapter, we will work under the following hypothesis regarding the drift:

$$\mu_1^+ > 0, \quad \mu_2^+ > 0, \quad \mu_1^- > 0, \quad \mu_2^- < 0, \tag{4.1.5}$$

see Figure 4.1. In particular, this hypothesis means that the process $Z_t = (A_t, B_t)$ is transient and eventually no longer touches the horizontal axis. The process B is a one dimensional skewed bang-bang process of parameter q_2 at 0 with positive drift on the nonnegative real axis and negative drift on the negative real axis.

4.1.3 Green's functions and functional equation

We have just seen that, when $q = q_0$, the generator can be expressed in divergence form. From Strook [113, Theorem II.3.8], we deduce that for all $t > 0$, the law of Z_t has a density $p_t^{z_0}(z)$ with respect to the Lebesgue measure on $\mathbb{R}^2$. Furthermore, the function $(t, z_0, z) \longmapsto p_t^{z_0}(z)$ is continuous on $(0, +\infty) \times \mathbb{R}^2 \times \mathbb{R}^2$.

Definition 4.3 (Green's measures and Laplace transforms)**.** *For $z_0 = (a_0, b_0) \in \mathbb{R}^2$, we define the Green's measure on the plane $\mathbb{R}^2$ by*

$$G(z_0, S) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} \mathbb{1}_S(Z_t)\, dt\right] = \iint_S g^{z_0}(a,b)\, da\, db, \quad \forall S \in \mathcal{B}(\mathbb{R}^2) \tag{4.1.6}$$

where its density g^{z_0} is called the Green's function and is equal to

$$g^{z_0}(a,b) = \int_0^{+\infty} p_t^{z_0}(a,b)\, dt, \quad \forall (a,b) \in \mathbb{R}^2. \tag{4.1.7}$$

For $x, y, z \in \mathbb{R}$, we also define the Laplace transforms of $G(z_0, \cdot)$ restricted to the half-planes $\mathbb{R}\times(0,+\infty)$ and $\mathbb{R} \times (-\infty, 0)$ by

$$\varphi_+^{z_0}(x,y) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} e^{xA_t + yB_t} \mathbb{1}_{B_t>0}\, dt\right] = \int_{\mathbb{R}}\int_0^{\infty} g^{z_0}(a,b) e^{xa+yb}\, da\, db,$$

$$\varphi_-^{z_0}(x,z) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} e^{xA_t + zB_t} \mathbb{1}_{B_t<0}\, dt\right] = \int_{\mathbb{R}}\int_{-\infty}^{0} g^{z_0}(a,b) e^{xa+zb}\, da\, db.$$

Finally, we define the Green's measure on the axis $\mathbb{R} \times \{0\}$ by

$$H(z_0, S) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} \mathbb{1}_S(A_t)\, dL_t^0(B)\right], \quad \forall S \in \mathcal{B}(\mathbb{R}) \tag{4.1.8}$$

and its Laplace transform φ^{z_0} as

$$\varphi^{z_0}(x) = \mathbb{E}_{z_0}\left[\int_0^{+\infty} e^{xA_t}\, dL_t^0(B)\right]. \tag{4.1.9}$$

For $x, y, z \in \mathbb{C}$, we set:

$$\begin{cases} \gamma_+(x,y) = \frac{1}{2}(x,y)\cdot\Sigma^+(x,y) + (x,y)\cdot\mu^+ = \frac{1}{2}(\Sigma_{11}^+ x^2 + 2\Sigma_{12}^+ xy + \Sigma_{22}^+ y^2) + \mu_1^+ x + \mu_2^+ y \\ \gamma_-(x,z) = \frac{1}{2}(x,z)\cdot\Sigma^-(x,z) + (x,z)\cdot\mu^- = \frac{1}{2}(\Sigma_{11}^- x^2 + 2\Sigma_{12}^- xz + \Sigma_{22}^- z^2) + \mu_1^- x + \mu_2^- z \\ \gamma(x,y,z) = q_1 x + \frac{1}{2}(y(1+q_2) + z(q_2-1)). \end{cases} \tag{4.1.10}$$

These polynomials are the coefficients of the following three-variable kernel functional equation, which is at the heart of this chapter. The following proposition is proven in Sections 4.2.3 and 4.2.4.

Proposition 4.4 (Functional Equation)**.** *Suppose that (4.1.5) holds. Then, there exists $\eta > 0$ such that for all $x \in (-\eta, 0)$, $y < 0$, and $z > 0$, the Laplace tranforms $\varphi_-^{z_0}(x,z)$, $\varphi_+^{z_0}(x,y)$, and $\varphi^{z_0}(x)$ are finite and satisfy the following functional equation:*

$$\gamma_-(x,z)\varphi_-^{z_0}(x,z) + \gamma_+(x,y)\varphi_+^{z_0}(x,y) + \gamma(x,y,z)\varphi^{z_0}(x) = -e^{xa_0 + yb_0\mathbb{1}_{b_0>0} + zb_0\mathbb{1}_{b_0<0}}. \tag{4.1.11}$$

The polynomials $\gamma_+(x,y)$ and $\gamma_-(x,z)$ are called the kernels. We are now going to define some important quantities that will be useful for stating the results. We first determine the complex branches

$Y^{\pm}(x)$ and $Z^{\pm}(x)$ which satisfy $\gamma_+(x, Y^{\pm}(x)) = 0$ and $\gamma_-(x, Z^{\pm}(x)) = 0$. By elementary considerations, these branches are given by

$$Y^{\pm}(x) = \frac{1}{\Sigma^+_{22}}\Big(-\Sigma^+_{12}x - \mu^+_2 \pm \sqrt{({\Sigma^+_{12}}^2 - \Sigma^+_{11}\Sigma^+_{22})x^2 + 2(\mu^+_2\Sigma^+_{12} - \mu^+_1\Sigma^+_{22})x + {\mu^+_2}^2}\Big) \tag{4.1.12}$$

and

$$Z^{\pm}(x) = \frac{1}{\Sigma^-_{22}}\Big(-\Sigma^-_{12}x - \mu^-_2 \pm \sqrt{({\Sigma^-_{12}}^2 - \Sigma^-_{11}\Sigma^-_{22})x^2 + 2(\mu^-_2\Sigma^-_{12} - \mu^-_1\Sigma^-_{22})x + {\mu^-_2}^2}\Big). \tag{4.1.13}$$

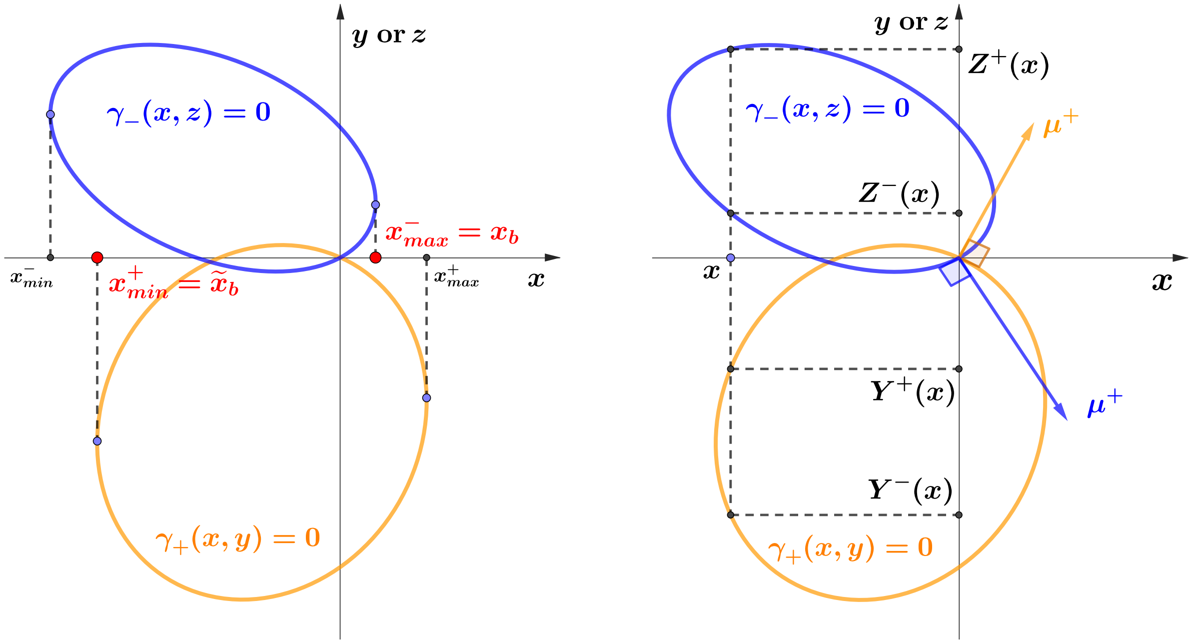
Furthermore, $Y^{\pm}(x)$ have branching points $x^+_{min} < 0$ and $x^+_{max} > 0$ given by

$$x^+_{min} = \frac{\mu^+_2\Sigma^+_{12} - \mu^+_1\Sigma^+_{22} - \sqrt{D_1}}{\det(\Sigma^+)}, \quad x^+_{max} = \frac{\mu^+_2\Sigma^+_{12} - \mu^+_1\Sigma^+_{22} + \sqrt{D_1}}{\det(\Sigma^+)} \tag{4.1.14}$$

with $D_1 = (\mu^+_2\Sigma^+_{12} - \mu_1{\Sigma^+_{22}}^2)^2 + {\mu^+_2}^2\det(\Sigma^+)$. Similarly, $Z^{\pm}(x)$ have branching points $x^-_{min} < 0$ and $x^-_{max} > 0$ with symmetric formulas. Finally, we denote the maximin and the minimax values of the branching points as follows:

$$\widetilde{x}_b = \max(x^+_{min}, x^-_{min}) < 0, \quad x_b = \min(x^+_{max}, x^-_{max}) > 0. \tag{4.1.15}$$

These points are illustrated in Figure 4.2.



(a) Branching points $\widetilde{x}_b$ and x_b.

(b) Branches $Y^{\pm}(x)$ and $Z^{\pm}(x)$.

Figure 4.2: Ellipses $\{(x,y) \in \mathbb{R}^2 : \gamma_+(x,y) = 0\}$ and $\{(x,z) \in \mathbb{R}^2 : \gamma_-(x,z) = 0\}$.

4.1.4 Main results

4.1.4.1 Explicit expression of the Laplace transforms

The first theorem provides explicit formulas for the Laplace transforms φ^{z_0} and $\varphi^{z_0}_{\pm}$. The proof is given in Section 4.3 by substituting $y = Y^-(x)$ and $z = Z^+(x)$ into the functional equation (4.1.11) to cancel the kernels.

Theorem 4.5. *The Laplace transform $\varphi^{z_0}(x)$ extends to a holomorphic function on $\mathbb{C}\setminus\big((-\infty,\widetilde{x}_b]\cup[x_b,+\infty)\big)$ and can be expressed as follows:*

$$\varphi^{z_0}(x) = -\frac{e^{xa_0+Y^-(x)b_0\mathbb{1}_{b_0>0}+Z^+(x)b_0\mathbb{1}_{b_0<0}}}{\gamma(x,Y^-(x),Z^+(x))}. \tag{4.1.16}$$

The points x_b and $\widetilde{x}_b$ are the branching points of φ^{z_0}. Furthermore,

$$\varphi_+^{z_0}(x,y) = \frac{-\gamma(x,y,Z^+(x))\varphi^{z_0}(x) - e^{xa_0+yb_0\mathbb{1}_{b_0>0}+zb_0\mathbb{1}_{b_0<0}}}{\gamma_+(x,y)}. \tag{4.1.17}$$

Similarly, a symmetric expression holds for $\varphi_-^{z_0}$.

4.1.4.2 Asymptotics of Green's functions

Before stating the main asymptotic theorems, a bit more notation needs to be introduced. For $\alpha\in(0,\pi)$, we define

$$(x(\alpha),y(\alpha)) = \underset{\gamma_+(x,y)=0}{\operatorname{argmax}}\,(\cos(\alpha)x+\sin(\alpha)y)\,, \tag{4.1.18}$$

and for $\alpha\in(\pi,2\pi)$, we set

$$(x(\alpha),z(\alpha)) = \underset{\gamma_-(x,z)=0}{\operatorname{argmax}}\,(\cos(\alpha)x+\sin(\alpha)z)\,. \tag{4.1.19}$$

See Figure 4.3 for a geometric interpretation. In the rest of this chapter, these will be the key saddle points we look at. Note that the functions

$$\alpha\in(0,\pi)\longmapsto(x(\alpha),y(\alpha))\in\{(x,Y^+(x))\ :\ x^+_{min}<x<x^+_{max}\}$$

and

$$\alpha\in(\pi,2\pi)\longmapsto(x(\alpha),z(\alpha))\in\{(x,Z^-(x))\ :\ x^-_{min}<x<x^-_{max}\}$$

are C^∞ diffeomorphisms. Note also that μ^+ is orthogonal to the ellipse $\{(x,y)\ :\ \gamma_+(x,y)=0\}$ at $(0,0)$, see Figure 4.2b. Hence, the direction $\alpha_{\mu^+}=\arctan(\mu_2^+/\mu_1^+)\in(0,\pi/2)$ of the drift μ^+ satisfies $(x(\alpha_{\mu^+}),y(\alpha_{\mu^+}))=(0,0)$. Similarly, if $\alpha_{\mu^-}:=\arctan(\mu_2^-/\mu_1^-)+2\pi\in(3\pi/2,2\pi)$ is the direction of the drift μ^-, then $(x(\alpha_{\mu^-}),z(\alpha_{\mu^-}))=(0,0)$.

If $x^+_{max}>x^-_{max}$, we define α_b as the unique angle $\alpha_b\in(0,\pi)$ such that

$$(x(\alpha_b),y(\alpha_b))=(x_b,Y^+(x_b)), \tag{4.1.20}$$

see Figure 4.4a. In this case, we have $0<\alpha_b<\alpha_{\mu^+}$ since $0=x(\alpha_{\mu^+})<x_b$.

If $x^+_{max}<x^-_{max}$, we define α_b as the unique angle $\alpha_b\in(\pi,2\pi)$ such that

$$(x(\alpha_b),z(\alpha_b))=(x_b,Z^-(x_b)). \tag{4.1.21}$$

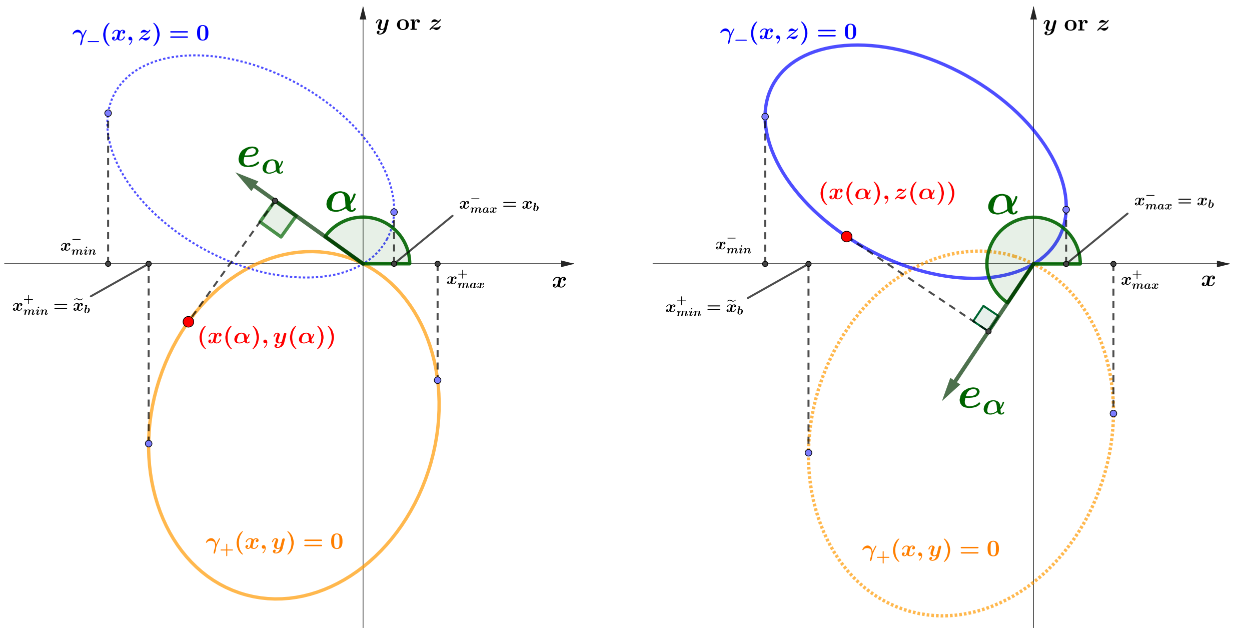
In this case, we have $\alpha_{\mu^-}<\alpha_b<2\pi$ since $0=x(\alpha_{\mu^-})<x_b$.

Next, considering $\widetilde{x}_b$, we similarly define $\widetilde{\alpha}_b$ by the condition $x(\widetilde{\alpha}_b)=\widetilde{x}_b$, where $\widetilde{\alpha}_b$ belongs to $(0,\pi)$ if $x^+_{min}<x^-_{min}$ and to $(\pi,2\pi)$ if $x^+_{min}>x^-_{min}$, see Figure 4.4b. Figure 4.5 provides an overview of all possible cases: A,B,C, and D. Then, we define

$$\mathcal{M}=\{\alpha\in[0,2\pi]:\widetilde{x}_b\leqslant x(\alpha)\leqslant x_b\}$$

which can be expressed as follows for the cases shown in Figure 4.5:

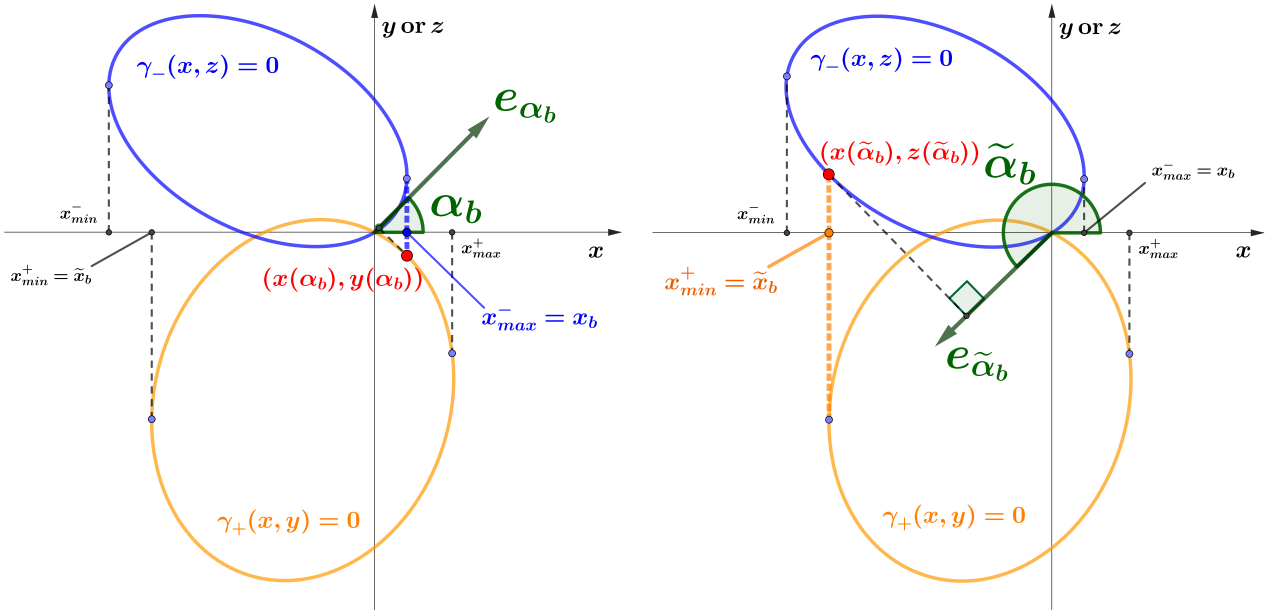
$$\mathcal{M}=\begin{cases}[\alpha_b,\widetilde{\alpha}_b]\cup[\pi,2\pi] & \text{in the case } A: \quad x^+_{max}>x^-_{max} \text{ and } x^+_{min}<x^-_{min},\\ [0,\widetilde{\alpha}_b]\cup[\pi,\alpha_b] & \text{in the case } B: \quad x^+_{max}<x^-_{max} \text{ and } x^+_{min}<x^-_{min},\\ [0,\pi]\cup[\widetilde{\alpha}_b,\alpha_b] & \text{in the case } C: \quad x^+_{max}<x^-_{max} \text{ and } x^+_{min}>x^-_{min},\\ [\alpha_b,\pi]\cup[\widetilde{\alpha}_b,2\pi] & \text{in the case } D: \quad x^+_{max}>x^-_{max} \text{ and } x^+_{min}>x^-_{min}.\end{cases}$$



(a) If $\alpha \in (0,\pi)$, the construction is done on the ellipse $\{(x,y) : \gamma_+(x,y)=0\}$.

(b) If $\alpha \in (\pi,2\pi)$, the construction is done on the ellipse $\{(x,z) : \gamma_-(x,z)=0\}$.

Figure 4.3: Geometric construction of $(x(\alpha),y(\alpha))$ and $(x(\alpha),z(\alpha))$. The vector e_α is defined by $e_\alpha=(\cos(\alpha),\sin(\alpha))$.



(a) Definition of α_b: here $x_b = x^-_{max} = x(\alpha_b)$.

(b) Definition of $\widetilde{\alpha}_b$: here $\widetilde{x}_b = x^+_{min} = x(\widetilde{\alpha}_b)$.

Figure 4.4: Definitions of α_b and $\widetilde{\alpha}_b$ in the cases $x_b = x^-_{max}$ and $\widetilde{x}_b = x^+_{min}$.

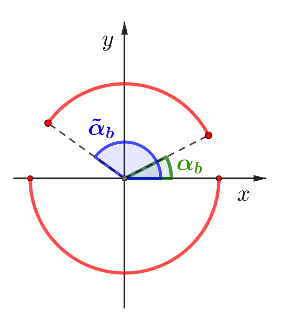
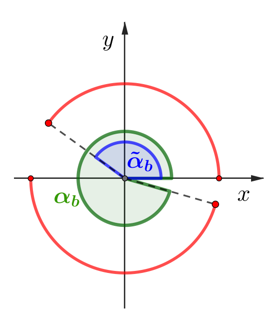
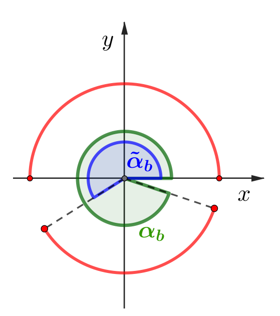
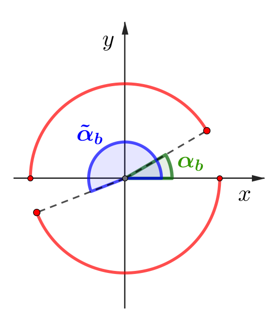
Case A	Case B	Case C	Case D
$x^+_{max} > x^-_{max}$ $x^+_{min} < x^-_{min}$	$x^+_{max} < x^-_{max}$ $x^+_{min} < x^-_{min}$	$x^+_{max} < x^-_{max}$ $x^+_{min} > x^-_{min}$	$x^+_{max} > x^-_{max}$ $x^+_{min} > x^-_{min}$
			

Figure 4.5: Angles α_b and $\widetilde{\alpha}_b$ depending on the parameters. The red arcs $\mathcal{M}$ correspond to the minimal Martin boundary.

The set $\mathcal{M}$ is represented by the red arcs in Figure 4.5. As we will see in Theorem 4.13, this is the minimal Martin boundary.

Main asymptotics theorems. We now state the asymptotic behaviour of the Green's functions, $g^{z_0}(r\cos(\alpha), r\sin(\alpha))$, as $r \to +\infty$ and $\alpha \to \alpha_0$. Figure 4.6 summarises the different cases covered by each theorem and indicates the associated colours. In Theorem 4.6 (brown arrows), we consider the case where the angle α is fixed at $\alpha_0 \in \{0, \pi\}$. In Theorem 4.7 (red arrows), we focus on directions where $\alpha \to \alpha_0$ with $\alpha_0 \in \mathring{\mathcal{M}}$. In Theorem 4.8 (purple arrows), we study the case when $\alpha \to 0$ or π but remaining within the red arc zones $\mathring{\mathcal{M}}$. Theorem 4.9 (blue arrows) treats the asymptotic behavior as $\alpha \to \alpha_b$ or $\alpha \to \widetilde{\alpha}_b$. Finally, Theorem 4.10 (green arrows) provides asymptotics as $\alpha \to \alpha_0$ with $\alpha_0 \in ([0, 2\pi] \backslash \mathcal{M})$, and for $\alpha \to 0$ or π while remaining outside $\mathcal{M}$.

The first theorem follows from a Tauberian theorem applied to Laplace transforms, based on the explicit expression (4.1.17). The proof is given in Section 4.3.

Theorem 4.6 (Asymptotics, case $\alpha = 0, \pi$)**.** *Let $z_0 \in \mathbb{R}^2$. The following asymptotics hold:*

$$g^{z_0}(r,0) \underset{r\to+\infty}{\sim} C_0 f_0(z_0) \frac{e^{-rx_b}}{r^{3/2}}, \qquad g^{z_0}(-r,0) \underset{r\to+\infty}{\sim} C_\pi f_\pi(z_0) \frac{e^{-r\widetilde{x}_b}}{r^{3/2}}, \tag{4.1.22}$$

where $C_0, C_\pi > 0$ and $f_0(z_0), f_\pi(z_0) > 0$ are defined as follows. The constant C_0 is given by:

$$C_0 = \begin{cases} -\dfrac{\sqrt{\det(\Sigma^+)(x_b - x^+_{min})}}{\sqrt{\pi}\Sigma^+_{22}(\Sigma^+_{22} + \Sigma^-_{22})\gamma(x_b, Y^-(x_b), Z^+(x_b))} > 0 & \textit{if } x_b = x^+_{max} < x^-_{max} \\ -\dfrac{\sqrt{\det(\Sigma^-)(x_b - x^-_{min})}}{\sqrt{\pi}\Sigma^-_{22}(\Sigma^+_{22} + \Sigma^-_{22})\gamma(x_b, Y^-(x_b), Z^+(x_b))} > 0 & \textit{if } x_b = x^-_{max} < x^+_{max}. \end{cases} \tag{4.1.23}$$

Moreover, $f_0(z_0)$ is defined as follows:

- *If $x_b = x^+_{max} < x^-_{max}$ (cases B and C of Figure 4.5), then*

$$f_0(a_0, b_0) = \begin{cases} \left(b_0 - \dfrac{\Sigma^+_{22}}{(\Sigma^+_{22} + \Sigma^-_{22})\gamma(x_b, Y^-(x_b), Z^+(x_b))} \right) e^{x_b a_0 + b_0 Y^\pm(x_b)} & \textit{if } b_0 \geq 0 \\ \dfrac{-\Sigma^+_{22}}{(\Sigma^+_{22} + \Sigma^-_{22})\gamma(x_b, Y^-(x_b), Z^+(x_b))} e^{x_b a_0 + b_0 Z^+(x_b)} & \textit{if } b_0 < 0 \end{cases} \tag{4.1.24}$$

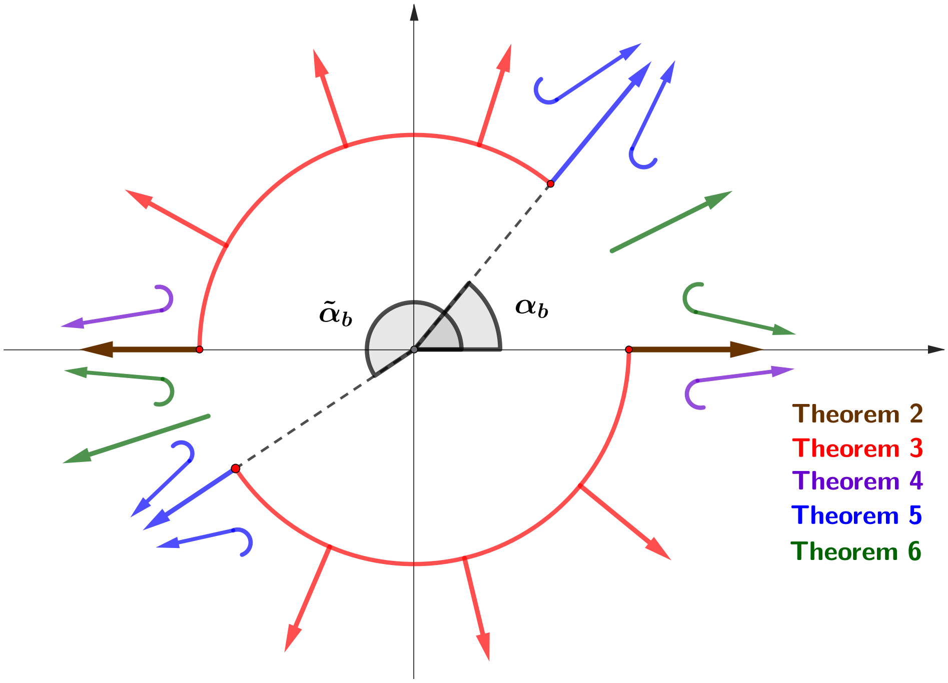


Figure 4.6: Direction of asymptotics: Theorems 4.6 to 4.10 (case D). Rounded arrows indicate that the angle α approaches the corresponding direction α_0 only from one side, i.e., either $\alpha > \alpha_0$ or $\alpha < \alpha_0$, depending on the arrow. For example, the green rounded arrow on the right means $\alpha \to 0$ with $\alpha > 0$.

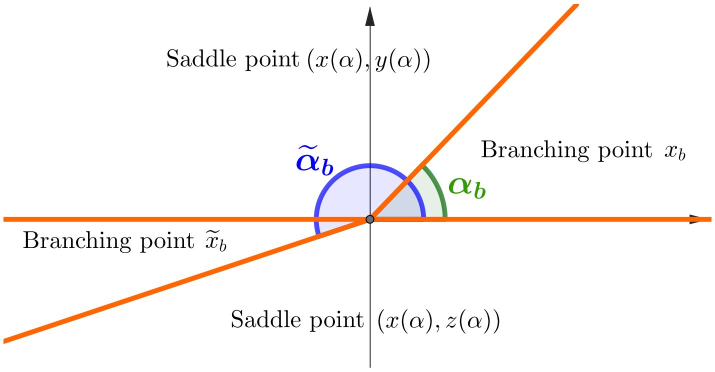


Figure 4.7: Type of asymptotics according to directions: saddle-point type or branching point.

- *If* $x_b = x^-_{max} < x^+_{max}$ *(cases A and D of Figure 4.5), then*

$$f_0(a_0,b_0) = \begin{cases} \dfrac{-\Sigma^-_{22}}{(\Sigma^+_{22}+\Sigma^-_{22})\gamma(x_b, Y^-(x_b), Z^+(x_b))} e^{x_b a_0 + b_0 Y^-(x_b)} & \text{if } b_0 \geq 0 \\ \left(-b_0 - \dfrac{\Sigma^-_{22}}{(\Sigma^+_{22}+\Sigma^-_{22})\gamma(x_b, Y^-(x_b), Z^+(x_b))}\right) e^{x_b a_0 + b_0 Z^\pm(x_b)} & \text{if } b_0 < 0 \end{cases} \tag{4.1.25}$$

The expressions for C_π *and* $f_\pi(z_0)$ *are symmetric.*

The following theorem gives the asymptotics in the most typical case when $\alpha \in \mathring{\mathcal{M}}$. It is proved at the end of Section 4.4.3 using the saddle point method on a Riemann surface after extending the Laplace transforms.

Theorem 4.7 (Asymptotics for $\alpha \to \alpha_0$, $\alpha_0 \in \mathring{\mathcal{M}}$)**.** *Let* $z_0 \in \mathbb{R}^2$*. Then, if* $\alpha_0 \in \mathring{\mathcal{M}} \cap (0,\pi)$*, we have*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} C^+(\alpha_0) h_{\alpha_0}(z_0) \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}} \tag{4.1.26}$$

and if $\alpha_0 \in \mathring{\mathcal{M}} \cap (\pi, 2\pi)$*, the following asymptotics hold*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} C^-(\alpha_0) h_{\alpha_0}(z_0) \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)z(\alpha))}}{\sqrt{r}} \tag{4.1.27}$$

where $C^\pm(\alpha)$ *and* $h_{\alpha_0}(z_0)$ *are defined as follows. The coefficient* $C^\pm(\alpha)$ *is given by:*

$$C^\varepsilon(\alpha) = \frac{1}{\sqrt{2\pi\left(\Sigma^\varepsilon_{11}\sin^2(\alpha) - 2\Sigma^\varepsilon_{12}\sin(\alpha)\cos(\alpha) + \Sigma^\varepsilon_{22}\cos^2(\alpha)\right)}}\sqrt{\frac{\sin(\alpha)}{\partial_y \gamma_\varepsilon(x(\alpha), y(\alpha))}}, \quad \varepsilon \in \{+,-\}. \tag{4.1.28}$$

Furthermore, $h_\alpha(z_0)$ *is defined as follows:*

- *If* $\alpha \in \mathcal{M} \cap (0,\pi)$*, then*

$$h_\alpha(z_0) = \begin{cases} e^{a_0 x(\alpha) + b_0 y(\alpha)} - \dfrac{\gamma(x(\alpha), y(\alpha), Z^+(x(\alpha)))}{\gamma(x(\alpha), Y^-(x(\alpha)), Z^+(x(\alpha)))} e^{a_0 x(\alpha) + b_0 Y^-(x(\alpha))} & \text{if } b_0 \geq 0 \\ \left(1 - \dfrac{\gamma(x(\alpha), y(\alpha), Z^+(x(\alpha)))}{\gamma(x(\alpha), Y^-(x(\alpha)), Z^+(x(\alpha)))}\right) e^{a_0 x(\alpha) + b_0 Z^+(x(\alpha))} & \text{if } b_0 < 0 \end{cases} \tag{4.1.29}$$

- *If* $\alpha \in \mathcal{M} \cap (\pi, 2\pi)$*, then*

$$h_\alpha(z_0) = \begin{cases} \left(1 - \dfrac{\gamma(x(\alpha), Y^-(x(\alpha)), z(\alpha))}{\gamma(x(\alpha), Y^-(x(\alpha)), Z^+(x(\alpha)))}\right) e^{a_0 x(\alpha) + b_0 Y^-(x(\alpha))} & \text{if } b_0 \geq 0 \\ e^{a_0 x(\alpha) + b_0 z(\alpha)} - \dfrac{\gamma(x(\alpha), Y^-(x(\alpha)), z(\alpha))}{\gamma(x(\alpha), Y^-(x(\alpha)), Z^+(x(\alpha)))} e^{a_0 x(\alpha) + b_0 Z^+(x(\alpha))} & \text{if } b_0 < 0 \end{cases} \tag{4.1.30}$$

The functions h_α are in fact harmonic functions, see Theorem 4.13 later.

The next theorem states the asymptotics of Green's functions as $\alpha \to 0, \pi$ and $\alpha \in \mathring{\mathcal{M}}$. It establishes the missing connection between Theorem 4.6 and Theorem 4.7. To clarify this point further, let us note that in (4.1.26) the constant term $C(\alpha_0)h_{\alpha_0}(z) \to 0$ as $\alpha_0 \to 0$. Hence, in the case $\alpha_0 = 0$, we need to expand the asymptotics (4.1.26) up to the second term. The saddle-point method yields :

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to 0,\, \alpha\geq 0}}{\sim} \left(C^{+}(\alpha)\, h_{\alpha}(z_0) + \frac{C_0 f_0(z_0)}{r} \right) \frac{e^{-r(\cos(\alpha)\, x(\alpha) + \sin(\alpha)\, y(\alpha))}}{\sqrt{r}},$$

where

$$C^{+}(\alpha)\, h_{\alpha}(z_0) \underset{\alpha\to 0}{\sim} \kappa C_0 f_0(z_0)\, \alpha$$

for some constant $\kappa > 0$ (see (4.4.28)). The first term corresponds to Theorem 4.7, while the second corresponds to Theorem 4.6. As $r \to \infty$ and $\alpha \to 0$, we observe a competition between these two contributions. It is this competition that is established in the following theorem, proved at the end of Section 4.4.3.

Theorem 4.8 (Asymptotics as $\alpha \to 0, \pi$ along $\mathcal{M}$)**.** *Let $z_0 \in \mathbb{R}^2$ and let $\alpha_0 = 0$.*

- *Assume that $x^{+}_{max} < x^{-}_{max}$ (cases B and C of Figure 4.5). Then, for $\alpha \geq 0$ we have*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to 0, \alpha\geq 0}}{\sim} C_0 f_0(z_0) \left(\kappa\alpha + \frac{1}{r} \right) \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}. \tag{4.1.31}$$

  where $\kappa > 0$ is an explicit constant (cf. (4.4.29) below), C_0 is given by (4.1.23), and $f_0(z_0)$ is given by (4.1.24). Moreover,

$$\partial_{\alpha} \left(C(\alpha) h_{\alpha}(z_0) \right)\big|_{\alpha = 0^{+}} = \kappa C_0 f_0(z_0).$$

- *Assume now that $x^{+}_{max} > x^{-}_{max}$ (cases A and D of Figure 4.5). Then, for $\alpha \leq 0$,*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to 0, \alpha\leq 0}}{\sim} C_0 f_0(z_0) \left(\kappa|\alpha| + \frac{1}{r} \right) \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)z(\alpha))}}{\sqrt{r}}, \tag{4.1.32}$$

  where $f_0(z_0)$ is given by (4.1.25). Moreover,

$$\partial_{\alpha} \left(C(\alpha) h_{\alpha}(z_0) \right)\big|_{\alpha = 0^{-}} = \kappa C_0 f_0(z_0).$$

The symmetric result holds for $\alpha_0 = \pi$.

Now, let us consider the asymptotics as $\alpha \to \alpha_b$ (and $\widetilde{\alpha}_b$). If α remains within $\mathcal{M}$, then the asymptotics are given by those of Theorem 4.7. However, this is no longer the case in the parts outside $\mathcal{M}$. The asymptotic behaviour arises from the competition between the saddle point $(x(\alpha), y(\alpha))$ and the branching point $(x(\alpha_b), y(\alpha_b))$ and thus depends on the speed of convergence of $\alpha \to \alpha_b$ relative to $r \to \infty$. The proof can be found at the end of Section 4.4.4.

Theorem 4.9 (Asymptotics for $\alpha \to \alpha_b, \widetilde{\alpha}_b$)**.** *Let $z_0 \in \mathbb{R}^2$ and let $\alpha_0 = \alpha_b$. Assume $x^{+}_{max} > x^{-}_{max}$, i.e., $\alpha_b \in (0, \pi)$ (cases A and D of Figure 4.5). Define*

$$K_{+}^{2} = \frac{\Sigma^{+}_{22}\big(y(\alpha_b) - Y^{-}(x(\alpha_b))\big)}{4|\sin(\alpha_b)| \left(\Sigma^{+}_{11} \sin^2(\alpha_b) - 2\Sigma^{+}_{12} \sin(\alpha_b)\cos(\alpha_b) + \Sigma^{+}_{22}\cos^2(\alpha_b) \right)}. \tag{4.1.33}$$

Then:

(i) If $\alpha > \alpha_b$, then (4.1.26) holds.

(ii) If $\alpha < \alpha_b$ and $r(\alpha_b - \alpha)^2$ is bounded, then (4.1.26) holds.

(iii) If $\alpha < \alpha_b$, $r(\alpha - \alpha_b)^2 \to +\infty$, and $\frac{e^{K_{+}^{2} r(\alpha_b - \alpha)^2}}{r(\alpha_b - \alpha)^{3/2}} \to 0$, then (4.1.26) holds.

(iv) If $\alpha < \alpha_b$, $r(\alpha-\alpha_b)^2 \to +\infty$, and $\frac{e^{K_+^2 r(\alpha_b-\alpha)^2}}{r(\alpha_b-\alpha)^{3/2}} \to +\infty$, then

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_b}}{\sim} C_{br} f_0(z_0) \frac{e^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b))}}{r^{3/2}(\alpha_b-\alpha)^{3/2}}, \tag{4.1.34}$$

where $f_0(z_0) > 0$ is given by (4.1.25) and $C_{br} = |\sin(\alpha_b)|^{3/2} C_0$ (see (4.1.23)).

(v) If $\alpha < \alpha_b$ (resp. $\alpha > \alpha_b$), $r(\alpha-\alpha_b)^2 \to +\infty$, and

$$\frac{e^{K_+^2 r(\alpha_b-\alpha)^2}}{r(\alpha_b-\alpha)^{3/2}} \to c > 0,$$

then

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_b}}{\sim} (C(\alpha_b)h_{\alpha_b}(z_0) + c\, C_{br} f_0(z_0)) \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}, \tag{4.1.35}$$

where $h_{\alpha_b}(z_0)$ is given by (4.1.29).

Now assume that $x_{max}^+ < x_{max}^-$ (cases B and C of Figure 4.5). Then, the analogous cases (i) to (v) hold. More precisely, let

$$K_-^2 = \frac{\Sigma_{22}^-\big(z(\alpha_b) - Z^+(x(\alpha_b))\big)}{4|\sin(\alpha_b)|\big(\Sigma_{11}^-\sin^2(\alpha_b) - 2\Sigma_{12}^-\sin(\alpha_b)\cos(\alpha_b) + \Sigma_{22}^-\cos^2(\alpha_b)\big)}.$$

(i)-(iii) In cases (i) to (iii), replace K_+^2 by K_-^2 and the conditions $\alpha < \alpha_b$ (resp. $\alpha > \alpha_b$) by $\alpha > \alpha_b$ (resp. $\alpha < \alpha_b$). Then, (4.1.27) holds in these cases.

(iv)-(v) In cases (iv) to (v), replace K_+^2 by K_-^2 and the condition $\alpha < \alpha_b$ by $\alpha > \alpha_b$. Then, the corresponding asymptotics hold replacing $y(\alpha)$ by $z(\alpha)$. Namely, in case (iv),

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_b}}{\sim} C_{br} f_0(z_0) \frac{e^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)z(\alpha_b))}}{r^{3/2}|\alpha_b-\alpha|^{3/2}}, \tag{4.1.36}$$

where $f_0(z_0) > 0$ is given by (4.1.24). In case (v),

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_b}}{\sim} (C(\alpha_b)h_{\alpha_b}(z_0) + c\, C_{br} f_0(z_0)) \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)z(\alpha))}}{\sqrt{r}}, \tag{4.1.37}$$

where $h_{\alpha_b}(z_0)$ is given by (4.1.30).

The symmetric result holds for $\alpha_0 = \widetilde{\alpha}_b$.

The direction α_b gives rise to two harmonic functions. The first of these is $h_{\alpha_b}(z_0)$, which is defined in (4.1.29) when $\alpha = \alpha_b$. This function can also be viewed as the limit of the functions $h_\alpha(z_0)$ obtained in Theorem 4.7 as $\alpha \to \alpha_b$. The second function is $f_0(z_0)$, given by (4.1.24), and corresponds to the direction $\alpha_0 = 0$. When $\alpha \to \alpha_b$ and $r \to \infty$ at the rate described in case (v), linear combinations of these two functions determine the asymptotic constants as in (4.1.35).

The last asymptotic theorem deals with the remaining cases where $\alpha_0 \in [0, 2\pi] \backslash \mathcal{M}$ or $\alpha \to 0, \pi$ as $\alpha \notin \mathcal{M}$. The asymptotics are then determined by the branching points. The proof is given at the end of Section 4.4.5.

Theorem 4.10 (Asymptotics for $\alpha \to \alpha_0$, $\alpha_0 \in [0, 2\pi]\backslash\mathcal{M}$, or $\alpha \to 0, \pi$ with $\alpha \in [0, 2\pi]\backslash\mathcal{M}$)**.** *Assume $x^+_{max} > x^-_{max}$ (cases A and D of Figure 4.5). Let $z_0 \in \mathbb{R}^2$ and let $0 \le \alpha_0 < \alpha_b$. Then:*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0, \alpha\ge 0}}{\sim} C'_{br}(\alpha_0) f_0(z_0) \frac{e^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b))}}{r^{3/2}} \tag{4.1.38}$$

where $C'_{br}(\alpha_0) = \left(\frac{\sin(\alpha_b)}{\sin(\alpha_b-\alpha_0)}\right)^{3/2} C_0$ (see (4.1.23)) and where $f_0(z_0)$ is given by (4.1.25).

Assume now $x^+_{max} < x^-_{max}$ (cases B and C of Figure 4.5) and let $\alpha_b < \alpha_0 \le 2\pi$. Then,

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0, \alpha\le 2\pi}}{\sim} C'_{br}(\alpha_0) f_0(z_0) \frac{e^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)z(\alpha_b))}}{r^{3/2}} \tag{4.1.39}$$

where $C'_{br}(\alpha_0)$ has the same expression and where $f_0(z_0)$ is given by (4.1.24).

The symmetric results hold for $\widetilde{\alpha}_b < \alpha_0 \le \pi$ in the cases A and B and for $\pi \le \alpha_0 < \widetilde{\alpha}_b$ in the cases C and D.

Note that the asymptotics (4.1.38) formally match case (*iv*) of Theorem 4.9 in the direction α_b, since $\sin(\alpha_b - \alpha) \sim \alpha_b - \alpha$ as $\alpha \to \alpha_b$. Since $C'_{br}(\alpha)|_{\alpha=0} = C_0$, the asymptotics given by (4.1.38) also formally match (4.1.22) for $\alpha = 0$. Furthermore, we observe that a single harmonic function, $f_0(z_0)$ (arising from the asymptotics at $\alpha = 0$ obtained in Theorem 4.6) corresponds to all directions described in Theorem 4.10.

4.1.4.3 Martin Boundary and positive harmonic functions

Before stating the results on the Martin boundary and harmonic functions, let us recall the definition of harmonic function.

Definition 4.11. *A measurable non-negative function h is called harmonic with respect to the Markov process Z if for all open set U relatively compact in $\mathbb{R}^2$ and $z_0 \in \mathbb{R}^2$,*

$$h(z_0) = \mathbb{E}_{z_0}\left[h\left(Z_{T_U}\right)\mathbb{1}_{T_U<+\infty}\right]$$

where $T_U = \inf\{t \ge 0 \mid X_t \in U\}$.

The general theory of Martin boundary [78, Theorem 3] justifies that the limits of the quotient of Green's functions $\frac{g(z_0,z)}{g(0,z)}$ as $|z| \to +\infty$ – called Martin functions – are generally *excessive* in z_0 (see [78, Section 3]), which is weaker than harmonic. In Theorems 4.6 to 4.10, we obtained all Martin functions. In particular, we obtained:

- For each direction $\alpha_0 \in \mathring{\mathcal{M}}$, one function $h_{\alpha_0}(z_0)$ (see Theorem 4.7);
- For the directions 0 and π, two functions $f_0(z_0)$ and $f_\pi(z_0)$ (see Theorems 4.6 and 4.8);
- For $\alpha_0 = \alpha_b$, all the convex combinations of $h_{\alpha_b}(z_0)$ and $f_0(z_0)$, see (4.1.35) (Theorem 4.9);
- Similarly, for $\alpha_0 = \widetilde{\alpha}_b$, with $h_{\widetilde{\alpha}_b}(z_0)$ and $f_\pi(z_0)$ (see Theorem 4.9);
- For each direction $\alpha_0 \in [0, 2\pi]\backslash\mathcal{M}$, the functions $f_0(z_0)$ and $f_\pi(z_0)$ (see Theorem 4.10).

We prove in Section 4.5 that those functions are harmonic.

Remark 4.12 (Partial differential equation). The partial differential equation associated with the harmonicity of a function h with respect to our process is given by

$$\begin{cases} \left(\frac{1}{2}\nabla\cdot\Sigma^+\nabla + \mu^+\cdot\nabla\right)h(z) = 0 & \text{for } z \in \mathbb{R}\times(0,+\infty), \\ \left(\frac{1}{2}\nabla\cdot\Sigma^-\nabla + \mu^-\cdot\nabla\right)h(z) = 0 & \text{for } z \in \mathbb{R}\times(-\infty,0), \\ (q_1, 1+q_2)\nabla h(x,0^+) = (-q_1, 1-q_2)\nabla h(x,0^-) & \text{for } x \in \mathbb{R}, \\ h(x,0^+) = h(x,^-) \text{ and } (1,0)\cdot\nabla h(x,0^+) = (1,0)\cdot\nabla h(x,0^-) & \text{for } x \in \mathbb{R}. \end{cases} \tag{4.1.40}$$

It is straightforward to verify that the harmonic functions f_0, f_π, and $h_\alpha, \alpha \in \mathcal{M}\backslash\{0,\pi\}$ obtained in our main results satisfy this equation.

Then, the structures of the Martin boundary and the minimal Martin boundary follow : the proof is detailed in Section 4.5. Let us just notice the fact that the non-minimality of the full Martin boundary comes from the Green's functions asymptotics (4.1.35) in Theorem 4.9.

Theorem 4.13 (Martin Boundary, minimal Martin Boundary and positive harmonic functions)**.** *All of the functions $h_\alpha, \alpha \in \mathcal{M}\backslash\{0,\pi\}$, f_0 and f_π are minimal harmonic functions. Furthermore, the Martin boundary Γ of the process is given by*

$$\Gamma = \left[\{e^{i\alpha}\}_{\alpha\in\mathcal{M}} \cup \{ue^{i\alpha}\}_{u\in[1,2],\, \alpha=\alpha_b,\widetilde{\alpha}_b}\right] / \mathcal{R} \ \sim \ \mathbb{S}^1 \tag{4.1.41}$$

where $\mathcal{R}$ is the equivalence relation defined by $e^{i.0}\,\mathcal{R}\,2e^{i\alpha_b}$, $e^{i\pi}\,\mathcal{R}\,2e^{i\widetilde{\alpha}_b}$ (see Figure 4.8).

The minimal Martin boundary, denoted by Γ_{min}, is given by

$$\Gamma_{min} = \{e^{i\alpha}\}_{\alpha\in\mathcal{M}} \tag{4.1.42}$$

(see Figure 4.5). Therefore, according to Martin boundary theory, for each positive harmonic function h (with respect to the process), there exists a unique Radon measure ν on $\mathcal{M}$ such that :

$$\forall z \in \mathbb{R}^2, \quad h(z) = \int_{\mathcal{M}} \frac{h_\alpha(z)}{h_\alpha(0)}\, d\nu(\alpha). \tag{4.1.43}$$

where we have used the notation $\frac{h_0(z_0)}{h_0(0)} = \frac{f_0(z_0)}{f_0(0)}$ and $\frac{h_\pi(z_0)}{h_\pi(0)} = \frac{f_\pi(z_0)}{f_\pi(0)}$ since

$$\frac{h_\alpha(z_0)}{h_\alpha(0)} \underset{\substack{\alpha\to0\\ \alpha\in\mathcal{M}}}{\longrightarrow} \frac{f_0(z_0)}{f_0(0)} \quad \text{and} \quad \frac{h_\alpha(z_0)}{h_\alpha(0)} \underset{\substack{\alpha\to\pi\\ \alpha\in\mathcal{M}}}{\longrightarrow} \frac{f_\pi(z_0)}{f_\pi(0)}.$$

Using the preceding representation theorem, we can derive the well-known one-dimensional result concerning the probabilities of escaping into the upper or lower half-plane. By assumption (4.1.5) on the drift (see Figure 4.1), the second component B of the process naturally satisfies $B_t \underset{t\to+\infty}{\longrightarrow} \pm\infty$. By standard arguments, the mappings

$$z_0 \longmapsto \mathbb{P}_{z_0}\big(B_t \underset{t\to+\infty}{\longrightarrow} +\infty\big) \quad \text{and} \quad z_0 \longmapsto \mathbb{P}_{z_0}\big(B_t \underset{t\to+\infty}{\longrightarrow} -\infty\big)$$

are harmonic functions. Looking at the bounded non negative harmonic functions (which reduces to convex combinations of $h_{\alpha_{\mu^+}}$ and $h_{\alpha_{\mu^-}}$), we deduce in Section 4.5 the following classical one-dimensional result.

Corollary 4.14 (Upper and lower escape probabilities)**.** *Let $(a_0,b_0) \in \mathbb{R}^2$. Then:*

$$\mathbb{P}_{(a_0,b_0)}\big(B_t \underset{t\to+\infty}{\longrightarrow} +\infty\big) = h_{\alpha_{\mu^+}}(b_0), \quad \text{and} \quad \mathbb{P}_{(a_0,b_0)}\big(B_t \underset{t\to+\infty}{\longrightarrow} -\infty\big) = h_{\alpha_{\mu^-}}(b_0).$$

In particular,

$$\mathbb{P}_{(a_0,b_0)}\big(B_t \underset{t\to+\infty}{\longrightarrow} +\infty\big) = 1 - \mathbb{P}_{(a_0,b_0)}\big(B_t \underset{t\to+\infty}{\longrightarrow} -\infty\big) \tag{4.1.44}$$

$$= \begin{cases} 1 + \frac{\mu_2^-}{\mu_2^+ - \mu_2^-} e^{-2b_0\mu_2^+/\Sigma_{22}^+} & \textit{if } b_0 \geq 0, \\ \frac{\mu_2^+}{\mu_2^+ - \mu_2^-} e^{-2b_0\mu_2^-/\Sigma_{22}^-} & \textit{if } b_0 < 0. \end{cases} \tag{4.1.45}$$

4.2 Process and functional equation

In Section 4.2.1, we prove Proposition 4.2. Then, we present some preliminary lemmas in Section 4.2.2, which are necessary to establish the functional equation (4.1.11). A short intuitive proof of the functional equation is given in Section 4.2.3, followed by a detailed one in Section 4.2.4. As a first reading, the reader may focus on the short proof.

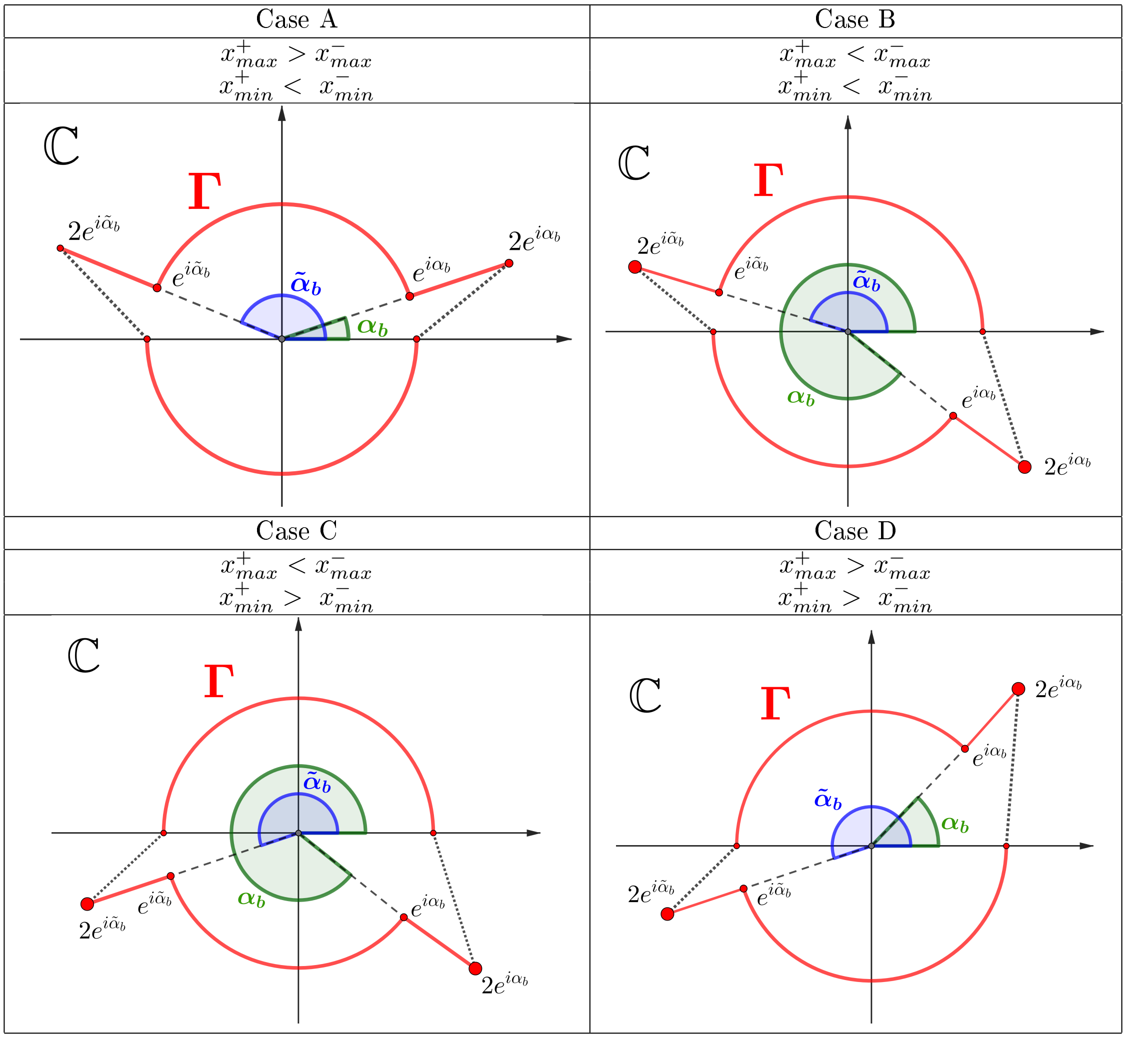


Figure 4.8: Representation of the Martin boundary Γ depending on the parameters.

4.2.1 Construction of the process

In this section, we prove Proposition 4.2. The proof is adapted from [85, Proposition 1], where the covariance matrix is assumed to be diagonal. We first establish the existence and pathwise uniqueness of the solution to equation (4.1.1), and then identify the corresponding generator.

Existence and uniqueness of (4.1.1)

First, note that (4.1.1) can be written as the following SDE:

$$\begin{cases} dA_t &= \sqrt{\Sigma_{11}(B_t) - \frac{\Sigma_{12}^2(B_t)}{\Sigma_{22}(B_t)}}\, dW_t^1 + \frac{\Sigma_{12}(B_t)}{\sqrt{\Sigma_{22}(B_t)}}\, dW_t^2 + \mu_1(B_t)\, dt + \frac{\Sigma_{12}^+ - \Sigma_{12}^-}{\Sigma_{22}^+ + \Sigma_{22}^-}\, dL_t^0(B), \quad (1) \\ dB_t &= \sqrt{\Sigma_{22}(B_t)}\, dW_t^2 + \mu_2(B_t)\, dt + \frac{\Sigma_{22}^+ - \Sigma_{22}^-}{\Sigma_{22}^+ + \Sigma_{22}^-}\, dL_t^0(B), \quad (2) \end{cases} \tag{4.2.1}$$

Here, $\big((W_t^1, W_t^2)\big)_{t\geq 0}$ is a standard two-dimensional Brownian motion, and $L^0(B)$ denotes the symmetric local time of B at 0. Indeed, the matrix

$$\sigma(B_t) = \begin{pmatrix} \sqrt{\Sigma_{11}(B_t) - \frac{\Sigma_{12}^2(B_t)}{\Sigma_{22}(B_t)}} & \frac{\Sigma_{12}(B_t)}{\sqrt{\Sigma_{22}(B_t)}} \\ 0 & \sqrt{\Sigma_{22}(B_t)} \end{pmatrix}$$

satisfies $\sigma(B_t)\,\sigma(B_t)^T = \Sigma(B_t)$.

Let (W^1, W^2) be a standard two-dimensional Brownian motion. By [83], there exists a pathwise unique solution $(B_t)_{t\geq 0}$ of equation (2) starting from $b_0 \in \mathbb{R}$. Then, equation (1) admits a pathwise unique solution starting from $a_0 \in \mathbb{R}$ given by:

$$A_t = a_0 + \int_0^t \sqrt{\Sigma_{11}(B_s) - \frac{\Sigma_{12}^2(B_s)}{\Sigma_{22}(B_s)}}\, dW_s^1 + \int_0^t \frac{\Sigma_{12}(B_s)}{\sqrt{\Sigma_{22}(B_s)}}\, dW_s^2 + \int_0^t \mu_1(B_s)\, ds + \frac{\Sigma_{12}^+ - \Sigma_{12}^-}{\Sigma_{22}^+ + \Sigma_{22}^-} L_t^0(B).$$

Using a generalisation of the Yamada Watanabe Engelbert theorem (see [82, Proposition 2.10]), uniqueness in law also holds for the process starting from (a_0, b_0). By standard arguments, we deduce that the family of probability laws indexed by $(a_0, b_0) \in \mathbb{R}^2$ defines a Markov process.

Shape of the generator

Let us prove that the generator can be written in divergence form. To this end, consider the continuous piecewise affine regularisations of Σ and μ defined by: $\Sigma^n(b) = \Sigma^+$ if $b \geq \frac{1}{n}$, $\Sigma^n(b) = \Sigma^-$ if $b \leq \frac{-1}{n}$, and Σ^n affine on $\left[-\frac{1}{n}, \frac{1}{n}\right]$ (similarly for μ^n).

Let (W^1, W^2) be a two-dimensional Brownian motion. By standard arguments, the SDE

$$\begin{cases} dA_t^n &= \sqrt{\Sigma_{11}^n(B_t^n) - \frac{{\Sigma_{12}^n}^2(B_t^n)}{\Sigma_{22}^n(B_t^n)}}\, dW_t^1 + \frac{\Sigma_{12}^n(B_t^n)}{\sqrt{\Sigma_{22}^n(B_t^n)}}\, dW_t^2 + \Big(\mu_1^n(B_t^n) + \frac{1}{2}\frac{d}{dy}\Sigma_{12}^n(B_t^n)\Big)\, dt, \\ dB_t^n &= \sqrt{\Sigma_{22}^n(B_t^n)}\, dW_t^2 + \Big(\mu_2^n(B_t^n) + \frac{1}{2}\frac{d}{dy}\Sigma_{22}^n(B_t^n)\Big)\, dt \end{cases}$$

defines a Markov process with generator $\mathcal{L}_n = \frac{1}{2}\nabla \cdot (\Sigma^n(y)\nabla) + \mu^n(y) \cdot \nabla$. By [113], this family of processes converges in law to the strong Markov process with generator given by (4.1.4). Moreover, using arguments similar to those in [85], one can show that (A_t^n, B_t^n) converges in the L^1 norm to (A_t, B_t) for all $t \geq 0$. By identifying the limits, we conclude that (4.1.4) is indeed the generator of (X, Y).

4.2.2 Preliminary lemmas

To derive the desired functional equation, we first establish a few preliminary lemmas. We begin by stating Aronson-type estimates for the transition density, along with some elementary consequences. These inequalities hold since the process admits a generator in divergence form.

Lemma 4.15 (Some estimates due to Aronson)**.** *There exists a constant $M > 0$ such that, for all $t > 0$ and $z_0 \in \mathbb{R}^2$,*

$$\frac{1}{Mt}\, e^{-M|z-z_0|^2/t-Mt} \leq p_t^{z_0}(z) \leq \frac{M}{t}\, e^{-|z-z_0|^2/Mt+Mt}. \tag{4.2.2}$$

Let $g_t^{z_0}(z)$ be defined by

$$g_t^{z_0}(z) := \int_0^t p_s^{z_0}(u,0)\, ds.$$

Then,

$$g_t^{z_0}(z) \leq \frac{M^2\, e^{-\frac{|z-z_0|^2}{Mt}+Mt}}{|z-z_0|^2}\, t. \tag{4.2.3}$$

Finally, for all $z_0 = (a_0, b_0) \in \mathbb{R}^2$ and $x \in \mathbb{R}$,

$$\mathbb{E}_{(a_0,b_0)}\left[e^{xA_t}\right] \leq e^{x^2Mt+a_0x}. \tag{4.2.4}$$

Proof. Inequalities (4.2.2) are proved in [113, Theorem II.3.8]. □

We now relate the occupation measure on the axis $\{y = 0\}$, defined in (4.1.8), to the values of the Green's function g^{z_0} restricted to $\{y = 0\}$. This connection is made precise in the following lemma.

Lemma 4.16 (Link between $H(z_0, \cdot)$ and $g^{z_0}(u, 0)$)**.** *Let $z_0 = (a_0, b_0) \in \mathbb{R}^2$ with $b_0 \neq 0$. Let $f : \mathbb{R} \to [0, +\infty)$ be a continuous nonnegative function. Then:*

$$\mathbb{E}_{z_0}\left[\int_0^{+\infty} f(A_s)\, dL_s^0(B)\right] = \left(\frac{\Sigma_{22}^- + \Sigma_{22}^+}{2}\right)\int_{\mathbb{R}} g^{z_0}(u,0)f(u)\, du. \tag{4.2.5}$$

In particular, combining this with the asymptotic behavior as $t \to +\infty$ and the definition (4.1.9), we obtain the identity

$$\varphi^{z_0}(x) = \left(\frac{\Sigma_{22}^- + \Sigma_{22}^+}{2}\right)\int_{\mathbb{R}} g^{z_0}(u,0)\, e^{-xu}\, du. \tag{4.2.6}$$

Proof. The proof reduces to showing that for all $t \geq 0$ and for every continuous, compactly supported, nonnegative function f, we have:

$$\mathbb{E}_{z_0}\left[\int_0^{t} f(A_s)\, dL_s^0(B)\right] = \left(\frac{\Sigma_{22}^- + \Sigma_{22}^+}{2}\right)\int_{\mathbb{R}} g_t^{z_0}(u,0)f(u)\, du. \tag{4.2.7}$$

Let f be such a function and let $\epsilon > 0$. Then:

$$\mathbb{E}\left[\frac{1}{2\epsilon}\int_0^t f(A_s)\mathbb{1}_{B_s\in[-\epsilon,\epsilon]}\, d\langle B\rangle_s\right] = \int_0^t\int_{\mathbb{R}} f(a)\cdot\frac{1}{2\epsilon}\left(\Sigma_{22}^-\int_{[-\epsilon,0]} p_s^{z_0}(a,b)\, db + \Sigma_{22}^+\int_{[0,\epsilon]} p_s^{z_0}(a,b)\, db\right) da\, ds.$$

Using the boundedness of f and the Aronson estimates, the right-hand side converges to

$$\frac{\Sigma_{22}^- + \Sigma_{22}^+}{2}\int_0^t\int_{\mathbb{R}} f(a)p_s^{z_0}(a,0)\, da\, ds = \frac{\Sigma_{22}^- + \Sigma_{22}^+}{2}\int_{\mathbb{R}} g_t^{z_0}(a,0)f(a)\, da.$$

as $\epsilon \to 0$. Moreover, we recall the almost sure identity:

$$\forall t \geq 0, \quad L_t^0(B) = \lim_{\epsilon\to 0}\int_0^t \frac{1}{2\epsilon}\mathbb{1}_{B_s\in[-\epsilon,\epsilon]}\, d\langle B\rangle_s. \tag{4.2.8}$$

If $s \mapsto H_s$ is a measurable step function with compact support, then:

$$\frac{1}{2\epsilon}\int_0^t H_s \mathbb{1}_{B_s\in[-\epsilon,\epsilon]}\, d\langle B\rangle_s \xrightarrow[\epsilon\to 0]{L^1,\ a.s.} \int_0^t H_s\, dL^0_s(B). \tag{4.2.9}$$

Since $s \mapsto f(A_s)$ is the uniform limit of step functions, the convergence in (4.2.9) remains valid when H_s is replaced with $f(A_s)$. □

For later technical reasons, it is convenient to establish the finiteness of some exponential moments of $L^0_\infty(B)$. To do so, we establish the following two lemmas. For any real-valued trajectory $b = (b_t)_{t\geq 0}$, we define:

$$T(b) := \inf\{t \geq 0 \mid |b_t| = 1\}.$$

Lemma 4.17 (Exponential moments, part 1)**.** *Let $T = T(B)$. Then, for all $\lambda \geq 0$ and $z_0 \in \mathbb{R}^2$, we have:*

$$\mathbb{E}_{z_0}[e^{\lambda L_T(B)}] < +\infty.$$

Proof. Let us recall that B satisfies:

$$dB_t = \Sigma_{22}(B_t)dW_t + \mu_2(B_t)dt + q_2 dL^0_t(B),$$

where W is a Brownian motion. By Girsanov's theorem, we can reduce the proof to the case $\mu_2 = 0$. Using the strong Markov property and the fact that the Stieltjes measure $dL^0(B)$ is supported on $\{t \geq 0 \mid B_t = 0\}$, we may further reduce the proof to the case $b_0 = 0$. By the Itô–Tanaka formula, we have:

$$|B_t| = \int_0^t \operatorname{sgn}(B_s)\left(\sigma_2(B_s)dW_s + q_2 dL^0_s(B)\right) + L^0_t(B) = \int_0^t \operatorname{sgn}(B_s)\sigma_2(B_s)dW_s + L^0_t(B) =: M_t + L^0_t(B). \tag{4.2.10}$$

Then, using the properties of exponential martingales:

$$\begin{aligned}\mathbb{E}\left[e^{\lambda L^0_{t\wedge T}(B)}\right] &\leq \mathbb{E}\left[e^{\lambda|B_{t\wedge T}|+\lambda M_t}\right] \leq \mathbb{E}\left[e^{\lambda+\frac{\lambda^2}{2}\langle M\rangle_{t\wedge T}}\right]\\ &\leq e^{\lambda}\mathbb{E}\left[e^{\frac{\lambda^2}{2}(t\wedge T)}\right] \leq e^{\lambda}\mathbb{E}\left[e^{\frac{\lambda^2}{2}T}\right].\end{aligned}$$

Define the time-change $C_t := \int_0^t \Sigma_{22}(B_s)^2 ds$ for $t \geq 0$. Then, for all $t \geq 0$, $\min(\Sigma^-_{22}, \Sigma^+_{22})t \leq C_t \leq \max(\Sigma^-_{22}, \Sigma^+_{22})t$. As a result,

$$T(B) \leq \max(\Sigma^-_{22}, \Sigma^+_{22}) \cdot T(B_{C_\cdot}).$$

Moreover, by the Dubins–Schwarz theorem, the process $(M_{C_t})_{t\geq 0}$ is a Brownian motion. Using (4.2.10), we deduce that $(|B_{C_t}|)_{t\geq 0}$ is a reflected Brownian motion. Finally,

$$\mathbb{E}\left[e^{\lambda L_T}\right] \leq e^{\lambda}\mathbb{E}\left[e^{\frac{\lambda^2}{2}\max(\Sigma^-_{22},\Sigma^+_{22})\cdot T(B_{C_\cdot})}\right] < +\infty,$$

by standard results on reflected Brownian motion. □

Let us recall that we are working under Assumption (4.1.5).

Lemma 4.18 (Exponential moments, part 2)**.** *There exists $\lambda^* > 0$ such that for all $0 \leq \lambda < \lambda^*$ and $z_0 \in \mathbb{R}^2$, we have:*

$$\mathbb{E}_{z_0}[e^{\lambda L^0_\infty(B)}] < +\infty.$$

Proof. For the proof, we use the strong Markov property and consider the back-and-forth excursions of the process between the sets $\{y=0\}$ and $\{|y|\geq 1\}$. As in the proof of Lemma 4.17, we may assume that $b_0 = 0$. Define the following stopping times:

$$\sigma_0 = 0, \quad T_n = \inf\{t \geq \sigma_{n-1} \mid |B_t| = 1\}, \quad \sigma_n = \inf\{t \geq T_n \mid B_t = 0\}, \quad n \geq 1.$$

Note that, almost surely, there exists a (random) rank n such that $\sigma_k = +\infty$ for all $k \geq n$. Then, by the strong Markov property,

$$\begin{aligned}\mathbb{E}_{z_0}\left[e^{\lambda L^0_\infty(B)}\right] &= \sum_{n=1}^{+\infty} \mathbb{E}_{z_0}\left[\mathbb{1}_{\sigma_{n-1}<+\infty,\ \sigma_n=+\infty} \prod_{k=1}^{n} e^{\lambda(L^0_{T_k}(B)-L^0_{\sigma_{k-1}}(B))}\right] \\ &= \sum_{n=1}^{+\infty} \mathbb{E}_0\left[\mathbb{1}_{\sigma_1=+\infty} e^{\lambda L^0_{T_1}(B)}\right] \cdot \mathbb{E}_0\left[\mathbb{1}_{\sigma_1<+\infty} e^{\lambda L^0_{T_1}(B)}\right]^{n-1}.\end{aligned}$$

Moreover, by Lemma 4.17 and the dominated convergence theorem, we have:

$$\mathbb{E}_0\left[\mathbb{1}_{\sigma_1<+\infty} e^{\lambda L^0_{T_1}(B)}\right] \underset{\lambda\to 0^+}{\longrightarrow} \mathbb{P}_0(\sigma_1 < +\infty) < 1.$$

This shows that $\mathbb{E}_{z_0}\left[e^{\lambda L^0_\infty(B)}\right]$ is finite for $\lambda > 0$ small enough. □

This lemma allows us to prove the final technical result before the proof of Proposition 4.4.

Lemma 4.19 (Convergence of exponential moments of A_t)**.** *Let $z_0 \in \mathbb{R}^2$. Then, the first component A of Z satisfies, $\mathbb{P}_{z_0}$-almost surely,*

$$A_t \underset{t\to\infty}{\longrightarrow} +\infty.$$

Furthermore, there exists a constant $\eta > 0$ such that for all $0 < u < \eta$,

$$\mathbb{E}_{z_0}\left[e^{-uA_t}\right] \underset{t\to+\infty}{\longrightarrow} 0.$$

Proof. The first statement follows directly from the fact that the process no longer touches the axis $\{y=0\}$ after a (random) time T. Indeed, after this time T, A evolves as a Brownian motion with drift $\mu_1^+ > 0$ or $\mu_1^- > 0$, and therefore $A_t \to +\infty$ almost surely. For the second statement, recall that

$$dA_t = \Sigma_{11}(B_t)\, d\widetilde{B}_t + \mu_1(B_t)\, dt + q_1\, dL^0_t(B),$$

where $\widetilde{B}$ is a Brownian motion (possibly correlated with B). Define the martingale $M_t := \int_0^t \Sigma_{11}(B_s)\, d\widetilde{B}_s$, $t \geq 0$. Then, by the Cauchy–Schwarz inequality, we obtain for all $u \geq 0$:

$$\begin{aligned}\mathbb{E}\left[e^{-uA_t}\right]^2 &\leq \mathbb{E}\left[e^{-2uM_t}\right] \cdot \mathbb{E}\left[e^{-2uq_1 L^0_t(B) - 2u\int_0^t \mu_1(B_s)\, ds}\right] \\ &\leq \mathbb{E}\left[e^{-2uq_1 L^0_\infty(B)}\right] \cdot e^{-2ut\left(\min(\mu_1^+,\mu_1^-) - u\max(\Sigma_{11}^+,\Sigma_{11}^-)\right)}.\end{aligned}$$

If we choose $u > 0$ small enough so that $\mathbb{E}\left[e^{-2uq_1 L^0_\infty(B)}\right] < +\infty$ by Lemma 4.18 and such that $\min(\mu_1^+,\mu_1^-) - u\max(\Sigma_{11}^+,\Sigma_{11}^-) > 0$, then the right-hand term tends to zero as $t \to +\infty$, which completes the proof. □

4.2.3 Short version of the proof of the functional equation

We now present a brief argument to establish Proposition 4.4. The idea of the proof is to apply Itô's formula to

$$f(a,b) = e^{ax+by}\mathbb{1}_{b\geq 0} + e^{ax+bz}\mathbb{1}_{b<0}.$$

Note that $f \notin \mathcal{C}^2(\mathbb{R}^2)$. In this section, we assume that all local martingales are true martingales, ignore convergence issues, and apply the standard Itô formula despite the lack of smoothness of f. We compute:

$$\nabla f(a,b) := \begin{pmatrix} xf(a,b) \\ ye^{ax+by}\mathbb{1}_{b\geq 0} + ze^{ax+bz}\mathbb{1}_{b<0} \end{pmatrix} \tag{4.2.11}$$

and

$$\nabla^2 f(a,b) = \begin{pmatrix} x^2 f(a,b) & xye^{ax+by}\mathbb{1}_{b\geq 0} + xze^{ax+bz}\mathbb{1}_{b<0} \\ xye^{ax+by}\mathbb{1}_{b\geq 0} + xze^{ax+bz}\mathbb{1}_{b<0} & y^2e^{ax+by}\mathbb{1}_{b\geq 0} + z^2e^{ax+bz}\mathbb{1}_{b<0} + (y-z)e^{ax}\delta_{b=0} \end{pmatrix}.$$

Formally, Itô's formula yields

$$f(Z_t) = f(Z_0) + \int_0^t \nabla f(Z_s)\,dZ_s + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\int_0^t \partial_i\partial_j f(Z_s)\,d\langle Z^i, Z^j\rangle_s \tag{4.2.12}$$

(with notation $Z^1 = A$ and $Z^2 = B$). Since $dZ_t = \sigma(B_t)dB_t + \mu(B_t)dt + q\,dL_t^0(B)$, we get

$$\mathbb{E}\left[\int_0^t \nabla f(Z_s)\,dZ_s\right] = (x\mu_1^+ + y\mu_2^+)\mathbb{E}\left[\int_0^t \mathbb{1}_{B_s\geq 0}e^{xA_s+yB_s}\,ds\right] + (x\mu_1^- + z\mu_2^-)\mathbb{E}\left[\int_0^t \mathbb{1}_{B_s<0}e^{xA_s+zB_s}\,ds\right]$$

$$+\left(q_1x + \frac{q_2(y+z)}{2}\right)\mathbb{E}\left[\int_0^t e^{xA_s}\,dL_s^0(B)\right].$$

Now, let us consider the second derivative terms. Similarly, we have:

$$\mathbb{E}\left[\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\int_0^t \partial_i\partial_j f(Z_s)\,d\langle Z^i, Z^j\rangle_s\right] = \frac{1}{2}(x^2\Sigma_{11}^+ + 2xy\Sigma_{12}^+ + y^2\Sigma_{22}^+)\mathbb{E}\left[\int_0^t \mathbb{1}_{B_s\geq 0}e^{xA_s+yB_s}\,ds\right]$$

$$+\frac{1}{2}(x^2\Sigma_{11}^- + 2xz\Sigma_{12}^- + z^2\Sigma_{22}^-)\mathbb{E}\left[\int_0^t \mathbb{1}_{B_s<0}e^{xA_s+zB_s}\,ds\right]$$

$$+\left(\frac{y-z}{2}\right)\mathbb{E}\left[\int_0^t e^{xA_s}\delta_{B_s=0}\,ds\right].$$

The last term can be interpreted as $\left(\frac{y-z}{2}\right)\mathbb{E}\left[\int_0^t e^{xA_s}\,dL_t^0(B)\right]$. Then, by rearranging the terms in the expected value of (4.2.12), we obtain

$$\gamma_-(x,z)\mathbb{E}\left[\int_0^t \mathbb{1}_{B_s<0}e^{xA_s+zB_s}\,ds\right] + \gamma_+(x,y)\mathbb{E}\left[\int_0^t \mathbb{1}_{B_s>0}e^{xA_s+yB_s}\,ds\right] + \gamma(x,y,z)\mathbb{E}\left[\int_0^t e^{xA_s}\,dL_s^0(B)\right] \tag{4.2.13}$$

$$= \mathbb{E}[f(Z_t)] - \left(e^{xa_0+yb_0}\mathbb{1}_{b_0\geq 0} + e^{xa_0+zb_0}\mathbb{1}_{b_0<0}\right).$$

Assuming that x, y, and z are chosen such that $\mathbb{E}[f(Z_t)] \to 0$ as $t \to +\infty$ and that the Laplace transforms converge, we obtain the desired equation.

4.2.4 Detailed version of the proof of the functional equation

In this section, we provide a detailed justification for the reasoning presented in Section 4.2.3 when $b_0 \neq 0$ (see Remark 4.20 for the case $b_0 = 0$). Note that an Itô–Tanaka formula exists in dimension 2 for non-smooth functions [43]. However, it is not straightforward to use, so instead we opted for an approximation using smooth functions. We then approximate the preceding function f by smooth functions f_ϵ defined by

$$f_\epsilon(a,b) = e^{ax+bz} + \psi(b/\epsilon)(e^{ax+by} - e^{ax+bz}), \tag{4.2.14}$$

where ψ is an increasing C^∞ function satisfying $\psi(u) := \begin{cases} 0 & \text{if} \quad u \leq -1, \\ 1 & \text{if} \quad u \geq 1, \\ \frac{1}{2} & \text{if} \quad u = 0. \end{cases}$ Note that $f_\epsilon(a,b) = e^{ax+by}$ if $b \geq \epsilon$ and $f_\epsilon(a,b) = e^{ax+bz}$ if $b \leq -\epsilon$. The Itô formula can now be applied to the C^∞ function f_ϵ: for all $t \geq 0$, we have

$$f_\epsilon(Z_t) = f_\epsilon(Z_0) + \int_0^t \nabla f_\epsilon(Z_s)\, dZ_s + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\int_0^t \partial_i\partial_j f_\epsilon(Z_s)\, d\langle Z^i, Z^j\rangle_s. \tag{4.2.15}$$

Step 1 : Study of integrals against dZ_s. Note that

$$\nabla f_\epsilon(a,b) := \begin{pmatrix} x f_\epsilon(a,b) \\ z e^{ax+bz} + \frac{1}{\epsilon}\psi'(b/\epsilon)(e^{ax+by} - e^{ax+bz}) + \psi(b/\epsilon)(y e^{ax+by} - z e^{ax+bz}) \end{pmatrix}. \tag{4.2.16}$$

Then,

$$\int_0^t \nabla f_\epsilon(Z_s) \mathbb{1}_{B_s\geq 0}\, dZ_s = x\int_0^t \mathbb{1}_{B_s \geq 0} f_\epsilon(Z_s)\, dA_s \tag{4.2.17}$$

$$+\int_0^t \mathbb{1}_{B_s\geq 0}\left(z e^{xA_s+zB_s} + \frac{1}{\epsilon}\psi'(B_s/\epsilon)(e^{xA_s+yB_s} - e^{xA_s+zB_s}) + \psi(B_s/\epsilon)(y e^{xA_s+yB_s} - z e^{xA_s+zB_s})\right) dB_s,$$

where $\mathbb{1}_{B_s\geq 0}\, dA_s = (\Sigma_{11}^{+} dB_s^1 + \Sigma_{12}^{+} dB_s^2 + \mu_1^{+} ds) + q_1\, dL_t^0(B)$ and $\mathbb{1}_{B_s \geq 0} dB_s = (\Sigma_{12}^{+} dB_s^1 + \Sigma_{22}^{+} dB_s^2 + \mu_2^{+} ds) + q_2 dL_t^0(B)$. Then, by Lemma 4.2.4, the Brownian integrals are martingales (not just local martingales), so their expectations vanish. Therefore,

$$\mathbb{E}_{z_0}\left[\int_0^t \nabla f_\epsilon(Z_s)\mathbb{1}_{B_s\geq 0}\, dZ_s\right] = x\mu_1^{+}\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s\geq 0} f_\epsilon(Z_s)\, ds\right] + y\mu_2^{+}\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s \geq \epsilon} e^{xA_s+yB_s}\, ds\right] \tag{4.2.18}$$

$$+\mu_2^{+}\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{0\leq B_s<\epsilon}\left(z e^{xA_s+zB_s} + \frac{1}{\epsilon}\psi'(B_s/\epsilon)(e^{xA_s+yB_s} - e^{xA_s+zB_s}) + \psi(B_s/\epsilon)(y e^{xA_s+yB_s} - z e^{xA_s+zB_s})\right) ds\right]$$

$$+\left(q_1 x + \frac{q_2(y+z)}{2}\right)\mathbb{E}_{z_0}\left[\int_0^t e^{xA_s}\, dL_t^0(B)\right].$$

By the dominated convergence theorem and inequality (4.2.4),

$$\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{0\leq B_s<\epsilon}\left(z e^{xA_s+zB_s} + \psi(B_s/\epsilon)(y e^{xA_s+yB_s} - z e^{xA_s+zB_s})\right) ds\right] \underset{\epsilon\to 0}{\longrightarrow} 0,$$

and

$$\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s\geq\epsilon} e^{xA_s+yB_s}\, ds\right] \underset{\epsilon\to 0}{\longrightarrow} \mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s\geq 0} e^{xA_s+yB_s}\, ds\right].$$

Moreover, by the definition of g_t, we have:

$$\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{0\leq B_s<\epsilon}\frac{1}{\epsilon}\psi'(B_s/\epsilon)(e^{xA_s+yB_s} - e^{xA_s+zB_s})\, ds\right] = \int_{\mathbb{R}}\int_0^{\epsilon}\frac{1}{\epsilon}\psi'(b/\epsilon)(e^{ax+by} - e^{ax+bz}) g_t(a,b)\, dbda.$$

Since $b_0 \neq 0$, estimates (4.2.3) yield the following limit for all $a \in \mathbb{R}$ as $\epsilon \to 0$:

$$\int_0^\epsilon \frac{1}{\epsilon}\psi'(b/\epsilon)(e^{ax+by} - e^{ax+bz})g_t(a,b)\,db = \int_0^1 \psi'(u)(e^{ax+\epsilon uy} - e^{ax+\epsilon uz})g_t(a,\epsilon u)\,du \tag{4.2.19}$$

$$\xrightarrow[\epsilon\to 0]{} (e^{ax} - e^{ax})g_t(a,0) = 0. \tag{4.2.20}$$

(Note that $\frac{1}{\epsilon}\psi'(\cdot/\epsilon) \xrightarrow[\epsilon\to 0]{} \delta_0$ in the sense of distributions.) Integrating in $a \in \mathbb{R}$ and using (4.2.3) again, we get

$$\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{0\le B_s<\epsilon}\frac{1}{\epsilon}\psi'(B_s/\epsilon)(e^{xA_s+yB_s} - e^{xA_s+zB_s})\,ds\right] \xrightarrow[\epsilon\to 0]{} 0. \tag{4.2.21}$$

By a similar analysis on $\int_0^t \nabla f_\epsilon(Z_s)\mathbb{1}_{B_s<0}\,dZ_s$, the following limit holds:

$$\begin{aligned}\mathbb{E}_{z_0}\left[\int_0^t \nabla f_\epsilon(Z_s)\,dZ_s\right] \xrightarrow[\epsilon\to 0]{} (x\mu_1^+ + y\mu_2^+)\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s\ge 0}e^{xA_s+yB_s}\,ds\right] + (x\mu_1^- + z\mu_2^-)\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s<0}e^{xA_s+zB_s}\,ds\right]\\ + \left(q_1x + \frac{q_2(y+z)}{2}\right)\mathbb{E}_{z_0}\left[\int_0^t e^{xA_s}\,dL_t^0(B)\right].\end{aligned} \tag{4.2.22}$$

Step 2: Study of integrals against $d\langle Z^i, Z^j\rangle_s$.
Since $\partial_1 f_\epsilon(a,b) = xf_\epsilon(a,b)$, we can similarly show that

$$\mathbb{E}_{z_0}\left[\int_0^t \partial_1\partial_1 f_\epsilon(Z_s)d\langle A,A\rangle_s\right] \xrightarrow[\epsilon\to 0]{} \Sigma_{11}^+x^2\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s\ge 0}e^{xA_s+yB_s}ds\right] + \Sigma_{11}^-x^2\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s<0}e^{xA_s+zB_s}ds\right] \tag{4.2.23}$$

and

$$\mathbb{E}_{z_0}\left[\int_0^t \partial_1\partial_2 f_\epsilon(Z_s)d\langle A,B\rangle_s\right] \xrightarrow[\epsilon\to 0]{} \Sigma_{12}^+xy\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s\ge 0}e^{xA_s+yB_s}ds\right] + \Sigma_{12}^-xz\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s<0}e^{xA_s+zB_s}ds\right]. \tag{4.2.24}$$

Now, let us study the last term $i = j = 2$. We have $d\langle B,B\rangle_t = \Sigma_{22}^+\mathbb{1}_{B_t\ge 0}dt + \Sigma_{22}^-\mathbb{1}_{B_t<0}dt$, and

$$\begin{aligned}\partial_2\partial_2 f_\epsilon(a,b) &= z^2e^{ax+bz} + \frac{1}{\epsilon^2}\psi''(b/\epsilon)(e^{ax+by} - e^{ax+bz})\\ &\quad + \frac{2}{\epsilon}\psi'(b/\epsilon)(ye^{ax+by} - ze^{ax+bz}) + \psi(b/\epsilon)(y^2e^{ax+by} - z^2e^{ax+bz}).\end{aligned}$$

By the dominated convergence theorem and inequality (4.2.4), we obtain similarly:

$$\begin{aligned}\mathbb{E}_{z_0}\left[\int_0^t \left(z^2e^{A_tx+B_tz} + \psi(b/\epsilon)(y^2e^{A_tx+bB_t} - z^2e^{A_tx+B_tz})\right)d\langle B,B\rangle_t\right] \xrightarrow[\epsilon\to 0]{} \Sigma_{22}^+y^2\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s\ge 0}e^{xA_s+yB_s}ds\right]\\ + \Sigma_{22}^-z^2\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{B_s<0}e^{xA_s+zB_s}ds\right].\end{aligned}$$

By the same arguments as for (4.2.21), we have

$$\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{-\epsilon\le B_s<\epsilon}\frac{2}{\epsilon}\psi'(B_s/\epsilon)(ye^{xA_s+yB_s} - ze^{xA_s+zB_s})d\langle B,B\rangle_s\right] \xrightarrow[\epsilon\to 0]{} (\Sigma_{22}^- + \Sigma_{22}^+)(y-z)\int_{\mathbb{R}} e^{ax}g_t(a,0)da. \tag{4.2.25}$$

Additionally,

$$\mathbb{E}_{z_0}\left[\int_0^t \mathbb{1}_{0\le B_s<\epsilon}(1/\epsilon^2)\psi''(B_s/\epsilon)(e^{xA_s+yB_s} - e^{xA_s+zB_s})ds\right] = \int_{\mathbb{R}}\int_0^\epsilon (1/\epsilon^2)\psi''(b/\epsilon)(e^{xa+yb} - e^{xa+zb})g_t(a,b)dbda$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \int_0^1 \psi''(b) \frac{e^{ax+b\epsilon y} - e^{ax+b\epsilon z}}{\epsilon} g_t(a, \epsilon b) db da \\
&\xrightarrow[\epsilon \to 0]{} (y-z) \int_{\mathbb{R}} e^{ax} g_t(a,0) da \int_0^1 b \psi''(b) db \\
&= (y-z)(\psi(0) - \psi(1)) \int_{\mathbb{R}} e^{ax} g_t(a,0) da = \frac{(z-y)}{2} \int_{\mathbb{R}} e^{ax} g_t(a,0) da
\end{aligned}
$$

Here, the limit is once again justified by the estimates in (4.2.3). Combining this with Lemma 4.16, we have shown that

$$
\begin{aligned}
\frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \mathbb{E}_{z_0} \left[\int_0^t \partial_i \partial_j f_\epsilon(Z_s) d\langle Z^i, Z^j \rangle_s \right] &\xrightarrow[\epsilon \to 0]{} \frac{1}{2} (\Sigma^+_{11} x^2 + 2\Sigma^+_{12} xy + \Sigma^+_{22} y^2) \mathbb{E}_{z_0} \left[\int_0^t \mathbb{1}_{B_s \geq 0} e^{xA_s + yB_s} ds \right] \qquad (4.2.26) \\
&+ \frac{1}{2} (\Sigma^-_{11} x^2 + 2\Sigma^-_{12} xy + \Sigma^-_{22} y^2) \mathbb{E} \left[\int_0^t \mathbb{1}_{B_s < 0} e^{xA_s + zB_s} ds \right] + (y-z) \mathbb{E}_{z_0} \left[\int_0^t e^{xA_s} dL^0_s(B) \right].
\end{aligned}
$$

Step 3: Sum up and take the limit as $t \to +\infty$.
Combining (4.2.15), (4.2.22), and (4.2.26), we obtain:

$$
\begin{aligned}
\mathbb{E}_{z_0}[f(Z_t)] = \gamma_-(x,z) \mathbb{E}_{z_0} \left[\int_0^t \mathbb{1}_{B_s < 0} e^{xA_s + zB_s} ds \right] + \gamma_+(x,y) \mathbb{E}_{z_0} \left[\int_0^t \mathbb{1}_{B_s > 0} e^{xA_s + yB_s} ds \right] \qquad (4.2.27) \\
+ \left(q_1 x + \frac{y(1+q_2) + z(q_2 - 1)}{2} \right) \mathbb{E}_{z_0} \left[\int_0^t e^{xA_t} dL^0_s(B) \right] + e^{xa_0 + yb_0 \mathbb{1}_{b_0 > 0} + zb_0 \mathbb{1}_{b_0 < 0}}.
\end{aligned}
$$

Moreover, note that $f(Z_t) \leq e^{xA_t}$ since $y < 0$ and $z > 0$. By Lemma 4.19, $\mathbb{E}_{z_0}[f(Z_t)] \xrightarrow[t \to +\infty]{} 0$ if $-\eta < x < 0$.

Note that $x < 0$, $y < 0$, $z > 0$ can be taken arbitrarily close to 0 such that the conditions $\gamma_+(x,y) < 0$, $\gamma_-(x,z) < 0$, $y - z < 0$, and $q_1 x + \frac{y(1+q_2)+z(q_2-1)}{2} < 0$ are satisfied. For such x, y, z, the quantity

$$
\begin{aligned}
\gamma_-(x,z) \mathbb{E}_{z_0} \left[\int_0^t \mathbb{1}_{B_s < 0} e^{xA_s + zB_s} ds \right] + \gamma_+(x,y) \mathbb{E}_{z_0} \left[\int_0^t \mathbb{1}_{B_s > 0} e^{xA_s + yB_s} ds \right] \\
+ \left(q_1 x + \frac{y(1+q_2) + z(q_2 - 1)}{2} \right) \mathbb{E}_{z_0} \left[\int_0^t e^{xA_t} dL^0_s(B) \right]
\end{aligned}
$$

converges to

$$
\gamma_-(x,z) \varphi^{z_0}_-(x,z) + \gamma_+(x,y) \varphi^{z_0}_+(x,y) + \left(q_1 x + \frac{y(1+q_2) + z(q_2 - 1)}{2} \right) \varphi^{z_0}(x) \in [-\infty, 0)
$$

as $t \to +\infty$ (if some terms diverge, they are equal to $-\infty$ so the sum remains well defined). By (4.2.27), the previous sum is finite when $x \in (-\eta, 0)$, so $\varphi^{z_0}_-(x,z)$, $\varphi^{z_0}_+(x,y)$, and $\varphi^{z_0}(x)$ are finite for $x \in (-\eta, 0)$, $y < 0$ and $z > 0$ arbitrarily close to 0. Now that we know the Laplace transforms converge for such x, y, z, the asymptotics of (4.2.27) as $t \to +\infty$ yield the desired functional equation.

Remark 4.20 (Choice of $b_0 \neq 0$). The choice $b_0 \neq 0$ for the initial point (a_0, b_0) is made for essentially technical reasons. Indeed, it ensures that g_t remains finite on the strip $\mathbb{R} \times [-\epsilon, \epsilon]$ for $\epsilon > 0$ sufficiently small; this avoids unnecessary difficulties when taking the limit $\epsilon \to 0$ in (4.2.20) and (4.2.25). The functional equation also holds for $b_0 = 0$, which can be proved by taking the limit of (4.1.11) as $z_0 \to (a_0, 0)$.

4.3 Proof of Theorem 4.5 and 4.6: Laplace transforms and asymptotics on the axis

Since the proof of Theorem 4.6 is quite independent of the proofs of the other theorems, it has been placed in a separate section. Let us recall the expressions for $Y^{\pm}(x)$ and $Z^{\pm}(x)$ given in (4.1.12) and (4.1.13).

Lemma 4.21. *There exists $\epsilon > 0$ such that for all $u \in [-\epsilon, 0]$ and $v \in \mathbb{R}$,*

$$\Re\big(Y^-(u+iv)\big) < 0 \quad \textit{(resp.} \quad \Re\big(Z^+(u+iv)\big) > 0\textit{).}$$

Proof. By elementary considerations (see Lemma 2.17, Chapter [**chap2**]), we have

$$\Re(Y^{\pm}(u+iv)) = \frac{1}{\Sigma_{22}^+}\Big(-\Sigma_{12}^+ u - \mu_2^+ \pm \frac{\sqrt{\det(\Sigma^+)}}{\sqrt{2}}\sqrt{(u-x_{min}^+)(x_{max}^+ - u) + v^2 + |(u+iv-x_{min}^+)(x_{max}^+ - u - iv)|}\Big). \tag{4.3.1}$$

Then, $\Re(Y^-(u+iv)) \leq \Re(Y^-(u))$ for all $v \in \mathbb{R}$. The inequality $Y^-(0) = -\frac{2\mu_2^+}{\Sigma_{22}^+} < 0$ yields the first inequality. The second one is obtained symmetrically since $\mu_2^- < 0$. ☐

We are now able to prove Theorem 4.5.

Proof of Theorem 4.5. For $x \in (-\epsilon, 0)$, we have $Y^-(x) < 0$ and $Z^+(x) > 0$. We can then substitute $(x, Y^-(x), Z^+(x))$ into the functional equation (4.1.11) to derive the equality (4.1.16) for $x \in (-\epsilon, 0)$. Since φ is a Laplace transform, equation (4.1.16) holds in the entire domain of convergence of φ by holomorphicity. Since the expression is meromorphic on $\mathbb{C} \setminus \big((-\infty, \widetilde{x}_b] \cup [x_b, +\infty)\big)$, it remains only to show that the equation $\gamma(x, Y^-(x), Z^+(x)) = 0$ has no solution on $\mathbb{C} \setminus \big((-\infty, \widetilde{x}_b) \cup (x_b, +\infty)\big)$. Using expressions (4.1.3), (4.1.12) and (4.1.13), we can write $\gamma(x, Y^-(x), Z^+(x))$ as:

$$\begin{aligned}\gamma(x, Y^-(x), Z^+(x)) &= \frac{1}{\Sigma_{22}^+ + \Sigma_{22}^-}\big((\Sigma_{12}^+ - \Sigma_{12}^-)x + \Sigma_{22}^+ Y^-(x) - \Sigma_{22}^- Z^+(x)\big) \qquad (4.3.2)\\ &= \frac{1}{\Sigma_{22}^+ + \Sigma_{22}^-}\Big(-\mu_2^+ + \mu_2^- - \sqrt{({\Sigma_{12}^+}^2 - \Sigma_{11}^+\Sigma_{22}^+)x^2 + 2(\mu_2^+\Sigma_{12}^+ - \mu_1^+\Sigma_{22}^+)x + (\mu_2^+)^2}\\ &\quad - \sqrt{({\Sigma_{12}^-}^2 - \Sigma_{11}^-\Sigma_{22}^-)x^2 + 2(\mu_2^-\Sigma_{12}^- - \mu_1^-\Sigma_{22}^-)x + (\mu_2^-)^2}\Big).\end{aligned}$$

Furthermore, by Assumption (4.1.5), $-\mu_2^+ + \mu_2^- < 0$ and by the same arguments as for (4.3.1),

$$\Re\left(\sqrt{({\Sigma_{12}^{\pm}}^2 - \Sigma_{11}^{\pm}\Sigma_{22}^{\pm})x^2 + 2(\mu_2^{\pm}\Sigma_{12}^{\pm} - \mu_1^{\pm}\Sigma_{22}^{\pm})x + (\mu_2^{\pm})^2}\right) \geq 0.$$

Considering the real part, we see that $\gamma(x, Y^-(x), Z^+(x))$ does not vanish on $\mathbb{C} \setminus \big((-\infty, \widetilde{x}_b] \cup [x_b, +\infty)\big)$, and the first result follows.

Equation (4.1.11), combined with the explicit expression of φ^{z_0} (see Theorem 4.5), provides an explicit formula for $\varphi_+^{z_0}(x, y)$. Indeed, by substituting $z = Z^+(x) > 0$ (with $x < 0$ small enough) and $y < 0$ into (4.1.11), we obtain

$$\varphi_+^{z_0}(x, y) = \frac{-\gamma(x, y, Z^+(x))\varphi^{z_0}(x) - e^{xa_0 + yb_0\mathbb{1}_{b_0 \geq 0} + zb_0\mathbb{1}_{b_0 < 0}}}{\gamma_+(x, y)}. \tag{4.3.3}$$

Using the holomorphic properties of the Laplace transforms, (4.3.3) holds for all (x, y) in the domain of convergence of $\varphi_+^{z_0}$. ☐

We now state a final preparatory lemma prior to the proof of Theorem 4.6.

Lemma 4.22 (Behaviour of φ^{z_0} at x_b)**.** *Let C_0 and $f_0(z_0)$ be defined as in Theorem 4.6. Then,*

$$\varphi^{z_0}(x) = \varphi^{z_0}(x_b) - 2\sqrt{\pi}\left(\frac{\Sigma_{22}^+ + \Sigma_{22}^-}{2}\right) C_0 f_0(z_0)\sqrt{x - x_b} + o(\sqrt{x - x_b})$$

as $x \to x_b$, where $f_0(z_0)$ is defined in Theorem 4.6.

Proof. Without loss of generality, we assume in this proof that $x_b = x^+_{max} < x^-_{max}$. We define C_y as $C_y = \frac{\sqrt{\det(\Sigma^+)(x_b - x^+_{min})}}{\Sigma^+_{22}}$ so that $Y^-(x) = Y^-(x_b) - C_y\sqrt{x_b - x} + o(\sqrt{x_b - x})$.

Now, let us perform straightforward computations using the explicit expression of φ^{z_0} given by (4.1.16). We have:

$$\begin{aligned}
\frac{1}{\gamma(x, Y^-(x), Z^+(x))} &= \frac{1}{\gamma(x_b, Y^-(x_b), Z^+(x_b)) - C_y\left(\frac{1+q_2}{2}\right)\sqrt{x_b - x}(1 + o(1))} \\
&= \frac{1}{\gamma(x_b, Y^-(x_b), Z^+(x_b))} \times \frac{1}{1 - \frac{C_y\left(\frac{1+q_2}{2}\right)}{\gamma(x_b, Y^-(x_b), Z^+(x_b))}\sqrt{x_b - x}(1 + o(1))} \\
&= \frac{1}{\gamma(x_b, Y^-(x_b), Z^+(x_b))}\left(1 + \frac{C_y\left(\frac{1+q_2}{2}\right)}{\gamma(x_b, Y^-(x_b), Z^+(x_b))}\sqrt{x_b - x}(1 + o(1))\right).
\end{aligned}$$

Furthermore,

$$e^{Y^-(x)b_0\mathbb{1}_{b_0>0}} = e^{(Y^-(x_b) - \sqrt{x_b - x}(C_y + o(1)))b_0\mathbb{1}_{b_0>0}} = e^{Y^-(x_b)b_0\mathbb{1}_{b_0>0}}(1 - \sqrt{x_b - x}(C_y + o(1))b_0\mathbb{1}_{b_0>0}).$$

Multiplying the two quantities together, we obtain:

$$\begin{aligned}
\varphi^{z_0}(x) - \varphi^{z_0}(x_b) = &\frac{C_y}{\gamma(x_b, Y^-(x_b), Z^+(x_b))} \\
&\times \left(b_0\mathbb{1}_{b_0>0} - \frac{\frac{1+q_2}{2}}{\gamma(x_b, Y^-(x_b), Z^+(x_b))}\right) e^{a_0 x_b + Y^-(x_b)b_0\mathbb{1}_{b_0>0} + Z^+(x_b)b_0\mathbb{1}_{b_0<0}}\sqrt{x - x_b} + o(\sqrt{x - x_b}),
\end{aligned}$$

which is the desired formula. □

Since φ^{z_0} is the Laplace transform of $\frac{\Sigma^+_{22} + \Sigma^-_{22}}{2} g(x, 0)$ (see Lemma 4.16), we can now derive the asymptotics of $g^{z_0}(r, 0)$ as $r \to +\infty$ (i.e., Theorem 4.6) using the Tauberian Theorem 4.A.1.

Proof of Theorem 4.6. By symmetry, we only consider the asymptotics as $r \to +\infty$. First, suppose that $a_0 = 0$. Note that the first and third conditions of Theorem 4.A.1 are satisfied by Theorem 4.5 and Lemma 4.22. Let us verify the second condition of Theorem 4.A.1.

From (4.3.1), there exist $\delta > 0$ and constants $M, c > 0$ such that $\Re(Y^-(u + iv)) \leq M - cv$ for all $u, v \in \mathbb{R}$ satisfying $|u| \leq \frac{1}{\tan(\delta)}|v|$. Then, $e^{b_0\Re(Y^-(x))\mathbb{1}_{b_0>0}}$ is bounded for $x \in G^+_\delta(x_b)$. The same reasoning applies to $e^{b_0\Re(Z^+(x))\mathbb{1}_{b_0<0}}$.

Furthermore, from (4.3.1), possibly after choosing δ close enough to $\pi/2$, we have

$$|\gamma(x, Y^-(x), Z^+(x))| \underset{\substack{|x| \to +\infty \\ x \in G^+_\delta(x_b)}}{\longrightarrow} +\infty.$$

We therefore obtain the second condition of Theorem 4.A.1 from the explicit expression of φ^{z_0} given in (4.1.17). Then, by Theorem 4.A.1, the following asymptotic holds:

$$g^{0,b_0}(r, 0) \underset{r \to +\infty}{\sim} \frac{-2\sqrt{\pi}}{\Gamma(-1/2)} C_0 f_0(z_0)\frac{e^{-rx_b}}{r^{3/2}} = C_0 f_0(z_0)\frac{e^{-rx_b}}{r^{3/2}}. \tag{4.3.4}$$

Now suppose that $a_0 \neq 0$. Note that if $Z \sim \mathbb{P}_{(0,b_0)}$, then $Z + (a_0, 0) \sim \mathbb{P}_{(a_0,b_0)}$. This follows from the pathwise uniqueness of the SDE (4.1.1). Therefore, $g^{(a_0,b_0)}(x, 0) = g^{(0,b_0)}(x - a_0, 0)$, which yields (4.3.4) using the identity $e^{a_0 x_b} f_0(0, b_0) = f_0(a_0, b_0)$ (see (4.1.24) and (4.1.25)). □

4.4 Proofs of Theorem 4.7 to 4.10: asymptotics along all directions

First, in Section 4.4.1, we establish a representation of the Green's functions g^{z_0} as simple integrals using an inverse Laplace theorem and some complex analysis. These integrals are of saddle-point type. Therefore, in Section 4.4.2, we define paths of steepest descent in a neighbourhood of the saddle points $x(\alpha)$ corresponding to different directions α. We should now shift the integration contours up to the saddle point, taking into account the singularities of the integrand. This procedure depends on the position of the saddle points $x(\alpha)$ relative to the branching points $\widetilde{x}_b$ and x_b of the integrand. In Section 4.4.3, we prove Theorems 4.7 and 4.8. In Section 4.4.4, we prove Theorem 4.9. Finally, we prove Theorem 4.10 in Section 4.4.5. In each section, we define a different path deformation for the integral representation of g^{z_0}. For each case, we identify the dominant contribution of the integral and determine which parts are negligible.

Without loss of generality, our proofs only consider angles $\alpha \in [0, \pi]$, since the remaining cases are symmetric.

4.4.1 Laplace inverse and reduction to simple integrals

Using the two-dimensional Laplace inversion theorem, we can express $g^{z_0}(a, b)$ in terms of its Laplace transform (see [33, Theorems 24.3 and 24.4] and [16]).

Lemma 4.23 (Inverse Laplace theorem). *Let $z_0 \in \mathbb{R}^2$ and $(a, b) \in \mathbb{R}^2$ such that $b > 0$ and $(a, b) \neq z_0$. Then, for $\epsilon_1 > 0, \epsilon_2 > 0$ sufficiently small,*

$$g^{z_0}(a, b) = \frac{1}{(2i\pi)^2} \int_{-\epsilon_1 - i\infty}^{-\epsilon_1 + i\infty} \int_{-\epsilon_2 - i\infty}^{-\epsilon_2 + i\infty} \varphi_+^{z_0}(x, y) e^{-ax - by} dy dx \tag{4.4.1}$$

where the convergence is understood in the sense of principal value.

Proof. This follows from the two-dimensional Laplace inversion theorem (see [33, Theorems 24.3 and 24.4] and [16]), provided that the double integral converges in the sense of principal value. This convergence is ensured by the explicit expression (4.1.17) and by standard computations based on integration by parts. ☐

By similar considerations as in Section 4.1.3, we define branches $X^{\pm}(y)$ by $\gamma_+(X^{\pm}(y), y) = 0$. These branches have branching points at y_{min} and y_{max}, defined by

$$y_{min} = \frac{\mu_1^+ \Sigma_{12}^+ - \mu_2^+ \Sigma_{11}^+ - \sqrt{D_2}}{\det(\Sigma^+)}, \quad y_{max} = \frac{\mu_1^+ \Sigma_{12}^+ + \mu_2^+ \Sigma_{11}^+ - \sqrt{D_2}}{\det(\Sigma^+)},$$

where $D_2 = (\mu_1^+ \Sigma_{12}^+ - \mu_2^+ \Sigma_{11}^+)^2 + {\mu_1^+}^2 \det(\Sigma^+)$.

Lemma 4.24 (Reduction to simple integrals). *Let $z_0 \in \mathbb{R}^2$ and $(a, b) \in \mathbb{R}^2$ such that $b > 0$ and $(a, b) \neq z_0$. Then, for $\epsilon > 0$ small enough, $g^{z_0}(a, b)$ can be written as $g^{z_0}(a, b) = I_1^{z_0}(a, b) + I_2^{z_0}(a, b)$, where*

$$I_1^{z_0}(a, b) = \frac{1}{2i\pi} \int_{-\epsilon - i\infty}^{-\epsilon + i\infty} \varphi^{z_0}(x) \frac{\gamma(x, Y^+(x), Z^+(x))}{\partial_y \gamma_+(x, Y^+(x))} e^{-ax - bY^+(x)} \, dx \tag{4.4.2}$$

and

$$I_2^{z_0}(a, b) = \frac{1}{2i\pi} \int_{-\epsilon - i\infty}^{-\epsilon + i\infty} \frac{e^{(a_0 - a)x} \left(e^{(b_0 - b)Y^+(x)} \mathbb{1}_{b_0 > 0} + e^{b_0 Z^+(x) - bY^+(x)} \mathbb{1}_{b_0 < 0} \right)}{\partial_y \gamma_+(x, Y^+(x))} \, dx \quad \text{if } b > b_0. \tag{4.4.3}$$

Furthermore, the following expressions hold for $I_2^{z_0}(a,b)$ *if* $b_0 > 0$*:*

$$I_2^{z_0}(a,b) = \frac{1}{2i\pi}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{e^{(a_0-a)X^+(y)+(b_0-b)y}}{\partial_x\gamma_+(X^+(y),y)}\,dy \quad \text{if } a > a_0 \tag{4.4.4}$$

$$I_2^{z_0}(a,b) = \frac{-1}{2i\pi}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{e^{(a_0-a)X^-(y)+(b_0-b)y}}{\partial_x\gamma_+(X^-(y),y)}\,dy \quad \text{if } a < a_0. \tag{4.4.5}$$

Proof. The proof is similar to that of Lemma 2.27, Chapter 2. We choose $\epsilon_1 = \epsilon_2 = \epsilon$ such that $Y^-(-\epsilon) < -\epsilon$ and $Y^+(-\epsilon) > 0$ (this is possible since $Y^-(0) = \frac{-2\mu_2^+}{\Sigma_{22}^+} < 0$, $Y^+(0) = 0$, and $(Y^+)'(0) = \frac{-\mu_1^+}{\mu_2^+} < 0$). Then, for all $v \in \mathbb{R}$, we have $\Re(Y^-(-\epsilon+iv)) \leq \Re(Y^-(-\epsilon)) < -\epsilon$ and $\Re(Y^+(-\epsilon+iv)) \geq Y^+(-\epsilon) > 0$ (see expression (4.3.1)).

Using (4.1.17) and (4.4.1), we derive the equality $g^{z_0}(a,b) = I_1^{z_0}(a,b) + I_2^{z_0}(a,b)$, where

$$I_1^{z_0}(a,b) = \frac{1}{(2i\pi)^2}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi^{z_0}(x)\left(\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{-\gamma(x,y,Z^+(x))}{\gamma_+(x,y)}e^{-ax-by}\,dy\right)dx$$

and

$$I_2^{z_0}(a,b) = \frac{1}{(2i\pi)^2}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{-e^{xa_0+yb_0\mathbb{1}_{b_0\geq 0}+zb_0\mathbb{1}_{b_0<0}}}{\gamma_+(x,y)}e^{-ax-by}\,dy\,dx.$$

It remains to show that $I_1^{z_0}(a,b)$ and $I_2^{z_0}(a,b)$ admit the desired integral representations. Let us focus on I_1. Fix $x \in -\epsilon + i\mathbb{R}$. For $R > 0$, consider the path $\Gamma_R = \{-\epsilon + Re^{i\theta} \mid \theta \in [0,\pi]\}$. Then, for sufficiently large $R > 0$, the point $Y^+(x)$ lies inside the closed contour formed by $\Gamma_R \cup [-\epsilon-iR, -\epsilon+iR]$, see Figure 4.9 (recall that $\Re(Y^+(x)) \geq \Re(Y^+(\epsilon)) > -\epsilon$). By the residue theorem,

$$\frac{1}{2i\pi}\left(\int_{-\epsilon-iR}^{-\epsilon+iR} + \int_{\Gamma_R}\right)\frac{-\gamma(x,y,Z^+(x))}{\gamma_+(x,y)}e^{-ax-by}\,dy = \frac{\gamma(x,Y^+(x),Z^+(x))}{\partial_y\gamma_+(x,Y^+(x))}e^{-ax-bY^+(x)}.$$

Furthermore,

$$\int_{\Gamma_R}\frac{\gamma(x,y,Z^+(x))}{\gamma_+(x,y)}e^{-ax-by}\,dy = e^{-ax}\int_0^\pi \frac{\gamma(x,-\epsilon+Re^{i\theta},Z^+(x))}{\gamma_+(x,-\epsilon+Re^{i\theta})}e^{-b\epsilon-bRe^{i\theta}}\,d\theta. \tag{4.4.6}$$

Moreover, $\sup_{\theta\in[0,\pi]}\left|\frac{\gamma(x,-\epsilon+Re^{i\theta},Z^+(x))}{\gamma_+(x,-\epsilon+Re^{i\theta})}\right| = O(1/R)$ as $R \to +\infty$, and $e^{-bRe^{i\theta}} \to 0$ as $R \to +\infty$, since $-b\cos(\theta) < 0$ for all $\theta \in (0,\pi)$. Therefore, (4.4.6) converges to 0 as $R \to +\infty$, which yields representation (4.4.2) for $I_1^{z_0}(a,b)$. The proof of the representations of $I_2^{z_0}(a,b)$ is analogous. □

4.4.2 Paths of the saddle point method

As the previous integrals are of the saddle-point type, we now introduce the associated paths. We recall the definition of $(x(\alpha), y(\alpha))$ for $\alpha \in [0,\pi]$ given by (4.1.18). Let us define

$$F(x,\alpha) = -\cos(\alpha)\,x - \sin(\alpha)\,Y^+(x) + \cos(\alpha)\,x(\alpha) + \sin(\alpha)\,y(\alpha). \tag{4.4.7}$$

By construction, for all $\alpha \in (0,\pi)$, we have $F(x(\alpha),\alpha) = 0$ and $F'_x(x(\alpha),\alpha) = 0$. It follows that $(Y^+)'(x(\alpha)) = -\frac{\cos(\alpha)}{\sin(\alpha)}$, and

$$x'(\alpha)\,F''_x(x(\alpha),\alpha) + F''_{x,\alpha}(x(\alpha),\alpha) = 0.$$

Then,

$$x'(\alpha)\,F''_x(x(\alpha),\alpha) = -\big(\sin(\alpha) - (Y^+)'(x(\alpha))\cos(\alpha)\big) = \frac{-1}{\sin(\alpha)}. \tag{4.4.8}$$

From computations carried out in Section 2.5, Chapter 2, we also have

$$F''_x(x(\alpha),\alpha) = \frac{\Sigma_{11}^+\sin^2(\alpha) - 2\Sigma_{12}^+\sin(\alpha)\cos(\alpha) + \Sigma_{22}^+\cos^2(\alpha)}{\partial_y\gamma_+(x(\alpha),y(\alpha))\sin(\alpha)} > 0, \quad \alpha \in (0,\pi). \tag{4.4.9}$$

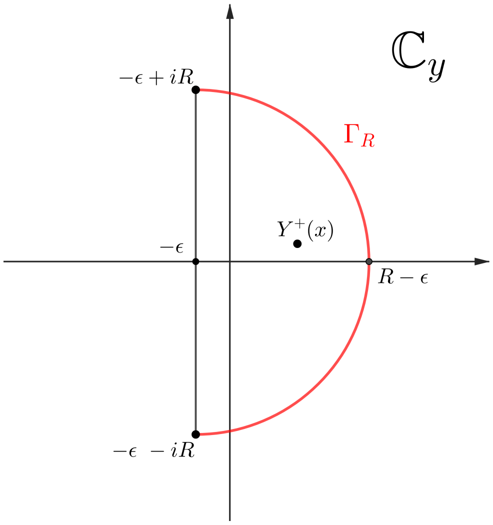


Figure 4.9: Changing path for representations (4.4.2) and (4.4.3) of $I_1^{z_0}(a,b)$ and $I_2^{z_0}(a,b)$.

Case $\alpha_0 \in (0,\pi)$**.**

Let us fix $\alpha_0 \in (0,\pi)$. By the parameterised Morse lemma (Lemma 2.A.1), there exists a neighbourhood of $(0,\alpha_0) \in \mathbb{C}\times\mathbb{R}$, given by

$$\Omega(0,\alpha_0) = \{(\omega,\alpha) \in \mathbb{C}\times[0,\pi] \mid |\omega| \leq L,\ |\alpha-\alpha_0| \leq \eta\} \tag{4.4.10}$$

for some $L, \eta > 0$, and a function $x : \Omega(0,\alpha_0) \longrightarrow \mathbb{C}$ of class $\mathcal{C}^\infty$ (and holomorphic in the first variable) such that

$$x(0,\alpha) = x(\alpha) \quad \text{for all } \alpha \text{ with } |\alpha-\alpha_0| \leq \eta,$$

and satisfying

$$F(x(\omega,\alpha),\alpha) = \omega^2, \quad \forall(\omega,\alpha) \in \Omega(0,\alpha_0). \tag{4.4.11}$$

Furthermore,

$$x'_\omega(0,\alpha) = \sqrt{\frac{2}{F''_x(x(\alpha),\alpha)}}. \tag{4.4.12}$$

Let $0 < \epsilon < K$. We define the contour of steepest descent as

$$\Gamma_{x,\alpha} = \{x(it,\alpha) \mid t \in [-\epsilon,\epsilon]\}.$$

Note that $F(x(it,\alpha),\alpha) = -t^2$. We denote by $x_\alpha^+ = x(i\epsilon,\alpha)$ and $x_\alpha^- = x(-i\epsilon,\alpha)$ the endpoints of $\Gamma_{x,\alpha}$. It can be shown (see Section 2.5, Chapter 2) that for any $\alpha_0 \in (0,\pi)$, there exist $\eta > 0$ and $\nu > 0$ such that

$$\operatorname{Im}(x_\alpha^+) > \nu, \quad \operatorname{Im}(x_\alpha^-) < -\nu \quad \text{for all } \alpha \text{ with } |\alpha-\alpha_0| < \eta. \tag{4.4.13}$$

Case $\alpha_0 \in \{0,\pi\}$**.**

We define

$$G(y,\alpha) = -\cos(\alpha)X^+(y) - \sin(\alpha)y + \cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha). \tag{4.4.14}$$

As before, we have $G(y(\alpha),\alpha) = 0$, $G'_y(y(\alpha),\alpha) = 0$, and $G''_y(y(\alpha),\alpha) > 0$ for $\alpha \in [0,\pi/2)\cup(\pi/2,\pi]$. Then, if $\alpha_0 \in [0,\pi/2)\cup(\pi/2,\pi]$ (and in particular if $\alpha_0 = 0$ or $\alpha_0 = \pi$), we define symmetrically the function $y(\omega,\alpha)$ corresponding to $G(y,\alpha)$ via the parameterised Morse Lemma. We then set

$$\Gamma_{y,\alpha} = \{y(it,\alpha) \mid t \in [-\epsilon,\epsilon]\},$$

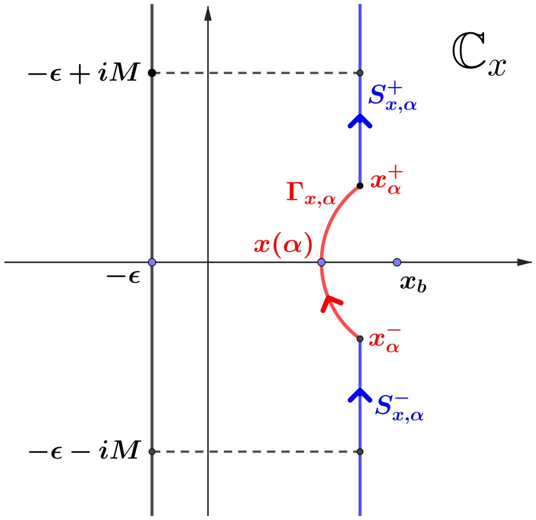


Figure 4.10: Changing path of Lemma 4.25

whose endpoints are $y^+_\alpha = y(i\epsilon, \alpha)$ and $y^-_\alpha = y(-i\epsilon, \alpha)$. The paths $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$ are linked by the relations

$$\Gamma_{x,\alpha} = \underrightarrow{\overleftarrow{X^+(\Gamma_{y,\alpha})}}, \quad \Gamma_{y,\alpha} = \underrightarrow{\overleftarrow{Y^+(\Gamma_{x,\alpha})}}, \quad \alpha \in (0, \pi/2) \cup (\pi/2, \pi). \tag{4.4.15}$$

where the arrows indicate that the paths are traversed in the reversed direction.

4.4.3 Lemmas for red-arc parts

Now, let us consider the asymptotics of Green's functions g^{z_0} in the directions corresponding to the red arc parts, excluding α_b and $\widetilde{\alpha}_b$ (see Figure 4.5). Recall the notation introduced in (4.1.15). We define

$$S^+_{x,\alpha} = \{x^+_\alpha + it \mid t \geq 0\}, \quad S^-_{x,\alpha} = \{x^-_\alpha - it \mid t \geq 0\},$$
$$S^+_{y,\alpha} = \{y^+_\alpha + it \mid t \geq 0\}, \quad S^-_{y,\alpha} = \{y^-_\alpha - it \mid t \geq 0\}.$$

The following lemma describes how to modify the contours of integration in (4.4.2) and (4.4.3) so that they pass through the saddle point along the steepest descent path.

Lemma 4.25 (Changing path through the saddle point)**.** *Let $z_0 \in \mathbb{R}^2$, $a \in \mathbb{R}$, $b > 0$ such that $(a, b) \neq z_0$, and let $\alpha \in (0, \pi)$ be such that $\widetilde{x}_b < x(\alpha) < x_b$. Then:*

$$I_1^{z_0}(a,b) = \frac{1}{2i\pi} \int_{S^-_{x,\alpha} \cup \Gamma_{x,\alpha} \cup S^+_{x,\alpha}} \varphi^{z_0}(x) \frac{\gamma(x, Y^+(x), Z^+(x))}{\partial_y \gamma_+(x, Y^+(x))} e^{-ax - bY^+(x)}\, dx, \tag{4.4.16}$$

and

$$I_2^{z_0}(a,b) = \frac{1}{2i\pi} \int_{S^-_{x,\alpha} \cup \Gamma_{x,\alpha} \cup S^+_{x,\alpha}} \frac{e^{(a_0 - a)x} \left(e^{(b_0 - b)Y^+(x)} \mathbb{1}_{b_0 > 0} + e^{b_0 Z^+(x) - bY^+(x)} \mathbb{1}_{b_0 < 0} \right)}{\partial_y \gamma_+(x, Y^+(x))}\, dx, \quad \textit{if } b > b_0. \tag{4.4.17}$$

Furthermore, if $b_0 > 0$, then

$$I_2^{z_0}(a,b) = \frac{1}{2i\pi} \int_{S^-_{y,\alpha} \cup \Gamma_{y,\alpha} \cup S^+_{y,\alpha}} \frac{e^{(a_0 - a)X^+(y) + (b_0 - b)y}}{\partial_x \gamma_+(X^+(y), y)}\, dy, \quad \textit{if } a > a_0, \tag{4.4.18}$$

and

$$I_2^{z_0}(a,b) = -\frac{1}{2i\pi}\int_{S^-_{y,\alpha}\cup\Gamma_{y,\alpha}\cup S^+_{y,\alpha}} \frac{e^{(a_0-a)X^-(y)+(b_0-b)y}}{\partial_x\gamma_+(X^-(y),y)}\,dy, \quad \text{if } a<a_0. \tag{4.4.19}$$

Proof. We prove (4.4.16); the other identities are analogous. The argument is similar to the one given in Lemma 2.28, Chapter 2. Using the Residue Theorem (see Figure 4.10) and Lemma 4.24, it suffices to verify that

$$\sup_{u\in[X^+(y_{\max})-\eta,\,x^{\max}+\eta]}\left|\frac{\varphi^{z_0}(u+iv)\,\gamma(u+iv,Y^+(u+iv),Z^+(u+iv))}{\partial_y\gamma_+(u+iv,Y^+(u+iv))}\,e^{-a(u+iv)-bY^+(u+iv)}\right| \longrightarrow 0 \tag{4.4.20}$$

as $|v|\to\infty$, for some $\eta>0$. This decay follows directly from (4.1.16) and the estimate on the real part in (4.3.1). ☐

The following two lemmas provide the asymptotic contribution of integrals along the steepest descent contours.

Lemma 4.26 (Asymptotics along $\Gamma_{x,\alpha}$ in the red arc region with $\alpha_0\neq 0,\pi$)**.** *Let* $\alpha_0\in(0,\pi)$ *such that* $\widetilde{x}_b < x(\alpha_0) < x_b$*. Let* $r=\sqrt{a^2+b^2}$ *and define* $\alpha=\alpha(a,b)\in(0,\pi)$ *such that* $\cos(\alpha)=\frac{a}{\sqrt{a^2+b^2}}$, $\sin(\alpha)=\frac{b}{\sqrt{a^2+b^2}}$*. Then, the following asymptotic holds as* $r\to+\infty$ *and* $\alpha\to\alpha_0$*:*

$$\frac{1}{2i\pi}\int_{\Gamma_{x,\alpha}}\left(\frac{\varphi^{z_0}(x)\gamma(x,Y^+(x),Z^+(x))e^{-ax-bY^+(x)}+e^{(a_0-a)x}\left(e^{(b_0-b)Y^+(x)}\mathbb{1}_{b_0>0}+e^{b_0Z^+(x)-bY^+(x)}\mathbb{1}_{b_0<0}\right)}{\partial_y\gamma_+(x,Y^+(x))}\right)dx \tag{4.4.21}$$

$$\underset{\substack{r\to+\infty\\ \alpha\to\alpha_0}}{\sim} C^+(\alpha_0)h_{\alpha_0}(z_0)\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}},$$

where $C^+(\alpha_0)$ *and* $h_{\alpha_0}(z_0)$ *are defined in* (4.1.28) *and* (4.1.29).

Proof. We have $F_x''(x(\alpha_0),\alpha_0)>0$ (see (4.4.9)). Then, by the parameterised saddle point method (see Lemma 2.33, Chapter 2), for all $n\geq 0$, the integral (4.4.21) admits the following asymptotic expansion as $\alpha=\alpha(a,b)\to\alpha_0$ and $r=\sqrt{a^2+b^2}\to+\infty$:

$$e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\sum_{k=0}^{n}\frac{c_k(\alpha)}{r^k\sqrt{r}}+o\left(\frac{1}{r^n\sqrt{r}}\right), \tag{4.4.22}$$

where the coefficients $c_0(\alpha),\dots,c_n(\alpha)$ are continuous functions of α and satisfy

$$c_0(\alpha)=\frac{h_\alpha(z_0)\,x'_\omega(0,\alpha)}{2\sqrt{\pi}\,\partial_y\gamma_+(x(\alpha),y(\alpha))}=C(\alpha)h_\alpha(z_0)$$

(see (4.4.12)). In particular, $c_0(\alpha_0)>0$ by expression (4.1.29). Therefore, the dominant asymptotic term is given by the $k=0$ term in (4.4.22), which concludes the proof. ☐

Lemma 4.27 (Asymptotics along $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$ in the red arc region at $\alpha_0=0,\pi$)**.** *Suppose* $x^+_{max}<x^-_{max}$ *and let* $\alpha_0=0$*. If* $r=\sqrt{a^2+b^2}$ *and* $\alpha=\alpha(a,b)\in(0,\pi)$ *is the angle such that* $\cos(\alpha)=\frac{a}{\sqrt{a^2+b^2}}$ *and* $\sin(\alpha)=\frac{b}{\sqrt{a^2+b^2}}$*, then the following asymptotics hold:*

$$\frac{1}{2i\pi}\int_{\Gamma_{x,\alpha}}\varphi^{z_0}(x)\frac{\gamma(x,Y^+(x),Z^+(x))}{\partial_y\gamma_+(x,Y^+(x))}e^{-ax-bY^+(x)}\,dx+\frac{1}{2i\pi}\int_{\Gamma_{y,\alpha}}\frac{e^{(a_0-a)X^+(y)+(b_0-b)y}}{\partial_x\gamma_+(X^+(y),y)}\,dy \tag{4.4.23}$$

$$\underset{r\to+\infty,\ \alpha\to0,\ \alpha\geq0}{\sim} C_0f_0(z_0)\left(\kappa\alpha+\frac{1}{r}\right)\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}},$$

where $\kappa > 0$ *is an explicit constant (see* (4.4.29)*) and* $f_0(z_0)$ *is given by* (4.1.24)*. Furthermore,*

$$\partial_\alpha\big(C(\alpha)h_\alpha(z_0)\big)\big|_{\alpha=0} = \kappa C_0 f_0(z_0).$$

If $x^+_{min} > x^-_{min}$*, the symmetric result holds for* $\alpha \to \pi$ *with* $\alpha \leq \pi$*.*

Proof. To prove this, we consider the integral representation along the contour $\Gamma_{y,\alpha}$. By performing the change of variables $x = X^+(y)$, we obtain:

$$\int\limits_{\Gamma_{x,\alpha}} \varphi^{z_0}(x)\frac{\gamma(x, Y^+(x), Z^+(x))}{\partial_y\gamma_+(x, Y^+(x))}e^{-ax-bY^+(x)}\,dx = \int\limits_{\Gamma_{y,\alpha}} \varphi^{z_0}(X^+(y))\frac{\gamma(X^+(y), y, Z^+(X^+(y)))}{\partial_x\gamma_+(X^+(y), y)}e^{-aX^+(y)-by}\,dy. \tag{4.4.24}$$

Recall that $(x(\alpha), y(\alpha)) \underset{\alpha\to 0}{\longrightarrow} (x^+_{max}, Y^\pm(x^+_{max}))$. Using (4.1.16), we have:

$$\varphi^{z_0}(X^+(y)) = -\frac{e^{X^+(y)a_0+Y^-(X^+(y))b_0\mathbb{1}_{b_0\geq 0}+Z^+(X^+(y))b_0\mathbb{1}_{b_0<0}}}{\gamma(X^+(y), Y^-(X^+(y)), Z^+(X^+(y)))}.$$

By definition, $X^+(y)$ is holomorphic in a neighborhood of $Y^\pm(x^+_{max})$. Moreover, by the Viète identities, the expression

$$Y^-(X^+(y)) = \frac{\Sigma^+_{11}(X^+(y))^2 + 2\mu^+_1 X^+(y)}{\Sigma^+_{22}y}$$

is holomorphic in a neighborhood of $Y^\pm(x^+_{max})$. Since we are in the case $x^+_{max} < x^-_{max}$, the function $Z^+(X^+(y))$ is also holomorphic in a neighborhood of $Y^\pm(x^+_{max})$. Thus, the integrand on the right-hand side of (4.4.24) is holomorphic near the saddle point.

Applying the parameterized saddle-point method to the contour $\Gamma_{y,\alpha}$, we deduce that for any $\alpha_0 \in [0, \pi/2)$, the expression in (4.4.23) admits the following asymptotic expansion as $\alpha = \alpha(a, b) \to \alpha_0$ and $r = \sqrt{a^2+b^2} \to \infty$:

$$\left(\widetilde{c}_0(\alpha) + \frac{\widetilde{c}_1(\alpha)}{r} + o\left(\frac{1}{r}\right)\right)\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}, \tag{4.4.25}$$

where the coefficients $\widetilde{c}_k(\alpha)$ are continuous in α. By uniqueness of the asymptotic expansion, we necessarily have $c_k(\alpha) = \widetilde{c}_k(\alpha)$ for all $\alpha \in (0, \pi/2]$ and for $k = 0, 1$. Recall that $X^+(y(\alpha)) = x(\alpha)$ and $Y^+(x(\alpha)) = y(\alpha)$ for all $\alpha \in (0, \pi/2)$. In particular, we have:

$$c_0(\alpha) = C^+(\alpha)\cdot\frac{e^{x(\alpha)a_0+b_0y(\alpha)\mathbb{1}_{b_0>0}+Z^+(x(\alpha))b_0\mathbb{1}_{b_0<0}}}{\gamma(x(\alpha), Y^-(x(\alpha)), Z^+(x(\alpha)))} \tag{4.4.26}$$

$$\times\left(-\gamma(x(\alpha), y(\alpha), Z^+(x(\alpha)))e^{(Y^-(x(\alpha))-y(\alpha))b_0\mathbb{1}_{b_0\geq 0}} + \gamma(x(\alpha), Y^-(x(\alpha)), Z^+(x(\alpha)))\right).$$

Furthermore, we observe that

$$\begin{aligned} y(\alpha) - Y^-(X^+(y(\alpha))) &= Y^+(X^+(y(\alpha))) - Y^-(X^+(y(\alpha))) \\ &= \frac{2}{\Sigma^+_{22}}\sqrt{\det(\Sigma_+)\cdot(x^+_{max} - X^+(y(\alpha)))\cdot(X^+(y(\alpha)) - x^+_{min})}. \end{aligned}$$

We also recall that $X^+(Y^\pm(x^+_{max})) = x^+_{max}$, and that

$$(X^+)'(y)\Big|_{y=Y^\pm(x^+_{max})} = 0, \quad (X^+)''(y)\Big|_{y=Y^\pm(x^+_{max})} = -\frac{\Sigma^+_{22}}{\partial_x\gamma_+(x^+_{max}, Y^\pm(x^+_{max}))}.$$

It follows that

$$y(\alpha) - Y^-(X^+(y(\alpha))) \sim \sqrt{\frac{2\det(\Sigma^+)(x^+_{max} - x^+_{min})}{\Sigma^+_{22}\partial_x\gamma_+(x^+_{max}, Y^\pm(x^+_{max}))}}\cdot\alpha \sim \Pi\alpha, \tag{4.4.27}$$

where we define

$$\Pi = \sqrt{\frac{2\det(\Sigma^+)(x^+_{max} - x^+_{min})}{\Sigma^+_{22}\partial_x\gamma_+(x^+_{max}, Y^\pm(x^+_{max}))}} = 2\sqrt{\frac{\det(\Sigma^+)}{\Sigma^+_{11}\Sigma^+_{22}}},$$

since $\partial_x\gamma_+(x^+_{max}, Y^\pm(x^+_{max})) = \frac{\Sigma^+_{11}(x^+_{max}-x^+_{min})}{2}$.

Injecting (4.4.27) into (4.4.26), and using the asymptotics

$$C^+(\alpha) \underset{\alpha\to 0^+}{\longrightarrow} \frac{1}{\sqrt{2\pi\Sigma^+_{22}\partial_x\gamma_+(x^+_{max}, Y^\pm(x^+_{max}))}} = \frac{1}{\sqrt{\pi\Sigma^+_{11}\Sigma^+_{22}(x^+_{max} - x^+_{min})}},$$

we obtain, after simplification,

$$C(\alpha)h_\alpha(z_0) = c_0(\alpha) \underset{\alpha\to 0^+}{\sim} \frac{2}{\Sigma^+_{11}\Sigma^+_{22}}\sqrt{\frac{\det(\Sigma^+)}{\pi(x^+_{max} - x^+_{min})}} f_0(z_0)\alpha = \kappa C_0 f_0(z_0)\alpha, \tag{4.4.28}$$

with

$$\kappa = \frac{-(\Sigma^+_{22} + \Sigma^-_{22})\gamma(x^+_{max}, Y^\pm(x^+_{max}), Z^+(x^+_{max}))}{\Sigma^+_{11}(x^+_{max} - x^+_{min})}. \tag{4.4.29}$$

Moreover, evaluating (4.4.25) at $\alpha = 0$ gives

$$g(r, 0) \underset{r\to\infty}{\sim} c_1(0) \cdot \frac{e^{-rx_b}}{r^{3/2}}.$$

Since $c_1(\alpha) = \widetilde{c}_1(\alpha)$ is continuous at $\alpha = 0^+$, we have $c_1(\alpha) \underset{\alpha\to 0}{\longrightarrow} C_0 f(z_0) > 0$, we conclude that the full expansion (4.4.23) holds as claimed. $\square$

The next lemma shows that in representations (4.4.16), (4.4.17), (4.4.18) and (4.4.19), the integrals along $S^\pm_{x,\alpha}$ and $S^\pm_{y,\alpha}$ are negligible compared to those along $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$.

Lemma 4.28 (Negligibility of the integrals along $S^\pm_{x,\alpha}$ and $S^\pm_{y,\alpha}$)**.** *Let $z_0 \in \mathbb{R}^2$. For $(a, b) \in \mathbb{R}\times(0, +\infty)$, let $\alpha(a, b) \in (0, \pi)$ be the angle such that $\cos(\alpha) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha) = \frac{b}{\sqrt{a^2+b^2}}$.*

- *Suppose $\alpha_0 \in (0, \pi)$. Then, for $\eta > 0$ small enough, there exist $r_0 > 0$ and $D > 0$ such that for all (a, b) satisfying $\sqrt{a^2 + b^2} > r_0$ and $|\alpha(a, b) - \alpha_0| < \eta$, we have:*

  $$\left| \int_{S^+_{x,\alpha}} \varphi^{z_0}(x)\frac{\gamma(x, Y^+(x), Z^+(x))}{\partial_y\gamma_+(x, Y^+(x))} e^{-ax-bY^+(x)}\, dx \right| \leq \frac{D}{b} e^{-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+b^2}}. \tag{4.4.30}$$

  If $b > b_0$, then the following estimate also holds:

  $$\left| \int_{S^+_{x,\alpha}} \frac{e^{(a_0-a)x}\left(e^{(b_0-b)Y^+(x)}\mathbb{1}_{b_0>0} + e^{b_0Z^+(x)-bY^+(x)}\mathbb{1}_{b_0<0}\right)}{\partial_y\gamma_+(x, Y^+(x))}\, dx \right| \leq \frac{D}{b-b_0} e^{-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+(b-b_0)^2}}. \tag{4.4.31}$$

- *Suppose $x_b = x^+_{\max}$ and let $\alpha_0 = 0$. Then, for $\eta > 0$ small enough, there exist $r_0 > 0$ and $D > 0$ such that for all (a, b) satisfying $\sqrt{a^2 + b^2} > r_0$ and $0 < \alpha(a, b) \leq \eta$, we have:*

  $$\left| \int_{S^+_{x,\alpha}} \varphi^{z_0}(x)\frac{\gamma(x, Y^+(x), Z^+(x))}{\partial_y\gamma_+(x, Y^+(x))} e^{-ax-bY^+(x)}\, dx \right| \leq De^{-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+b^2}}, \tag{4.4.32}$$

$$\left|\int\limits_{S^+_{y,\alpha}} \frac{e^{(a_0-a)X^+(y)+(b_0-b)y}}{\partial_x\gamma_+(X^+(y),y)}\,dy\right| \leq \frac{D}{a}e^{-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+b^2}}. \tag{4.4.33}$$

- *Symmetric estimates hold for* $\alpha_0=\pi$ *when* $\widetilde{x}_b = x^+_{\min}$.

The same estimates hold for $S^-_{x,\alpha}$ *and* $S^-_{y,\alpha}$.

Proof. First, let us consider the case $0<\alpha_0<\pi$. Using the notation (4.4.7), inequality (4.4.30) can be written as

$$\left|\int\limits_{v>0} \frac{\varphi^{z_0}(x^+_\alpha+iv)\gamma(x^+_\alpha+iv,Y^+(x^+_\alpha+iv),Z^-(x^+_\alpha+iv))}{\partial_y\gamma(x^+_\alpha+iv,Y^+(x^+_\alpha+iv))}\exp\left(\sqrt{a^2+b^2}(F(x^+_\alpha+iv,\alpha)-F(x^+_\alpha,\alpha))\right)dv\right| \leq \frac{D}{b}. \tag{4.4.34}$$

Furthermore,

$$\left|\exp\left(\sqrt{a^2+b^2}(F(x^+_\alpha+iv,\alpha)-F(x^+_\alpha,\alpha))\right)\right| = \exp\left(-b\big(\Re Y^+(x^+_\alpha+iv)-\Re Y^+(x^+_\alpha)\big)\right). \tag{4.4.35}$$

By (4.1.12), the quantity $\Re Y^+(x^+_\alpha+iv)-\Re Y^+(x^+_\alpha)$ vanishes at $v=0$ and grows linearly as $v\to+\infty$. Therefore, there exist constants $c>0$ and $\eta>0$ such that for all α satisfying $|\alpha-\alpha_0|\leq\eta$ and all $v\geq 0$,

$$\Re Y^+(x^+_\alpha+iv)-\Re Y^+(x^+_\alpha)\geq cv. \tag{4.4.36}$$

By (4.1.12) and (4.1.13), the quantity $\gamma(u+iv,Y^\pm(u+iv),Z^+(u+iv))$ grows linearly as $v\to+\infty$, uniformly in $u\in[-\widetilde{x}_b-\epsilon,x_b+\epsilon]$ for a given $\epsilon>0$. Similarly, $\partial_y\gamma_+(u+iv,Y^+(u+iv))$ tends to 0 as $v\to+\infty$, also uniformly in $u\in[-\widetilde{x}_b-\epsilon,x_b+\epsilon]$. Then, by (4.1.16), we have

$$\sup_{v\geq 0,|\alpha-\alpha_0|\leq\eta}\left|\frac{\varphi^{z_0}(x^+_\alpha+iv)\gamma(x^+_\alpha+iv,Y^+(x^+_\alpha+iv),Z^+(x^+_\alpha+iv))}{\partial_y\gamma_+(x^+_\alpha+iv,Y^+(x^+_\alpha+iv))}\right|<+\infty. \tag{4.4.37}$$

Then, for all (a,b) such that $|\alpha(a,b)-\alpha_0|\leq\eta$, the left-hand side of (4.4.34) is bounded by

$$C\int_0^\infty e^{-bcv}\,dv=\frac{C}{cb},$$

where $C>0$ is a constant independent of (a,b). This proves (4.4.30). Inequalities (4.4.31) and (4.4.33) follow by analogous symmetric arguments.

Now, let us consider the case $\alpha_0=0$ and prove (4.4.32). This inequality can be written as (4.4.34), but with the right-hand side $\frac{D}{b}$ replaced by D. The exponential term

$$\exp\left(-\sqrt{a^2+b^2}\left(F(x^+_\alpha+iv,\alpha)-F(x^+_\alpha,\alpha)\right)\right)$$

can be bounded by 1. By similar arguments, there exists $\eta>0$ such that

$$C':=\sup_{v\geq 0,\,0<\alpha\leq\eta}\left|\frac{\gamma(x^+_\alpha+iv,Y^+(x^+_\alpha+iv),Z^+(x^+_\alpha+iv))}{\partial_y\gamma_+(x^+_\alpha+iv,Y^+(x^+_\alpha+iv))}\right|<\infty. \tag{4.4.38}$$

If $b_0\neq 0$, then by (4.3.1) and (4.1.16), we have

$$C'':=\sup_{0<\alpha\leq\eta}\int_0^{+\infty}\left|\varphi^{z_0}(x^+_\alpha+iv)\right|dv<+\infty, \tag{4.4.39}$$

which gives the desired result with $D=C'C''$. If $b_0=0$, an integration by parts argument justifies the same estimate, using the explicit expression (4.1.16) (see Lemma 2.31, Chapter 2 for similar estimates). □

We now have the tools to prove Theorems 4.7 and 4.8.

Proof of Theorem 4.7. Let $\alpha_0 \in (0,\pi)$ be such that $\widetilde{x}_b < x(\alpha_0) < x_b$, and let $z_0 \in \mathbb{R}^2$. By Lemmas 4.24 and 4.25, the function $g^{z_0}(a,b)$ can be written as $g^{z_0}(a,b) = I_1^{z_0}(a,b) + I_2^{z_0}(a,b)$, where $I_1^{z_0}(a,b)$ and $I_2^{z_0}(a,b)$ are given by (4.4.16) and (4.4.17). Using the asymptotic expansion given in (4.4.21) from Lemma 4.26, together with the bounds (4.4.30) and (4.4.31) from Lemma 4.28, we see that the contributions of the integrals along the segments $S^{\pm}_{x,\alpha}$ are negligible compared to those along $\Gamma_{x,\alpha}$. Therefore, the asymptotics of $g^{z_0}(a,b)$ are given by (4.4.21). ☐

Proof of Theorem 4.8. Without loss of generality, we consider the case $\alpha_0 = 0$. We use the representations of $I_1^{z_0}(a,b)$ and $I_2^{z_0}(a,b)$ given by (4.4.16) and (4.4.18). Then, applying the asymptotic expansion (4.4.23) from Lemma 4.27 and the estimates (4.4.32) and (4.4.33) from Lemma 4.28, we conclude the proof in the same way as for Theorem 4.7. ☐

4.4.4 Lemmas for directions α_b and $\widetilde{\alpha}_b$

Without loss of generality, we consider the asymptotic behaviour of g^{z_0} in the direction α_b. Since the integrands in (4.4.2) and (4.4.3) are no longer holomorphic at points $x > x_b$, we must introduce a different deformation of the integration path; see Figure 4.11b. Note that the integrands of (4.4.2) and (4.4.3) admit limits at the top and bottom edges of the branch cut $[x_b,+\infty)$. To define the new contour, we introduce the following notation.

Notation 4.29. *Let $M > x_b$, and let f be a continuous function defined in a neighbourhood of $[x_b, M]$ within $\{x \in \mathbb{C} : \Im(x) \geq 0\}$. We define*

$$\int_{\Omega_{+,M}} f(x)\,dx := \lim_{\eta\to 0} \int_{\Omega^{\eta}_{+,M}} f(x)\,dx,$$

where $\Omega^{\eta}_{+,M}$ denotes the straight segment connecting $M + i\eta$ to $x_b + i\eta$, oriented as in Figure 4.11a. Integration along $\Omega_{-,M}$ is defined analogously, following the orientation shown in Figure 4.11a.

If $\alpha < \alpha_b$ (i.e., $x(\alpha) > x_b$), we set $\Omega_{\pm,\alpha} = \Omega_{\pm,x(\alpha)}$. If $\alpha > \alpha_b$ (i.e., $x(\alpha) < x_b$), we adopt the convention $\Omega_{+,\alpha} = \Omega_{-,\alpha} = \{x(\alpha)\}$, i.e., the constant path reduced to the point $x(\alpha)$. We also define

$$\Gamma^-_{x,\alpha} = \{x(it,\alpha) \mid t \in [-\epsilon, 0)\}, \qquad \Gamma^+_{x,\alpha} = \{x(it,\alpha) \mid t \in (0,\epsilon]\}.$$

We can thus consider integrals of the relevant quantities along $\Omega_{\pm,\alpha}$. Let us now define the function Φ as

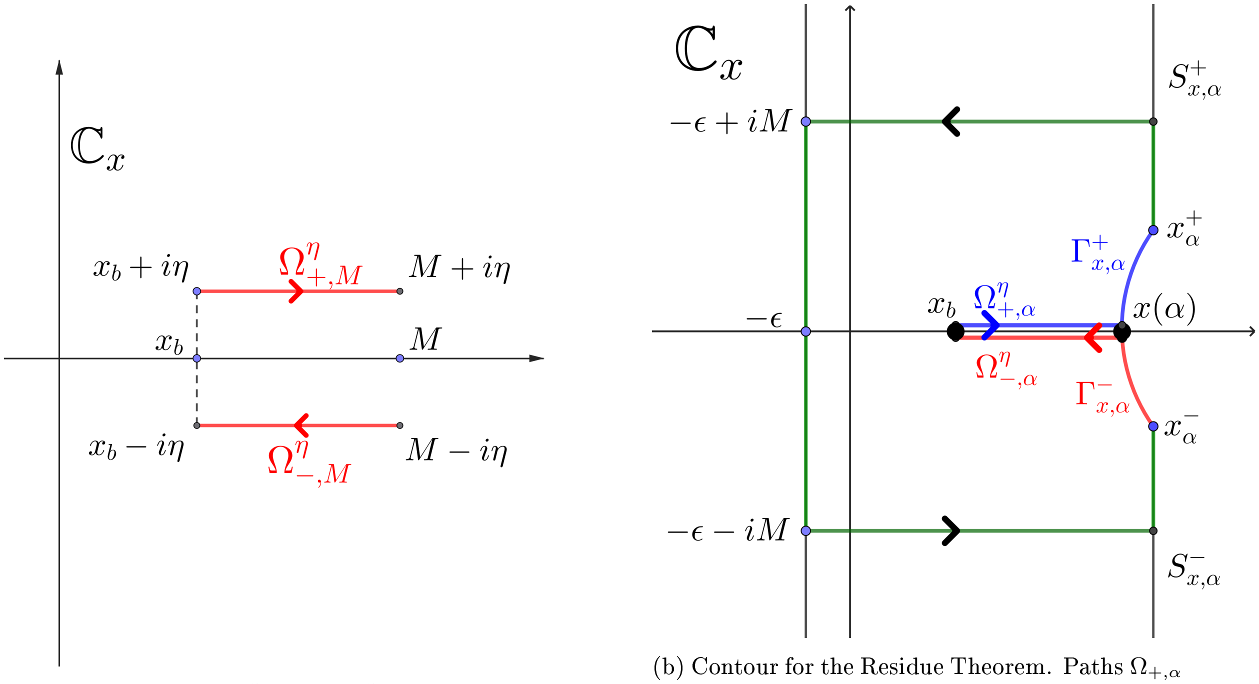
$$\begin{aligned}\Phi(x) &= \varphi^{z_0}(x)\frac{\gamma(x,Y^+(x),Z^+(x))}{\partial_y\gamma_+(x,Y^+(x))} + \frac{e^{a_0x+b_0Y^+(x)\mathbb{1}_{b_0>0}+b_0Z^+(x)\mathbb{1}_{b_0<0}}}{\partial_y\gamma_+(x,Y^+(x))} &(4.4.40)\\ &= \frac{e^{a_0x+Z^+(x)b_0\mathbb{1}_{b_0<0}}}{\partial_y\gamma_+(x,Y^+(x))}\left(-\frac{\gamma(x,Y^+(x),Z^+(x))}{\gamma(x,Y^-(x),Z^+(x))}e^{b_0Y^-(x)\mathbb{1}_{b_0>0}} + e^{b_0Y^+(x)\mathbb{1}_{b_0>0}}\right). &(4.4.41)\end{aligned}$$

Observe that Φ is discontinuous along the cut $[x_b,+\infty)$. The following lemma provides the appropriate contour deformation when $\alpha_0 = \alpha_b$.

Lemma 4.30 (Change of contour for direction α_b)**.** *Assume that $x^+_{max} < x^-_{max}$, i.e., $\alpha_b \in (0,\pi)$. Let $z_0 \in \mathbb{R}^2$, $a \in \mathbb{R}$, and $b > b_0$ such that $(a,b) \neq z_0$. Then, for $0 < \alpha < \alpha_b$ (i.e., $x^+_{max} < x(\alpha)$), we have*

$$I_1^{z_0}(a,b) + I_2^{z_0}(a,b) = \frac{1}{2i\pi}\int_{S^-_{x,\alpha}\cup\Gamma^-_{x,\alpha}\cup\Omega_{-,\alpha}\cup\Omega_{+,\alpha}\cup\Gamma^+_{x,\alpha}\cup S^+_{x,\alpha}} \Phi(x)e^{-ax-bY^+(x)}\,dx. \tag{4.4.42}$$

Proof. We apply the Residue theorem to the representations (4.4.2) and (4.4.3) of $I_1^{z_0}(a,b)$ and $I_2^{z_0}(a,b)$, using the contour described in Figure 4.11b. The asymptotic as $M \to +\infty$ is justified by (4.4.20), and the limit as $\eta \to 0$ follows from the dominated convergence theorem. This yields the desired identity (4.4.42). ☐



(a) Paths $\Omega^\eta_{+,M}$ and $\Omega^\eta_{-,M}$.

(b) Contour for the Residue Theorem. Paths $\Omega_{+,\alpha}$ and $\Omega_{-,\alpha}$ coincide in $\mathbb{C}_x$.

Figure 4.11: Illustration of the integration paths $\Omega^\eta_{+,\alpha}$ and the contour deformation for direction α_b.

For the asymptotic behaviour of Green's functions as $\alpha \to \alpha_b$, both contours $\Omega_{\pm,\alpha}$ and $\Gamma^\pm_{x,\alpha}$ contribute to the leading term. Since the function Φ is not holomorphic at x_b, the standard saddle-point expansion does not apply. However, we present an adapted version of the method.

Recall the notation (4.4.10). From the expressions (4.1.12) and (4.1.16), there exists a neighbourhood V of x_b, containing the disc $\mathbb{D}(0, L)$ for some $L > 0$, and a holomorphic function ψ on V such that

$$\Phi(x) = \psi\left(\sqrt{x_b - x}\right), \quad \text{for all } x \in V \setminus [x_b, +\infty).$$

By separating the even and odd parts in the power series expansion of ψ around x_b, we may write

$$\Phi(x) = \Phi_1(x) + \sqrt{x_b - x}\,\Phi_2(x) \tag{4.4.43}$$

for $x \in V \setminus [x_b, +\infty)$, where Φ_1 and Φ_2 are holomorphic in a neighbourhood of x_b. This decomposition allows us to establish the following lemma.

Lemma 4.31 (Contribution of $\Gamma^\pm_{x,\alpha}$ in direction α_b). *Let $z_0 \in \mathbb{R}^2$. Suppose $x^+_{\max} < x^-_{\max}$. Then, the following asymptotic expansion holds:*

$$\frac{1}{2i\pi} \int_{\Gamma^-_{x,\alpha} \cup \Gamma^+_{x,\alpha}} \Phi(x)\, e^{-r(\cos(\alpha)x + \sin(\alpha)Y^+(x))}\, dx \underset{\substack{r \to +\infty \\ \alpha \to \alpha_b}}{\sim} C^+(\alpha_b)\, h_{\alpha_b}(z_0)\, \frac{e^{-r(\cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha))}}{\sqrt{r}} \tag{4.4.44}$$

where $C^+(\alpha_b)$ and $h_{\alpha_b}(z_0)$ are given by (4.1.28) and (4.1.29).

Proof. Applying the saddle-point method to the holomorphic function Φ_1, as developed in Lemma 2.33 (Chapter 2), we obtain:

$$\frac{1}{2i\pi} \int_{\Gamma_{x,\alpha}} \Phi_1(x)\, e^{-r(\cos(\alpha)x + \sin(\alpha)Y^+(x))}\, dx \underset{\substack{r \to +\infty \\ \alpha \to \alpha_b}}{\sim} C^+(\alpha_b)\, h_{\alpha_b}(z_0)\, \frac{e^{-r(\cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha))}}{\sqrt{r}}, \tag{4.4.45}$$

since $\Phi(x_b) = \Phi_1(x_b)$. Moreover, for all $(t, \alpha) \in \Omega(0, \alpha_b)$, we have

$$\left|\sqrt{x_b - x(it,\alpha)}\right| \leq \sqrt{|x(0,\alpha_b) - x(0,\alpha)| + |x(0,\alpha) - x(it,\alpha)|} \leq C\left(\sqrt{|\alpha_b - \alpha|} + \sqrt{|t|}\right),$$

for some constant $C > 0$ independent of t and α. Since Φ_2 is holomorphic, it is bounded on every compact sets. Therefore, there exist constants $M, M' > 0$ such that, for all α with $|\alpha - \alpha_b| < \eta$, we have:

$$\begin{aligned}\left|\int_{\Gamma^+_{x,\alpha}} \sqrt{x_b - x}\, \Phi_2(x)\, e^{-r(\cos(\alpha)x + \sin(\alpha)Y^+(x))} dx\right| &= e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))} \\ &\quad \times \left|\int_0^\epsilon \sqrt{x_b - x(it,\alpha)}\, \Phi_2(x(it,\alpha))\, e^{-rt^2} i x'_\omega(it,\alpha)\, dt\right| \\ &\leq M\, e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))} \int_0^\epsilon \left(\sqrt{|\alpha_b - \alpha|} + \sqrt{t}\right) e^{-rt^2} dt \\ &\underset{\substack{r\to+\infty \\ \alpha\to\alpha_b}}{\sim} M'\, e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\left(\frac{\sqrt{|\alpha_b - \alpha|}}{\sqrt{r}} + \frac{1}{r}\right),\end{aligned} \tag{4.4.46}$$

which is negligible compared to (4.4.45) as $r \to +\infty$ and $\alpha \to \alpha_b$. The same estimate holds for the integral over $\Gamma^-_{x,\alpha}$. This completes the proof of (4.4.44). □

We split the proof of the contribution of the integrals along $\Omega_{\pm,\alpha}$ into two lemmas.

Lemma 4.32 (Value of Φ_2 at x_b)**.** *Recall the notation from* (4.4.43)*. Suppose* $x^-_{max} < x^+_{max}$*. Then:*

$$\Phi_2(x_b) = -2\frac{\sqrt{\det(\Sigma^-)(x_b - x^-_{min})}}{\Sigma^-_{22}(\Sigma^+_{22} + \Sigma^-_{22})} f_0(z_0)$$

where $f_0(z_0)$ *is given by* (4.1.25)*.*

Proof. The proof reduces to showing that

$$\Phi(x) = \Phi(x_b) - 2\frac{\sqrt{\det(\Sigma^-)(x_b - x^-_{min})}}{\Sigma^-_{22}(\Sigma^+_{22} + \Sigma^-_{22})} f_0(z_0)\sqrt{x_b - x} + o(\sqrt{x_b - x}) \tag{4.4.47}$$

as $x \to x_b$. To prove the above formula, we use the explicit expressions of Φ and φ^{z_0} given by (4.4.40) and (4.1.16). Let us denote $C_z = \frac{\sqrt{\det(\Sigma^-)(x_b - x^-_{min})}}{\Sigma^-_{22}}$ so that $Z^+(x) = Z^+(x_b) + C_z\sqrt{x_b - x} + o(\sqrt{x_b - x})$. We have:

$$\begin{aligned}\frac{\gamma(x, Y^+(x), Z^+(x))}{\gamma(x, Y^-(x), Z^+(x))} &= 1 + \frac{\left(\frac{1+q_2}{2}\right)(Y^+(x) - Y^-(x))}{\gamma(x, Y^-(x), Z^+(x))} \\ &= 1 + \left(\frac{1+q_2}{2}\right)\frac{Y^+(x) - Y^-(x)}{\gamma(x_b, Y^-(x_b), Z^+(x_b)) + \frac{q_2-1}{2}C_z\sqrt{x_b - x} + o(\sqrt{x_b - x})} \\ &= 1 + \left(\frac{1+q_2}{2}\right)\frac{Y^+(x_b) - Y^-(x_b)}{\gamma(x_b, Y^-(x_b), Z^+(x_b))}\left(1 - \frac{\frac{q_2-1}{2}C_z\sqrt{x_b - x}(1 + o(1))}{\gamma(x_b, Y^-(x_b), Z^+(x_b))}\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}e^{Z^+(x)b_0\mathbb{1}_{b_0<0}} &= e^{(Z^+(x_b) + \sqrt{x_b - x}(C_z + o(1)))b_0\mathbb{1}_{b_0<0}} \\ &= e^{Z^+(x_b)b_0\mathbb{1}_{b_0<0}}(1 + \sqrt{x_b - x}(C_z + o(1))b_0\mathbb{1}_{b_0<0}).\end{aligned}$$

Then, if $b_0 < 0$, (4.4.41) can be written as

$$\partial_y \gamma_+(x, Y^+(x))\Phi(x) = e^{x_b a_0 + Z^+(x_b) b_0} \left(1 + C_z \sqrt{x_b - x}(b_0 + o(1))\right)$$
$$\times - \left(\frac{1+q_2}{2}\right) \frac{Y^+(x_b) - Y^-(x_b)}{\gamma(x_b, Y^-(x_b), Z^+(x_b))} \left(1 - \frac{\frac{q_2-1}{2} C_z \sqrt{x_b - x}(1 + o(1))}{\gamma(x_b, Y^-(x_b), Z^+(x_b))}\right)$$

which yields (4.4.47), using the identity $\partial_y \gamma_+(x_b, Y^+(x_b)) = \frac{\Sigma_{22}^+}{2}(Y^+(x_b) - Y^-(x_b))$. If $b_0 > 0$, the computations are analogous. ☐

Before setting the asymptotics along $\Omega_{\pm,\alpha}$, we need to introduce some notation. We recall that $x(0, \alpha_b) = x_b$ and that $x'_\omega(0, \alpha_b) = \sqrt{\frac{2}{F''_x(x(\alpha_b),\alpha)}} \neq 0$ by (4.4.9). Then, by the implicit function theorem, there exists a C^∞ function, denoted by τ, and a neighbourhood W of $(0, \alpha_b)$ in $\mathbb{R}_\omega \times \mathbb{R}_\alpha$ (contained in $\Omega(0, \alpha_b)$, see (4.4.10)) such that

$$\forall (\omega, \alpha) \in W, \quad [x(\omega, \alpha) = x_b] \Longleftrightarrow [\omega = \tau(\alpha)].$$

In particular, $x(\tau(\alpha), \alpha) = x_b$, and $\tau'(\alpha_b) = \frac{-x'(\alpha_b)}{x'_\omega(0,\alpha_b)} > 0$. Then, for $\alpha < \alpha_b$, we have $\tau(\alpha) < 0$ and

$$\tau(\alpha) = \frac{-x'(\alpha_b)}{x'_\omega(0, \alpha_b)}(\alpha - \alpha_b)(1 + o_{\alpha \to \alpha_b}(1)).$$

Note that, by (4.4.8) and (4.4.12), we have $|K_+| = \frac{-x'(\alpha_b)}{x'_\omega(0,\alpha_b)}$ with the notation (4.1.33). Since, by (4.4.11), $F(x_b, \alpha) = F(x(\tau(\alpha), \alpha), \alpha) = \tau^2(\alpha)$, we deduce from (4.4.7) the following identity:

$$\cos(\alpha) x(\alpha_b) + \sin(\alpha) y(\alpha_b) = \cos(\alpha) x(\alpha) + \sin(\alpha) y(\alpha) + \tau^2(\alpha). \tag{4.4.48}$$

Lemma 4.33 (Contribution of $\Omega_{\pm,\alpha}$ in direction α_b)**.** *Suppose that $x^+_{max} < x^-_{max}$. Then,*

- *If $r(\alpha - \alpha_b)^2$ remains bounded, then the asymptotic contribution of the integrals along $\Omega_{\pm,\alpha}$ in (4.4.42) is negligible compared to (4.4.44) as $r \to +\infty$ and $\alpha \to \alpha_b$.*
- *If $r(\alpha - \alpha_b)^2 \to +\infty$ and $\alpha \leq \alpha_b$, the following asymptotics hold:*

$$\frac{1}{2i\pi} \int_{\Omega_{\pm,\alpha}} \Phi(x) e^{-r(\cos(\alpha)x + \sin(\alpha)Y^+(x))} dx \tag{4.4.49}$$

$$\underset{\substack{r\to+\infty \\ \alpha\to\alpha_b, \alpha\leq\alpha_b}}{\sim} \frac{\sin^{3/2}(\alpha_b)\sqrt{\det(\Sigma^-)(x_b - x^-_{min})}}{\sqrt{\pi}\Sigma^-_{22}(\Sigma^+_{22} + \Sigma^-_{22})} f_0(z_0) e^{-r(\cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha))} \frac{e^{r\tau(\alpha)^2}}{(r(\alpha_b - \alpha))^{3/2}}.$$

Proof. First, since Φ_1 is holomorphic, we have

$$\frac{1}{2i\pi} \int_{\Omega^-_{x,\alpha} \cup \Omega^+_{x,\alpha}} \Phi_1(x) e^{-r(\cos(\alpha)x + \sin(\alpha)Y^+(x))} dx = 0.$$

For Φ_2, we note that

$$\int_{\Omega^-_{x,\alpha}} \sqrt{x_b - x}\Phi_2(x) e^{-ax - bY^+(x)} dx = \int_{\Omega^+_{x,\alpha}} \sqrt{x_b - x}\Phi_2(x) e^{-ax - bY^+(x)} dx$$

by the definition of integration over $\Omega_{\pm,\alpha}$ and the convention for the complex square root. We parameterise the path $\Omega_{+,\alpha}$ as $\Omega_{+,\alpha} = \{x(t, \alpha) \mid t \in [\tau(\alpha), 0]\}$. Then,

$$\int_{\Omega^+_{x,\alpha}} \sqrt{x_b - x}\Phi_2(x) e^{-r(\cos(\alpha)x + \sin(\alpha)Y^+(x))} dx \tag{4.4.50}$$

$$= -ie^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))} \int_{\tau(\alpha)}^{0} \sqrt{x(t,\alpha)-x_b}\Phi_2(x(t,\alpha))e^{rt^2}x'_\omega(t,\alpha)dt.$$

Moreover, for any $t \in (\tau(\alpha), 0)$, there exists $\xi_{t,\alpha} \in (\tau(\alpha), t)$ such that

$$x(t,\alpha) - x_b = x(t,\alpha) - x(\tau(\alpha),\alpha) = x'_\omega(\xi_{t,\alpha},\alpha)(t-\tau(\alpha)).$$

Therefore,

$$\left|\int_{\tau(\alpha)}^{0} \sqrt{x(t,\alpha)-x_b}\Phi_2(x(t,\alpha))e^{rt^2}x'_\omega(t,\alpha)dt - \Phi_2(x(0,\alpha))x'_\omega(0,\alpha)^{3/2}\int_{\tau(\alpha)}^{0}\sqrt{t-\tau(\alpha)}e^{rt^2}dt\right| \tag{4.4.51}$$
$$\leq \epsilon(\alpha)\int_{\tau(\alpha)}^{0}\sqrt{t-\tau(\alpha)}e^{rt^2}dt$$

where

$$\epsilon(\alpha) := \sup_{t\in[\tau(\alpha),0]}\left|\sqrt{x'_\omega(\xi_{t,\alpha},\alpha)}\Phi_2(x(t,\alpha))x'_\omega(t,\alpha) - \sqrt{x'_\omega(0,\alpha)}\Phi_2(x(0,\alpha))x'_\omega(0,\alpha)\right|$$
$$\leq \sup_{\substack{t\in[\tau(\alpha),0]\\ \xi\in[\tau(\alpha),0]}}\left|\sqrt{x'_\omega(\xi,\alpha)}\Phi_2(x(t,\alpha))x'_\omega(t,\alpha) - \sqrt{x'_\omega(0,\alpha)}\Phi_2(x(0,\alpha))x'_\omega(0,\alpha)\right| \underset{\alpha\to\alpha_b}{\longrightarrow} 0.$$

Using the change of variables $t = \tau(\alpha)s$, we have

$$\int_{\tau(\alpha)}^{0}\sqrt{t-\tau(\alpha)}\,e^{rt^2}dt = |\tau(\alpha)|^{3/2}\int_0^1\sqrt{1-s}\,e^{r\tau(\alpha)^2s^2}ds. \tag{4.4.52}$$

Assume first that $r\tau(\alpha)^2$ remains bounded.Then, (4.4.52) is also bounded. By (4.4.51), (4.4.50) is then bounded by $|\tau(\alpha)|^{3/2}e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))} \leq M\sqrt{|\tau(\alpha)|}\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}$ for some constant $M > 0$, which is negligible compared to (4.4.44). This proves (ii).

Now, suppose that $r\tau(\alpha)^2 \to +\infty$. By Lemma 4.B.1, the following asymptotics hold:

$$|\tau(\alpha)|^{3/2}\int_0^1\sqrt{1-s}e^{r\tau(\alpha)^2s^2}ds \sim |\tau(\alpha)|^{3/2}\frac{\sqrt{\pi}}{4\sqrt{2}}\frac{e^{r\tau(\alpha)^2}}{(r\tau(\alpha)^2)^{3/2}} = \frac{\sqrt{\pi}}{4\sqrt{2}}\frac{e^{r\tau(\alpha)^2}}{(r|\tau(\alpha)|)^{3/2}}$$

Hence,

$$\frac{1}{2i\pi}\int_{\Omega^-_{x,\alpha}\cup\Omega^+_{x,\alpha}}\Phi(x)e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}dx \sim \frac{-i}{i\pi}\frac{\sqrt{\pi}}{4\sqrt{2}}\Phi_2(x_b)x'_\omega(0,\alpha_b)^{3/2}e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{e^{r\tau(\alpha)^2}}{(r\tau(\alpha))^{3/2}}$$
$$\sim \frac{-\sin^{3/2}(\alpha_b)}{2\sqrt{\pi}}\Phi_2(x_b)e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{e^{r\tau(\alpha)^2}}{(r\sin(\alpha_b-\alpha))^{3/2}}$$

since

$$\frac{x'_\omega(0,\alpha_b)^3}{2\sqrt{2}|x'(\alpha_b)|^{3/2}} = \frac{1}{|F''(x(\alpha_b),\alpha_b)x'(\alpha_b)|^{3/2}} = |\sin(\alpha_b)|^{3/2}$$

(see (4.4.8) and (4.4.12)). The conclusion follows from Lemma 4.32. □

Proof of Theorem 4.9. By Lemmas 4.24 and 4.30, the function g^{z_0} can be written as $g^{z_0} = I_1^{z_0}(a,b) + I_2^{z_0}(a,b)$, with representation given by (4.4.42). According to Lemmas 4.28, 4.31, and 4.33, the contributions of the integrals along the paths $S^\pm_{x,\alpha}$ are negligible compared to those along $\Gamma^\pm_{x,\alpha}$ and $\Omega^\pm_{x,\alpha}$. Thus, we are left with a competition between the contributions from $\Gamma^\pm_{x,\alpha}$ and $\Omega^\pm_{x,\alpha}$.

- (i) If $\alpha > \alpha_b$, then the integrals along $\Omega_{\pm,\alpha}$ vanish (since $\Omega_{\pm,\alpha} = \{x(\alpha)\}$), and the result (4.4.44) follows from Lemma 4.31.
- (ii) The statement in point (ii) directly follows from Lemma 4.33.
- $(iii)-(iv)-(v)$ Now suppose that $\alpha < \alpha_b$ and $r(\alpha-\alpha_b)^2 \to +\infty$. Then we have: We have

$$\frac{e^{r\tau^2(\alpha)}}{r|\tau(\alpha)|^{3/2}} = \frac{e^{r\tau^2(\alpha)}}{r|\tau(\alpha)|^2}\sqrt{|\tau(\alpha)|} \tag{4.4.53}$$

$$= \exp\left(r\tau^2(\alpha) - \ln(r\tau^2(\alpha)) + \frac{1}{2}\ln(|\tau(\alpha)|)\right)$$

$$= \exp\left(K_+^2 r(\alpha_b-\alpha)^2(1+o(1)) + \frac{1}{2}\ln(\alpha_b-\alpha) + \frac{1}{2}\ln(|K_+|)\right)$$

$$= \sqrt{|K_+|}\exp\left(K_+^2 r(\alpha_b-\alpha)^2\left(\frac{\ln(\alpha_b-\alpha)}{2K_+^2 r(\alpha_b-\alpha)^2} + 1 + o(1)\right)\right). \tag{4.4.54}$$

If $\frac{\ln(\alpha_b-\alpha)}{2K_+^2 r(\alpha_b-\alpha)^2} \to l \in [-\infty,-1)$, then (4.4.53) tends to 0, and the asymptotics of $g(r\cos(\alpha), r\sin(\alpha))$ are governed by (4.4.44). Note that in this case, $\frac{e^{K_+^2 r(\alpha_b-\alpha)^2}}{r(\alpha_b-\alpha)^{3/2}} \to 0$. If $\frac{\ln(\alpha_b-\alpha)}{2K_+^2 r(\alpha_b-\alpha)^2} \to l \in (-1,0]$, then (4.4.53) diverges to $+\infty$, and the asymptotics of $g(r\cos(\alpha), r\sin(\alpha))$ are given by (4.4.49), which coincides with (4.1.34) thanks to (4.4.48). In this case, $\frac{e^{K_+^2 r(\alpha_b-\alpha)^2}}{r(\alpha_b-\alpha)^{3/2}} \to +\infty$. It remains to analyze the critical case where $\frac{\ln(\alpha_b-\alpha)}{2K_+^2 r(\alpha_b-\alpha)^2} \to -1$. In this situation, we have $r \sim \frac{-\ln(\alpha_b-\alpha)}{2K_+^2(\alpha_b-\alpha)^2}$ so that

$$r\tau^2(\alpha) = rK_+^2(\alpha_b-\alpha)^2 + O\left(r(\alpha_b-\alpha)^3\right) = rK_+^2(\alpha_b-\alpha)^2 + o(1)$$

and consequently,

$$\frac{e^{r\tau^2(\alpha)}}{r|\tau(\alpha)|^{3/2}} \sim \frac{e^{K_+^2 r(\alpha_b-\alpha)^2}}{r|K_+(\alpha-\alpha_b)|^{3/2}}. \tag{4.4.55}$$

Hence, both contributions (4.4.44) and (4.4.49) may be significant, and expression (4.4.55) provides the conclusion for points (iii) to (v), depending on the possible limiting behavior of $\frac{e^{K_+^2 r(\alpha_b-\alpha)^2}}{r|K_+(\alpha-\alpha_b)|^{3/2}}$.

□

4.4.5 Lemmas for the non-red-arc zones

For angles outside of the red zones, we use a different changing path again. For $\delta > 0$, we set

$$R_\delta^+ = \{x_b+\delta+iv \mid v>0\}, \quad R_\delta^- = \{x_b+\delta-iv \mid v>0\}. \tag{4.4.56}$$

The following lemma performs the appropriate path deformation; see Figure 4.12.

Lemma 4.34 (Changing path for directions $0 \le \alpha < \alpha_b$)**.** *Suppose that $x_{max}^+ < x_{max}^-$ (i.e., $\alpha_b \in (0,\pi)$). Let $\delta > 0$ and let $z_0 \in \mathbb{R}^2$, $a \in \mathbb{R}$, and $b > 0$ such that $(a,b) \neq z_0$. Then, for $0 \le \alpha < \alpha_b$,*

$$I_1^{z_0}(a,b) = \frac{1}{2i\pi}\int_{R_\delta^- \cup \Omega_{-,x_b+\delta} \cup \Omega_{+,x_b+\delta} \cup R_\delta^-} \varphi^{z_0}(x)\frac{\gamma(x,Y^+(x),Z^+(x))}{\partial_y\gamma_+(x,Y^+(x))}e^{-ax-bY^+(x)}dx. \tag{4.4.57}$$

Furthermore, if $b_0 > 0$ and $a > a_0$, then (4.4.18) holds. Finally, if $b_0 \le 0$, then

$$I_2^{z_0}(a,b) = \frac{1}{2i\pi}\int_{R_\delta^- \cup \Omega_{-,x_b+\delta} \cup \Omega_{+,x_b+\delta} \cup R_\delta^+} \frac{e^{(a_0-a)x+b_0Z^+(x)-bY^+(x)}}{\partial_y\gamma_+(x,Y^+(x))}dx. \tag{4.4.58}$$

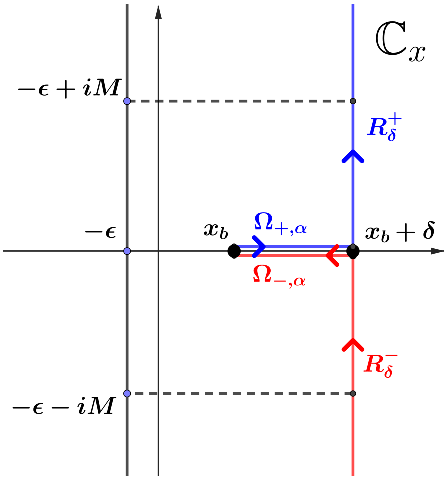


Figure 4.12: Path deformation in Lemma 4.34

Proof. The proof is identical to that of Lemma 4.30, except that the paths $\Gamma^\pm_{x,\alpha}$ are replaced by straight lines. ☐

Once again, we split the analysis of the asymptotics. We recall the notation from (4.4.40). The following lemma provides the main asymptotic contribution.

Lemma 4.35 (Contribution of $\Omega_{-,x_b+\delta}$)**.** *Let $z_0\in\mathbb{R}^2$. Suppose that $x^+_{max}<x^-_{max}$ (i.e., $\alpha_b\in(0,\pi)$), and let $\alpha_0\in[0,\alpha_b)$. If $b_0\leq 0$, then*

$$\frac{1}{2i\pi}\int_{\Omega_{-,x_b+\delta}\cup\Omega_{+,x_b+\delta}}\Phi(x)e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}dx \underset{\substack{r\to\infty\\ \alpha\to\alpha_0,\alpha\geq0}}{\sim} C_{br}f_0(z_0)\frac{e^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b))}}{r^{3/2}\sin(\alpha_b-\alpha)^{3/2}} \tag{4.4.59}$$

where $C_{br}=|\sin(\alpha_b)|^{3/2}C_0$ (see (4.1.23)), and where $f_0(z_0)$ is defined in (4.1.25). If $b_0>0$, then

$$\frac{1}{2i\pi}\int_{\Omega_{-,x_b+\delta}\cup\Omega_{+,x_b+\delta}}\varphi^{z_0}(x)\frac{\gamma(x,Y^+(x),Z^+(x))}{\partial_y\gamma_+(x,Y^+(x))}e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}dx \tag{4.4.60}$$

$$\underset{\substack{r\to\infty\\ \alpha\to\alpha_0,\alpha\geq0}}{\sim} C_{br}f_0(z_0)\frac{e^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b))}}{r^{3/2}\sin(\alpha_b-\alpha)^{3/2}}.$$

Proof. First, assume that $b_0\leq 0$. Once again, by the convention on the complex square root,

$$\begin{aligned}\frac{1}{2i\pi}\int_{\Omega_{-,x_b+\delta}\cup\Omega_{+,x_b+\delta}}\Phi(x)e^{-ax-bY^+(x)}dx &= \frac{2}{2i\pi}\int_{\Omega_{+,x_b+\delta}}\sqrt{x_b-x}\,\Phi_2(x)e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}dx\\ &=\frac{e^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b))}}{i\pi}\int_{\Omega_{+,x_b+\delta}}\sqrt{x_b-x}\,\Phi_2(x)e^{-rU(x,\alpha)}dx\end{aligned}$$

where

$$U(x,\alpha)=\cos(\alpha)(x-x(\alpha_b))+\sin(\alpha)(Y^+(x)-y(\alpha_b)). \tag{4.4.61}$$

By the definition of $(x(\alpha_b),y(\alpha_b))$, we have $(Y^+)'(x_b)=-\frac{\cos(\alpha_b)}{\sin(\alpha_b)}$. Therefore,

$$U'_x(x_b,\alpha_0)=\cos(\alpha_0)-\sin(\alpha_0)\frac{\cos(\alpha_b)}{\sin(\alpha_b)}=\frac{\sin(\alpha_b-\alpha_0)}{\sin(\alpha_b)}>0.$$

Moreover, $U(x_b, \alpha_0) = 0$. Then, by the implicit function theorem, there exists a neighborhood $V \subset \mathbb{R} \times [0, 2\pi]$ of $(0, \alpha_0)$ given by

$$V = \{(t, \alpha), \quad |t| \leq L, |\alpha - \alpha_0| \leq \eta\} \tag{4.4.62}$$

and a map $z : V \to \mathbb{R}$ such that, for all $(t, \alpha) \in V$,

$$U(x, \alpha) = t \Longleftrightarrow x = z(t, \alpha) \tag{4.4.63}$$

with $x_b = z(0, \alpha_0)$. This means that $U(z(t, \alpha), \alpha) = t$. In particular, we obtain $U(z(0, \alpha), \alpha) = 0$. Moreover, by the definition of $U(x, \alpha)$, we have $U(x_b, \alpha) = 0$. By the uniqueness stated in (4.4.63), we then deduce that $z(0, \alpha) = x_b$ for every α satisfying $|\alpha - \alpha_0| \leq \eta$. Furthermore,

$$z_t'(0, \alpha_0) = \frac{1}{U_x'(x_b, \alpha_0)} = \frac{\sin(\alpha_b)}{\sin(\alpha_b - \alpha_0)} > 0. \tag{4.4.64}$$

Again by the implicit function theorem, if $\delta > 0$ is sufficiently small, there exists a smooth function $t(\alpha)$ (possibly requiring a smaller η in (4.4.62)), defined for $|\alpha - \alpha_0| \leq \eta$, such that

$$z(t(\alpha), \alpha) = x_b + \delta \quad \forall |\alpha - \alpha_0| \leq \eta.$$

By choosing $L > 0$ and $\eta > 0$ sufficiently small in (4.4.62), we can assume that $z_t'(t, \alpha) > 0$ for all $(t, \alpha) \in V$. Since $z(0, \alpha) = x_b$, it follows that $t(\alpha) > 0$. In particular, $t(\alpha) \to t(\alpha_0) > 0$ as $\alpha \to \alpha_0$. We now parametrize the segment $\Omega_{+, x_b + \delta}$ by $\{z(t, \alpha) \mid t \in [0, t(\alpha)]\}$. Then,

$$\int_{\Omega_{+, x_b+\delta}} \sqrt{x_b - x}\,\Phi_2(x) e^{-rU(x,\alpha)} dx = -i \int_0^{t(\alpha)} \sqrt{z(t, \alpha) - x_b}\,\Phi_2(z(t, \alpha)) z_t'(t, \alpha) e^{-rt} dt.$$

Then, by the Taylor–Lagrange inequality,

$$|z(t, \alpha) - z(0, \alpha) - t z_t'(0, \alpha)| \leq C_1 t^2, \quad \forall (t, \alpha) \in V$$

where $C_1 = \frac{1}{2} \sup_{(t,\alpha) \in V} |z_t''(0, \alpha)|$. Since $|\sqrt{a} - 1| = \frac{|a-1|}{\sqrt{a}+1} \leq |a - 1|$ for $a \geq 0$, we obtain

$$\left| \sqrt{\frac{z(t, \alpha) - z(0, \alpha)}{t z_t'(0, \alpha)}} - 1 \right| \leq C_2 |t|$$

for some constant $C_2 > 0$ (since V can be chosen so that $1/z_t'(0, \alpha)$ is bounded on V). Hence,

$$\left| \sqrt{z(t, \alpha) - z(0, \alpha)} - \sqrt{t z_t'(0, \alpha)} \right| \leq C_3 |t|^{3/2} \quad \forall (t, \alpha),\ 0 \leq t \leq L,\ \forall |\alpha - \alpha_0| \leq \eta$$

for some constant $C_3 > 0$. Similarly, there exists a constant $C_4 > 0$ such that

$$|\Phi_2(z(t, \alpha)) z_t'(t, \alpha) - \Phi_2(x_b) z_t'(0, \alpha)| \leq C_4 |t| \quad \forall (t, \alpha),\ 0 \leq t \leq L,\ \forall |\alpha - \alpha_0| \leq \eta.$$

Thus, there exists $C > 0$ such that

$$\left| \sqrt{z(t, \alpha) - z(0, \alpha)}\,\Phi_2(z(t, \alpha)) z_t'(t, \alpha) - \Phi_2(x_b) z_t'(0, \alpha) \sqrt{t z_t'(0, \alpha)} \right| \leq C |t|^{3/2} \quad \forall (t, \alpha),\ 0 \leq t \leq K,\ \forall |\alpha - \alpha_0| \leq \eta.$$

It follows that

$$-i \int_0^{t(\alpha)} \sqrt{z(t, \alpha) - x_b}\,\Phi_2(z(t, \alpha)) z_t'(t, \alpha) e^{-rt} dt = -i \Phi_2(x_b) |z_t'(0, \alpha)|^{3/2} \int_0^{t(\alpha)} \sqrt{t} e^{-rt} dt + R(r, \alpha)$$

where

$$|R(r,\alpha)| \le C\int_0^{t(\alpha)} t^{3/2}e^{-rt}dt = r^{-5/2}\int_0^{rt(\alpha)} t^{3/2}e^{-t}dt = O(r^{-5/2})$$

as $r \to +\infty$ and $\alpha \to \alpha_0$. Furthermore,

$$\int_0^{t(\alpha)} \sqrt{t}e^{-rt}dt \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} r^{-3/2}\int_0^{+\infty} \sqrt{t}e^{-t}dt = \frac{\sqrt{\pi}}{2r^{3/2}}.$$

Hence,

$$\frac{1}{2i\pi}\int_{\Omega_{-,x_b+\delta}\cup\Omega_{+,x_b+\delta}} \Phi(x)e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}dx \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} \frac{-ie^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b))}}{i\pi}\Phi_2(x_b)|z_t'(0,\alpha_0)|^{3/2}\frac{\sqrt{\pi}}{2r^{3/2}} \tag{4.4.65}$$

$$\underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} C_{br}f_0(z_0)\frac{e^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b))}}{r^{3/2}\sin(\alpha_b-\alpha_0)^{3/2}}. \tag{4.4.66}$$

by (4.4.64) and Lemma 4.32.

If $b_0 > 0$, the same argument applies, replacing $\Phi(x)$ with $\varphi^{z_0}(x)\frac{\gamma(x,Y^+(x),Z^+(x))}{\partial_y\gamma_+(x,Y^+(x))}$ which yields the result in (4.4.60). ☐

To show that the asymptotics of the previous integrals yield the leading term in the asymptotics of the Green's function, we establish the following lemma.

Lemma 4.36 (Negligibility of remaining integrals)**.** *Suppose $x^+_{max} < x^-_{max}$ (i.e., $\alpha_b \in (0,\pi)$). Let $0 \le \alpha_0 < \alpha_b$. Then, for $\delta > 0$ small enough, there exist constants $D > 0$, $\epsilon' > 0$, $\eta > 0$, and $r_0 > 0$ such that for all $r \ge r_0$ and all $\alpha \ge 0$ satisfying $|\alpha-\alpha_0| < \eta$, the following estimates hold:*

$$\Big|\int_{R^\pm_\delta} \varphi^{z_0}(x)\frac{\gamma(x,Y^+(x),Z^+(x))}{\partial_y\gamma_+(x,Y^+(x))}e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}dx\Big| \le De^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b)+\epsilon')}, \tag{4.4.67}$$

and:

- *If $b_0 > 0$, then*
$$I_2(r\cos(\alpha), r\sin(\alpha)) \le De^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b)+\epsilon')} \tag{4.4.68}$$
- *If $b_0 \le 0$, then*
$$\Big|\int_{R^\pm_\delta} \frac{e^{(a_0-a)x+b_0Z^+(x)-bY^+(x)}}{\partial_y\gamma_+(x,Y^+(x))}e^{-r(\cos(\alpha)x+\sin(\alpha)Y^+(x))}dx\Big| \le De^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b)+\epsilon')} \tag{4.4.69}$$

Proof. We begin by proving (4.4.67). The argument is analogous to that used in the proof of Lemma 4.28. Suppose first that $\alpha_0 > 0$. Then (4.4.67) can be rewritten as

$$\Big|\int_0^{+\infty} \varphi^{z_0}(x_b+\delta+iv)\frac{\gamma(x_b+\delta+iv, Y^+(x_b+\delta+iv), Z^+(x_b+\delta+iv))}{\partial_y\gamma_+(x_b+\delta+iv, Y^+(x_b+\delta+iv))}e^{-r(F(x_b,\alpha)-F(x_b+\delta+iv)-\epsilon')}dx\Big| \le D. \tag{4.4.70}$$

We decompose the exponent as follows:

$$F(x_b,\alpha)-F(x_b+\delta+iv)-\epsilon' = (F(x_b,\alpha)-F(x_b+\delta,\alpha)-\epsilon') + (F(x_b+\delta,\alpha)-F(x_b+\delta+iv)).$$

Now, observe that the function $x \mapsto F(x,\alpha_0)$ is strictly decreasing on $[-\pi, x(\alpha_0)]$ and strictly increasing on $[x(\alpha_0), x^+_{max}]$. Since $\alpha_0 < \alpha_b$, it follows that $x(\alpha_0) > x_b$. Therefore, for some $\delta > 0$ and $\epsilon' > 0$

small enough, there exists $\eta > 0$ such that $F(x_b, \alpha) - F(x_b + \delta, \alpha) - \epsilon' \geq 0$ for all $\alpha \geq 0$ such that $|\alpha - \alpha_0| < \eta$. Moreover, using the same reasoning as in the proof of Lemma 4.28, we find that there exists $c > 0$ such that for all $v \geq 0$,

$$F(x_b + \delta, \alpha) - \Re F(x_b + \delta + iv) \geq c \sin(\alpha) v. \tag{4.4.71}$$

Then, by applying (4.4.37), the left-hand side of (4.4.70) is bounded by $\frac{C}{r \sin(\alpha)} \leq D$, for some $D > 0$ independent of r and $\alpha \geq \alpha_0 - \eta > 0$ (for $\eta > 0$ sufficiently small). This proves (4.4.70) when $\alpha_0 > 0$.

If $\alpha_0 = 0$, (4.4.67) can be obtained similarly by integration by parts, see Lemma 2.31 (Chapter 2) for similar considerations. The proof of (4.4.69) is symmetrical.

We now turn to the proof of (4.4.68). Suppose $b_0 \geq 0$ and consider the representation of $I_2^{z_0}(a, b)$ given by (4.4.18). Applying the saddle-point method to (4.4.18) (see Lemmas 4.26, 4.27, and 4.28), we obtain

$$I_2^{z_0}(r\cos(\alpha), r\sin(\alpha)) = O\left(\frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}}\right) = o\left(e^{-r(\cos(\alpha)x(\alpha_b)+\sin(\alpha)y(\alpha_b)+\epsilon')}\right)$$

as $r \to +\infty$ and $\alpha \to \alpha_0$, for $\epsilon' > 0$ small enough by the definition of $(x(\alpha), y(\alpha))$. This proves (4.4.68) for some constant $D > 0$. □

Proof of Theorem 4.10. If $b_0 \leq 0$, we consider the representations of $I_1^{z_0}(a, b)$ and $I_2^{z_0}(a, b)$ given by (4.4.57) and (4.4.58), respectively, in Lemma 4.34. The contributions of these integrals along $\Omega_{\pm, x_b+\delta}$ are exactly given by (4.4.59) from Lemma 4.35. By Lemma 4.36, and more precisely by (4.4.67) and (4.4.69), the asymptotic contributions of the integrals along $R_\delta^{\pm}$ in the representations (4.4.57) and (4.4.58) are negligible compared to those along $\Omega_{\pm, x_b+\delta}$. It follows that the asymptotics of $g^{z_0}(r\cos(\alpha), r\sin(\alpha))$ are given by (4.4.59), as claimed.

If $b_0 > 0$, we consider the representation of $I_1^{z_0}(a, b)$ given by (4.4.57), and the inequality for $I_2^{z_0}(a, b)$ provided in (4.4.68). The contribution of the integral (4.4.57) along $\Omega_{\pm, x_b+\delta}$ is given by (4.4.60). By (4.4.67), the integrals along $R_\delta^{\pm}$ from the representation (4.4.57) are asymptotically negligible compared to (4.4.60). Moreover, by (4.4.68), the asymptotic contribution of $I_2^{z_0}(r\cos(\alpha), r\sin(\alpha))$ is also negligible compared to (4.4.60). Therefore, the asymptotics of $g^{z_0}(r\cos(\alpha), r\sin(\alpha))$ are given by (4.4.60), which completes the proof. □

4.5 Proof of Theorem 4.13: full and minimal Martin boundary

In this section, we prove the result concerning the Martin boundary using Theorems 4.6 to 4.10. In the general theory of Martin boundaries, Martin functions are usually *excessive* (see [78, Theorem 3]). Nevertheless, for Markov chains with a finite number of steps from each state, one can immediately verify that Martin functions are harmonic. They are also harmonic for (drifted) Brownian motion in $\mathbb{R}^d$. In the next lemmas, we show that the functions f_0, f_π, and h_α, $\alpha \in \mathcal{M} \setminus \{0, \pi\}$, in our model are harmonic as well. Due to the fundamental role of z_0 in this section, we write $g(z_0, z)$ for $g^{z_0}(z)$.

Lemma 4.37 (Harmonicity)**.** *The functions f_0, f_π, and h_α, $\alpha \in \mathcal{M}\setminus\{0, \pi\}$, are harmonic.*

Proof. Let U be an open set that is relatively compact in $\mathbb{R}^2$. By [78, Proposition 6.2], if $z \in \mathbb{R}^2$, then $g(\cdot, z)$ is harmonic on $\mathbb{R}^2 \setminus \{z\}$. Hence the Martin kernel $K(\cdot, z)$ defined by $K(z_0, z) = \frac{g(z_0,z)}{g(0,z)}$ is harmonic as well on $\mathbb{R}^2 \setminus \{z\}$. Therefore, if $z \notin U$ and $z_0 \in \mathbb{R}^2$ satisfy $z \neq z_0$, then

$$K(z_0, z) = \mathbb{E}_{z_0}\left[K(Z_{\tau_{U^c}}, z)\right]. \tag{4.5.1}$$

Let $\alpha_0 \in \mathcal{M}$. We consider $z = (r\cos(\alpha), r\sin(\alpha))$ with $\alpha \in \mathcal{M}$, and let $r \to \infty$ and $\alpha \to \alpha_0$." By Theorems 4.7 to 4.9, we have

$$K(z_0, (r\cos(\alpha), r\sin(\alpha))) = \frac{h_{\alpha_0}(z_0) + \varepsilon(z_0, r, \alpha)}{h_{\alpha_0}(0) + \varepsilon(0, r, \alpha)}$$

where $\varepsilon(z_0, r, \alpha) \underset{\substack{r\to+\infty \\ \alpha\to\alpha_0}}{\longrightarrow} 0$ (and where, for consistency of notation, we have set $h_0(z_0) = f_0(z_0)$ and $h_\pi(z_0) = f_\pi(z_0)$). If $\varepsilon(z_0, r, \alpha)$ converges to zero as $r \to +\infty, \alpha \to \alpha_0$ *uniformly in* $z_0 \in U$, the dominated convergence theorem applied to (4.5.1) yields

$$h_{\alpha_0}(z_0) = \mathbb{E}_{z_0}\left[h_{\alpha_0}(Z_{\tau_{U^c}})\right]$$

and completes the proof. The uniform convergence in $z_0 \in U$ can be established using arguments similar to those in [101, Lemmas 5.3 and Section 6]. The only directions specific to our model to which these arguments do not apply directly are α_b and $\widetilde{\alpha}_b$ corresponding to branching points, cf. (4.1.15), (4.1.20), and (4.1.21). It remains to establish uniform convergence for these directions. Let us consider case (i) of Theorem 4.9. We prove that the equivalence (4.4.45) is uniform in $z_0 \in U$ and that the constant M in (4.4.46) can be chosen independently of $z_0 \in U$. Recall the definitions (4.4.43) of Φ_1 and Φ_2, which we denote by $\Phi_1^{z_0}$ and $\Phi_2^{z_0}$. For (4.4.45), it suffices (see the proof of Lemma 2.33, Chapter 2) to show that

$$\sup_{\substack{z_0\in U \\ x\in K}} |\Phi_1^{z_0}(x)| < +\infty \tag{4.5.2}$$

for some compact neighborhood $K \subset \mathbb{C}$ of x_b. For (4.4.46), it suffices to prove the analogue of (4.5.2) with $\Phi_2^{z_0}$ instead of $\Phi_1^{z_0}$. To do this, we use the explicit expression (4.4.41) of $\Phi^{z_0}(x)$. Note that $\Phi^{z_0}(x)$ can be written as $(\Psi_1(x, z_0) + \Psi_2(x, z_0)\sqrt{x_b - x})e^{f(x)\sqrt{x_b-x}b_0\mathbb{1}_{b_0<0}}$ where $\Psi_i(x, z_0), i = 1, 2$ are locally bounded in (x, z_0) and both $\Psi_i(x, z_0), i = 1, 2$ and $f(x)$ are holomorphic in x in a neighborhood of x_b. Expanding the exponential into a Taylor series gives

$$e^{f(x)\sqrt{x_b-x}b_0\mathbb{1}_{b_0<0}} = \underbrace{\sum_{k=0}^{+\infty} \frac{f(x)^{2k}}{(2k)!}(x_b - x)^k b_0^{2k}\mathbb{1}_{b_0<0}}_{=\Xi_1(x,b_0)} + \sqrt{x_b - x}\underbrace{\sum_{k=0}^{+\infty} \frac{f(x)^{2k+1}}{(2k+1)!}(x_b - x)^k b_0^{2k+1}\mathbb{1}_{b_0<0}}_{=\Xi_2(x,b_0)}.$$

A straightforward argument shows that $\Xi_1(x, b_0)$ and $\Xi_2(x, b_0)$ are bounded when x lies in a sufficiently small neighborhood of x_b and b_0 ranges over a compact set. Since $\Phi_1(x, z_0) = \Psi_1(x, z_0)\,\Xi_1(x, b_0) + (x_b - x)\Psi_2(x, z_0)\,\Xi_2(x, b_0)$ and $\Phi_2(x, z_0) = \Psi_1(x, z_0)\,\Xi_2(x, b_0) + \Psi_2(x, z_0)\,\Xi_1(x, b_0)$, this yields the desired uniform bounds. □

To prove Theorem 4.13, we state the following technical lemma.

Lemma 4.38 (Asymptotic inequalities for h_α, f_0, f_π)**.** *Let* $\alpha, \beta \in \mathcal{M}$ *with* $\alpha \neq \beta$. *Then:*

- *Suppose that* $\beta \in \mathcal{M} \cap (0, \pi)$. *Then there exist* $\epsilon > 0$, $\eta > 0$, $r_0 > 0$, *and* $C > 0$ *such that for all* $\gamma \in \mathcal{M} \cap (\beta - \epsilon, \beta + \epsilon)$ *and all* $r \geq r_0$,

  $$h_\gamma(r\cos(\beta), r\sin(\beta)) \geq C^{-1}e^{r(x(\beta)\cos(\alpha)+y(\beta)\sin(\beta)-\eta)} \tag{4.5.3}$$

  and such that for all $r \geq r_0$,

  $$h_\alpha(r\cos(\beta), r\sin(\beta)) \leq Ce^{r(x(\beta)\cos(\alpha)+y(\beta)\sin(\beta)-2\eta)}. \tag{4.5.4}$$

- *Suppose that* $\beta \in \mathcal{M} \cap (\pi, 2\pi)$. *Then there exist* $\epsilon > 0$, $\eta > 0$, $r_0 > 0$, *and* $C > 0$ *such that for all* $\gamma \in \mathcal{M} \cap (\beta - \epsilon, \beta + \epsilon)$ *and all* $r \geq r_0$,

  $$h_\gamma(r\cos(\beta), r\sin(\beta)) \geq C^{-1}e^{r(x(\beta)\cos(\alpha)+z(\beta)\sin(\beta)-\eta)} \tag{4.5.5}$$

  and such that for all $r \geq r_0$

  $$h_\alpha(r\cos(\beta), r\sin(\beta)) \leq Ce^{r(x(\beta)\cos(\alpha)+z(\beta)\sin(\beta)-2\eta)}. \tag{4.5.6}$$

Inequalities (4.5.4) *and* (4.5.6) *also hold if* α *is replaced by* 0 *(resp.* π*) and* h_α *by* f_0 *(resp.* f_π*).*

Proof. This is a direct consequence of expressions (4.1.25) and (4.1.29). ☐

Proof of Theorem 4.13. By Theorems 4.6 to 4.10 and Lemma 4.37, it remains to show that functions h_α and f_0, f_π are minimal. The main idea of the proof is that $h_\beta(r\cos(\beta), r\sin(\beta))$ is asymptotically of greater order than $h_\alpha(r\cos(\beta), r\sin(\beta))$ as $r \to +\infty$ for any fixed β such that $\beta \neq \alpha$.

Let $\alpha \in \mathcal{M}\backslash\{0,\pi\}$. We now rigorously prove that h_α is minimal. To do so, we show that if $c_0, c_\pi \geq 0$ and if μ is a Borel measure on $\mathcal{M}\backslash\{0,\pi\}$ such that

$$h_\alpha = \int_{\mathcal{M}\backslash\{0,\pi\}} h_\alpha \, d\mu(\alpha) + c_0 f_0 + c_\pi f_\pi, \tag{4.5.7}$$

then $c_0 = c_\pi = 0$ and $\mu = \delta_\alpha$ (we excluded 0 and π in the integral term to avoid problems of notation). Let $\beta \in \mathcal{M} \cap (0,\pi)$ with $\alpha \neq \beta$, and let $\epsilon, \eta, C, r_0 > 0$ be the constants provided by Lemma 4.38. Then, using inequalities (4.5.3), (4.5.4), and the representation (4.5.7), for all $r \geq r_0$ we have

$$\begin{aligned} Ce^{r(x(\beta)\cos(\alpha)+y(\beta)\sin(\beta)-2\eta)} &\geq h_\alpha(r\cos(\beta), r\sin(\beta)) \\ &\geq \int_{(\beta-\epsilon,\beta+\epsilon)\cap\mathcal{M}} h_\beta(r\cos(\gamma), r\sin(\gamma)) d\mu(\gamma) \\ &\geq C^{-1}\mu((\beta-\epsilon,\beta+\epsilon)\cap\mathcal{M})e^{r(x(\beta)\cos(\alpha)+y(\beta)\sin(\beta)-\eta)}. \end{aligned}$$

Hence, for $r \geq r_0$,

$$C^2 e^{-\eta r} \geq \mu((\beta-\epsilon,\beta+\epsilon)\cap\mathcal{M}).$$

Taking the limit as $r \to +\infty$, we obtain $\mu((\beta-\epsilon,\beta+\epsilon)\cap\mathcal{M}) = 0$. By a standard property of Borel measures, it follows that $\mu(\mathcal{M}\cap(0,\pi)\setminus\{\alpha\}) = 0$. Using symmetric arguments along with inequalities (4.5.5) and (4.5.6), we similarly get $\mu(\mathcal{M}\cap(\pi,2\pi)\setminus\{\alpha\}) = 0$. Hence, $\mu = c\delta_\alpha$, where δ_α denotes the Dirac measure at α. By (4.5.7), we get $h_\alpha = ch_\alpha + c_0 f_0 + c_\pi f_\pi$. Since h_α, f_0, and f_π are clearly linearly independent, it follows that $c_0 = c_\pi = 0$ and $c = 1$. The proofs of the minimality of f_0 and f_π are analogous. ☐

Let us now prove Corollary 4.14.

Proof of Corollary 4.14. By standard arguments, the mapping

$$h : (a_0, b_0) \longmapsto \mathbb{P}_{(a_0,b_0)}\big(B_t \xrightarrow[t\to+\infty]{} +\infty\big)$$

is harmonic. Moreover, note that $h_{\alpha_{\mu^+}}$ and $h_{\alpha_{\mu^-}}$ are the only bounded harmonic functions among f_0, f_π, and h_α for $\alpha \in \mathcal{M}\backslash\{0,\pi\}$ (see (4.1.24), (4.1.25), (4.1.29), and (4.1.30)). Hence, applying the representation (4.1.43) to the bounded harmonic function h yields $h = c_+ h_{\alpha_{\mu^+}} + c_- h_{\alpha_{\mu^-}}$ for some nonnegative constants c_+ and c_-. By properties of standard Brownian motion (denoted here by W), we have

$$h(a_0,b_0) \geq \mathbb{P}\left[\left\{b_0 + \sqrt{\Sigma_{22}}W_t + \mu_2^+ t \xrightarrow[t\to\infty]{} +\infty\right\} \cap \left\{b_0 + \sqrt{\Sigma_{22}}W_t + \mu_+ t > 0 \quad \forall t \geq 0\right\}\right] \xrightarrow[b_0\to+\infty]{} 1$$

by Hypothesis (4.1.5) on the drift. Since $h_{\alpha_{\mu^-}}(a_0,b_0) \xrightarrow[b_0\to+\infty]{} 0$ and $h_{\alpha_{\mu^+}}(a_0,b_0) \xrightarrow[b_0\to+\infty]{} 1$ (see (4.1.29) and (4.1.30)), we obtain $c_+ = 1$. Considering now the limit as $b_0 \to -\infty$, we deduce that $c_- = 0$, and the desired result follows. ☐

4.6 Generalization to skew diffusions acting in any oblique direction q

Up to this point, we have imposed the condition $q = q_0$ (see (4.1.3)) to have a divergence form generator. The goal of this section is to study the extension of our results for $q \in \mathbb{R} \times (-1,1)$.

In Section 4.6.1, we state hypotheses 4.39 and 4.40 on q under which this analysis can be fruitful. Section 4.6.2 describes new properties satisfied by the Laplace transforms. In particular, the Laplace transform φ^{z_0} may now have a pole. In Section 4.6.3, we introduce a new angular region (arising from the pole of φ^{z_0}) and generalise Theorems 4.7 – 4.10 to this broader range of parameters q. Sections 4.6.4 and 4.6.5 are devoted to the study of the asymptotic behaviour of the Green's functions g^{z_0} within this new region. Finally, in Section 4.6.6, we identify the Martin boundary and the minimal Martin boundary for the generalised processes.

Let Σ^+, Σ^- be covariance matrices, and let $\mu^+, \mu^- \in \mathbb{R}^2$ satisfy (4.1.5). Let $q \in \mathbb{R} \times (-1,1)$. Using the same method as in Section 4.2.1, one can show that for every $(a_0, b_0) \in \mathbb{R}^2$, there exists a unique pathwise solution $Z = (A, B)$ to equation (4.1.1), driven by a Brownian motion $(W_t)_{t \geq 0}$.

4.6.1 Hypotheses

We now state two technical hypotheses on the parameter $q \in \mathbb{R} \times (-1,1)$ under which our extended results hold.

Hypothesis 4.39 (Markov property and density). The process Z defines a strong Markov process. Moreover, for any $z_0 \in \mathbb{R}^2$ and any $t > 0$, the law of Z_t starting from z_0 admits a density $p_t^{z_0}(z)$ with respect to the Lebesgue measure on $\mathbb{R}^2$. Finally, the function $z \longmapsto p_t^{z_0}(z)$ has limits at the top and bottom of the axis $\{y = 0\}$.

Hypothesis 4.40 (Aronson-like inequality). There exists a constant $M > 0$ such that for all $t > 0$ and all $z_0 \in \mathbb{R}^2$,

$$p_t^{z_0}(z) \leq \frac{M}{t} e^{-|z-z_0|^2/Mt + Mt}. \tag{4.6.1}$$

Assuming Hypothesis 4.39, we may define the Green's functions by $g^{z_0}(z) = \int_0^{+\infty} p_t^{z_0}(z)\, dt$ for $z = (x,y) \in \{y \neq 0\}$ and $z_0 \in \mathbb{R}^2$. Under Hypothesis 4.40, inequality (4.2.4) holds, and the functions g^{z_0} satisfy inequalities (4.2.3).

4.6.2 Analogous results for Laplace transforms.

In this section, we investigate properties of the Laplace transforms for a general parameter q. The functional equation (4.1.9) and the continuation formula for φ^{z_0} given in (4.1.16) remain valid and can be proved using the same arguments.

Proposition 4.41 (Functional equation and continuation of φ^{z_0})**.** *There exists $\eta > 0$ such that for all $x \in (-\eta, 0)$, $y < 0$, and $z > 0$, the functions $\varphi_-^{z_0}(x,z)$, $\varphi_+^{z_0}(x,y)$, and $\varphi^{z_0}(x)$ are finite and satisfy (4.1.11). Furthermore, φ^{z_0} admits a meromorphic continuation on $\mathbb{C} \setminus \big((-\infty, \widetilde{x}_b] \cup [x_b, +\infty)\big)$ given by (4.1.16).*

The main difference with the case $q = q_0$ is that φ^{z_0} may now have a pole.

Proposition 4.42 (Poles of φ^{z_0} on $(\widetilde{x}_b, x_b)$)**.** *On $[\widetilde{x}_b, x_b]$, the equation $(E) : \gamma(x, Y^-(x), Z^+(x)) = 0$ has at most one solution, denoted by x^*. If this equation has no solution, we adopt the convention $x^* = +\infty$. Then, on $(\widetilde{x}_b, x_b)$, the extended function φ^{z_0} has a pole if and only if (E) has a solution, and the pole is located at x^* whenever it exists. In that case, $(x^*, Y^-(x^*), Z^+(x^*))$ solves the system*

$$\begin{cases} \gamma(x,y,z) = 0 \\ \gamma_+(x,y) = 0 \\ \gamma_-(x,z) = 0 \end{cases} \tag{4.6.2}$$

and

$$res_{x=x^*}\varphi^{z_0}(x) = -\frac{e^{x^* a_0 + Y^-(x^*) b_0 \mathbb{1}_{b_0>0} + Z^+(x^*) b_0 \mathbb{1}_{b_0<0}}}{\frac{d}{dx}\gamma(x, Y^-(x), Z^+(x))\big|_{x=x^*}}.$$

Furthermore, if $x^* > 0$ *(resp.* $x^* < 0$*), then* x^* *is the unique pole of* φ^{z_0} *in the strip* $\{\widetilde{x}_b - \epsilon < \Re(x) < x^* + \epsilon\}$ *(resp.* $\{x^* - \epsilon < \Re(x) < x_b + \epsilon\}$*) for sufficiently small* $\epsilon > 0$.

Proof. From (4.1.16), it is clear that x is a pole of φ^{z_0} if and only if $(x, Y^-(x), Z^+(x))$ satisfies (4.6.2). Note that the function $\gamma(x, Y^-(x), Z^+(x))$ is convex, being the sum of convex functions. Moreover, its explicit expression shows that at least one of the inequalities $\gamma(x, Y^-(x), Z^+(x))|_{x=\widetilde{x}_b} < 0$ or $\gamma(x, Y^-(x), Z^+(x))|_{x=x_b} < 0$ must hold. Hence, $\gamma(x, Y^-(x), Z^+(x))$ has at most one zero on $(\widetilde{x}_b, x_b)$, and φ^{z_0} has at most one pole in this interval. The last statement of the lemma follows from a straightforward analysis of $\gamma(x, Y^-(x), Z^+(x))$. □

Note that $x^* \neq 0$ since $\gamma(0, Y^-(0), Z^+(0)) = \left(\frac{1+q_2}{2}\right) Y^-(0) + \left(\frac{q_2-1}{2}\right) Z^+(0) < 0$.

Remark 4.43 (Values of Green's functions at the boundary). In the case $q = q_0$, the Green's function is continuous on the axis, meaning that the upper and lower limits $g^{z_0}(u, 0^+)$ and $g^{z_0}(u, 0^-)$ are equal. When $q \neq q_0$, this is no longer the case. Using the initial value theorem for Laplace transforms together with (4.1.11), we obtain for all $u \in \mathbb{R}$ with $z_0 \neq (u, 0)$,

$$\Sigma_{22}^+ \left(\frac{1-q_2}{2}\right) g^{z_0}(u, 0^+) = \Sigma_{22}^- \left(\frac{1+q_2}{2}\right) g^{z_0}(u, 0^-). \tag{4.6.3}$$

By the same argument as in the proof of Lemma 4.16, we also have

$$\varphi^{z_0}(x) = \int_{\mathbb{R}} \frac{\Sigma_{22}^- g^{z_0}(u, 0^-) + \Sigma_{22}^+ g^{z_0}(u, 0^+)}{2} e^{-xu} du \tag{4.6.4}$$

which can be interpreted as the Laplace transform of the Green's functions on the axis. In fact, if we adopt the convention $g^{z_0}(u, 0) = \frac{\Sigma_{22}^- g^{z_0}(u,0^-) + \Sigma_{22}^+ g^{z_0}(u,0^+)}{\Sigma_{22}^- + \Sigma_{22}^+}$, then φ^{z_0} is the Laplace transform of $\frac{\Sigma_{22}^- + \Sigma_{22}^+}{2} g^{z_0}(u, 0)$ exactly as in the case $q = q_0$. Theorem 4.6 for $\alpha = \alpha_0 \in \{0, \pi\}$ could be extended to the case of any $q \in \mathbb{R} \times (-1, 1)$ in this sense.

4.6.3 Generalisation of Theorems 4.7 to 4.10 and introduction of a new region for angles

4.6.3.1 Case without pole

If $x^* = +\infty$, then all the arguments remain unchanged, and Theorems 4.7 to 4.10 hold for general $q \in \mathbb{R} \times (-1, 1)$.

4.6.3.2 Notation if φ^{z_0} has a pole

Now suppose that φ^{z_0} has a pole x^*, and let us introduce a new region of angles for the asymptotic analysis. For simplicity, we assume that $x^* \neq \widetilde{x}_b$ and $x^* \neq x_b$.

We recall the notation $\alpha_{\mu^+} = \arctan(\mu_2^+/\mu_1^+) \in (0, \pi)$ and $\alpha_{\mu^-} = \arctan(\mu_2^-/\mu_1^-) + 2\pi \in (3\pi/2, 2\pi)$. Note that $x(\alpha) > 0$ for $\alpha \in (0, \alpha_{\mu^+}) \cup (\alpha_{\mu^-}, 2\pi)$, and that $x(\alpha) < 0$ for $\alpha \in (\alpha_{\mu^+}, \pi) \cup (\pi, \alpha_{\mu^-})$ (see Section 4.1.4.2). Let $\alpha_-^* \in (0, \pi)$ and $\alpha_+^* \in (\pi, 2\pi)$ be the angles defined by

$$(x(\alpha_+^*), y(\alpha_+^*)) = (x^*, Y^+(x^*)), \quad (x(\alpha_-^*), y(\alpha_-^*)) = (x^*, Z^-(x^*)). \tag{4.6.5}$$

(see Figure 4.13). From the above remarks on the sign of $x(\alpha)$, we observe that:

- If $x^* \in (0, x_b)$, then $\alpha_+^* \in (0, \alpha_{\mu^+})$ and $\alpha_-^* \in (\alpha_{\mu^-}, 2\pi)$.
- If $x^* \in (\widetilde{x}_b, 0)$, then $\alpha_+^* \in (\alpha_{\mu^+}, \pi)$ and $\alpha_-^* \in (\pi, \alpha_{\mu^-})$.

This defines a new (pink) region of angles; see Figure 4.14 for the case $x^* \in (0, x_b)$.

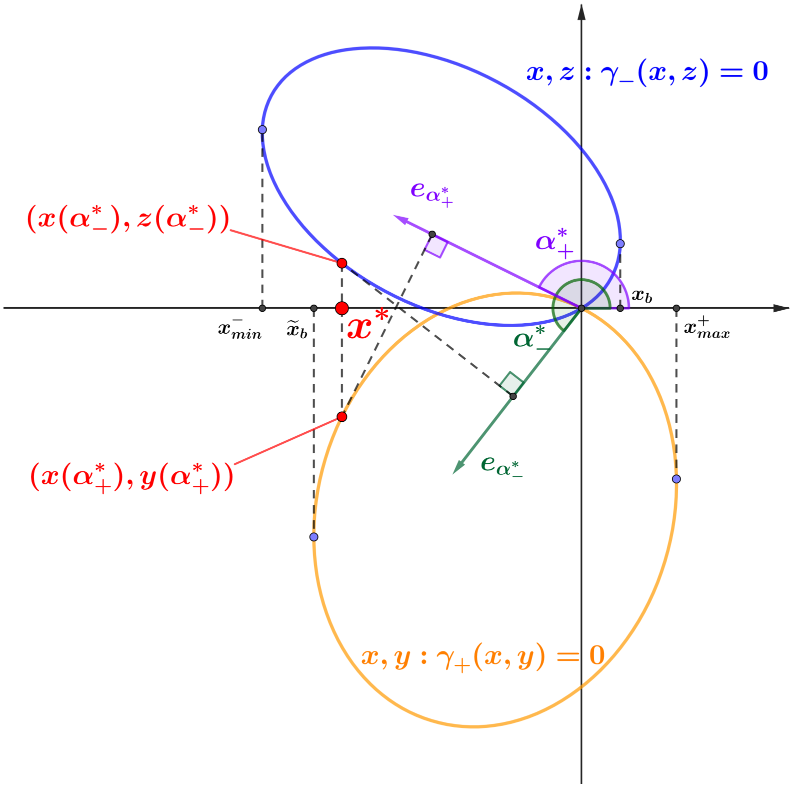


Figure 4.13: Definition of $\alpha_\pm^*$ from the data x^*. In this case, $x^* \in (\widetilde{x}_b, 0)$.

4.6.3.3 Generalisation of Theorems 4.7 to 4.10 when φ^{z_0} has a pole

Outside the pink region, the same method as in the case $q = q_0$ applies. In these areas, the contour deformations do not involve the pole of φ^{z_0}, so the asymptotics in Theorems 4.7 to 4.10 remain valid in the corresponding non-pink regions (see Figure 4.14 for the case $x^* > 0$ and $\widetilde{x}_b = x^+_{min}$).

4.6.4 Study of asymptotics (strictly) inside the dotted pink region

Without loss of generality, we focus on the case where $x^* \in (0, x_b)$. In this section, we study the asymptotics of the Green's functions g^{z_0} in the directions $\alpha_0 \in (\alpha^*_-, 2\pi) \cup [0, \alpha^*_+)$.

Note that Lemmas 4.23, 4.24, and the constructions in Section 4.4.2 still hold, namely that $g^{z_0} = I_1^{z_0} + I_2^{z_0}$.

Lemma 4.44 (Changing path for directions $\alpha_0 \in [0, \alpha^*_+)$)**.** *Suppose $x^* \in (0, x_b)$. Let $x^* < x_c < x_b$. Then, for $a \in \mathbb{R}$, $b > 0$, and $z_0 \in \mathbb{R}^2$ with $(a, b) \neq z_0$, we have:*

$$I_1^{z_0}(a,b) = res_{x=x^*} \varphi^{z_0}(x) \frac{\gamma(x^*, Y^+(x^*), Z^+(x^*))}{\partial_y \gamma_+(x^*, Y^+(x^*))} e^{-ax^* - bY^+(x^*)}. \tag{4.6.6}$$

$$+ \frac{1}{2i\pi} \int_{x_c + i\mathbb{R}} \frac{\varphi^{z_0}(x)\gamma(x, Y^+(x), Z^+(x))e^{-ax-bY^+(x)}}{\partial_y \gamma_+(x, Y^+(x))} dx$$

where $\gamma(x^*, Y^+(x^*), Z^+(x^*)) \neq 0$. Furthermore,

$$I_2^{z_0}(a,b) = \frac{1}{2i\pi} \int_{x_c + i\mathbb{R}} \frac{e^{(a_0 - a)x} \left(e^{(b_0 - b)Y^+(x)} \mathbb{1}_{b_0 > 0} + e^{b_0 Z^+(x) - bY^+(x)} \mathbb{1}_{b_0 < 0} \right)}{\partial_y \gamma_+(x, Y^+(x))} dx \quad \text{if } b > b_0 \tag{4.6.7}$$

and

$$I_2^{z_0}(a,b) = \frac{1}{2i\pi} \int_{Y^+(x_c) + i\mathbb{R}} \frac{e^{(a_0 - a)X^+(y) + (b_0 - b)y}}{\partial_x \gamma_+(X^+(y), y)} dy \quad \text{if } a > a_0. \tag{4.6.8}$$

Proof. The deformation of the integration path is justified using the same arguments as in Lemma 4.34. Equation $\gamma(x^*, Y^+(x^*), Z^+(x^*)) \neq 0$ follows from the fact that it is not possible to have both $\gamma(x, Y^+(x), Z^+(x)) = 0$ and $\gamma(x, Y^-(x), Z^+(x)) = 0$ simultaneously, except possibly at $x = x^+_{min}$ and $x = x^+_{max}$. □

The term $I_1^{z_0}$ is decomposed into the sum of two terms, namely the residue at the pole and a remaining integral term. In the next lemma, we show that this integral term and also $I_2(a, b)$ are negligible in the asymptotics compared with the residue at the pole.

Lemma 4.45 (Negligibility of integrals compared to the pole)**.** *Suppose $x^* \in (0, x_b)$. Let $z_0 \in \mathbb{R}^2$, $\alpha_0 \in [0, \alpha^*_+)$, and let x_c be such that $x^* < x_c < \min(x(\alpha_0), x_b)$. Let $r = \sqrt{a^2 + b^2}$ and let $\alpha(a, b) \in (0, \pi)$ denote the angle satisfying $\cos(\alpha) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha) = \frac{b}{\sqrt{a^2+b^2}}$. Then, for $\eta > 0$ small enough, there exist constants $r_0 > 0$ and $D > 0$ such that for all (a, b) satisfying $r > r_0$, $\alpha(a, b) > 0$, and $|\alpha(a, b) - \alpha_0| < \eta$, we have:*

$$\left| \frac{1}{2i\pi} \int_{x_c + i\mathbb{R}} \frac{\varphi^{z_0}(x)\gamma(x, Y^+(x), Z^+(x))e^{-ax-bY^+(x)}}{\partial_y \gamma_+(x, Y^+(x))} dx \right| + |I_2(a,b)| \leq De^{-r(\cos(\alpha)x_c + \sin(\alpha)Y^+(x_c))}$$

Proof. The proof is identical to that of Lemma 4.36. □

We now prove the corresponding theorem for the dotted pink region (see Figure 4.14).

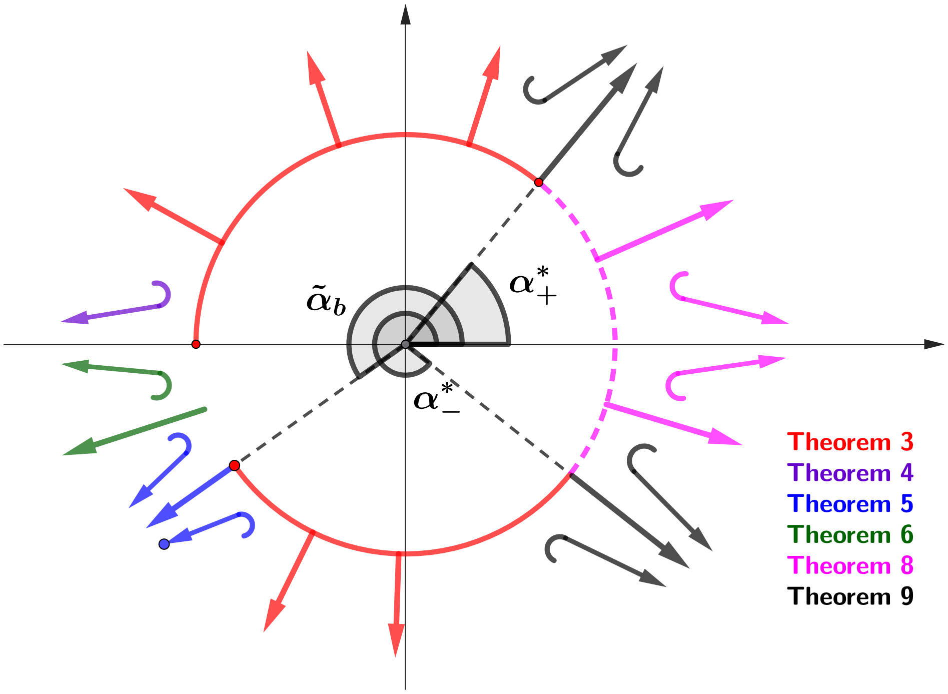


Figure 4.14: Summary of asymptotics for the extended results. In this case, $\widetilde{x}_b = x_{min}^+$ and $x^* \in (0, x_b)$. We omit α_b for readability.

Theorem 4.46 (Asymptotics strictly inside the dotted pink region)**.** *Suppose that* $x^* \in (0, x_b)$. *Let* $z_0 \in \mathbb{R}^2$ *and* $\alpha_0 \in [0, \alpha_+^*)$. *Then:*

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to+\infty\\ \alpha\to\alpha_0,\alpha>0}}{\sim} C_*^+ f_*(z_0) e^{-r(\cos(\alpha)x^*+\sin(\alpha)Y^+(x^*))} \tag{4.6.9}$$

where

$$f_*(a_0, b_0) = \begin{cases} e^{x^* a_0 + Y^-(x^*) b_0} & \text{if } b_0 \geq 0 \\ e^{x^* a_0 + Z^+(x^*) b_0} & \text{if } b_0 < 0. \end{cases} \tag{4.6.10}$$

and

$$C_*^+ = \frac{\gamma(x^*, Y^+(x^*), Z^+(x^*))}{\partial_y \gamma_+(x^*, Y^+(x^*)) \frac{d}{dx}\gamma(x, Y^-(x), Z^+(x))\big|_{x=x^*}}. \tag{4.6.11}$$

The symmetric result holds for angles $\alpha_0 \in (\alpha_-^*, 2\pi]$.

Proof. Choose $x^* < x_c < \min(x(\alpha_0), x_b)$ and consider the representation $g^{z_0} = I_1^{z_0} + I_2^{z_0}$ from Lemma 4.44. By the choice of x_c, we have $\cos(\alpha_0)x^* + \sin(\alpha_0)Y^+(x^*) < \cos(\alpha_0)x_c + \sin(\alpha_0)Y^+(x_c)$. Then, by Lemma 4.45, both the integral term in (4.6.6) and the integral in (4.6.7) are negligible compared with the residue as $\alpha \to \alpha_0$ and $r \to +\infty$. The result follows immediately. ☐

4.6.5 Asymptotics in directions $\alpha_\pm^*$

We now consider the asymptotics in the directions $\alpha_\pm^*$ when $x^* \neq +\infty$. For simplicity, we assume that $x^* \in (0, x_b)$. This "pole-type" asymptotics has been studied in detail in Section 2.10 (Chapter 2), and the same arguments apply here. Therefore, we state the following theorem without providing a proof. Let erf denote the function defined by $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2) ds$.

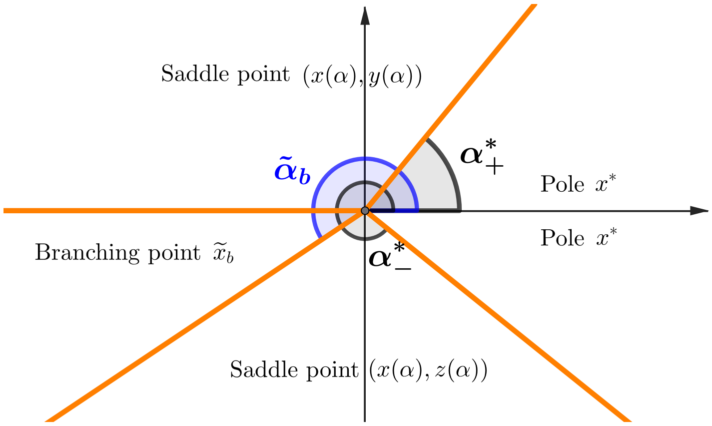


Figure 4.15: Type of asymptotics according to directions: saddle point type, pole or branching point.

Theorem 4.47 (Asymptotics as $\alpha \to \alpha^*_\pm$). *Suppose that $x^* \in (0, x_b)$. Then, the following asymptotics hold:*

(i) If $\alpha > \alpha^*_+$ and $r(\alpha - \alpha^*_+)^2 \to \infty$, then

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to+\infty \\ \alpha\to\alpha^*_+}}{\sim} C'_* f_*(z_0) \frac{e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}}{\sqrt{r}(x(\alpha) - x^*)} \tag{4.6.12}$$

where

$$C'_* = -C^+(\alpha^*_+) \frac{\gamma(x^*, Y^+(x^*), Z^+(x^*))}{\frac{d}{dx}\gamma(x, Y^-(x), Z^+(x))\big|_{x=x^*}} > 0$$

and where $C^+(\alpha^*_+)$ is given by (4.1.28).

(ii) If $\alpha > \alpha^*_+$ and $r(\alpha - \alpha^*_+)^2 \to c > 0$, then

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to+\infty \\ \alpha\to\alpha^*_+}}{\sim} \frac{1}{2} C^+_* \left(1 - \mathrm{erf}\left(\sqrt{c}A(\alpha^*_+)\right)\right) f_*(z_0) e^{-r(\cos(\alpha)x^* + \sin(\alpha)y^*)}. \tag{4.6.13}$$

where C^+_* is given by (4.6.11), $A(\alpha^*_+) = \frac{-x'(\alpha^*_+)}{x'_\omega(0,\alpha^*_+)}$ and where $x'_\omega(0, \alpha^*_+)$ is defined in (4.4.12).

(iii) If $r(\alpha - \alpha^*_+)^2 \to 0$, then

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to+\infty \\ \alpha\to\alpha^*_+}}{\sim} \frac{1}{2} C^+_* f_*(z_0) e^{-r(\cos(\alpha)x^* + \sin(\alpha)y^*)}. \tag{4.6.14}$$

(iv) If $\alpha < \alpha^*_+$ and $r(\alpha - \alpha^*_+)^2 \to c > 0$, then

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to+\infty \\ \alpha\to\alpha^*_+}}{\sim} \frac{1}{2} C^+_* \left(1 + \mathrm{erf}\left(\sqrt{c}A(\alpha^*_+)\right)\right) f_*(z_0) e^{-r(\cos(\alpha)x^* + \sin(\alpha)y^*)}. \tag{4.6.15}$$

(v) If $\alpha < \alpha^*_+$ and $r(\alpha - \alpha^*_+)^2 \to \infty$, then

$$g^{z_0}(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to+\infty \\ \alpha\to\alpha^*_+}}{\sim} C^+_* f_*(z_0) e^{-r(\cos(\alpha)x^* + \sin(\alpha)y^*)}. \tag{4.6.16}$$

The symmetric asymptotics hold as $\alpha \to \alpha^*_-$.

4.6.6 Martin boundary, general case

The study of the asymptotics carried out in Sections 4.6.4 and 4.6.5, together with the results established in Sections 4.6.3.3 and 4.6.3.1, leads to the following theorem on the Martin boundary.

Theorem 4.48 (Martin Boundary and harmonic functions for general q)**.** *If* $x^* = +\infty$, *then Theorem 4.13 applies. Assume now that* $x^* \in (0, x_b)$. *Then, the Martin boundary* Γ *is given by*

$$\Gamma = \left[\{e^{i\alpha}\}_{\alpha\in[\alpha_+^*,\alpha_-^*]\cap\mathcal{M}} \cup \{ue^{i\widetilde{\alpha}_b}\}_{u\in[1,2]}\right]/\mathcal{R}_1 \ \sim\ \mathbb{S}^1 \tag{4.6.17}$$

where $\mathcal{R}_1$ *is the equivalence relation given by* $e^{i\alpha_-^*}\,\mathcal{R}_1\,e^{i\alpha_+^*}$, $e^{i\pi}\,\mathcal{R}_1\,2e^{i\widetilde{\alpha}_b}$, *and* $[x\mathcal{R}_1 y \iff x = y]$ *for the other points (see Figure 4.16). Furthermore, the minimal Martin boundary, denoted by* Γ_{min}, *is given by*

$$\Gamma_{min} = \left[\{e^{i\alpha}\}_{\alpha\in[\alpha_+^*,\alpha_-^*]\cap\mathcal{M}}\right]/\mathcal{R}_2 \ \sim\ [0,1] \tag{4.6.18}$$

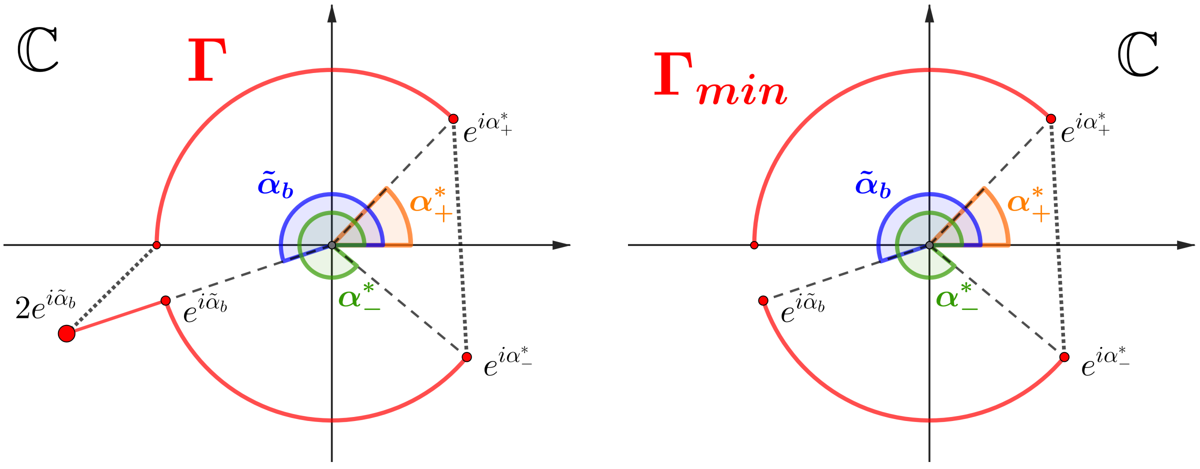
where $\mathcal{R}_2$ *is the equivalence relation given by* $e^{i\alpha_-^*}\,\mathcal{R}_2\,e^{i\alpha_+^*}$ *and* $[x\mathcal{R}_2 y \iff x = y]$ *for the other points (see again Figure 4.16). Moreover, for each positive harmonic function* h *(with respect to the process), there exists a unique Radon measure* ν *on* $[\alpha_+^*, \alpha_-^*) \cap \mathcal{M}$ *such that*

$$\forall z \in \mathbb{R}^2, \quad h(z) = \int_{[\alpha_+^*,\alpha_-^*)\cap\mathcal{M}} \frac{h_\alpha(z)}{h_\alpha(0)}\, d\nu(\alpha) \tag{4.6.19}$$

where the conventions $\frac{h_{\alpha_+^*}(z_0)}{h_{\alpha_+^*}(0)} = \frac{f_*(z_0)}{f_*(0)}$ *and* $\frac{h_\pi(z_0)}{h_\pi(0)} = \frac{f_\pi(z_0)}{f_\pi(0)}$ *are used, since*

$$\lim_{\substack{\alpha\to\alpha_+^*\\ \alpha\in\mathcal{M}}} \frac{h_\alpha(z_0)}{h_\alpha(0)} \left(= \lim_{\substack{\alpha\to\alpha_-^*\\ \alpha\in\mathcal{M}}} \frac{h_\alpha(z_0)}{h_\alpha(0)} \right) = \frac{f_*(z_0)}{f_*(0)} \quad \text{and} \quad \frac{h_\alpha(z_0)}{h_\alpha(0)} \xrightarrow[\alpha\in\mathcal{M}]{\alpha\to\pi} \frac{f_\pi(z_0)}{f_\pi(0)}.$$

Finally, the case $x^* \in (\widetilde{x}_b, 0)$ *is symmetric.*



(a) Martin Boundary.

(b) Minimal Martin Boundary.

Figure 4.16: Martin Boundary and Minimal Martin Boundary for the general process in the case $x^* \in (0, x_b)$ and $x_{min}^- < x_{min}^+$.

4.A Two-sided Tauberian theorem

Tauberian theorems are known for Laplace transforms on $\mathbb{R}^+$, giving a link between the asymptotics of a function and its Laplace transform (cf [33, Theorem 37.1] or [24, Lemma C.2]). In fact, the proof can be directly adapted to two-sided Laplace transforms without any technical issue.

Lemma 4.A.1. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ *be a continuous function and* $\varphi(z) := \int_{\mathbb{R}} e^{xz} f(x)dx$*. Suppose that there are* $a < 0 < b$ *such that* φ *converges absolutely in* $a < \Re(z) < b$*. For* $\delta > 0$ *and* $s \in \mathbb{C}$*, let* $G_\delta^+(s) = \{z \in \mathbb{C}, \Re(z) > 0, z \neq s, |\arg(z-s)| > \delta\}$ *(where the complex argument is taken in* $(-\pi, \pi]$*). Suppose that there exists* $0 \leq \delta < \pi/2$ *such that*

- *Function* φ *has a holomorphic continuation on* $G_\delta^+(b)$*.*
- *The following asymptotics hold:*
$$\varphi(z) \underset{\substack{|z|\to+\infty \\ z\in G_\delta^+(b)}}{\longrightarrow} 0.$$
- *There exists* $\lambda \in \mathbb{R}$ *and constants* $c \in \mathbb{C}$*,* $A \neq 0$ *such that*
$$(\varphi(z) - c)(b-z)^\lambda \underset{\substack{z\to b \\ z\in G_\delta^+(b)}}{\longrightarrow} A. \tag{4.A.1}$$

Then:

$$f(x) \underset{x\to+\infty}{\sim} \frac{A}{\Gamma(\lambda)} e^{-bx} x^{\lambda-1}$$

Obviously, the symmetric lemma for the asymptotics of f in $-\infty$ holds.

4.B An integral asymptotic lemma

Lemma 4.B.1. *The following asymptotics hold as* $q \to +\infty$

$$\int_0^1 \sqrt{1-s}e^{qs^2}ds \underset{q\to+\infty}{\sim} \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{e^q}{q^{3/2}}.$$

Proof. It suffices to establish that, for some $\epsilon > 0$, the integral from over $[1-\epsilon, 1]$ has the same asymptotic. Since the asymptotic of the integral over $[0, 1-\epsilon]$ is bounded by $O(\exp(q(1-\epsilon)^2)))$, as $q \to \infty$. Let us look at

$$J(q,\epsilon) = \exp(-q) \int_{1-\epsilon}^1 \sqrt{1-s}\exp(qs^2)ds$$

for some $\epsilon > 0$ small enough. After the change of variables $t = q - qs^2$, it becomes

$$J(q,\epsilon) = \frac{1}{2q} \int_0^{q(1-(1-\epsilon)^2)} \frac{\sqrt{1-\sqrt{1-t/q}}}{\sqrt{1-t/q}} \exp(-t)dt.$$

For any $\epsilon > 0$ small enough there exists a constant $C > 0$ such that

$$\Big|\frac{\sqrt{1-\sqrt{1-s}}}{\sqrt{1-s}} - \sqrt{\frac{s}{2}}\Big| \leq Cs\sqrt{\frac{s}{2}}, \quad \forall s \in [0, 1-(1-\epsilon)^2].$$

Then

$$J(q,\epsilon) = \frac{1}{2q} \int\limits_0^{q(1-(1-\epsilon)^2)} \sqrt{\frac{t}{2q}} \exp(-t)dt + R(q,\epsilon)$$

where

$$|R(q,\epsilon)| \leq \frac{1}{2q}\int_0^\infty \sqrt{\frac{t}{2q}}\frac{t}{q}\exp(-t)dt = O(q^{-5/2}), \quad q \to \infty$$

is of smaller order than the first term. It remains to compute

$$\int_0^{q(1-(1-\epsilon)^2)} \sqrt{t}\exp(-t)dt \to \frac{\sqrt{\pi}}{2}, \quad q \to \infty.$$

□

Annexe A

Présentation de la théorie de la frontière de Martin

Contents

A.1 Contexte et références

La théorie de la frontière de Martin pour les *chaînes* de Markov est à présent bien établie et synthétisée : on pourra se référer à l'article de Dynkin [36] pour les chaînes de Markov générales ou à Sawyer [111] qui résume le cas des marches aléatoires sur les groupes. En ce qui concerne les *processus* de Markov, le livre de Pinsky [102] résume de manière claire la théorie de la frontière de Martin pour les diffusions au sens usuel, mais ne couvre pas les processus étudiés dans ce manuscrit. Comme présenté dans le paragraphe suivant, la théorie générale dans le cadre continu est maintenant bien connue mais – à notre connaissance – pas encore synthétisée. Pour cette raison, on propose dans cette annexe un résumé de la théorie de la frontière de Martin pour la classe des processus dits *standards*. Nous ferons également le lien avec la théorie du potentiel analytique en discutant du problème de Dirichlet et en liant les définitions analytique et probabiliste des fonctions de Green. Quelques preuves courtes et éclairantes sont reproduites dans cette partie, mais on renvoie aux références bibliographiques pour la majorité des démonstrations pour des raisons techniques.

Références principales

La théorie du potentiel, des fonctions harmoniques et de la frontière de Martin s'est construite à l'interface entre les probabilités et l'analyse. Elle doit son nom à l'article de Martin [92] (1941), qui fournit un théorème de représentation de l'ensemble des fonctions harmoniques positives dans un domaine. On trouvera dans l'ouvrage de Doob [34] une présentation de la théorie sous ses aspects analytique et probabiliste (dans le cas du mouvement brownien). Le cadre probabiliste de la théorie du potentiel s'est construit notamment à partir des travaux de Hunt [69] et de Kunita et Watanabe [78]. On adoptera l'approche introduite dans ce dernier article pour décrire la compactification de Martin. Une présentation entièrement probabiliste de la théorie du potentiel et de la frontière de Martin est donnée par Blumenthal et Getoor [106], Dellacherie et Meyer [30], ainsi que dans l'ouvrage de Chung et Walsh [20]. Pour le cas particulier des diffusions, la description de la frontière de Martin est développée dans Pinsky [102]. Le lecteur intéressé pourra également se référer à Murata [99] pour une construction analytique de la frontière de Martin pour une famille d'opérateurs elliptiques du second ordre.

Plan

On commence en Section A.2 par introduire la notion de fonction harmonique. Celle-ci est d'abord définie au sens usuel pour le Laplacien en Section A.2.1, puis généralisée via la propriété de la moyenne associée à un processus de Markov en Section A.2.2. La Section A.3 est ensuite consacrée aux fonctions de Green analytiques (Section A.3.1) et probabilistes (Sections A.3.2 et A.3.3), deux notions qui coïncident dans le cas du mouvement brownien tué au bord d'un domaine. En Section A.4, on esquisse la construction théorique de la frontière de Martin à partir des fonctions de Green (Section A.4.1). On y présente également une méthode pratique de calcul (Section A.4.2), ainsi que le théorème de représentation des fonctions harmoniques et excessives à l'aide du noyau de Martin (Section A.4.3). Enfin, la Section A.5 introduit la notion de h-transformée de Doob (Section A.5.1) et en propose une interprétation trajectorielle en termes de convergence vers un point de la frontière de Martin (Section A.5.2).

A.2 Fonctions harmoniques

On présente dans la Section A.2.1 la notion de fonction harmonique au sens usuel (pour le Laplacien), en le liant au mouvement brownien et au problème de Dirichlet. On généralise ensuite cette notion en Section A.2.2 pour un processus de Markov général.

A.2.1 Fonction harmonique pour le Laplacien et problème de Dirichlet

En physique, l'équation $\Delta u = 0$ avec *conditions frontières* au bord d'un domaine décrit la répartition de la température dans un milieu dont la température a été fixée sur le bord. Dans cette section, on présente une solution probabiliste du problème de Dirichlet. Plus généralement, on peut considérer des équations du type $\mathcal{L}u = 0$ avec conditions frontières où $\mathcal{L} = \sum_{i,j=1}^{d} a_{i,j}(x)\partial_{ij}^2 + \sum_{i=1}^{d} b_i(x)\partial_i$, et relier les solutions du problème de Dirichlet associé à des diffusions de générateur $\mathcal{L}$. On renvoie le lecteur au livre de Pinsky [102].

Fonction harmonique au sens usuel

Soit D un domaine de $\mathbb{R}^d$ ($d \geq 1$). On dit qu'une fonction $h \in C^2(D)$ est harmonique (au sens usuel) si $\Delta h = 0$ sur D où Δ désigne $\Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$. On rappelle une autre caractérisation des fonctions harmoniques via une formule de la moyenne. Ce résultat nous sera utile plus tard pour relier les fonctions de Green probabilistes et analytiques.

Théorème A.1 (de Gauss-Koebe [34, Section 4.3])**.** *Soient D un ouvert et $h : D \longrightarrow \mathbb{R}$ une fonction continue telle que pour tout $x \in D$ et $r > 0$ tel que $\overline{B}(0, r) \subset D$,*

$$h(x) = \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} h(y)\sigma(dy) \tag{A.2.1}$$

où $d\sigma$ désigne la mesure de surface sur $\partial B(x, r)$. Alors, h est de classe C^∞ sur D et $\Delta h = 0$ sur D. Réciproquement, toute fonction $h \in C^2(D)$ satisfaisant $\Delta h = 0$ satisfait (A.2.1) (et est donc de classe $C^\infty(D)$).

Le problème de Dirichlet

Soient D un domaine et $f : \partial D \longrightarrow \mathbb{R}$. Le problème de Dirichlet associé à (D, f) est le système d'inconnue $h \in C^0(\overline{D})$ donné par :

$$\begin{cases} \Delta h(x) = 0, & x \in D \\ h(x) = f(x), & x \in \partial D. \end{cases} \tag{A.2.2}$$

Si une solution continue sur $\overline{D}$ existe et que D est borné, alors elle est unique : c'est une application du principe du maximum [34, Section 1.I.4].

Lemme A.2 (Principe du maximum et unicité du problème de Dirichlet)**.** *Soient D un ouvert borné et u une fonction continue sur $\overline{D}$ et harmonique sur D. Alors,*

$$\sup_{x \in \overline{D}} u(x) = \sup_{x \in \partial D} u(x). \tag{A.2.3}$$

De plus, si u, v sont continues sur $\overline{D}$, harmoniques sur D et solutions de (A.2.2), alors $u = v$.

Résolution probabiliste

Donnons à présent la solution probabiliste du problème de Dirichlet avec une condition aux bords f. Soient D un ouvert de $\mathbb{R}^d$ et $(W_t)_{t \geq 0}$ un mouvement brownien d-dimensionnel. Si A est un borélien de D, on note $\tau_A = \inf\{t > 0,\ W_t \in A\}$.

Lemme A.3. *Soient $f : \partial D \longrightarrow \mathbb{R}$ une fonction mesurable bornée et $H_D f : \mathbb{R}^d \longrightarrow \mathbb{R}$ la fonction définie par*

$$H_D f : x \longmapsto \mathbb{E}_x[f(W_{\tau_{\partial D}})\mathbb{1}_{\tau_{\partial D} < +\infty}]. \tag{A.2.4}$$

Alors, $H_D f$ est harmonique sur $\mathbb{R}^d \backslash \partial D$.

Démonstration. Soient $x \in \mathbb{R}^d \backslash \partial D$ et $r > 0$ est assez petit pour avoir $\overline{B}(x,r) \subset D$. Par la propriété de Markov forte du mouvement brownien,

$$\begin{aligned} H_D f(x) &= \mathbb{E}_x[\mathbb{E}[f(W_{\tau_{\partial D}\circ\theta_{\tau_{B(x,r)}}})\mathbb{1}_{\tau_{\partial D}\circ\theta_{\tau_{B(x,r)}}<+\infty}|\mathcal{F}_{\tau_{B(x,r)}}]] \\ &= \mathbb{E}_x\left[\mathbb{E}_{W_{\tau_{B(x,r)}}}[f(W_{\tau_{\partial D}})\mathbb{1}_{\tau_{\partial D}<+\infty}]\right] \\ &= \frac{1}{\sigma(\partial B(x_0,r))}\int_{\partial B(x,r)} H_D f(u)d\sigma(u) \end{aligned}$$

car la densité de sortie d'une boule d'un mouvement brownien issu du centre est uniformément répartie sur le bord de cette boule. Ainsi, $H_D f$ est harmonique par le Théorème A.1. □

Sous de bonnes conditions, une fonction harmonique sur D peut être représentée comme la moyenne de ses valeurs sur ∂D, pondérée par la loi de $B_{\tau_{\partial D}}$. Cela ne peut avoir lieu que si f est continue, mais ce résultat sera généralisé en Section A.5.2 dans le cadre de limites *fines*. Par ailleurs, la géométrie du bord intervient dans l'existence d'une solution, comme le montre l'exemple suivant.

Exemple A.4 (de non existence d'une solution). Soient $d \geq 2$, $D = B(0,1)\backslash\{0\}$ et $f\ :\partial D \longrightarrow \mathbb{R}_+$ donnée par $f(0) = 0$ et $f(x) = 1$ si $|x| = 1$. Alors, le problème de Dirichlet (A.2.2) n'admet pas de solution. En effet, si une telle solution u existe, alors $H_D f = 1$ sur D car le mouvement brownien issu de $x \neq 0$ ne touche presque sûrement pas 0. On ne peut donc pas avoir $u(x) \underset{x\to 0}{\longrightarrow} 0$.

La raison de ce contre-exemple est qu'il ne rentre pas dans le cadre des domaines dits *réguliers*.

Définition A.5 (Domaine régulier)**.** *Un domaine D est dit régulier si, pour tout $z \in \partial D$, $\mathbb{P}_z(\tau_{D^c} = 0) = 1$, i.e. si le mouvement brownien issu de tout point du bord revient instantanément dans D^c.*

Exemple A.6. Les domaines $B(0,1)$, $\mathbb{C}\backslash[0,+\infty)$, sont réguliers. Tout domaine de $\mathbb{R}^d, d \geq 2$ dont le bord est localement une sous-variété C^1 de dimension $d-1$ est régulier par propriété du mouvement brownien. En revanche, $B(0,1)\backslash\{0\}$ ne l'est pas pour $d \geq 2$.

Si le domaine D est régulier, on a le résultat suivant [20, 4.4, Proposition 5].

Théorème A.7 (Solution probabiliste du problème de Dirichlet)**.** *Soient D un ouvert borné régulier (au sens de la Définition A.5) et $f\ :\partial D \longrightarrow \mathbb{R}$ une fonction continue. Alors, la fonction $H_D f$ définie par* (A.2.4) *est l'unique solution de* (A.2.2).

On pourra se référer à [20, Chapitre 4.4] et [34, Chapitre VIII] pour une présentation plus exhaustive de résultats, notamment concernant la régularité au bord de D concernant $H_D f$ lorsque f n'est pas continue.

A.2.2 Fonction harmonique pour un processus de Markov

On décrit dans cette section la classe générale des processus considérés. On présente également les notions de fonction excessive et de fonction harmonique pour le semi-groupe d'un processus de Markov : ce sont ces fonctions que la théorie de la frontière de Martin cherche à classifier.

Cadre des processus

On présente ici la théorie de la frontière de Martin pour des processus dits *standards*. Pour un cadre plus général, on pourra se référer à [20]. Considérons un espace localement compact à base dénombrable E, non compact (pour les résultats de cette thèse $E = \mathbb{R}^2$ ou $E = \mathbb{R}^2_+$), auquel on ajoute un point cimetière ∂ placé topologiquement à l'infini. On note $\mathcal{B}$ la tribu borélienne sur $E \cup \{\partial\}$. Pour toute fonction $x\ :[0,+\infty) \longrightarrow E \cup \{\partial\}$, on note

$$\zeta(x) = \inf(t \geq 0, x_t = \partial)$$

(avec la convention $\inf(\varnothing) = +\infty$). Soit Ω l'ensemble des fonctions $[0, +\infty) \longrightarrow S \cup \{\partial\}$ continues à droite, ayant des limites à gauche sur $(0, \zeta)$ et constantes à ∂ sur $[\zeta, +\infty)$. Remarquons que l'existence de la limite à gauche $x_{\zeta-}$ en ζ n'est pas supposée. On considère la tribu canonique sur Ω donnée par $\mathcal{F} = \sigma\left(\{x_s \in A,\ A \in \mathcal{B},\ 0 \leq s < \zeta(x)\}\right)$ ainsi que la filtration canonique $(\mathcal{F}_t)_{t\geq 0}$ engendrée par les cylindres : $\mathcal{F}_t = \sigma\left(\{x_s \in A,\ A \in \mathcal{B},\ s \leq t < \zeta(x)\}\right)$. On note $\theta_t \ : x \in \Omega \longmapsto (x_{t+s})_{s\geq 0}$ l'opérateur canonique de décalage. Soient $(\mathbb{P}_x)_{x\in E}$ la donnée d'un processus de Markov sur $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ et $(\mathcal{G}, (\mathcal{G}_t)_{t\geq 0})$ la complétion habituelle continue à droite associée à $(\mathbb{P}_x)_{x\in E}$. On note $X \ : \Omega \longrightarrow \Omega$ le processus canonique (donné par l'identité). On suppose alors que le processus est un processus standard au sens de la définition suivante.

Définition A.8 (Processus standard)**.** *On dit qu'un processus de Markov* $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, X, (\mathbb{P}_x)_{x\in E})$ *est standard si les conditions suivantes sont satisfaites :*

- $\mathbb{P}_x - p.s.,\ X_0 = x,$
- *X est un processus de Markov fort*
- *X est quasi-continu à gauche : pour toute suite croissante $(T_n)_{n\geq 0}$ de temps d'arrêt convergeant vers T, $X_{T_n} \underset{n\to+\infty}{\longrightarrow} X_T$ sur $\{T < \zeta\}$ $\mathbb{P}_x - p.s.$ pour tout $x \in E$.*

On note $(P_t)_{t\geq 0}$ le semi-groupe associé au Markov X, i.e. $P_t h(x) = \mathbb{E}_x[h(X_t)\mathbb{1}_{t<\zeta}]$.

Fonction harmonique ou excessive au sens du processus

Définissons tout d'abord les notions de fonctions harmonique et excessive pour le semi-groupe $(P_t)_{t\geq 0}$.

Définition A.9 (Fonction excessive)**.** *Une fonction mesurable positive $h \ : E \longrightarrow [0, +\infty]$ est dite excessive pour le processus de Markov X si $P_t h \leq h$ pour tout $t \geq 0$ et si $P_t h \underset{t\to 0}{\longrightarrow} h$ ponctuellement.*

On pourra se référer à [78, Proposition 3.1] pour plusieurs caractérisations de l'excessivité. En ce qui concerne l'harmonicité, on la définit par la propriété de la moyenne (par analogie à (A.2.1) vue pour le mouvement brownien, voir aussi (A.2.4)).

Définition A.10 (Fonction harmonique)**.** *Une fonction mesurable positive $h \ : E \longrightarrow [0, +\infty]$ est dite harmonique sur un ouvert G de E pour le processus de Markov X si pour tout ouvert U relativement compact dans G et pour tout $x \in E$,*

$$\mathbb{E}_x[h(X_{\tau_{U^c}})\mathbb{1}_{\tau_{U^c}<\zeta}] = h(x). \tag{A.2.5}$$

La fonction h est dite harmonique si elle est harmonique sur E.

On pourra trouver dans certaines références une définition dans laquelle U est pris borélien relativement compact (non nécessairement ouvert) : ces définitions sont en réalité équivalentes [93, Chapitre I.3, Théorème T12]. On peut montrer [78, Proposition 3.4] qu'une fonction harmonique est nécessairement excessive.

Lorsqu'un processus Markovien est récurrent, l'étude des fonctions excessives est triviale car ces fonctions sont toutes constantes (voir [20, Section 3.7]). Il est donc naturel d'imposer la transience du processus pour l'étude des fonctions excessives, dont on rappelle ci-dessous la définition.

Définition A.11 (Transience)**.** *On dit que X est transient si pour tout compact K de E et $x \in E$, X ne revient plus dans K à partir d'un certain temps $\mathbb{P}_x$ presque sûrement, c'est à dire*

$$\sup\{t \geq 0 \ : \ X_t \in K\} < +\infty \quad \mathbb{P}_x - p.s.$$

On pourra trouver dans [59, Corollaire 2.3] et [20, Section 3.7] des caractérisations de la transience pour des processus de Hunt. En particulier, si les fonctions excessives sont semi-continues inférieurement, alors la transience est équivalente à la condition : pour tout compact K de E,

$$G(x,K) \ := \int_0^{+\infty} P_x(X_t \in K)dt < +\infty.$$

Une condition suffisante à la semi-continuité inférieure des fonctions excessives est que les résolvantes $(R_\alpha)_{\alpha>0}$ – définies par $R_\alpha f(x) = \mathbb{E}_x\left[\int_0^{+\infty} e^{-\alpha t} f(X_t)dt\right]$ pour toute fonction f mesurable positive sur E – soient fortement felleriennes (voir [106, Exercice II(2.16)]). Cela signifie que pour tout $\alpha > 0$ et toute fonction f mesurable positive sur E, $R_\alpha f$ est continue sur E. Le lecteur intéressé par des considérations générales de la théorie du potentiel (et, en particulier, les conditions de transience) pourra également se pencher sur la thèse de Baguley [9, Sections 2.4 à 2.6].

On se place à présent sous l'hypothèse où X est transient. Rappelons le lien entre l'harmonicité (au sens de la Définition A.10) et le générateur dans le cas où le processus est de Feller Dynkin dont les résolvantes sont fortement felleriennes. On rappelle qu'un processus de Markov X est de Feller-Dynkin si pour toute fonction f continue sur E tendant vers 0 en ∂, $\sup_{x\in E} |\mathbb{E}_x[f(X_t)] - f(x)| \underset{t\to 0}{\longrightarrow} 0$.

Proposition A.12. *Supposons que X est un processus de Feller Dynkin transient càdlàg dont les résolvantes sont fortement felleriennes. On note $(\mathcal{G}, D(\mathcal{G}))$ son générateur. Soit $h \ : E \longrightarrow [0,+\infty]$ dans $D(\mathcal{G})$. Alors, les assertions suivantes sont équivalentes*

- *(i) h est harmonique au sens de la Définition A.10*
- *(ii) $\mathcal{G}h = 0$*
- *(iii) Pour tout $t \geq 0$ et $x \in E$, $P_t h(x) = h(x)$*

Démonstration. L'équivalence entre (ii) et (iii) est classique et vient de l'équation de Kolmogorov $\frac{d}{dt}P_t = \mathcal{G}P_t = P_t\mathcal{G}$. Rappelons que si Abs désigne l'ensemble des points absorbants pour le processus, on a l'équivalence

$$[f \in D(\mathcal{G}) \text{ et } \mathcal{G}f = g] \Longleftrightarrow$$
$$\left[\forall x \in Abs,\ g(x) = 0 \text{ et } \forall x \in E\backslash Abs,\ \frac{\mathbb{E}_x[f(X_{T_{B(x,\epsilon)^c}})\mathbb{1}_{T_{B(x,\epsilon)^c}<\zeta}] - f(x)}{\mathbb{E}_x[T_{B(x,\epsilon)^c} \wedge \zeta]} \underset{\epsilon\to 0}{\longrightarrow} g(x)\right]$$

où $B(x,\epsilon)$ désigne la boule centrée en x et de rayon ϵ. Ainsi, une fonction satisfaisant (i) satisfait nécessairement (ii). Réciproquement, supposons (ii). Comme X est transient, on a

$$\mathbb{E}_x[T_{K^c}] \leq \mathbb{E}_x\left[\int_0^{+\infty} \mathbb{1}_K(X_t)dt\right] = G(x,K) < +\infty \tag{A.2.6}$$

pour tout compact K de E et tout $x \in E$ car les résolvantes de X sont fortement felleriennes. On peut donc appliquer la formule de Dynkin avec le temps d'arrêt $T_{B(x,\epsilon)^c} \wedge \zeta$ d'espérance finie :

$$\mathbb{E}_x[h(X_t)\mathbb{1}_{t<\zeta}] = h(x) + \mathbb{E}\left[\int_0^{T_{B(x,\epsilon)^c}\wedge\zeta} \mathcal{G}h(X_s)ds\right] = h(x)$$

i.e. h satisfait (i). □

Exemple A.13 (Mouvement brownien avec drift). Soient $d \geq 1$, $\mu \in \mathbb{R}^d$ et W un mouvement brownien d-dimensionnel. Une fonction $h \ : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ est harmonique pour $(W_t + \mu t)_{t\geq 0}$ si et seulement si $\left(\frac{1}{2}\Delta + \mu\nabla\right) h = 0$.

Exemple A.14 (Diffusion dans un domaine). Pour des diffusions sous la forme $dX_t = a(X_t)dW_t + b(X_t)dt$, le générateur s'écrit [102] : $\mathcal{L}h = \left(\frac{1}{2}\nabla \cdot a(x)\nabla + b(x)\nabla\right) h$.

A.3 Fonctions de Green

On commence en Section A.3.1 par introduire la définition analytique de la fonction de Green g_D dans un domaine D. On établit ensuite en Section A.3.2 le lien entre g_D et la mesure d'occupation (ou mesure de Green) du mouvement brownien dans D tué sur le bord de celui-ci. On présente enfin en Section A.3.3 les fonctions de Green probabilistes, dont la définition nécessite des considérations techniques liées à la dualité.

A.3.1 Fonctions de Green en analyse

On présente dans cette section les fonctions de Green analytiques par le prisme du problème de Dirichlet (A.2.2). A un domaine D donné, on définit une fonction $g_D(x,y)$, continue en la variable y sur $\overline{D}\backslash\{x\}$, qui soit solution du problème suivant

$$\begin{cases} \Delta_y g_D(x,\cdot) = \delta_x \\ g_D(x,y) = 0, \qquad y \in \partial D; \end{cases} \tag{A.3.1}$$

dans l'optique de construire ensuite une solution de (A.2.2) à partir de g_D. Cette fonction est appelée *fonction de Green sur D*. On a le résultat suivant [34, Section 1.I.8], dont nous donnerons la preuve en fin de section.

Proposition A.15 (Théorème de représentation avec fonction de Green)**.** *Soit D un ouvert lisse borné telle que g_D existe. Soit $u \in C^2(\overline{D})$ une fonction harmonique sur D. Alors,*

$$u(x) = \int_{\partial D} -\partial_n^y g_D(x,y)u(y)\sigma(dy). \tag{A.3.2}$$

En particulier, toute fonction harmonique est représentée par ses valeurs au bord du domaine via g_D. Avant de justifier la proposition précédente, décrivons la construction de g_D. On définit la fonctions de Green dans $\mathbb{R}^d$ par :

$$g_{\mathbb{R}^d}(x,y) = \begin{cases} \frac{-1}{2\pi}\ln\left(\frac{1}{|x-y|}\right) & \text{si } d = 2 \\ \frac{1}{d(d-2)\alpha(d)}\frac{1}{|x-y|^{d-2}} & \text{si } d \geq 3. \end{cases} \tag{A.3.3}$$

où $\alpha(d)$ désigne le volume de la boule unité $d-$dimensionnelle et où $g_{\mathbb{R}^d}(x,x) = +\infty$. Les fonctions $g_{\mathbb{R}^d}(x,\cdot)$ sont appelées aussi solutions fondamentales du Laplacien car pour tout $x \in \mathbb{R}^d$,

$$\Delta g_{\mathbb{R}^d}(x,\cdot) = \delta_x \tag{A.3.4}$$

au sens des distributions. Supposons que pour tout $x \in D$, il existe une fonction h_D^x harmonique sur D telle que $h_D^x|_{\partial D} = g_{\mathbb{R}^d}(x,\cdot)|_{\partial D}$ (notons que h_D^x est unique par le principe du maximum). Ainsi,

$$g_D(x,y) \;:= g_{\mathbb{R}^2}(x,y) - h_D^x(y) \tag{A.3.5}$$

est solution de (A.3.1), et on définit par cette dernière formule la *fonction de Green sur D*.

Remarque A.16 (Domaine ayant une fonction de Green)*.* Si D est lisse, borné, et régulier au sens de la Définition A.5, le Théorème A.7 fournit une telle fonction h^x pour tout $x \in D$, et l'existence de g_D est assurée. En toute généralité, la notion de fonction de Green pour un ouvert quelconque est donnée par $g_D(x,\cdot) = g_{\mathbb{R}^d}(x,\cdot) - GM_D g_{\mathbb{R}^d}(x,\cdot)$ où $GM_D g_{\mathbb{R}^d}(x,\cdot)$ est la plus grande fonction sous-harmonique sur D bornée par $g_{\mathbb{R}^d}(x,\cdot)$. Ces fonctions $GM_D(x,\cdot)$ existent dans le cadre où D est un ensemble de Green (ou *Greenian set* en anglais, cf [34, Partie 1, ChapitreVII]), ce qui est en particulier le cas si $d \geq 3$ ou si $d = 2$ et que D est borné.

Pour la preuve de la Proposition A.15, on rappelle la formule suivante [34, Section 1.I.8].

Lemme A.17 (Formule de représentation avec fonction de Green)**.** *Soient D un ouvert lisse borné et $u \in C^2(\overline{D})$. Alors,*

$$u(x) = \int_{\partial D} (g_{\mathbb{R}^d}(x,y)\partial_n u(y) - u(y)\partial_n^y g_{\mathbb{R}^d}(x,y))\sigma(dy) - \int_D g_{\mathbb{R}^d}(x,y)\Delta u(y)dy. \tag{A.3.6}$$

De plus, (A.3.6) *est valable en remplaçant $g_{\mathbb{R}^d}$ par $g_{\mathbb{R}^d} - h$ où $h \in C^0(\overline{D})$ est une fonction harmonique sur D.*

Le Lemme A.17 est une conséquence de la formule de Green appliquée aux fonctions u et $g_{\mathbb{R}^d}$ (explicite, donnée par (A.3.3)) et au domaine $D\backslash\overline{B}(x,\delta)$; puis à l'asymptotique $\delta \to 0$. La preuve de la Proposition A.15 en découle directement.

Preuve de la Proposition A.15. Il suffit d'appliquer le Lemme A.17 à la fonction $g_{\mathbb{R}^d} - h_D^x$ qui se trouve être g_D par la relation (A.3.5). □

En général, il est compliqué de calculer la fonction de Green d'un domaine. Cependant, on peut en donner une expression explicite pour certains domaines.

Exemple A.18 (Cas de la boule unité, formule de Poisson). Posons $D = B(0,1)$. Si $x \in D(0,1)$ et $x \neq 0$, on note $x' = \frac{x}{|x|^2}$ l'inversion de x par rapport à la sphère unité. Alors (cf [34], ou par une simple vérification) on a $g_D(x,y) = g_{\mathbb{R}^d}(x,y) - g_{\mathbb{R}^d}(x',y)$; i.e.

$$g_D(x,y) = \begin{cases} \frac{1}{2\pi}\log\frac{|x'-y|\,|x|}{|x-y|} & \text{pour } d=2, \\ \frac{1}{d(d-2)\alpha(d)}\Big(|x-y|^{2-d} - (|x||x'-y|)^{2-d}\Big) & \text{pour } d \geq 3. \end{cases}$$

avec par convention $g_D(x,0) = \log\dfrac{1}{|y|}$ si $d=2$ et $g_D(x,0) = |y|^{2-d} - 1$ si $d \geq 3$. La formule (A.3.2) avec $d \geq 3$ pour une fonction u harmonique sur D et continue sur $\overline{D}$ s'écrit alors

$$u(x) = \frac{1-|x|^2}{d\alpha(d)}\int_{\partial D}\frac{u(y)}{|y-x|^d}\sigma(dy). \tag{A.3.7}$$

La formule analogue pour $d=2$ est également valide.

Exemple A.19 (Cas du demi-espace). Soit $D = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n > 0\}$. Si $x = (x_1,\ldots,x_n) \in D$, on note $\widetilde{x} = (x_1,\ldots,x_{n-1},-x_n)$. De même, on vérifie directement que $g_D(x,y) = g_{\mathbb{R}^d}(x,y) - g_{\mathbb{R}^d}(\widetilde{x},y)$.

On évoque un dernier résultat sur la symétrie des fonctions de Green, qui se démontre avec des considérations élémentaires [40, Section 2.2, Théorème 13]. Ce résultat nous sera utile dans la section suivante.

Lemme A.20 (Symétrie de G_D)**.** *Soit D un domaine borné tel que G_D existe. Alors, $G_D(x,y) = G_D(y,x)$ pour tout $x,y \in D$. En particulier, on a $h_D^x(y) = h_D^y(x)$ pour $x,y \in D$.*

A.3.2 Lien entre fonctions de Green analytiques et brownien tué

La Proposition A.15 et le Théorème A.7 relient la fonction de Green g_D à la loi de $W_{\tau_{\partial D}}$, qui est ainsi donnée par $-\partial_n^y g_D(x,y)\sigma_D(dy)$ (du moins, pour D régulier, lisse et borné). Le but de cette section est de donner un lien plus large entre les aspects analytiques et probabilistes à l'aide de la mesure de Green probabiliste.

Soient W un mouvement brownien dans $\mathbb{R}^d$ où $d \geq 2$, D un domaine de $\mathbb{R}^d$ et soit X le processus W tué lorsqu'il atteint D^c, c'est à dire :

$$X|_{[0,\tau_{D^c})} = W|_{[0,\tau_{D^c})} \quad \text{et} \quad X|_{[\tau_{D^c},+\infty)} = \partial$$

où ∂ désigne le point cimetière et où $\tau_{D^c} = \inf\{t \geq 0 \mid W_t \in D^c\}$. On définit la mesure de Green du mouvement brownien tué sur ∂D, issu de $x \in D$ par :

$$\mathbf{G}_D(x, A) = \mathbb{E}_x\left[\int_0^{+\infty} \mathbb{1}_A(X_t)dt\right] \left(= \mathbb{E}_x\left[\int_0^{\tau_{D^c}} \mathbb{1}_A(W_t)dt\right]\right), \quad A \in \mathcal{B}(D). \tag{A.3.8}$$

Le résultat suivant relie la mesure de Green du mouvement Brownien tué à la fonction de Green analytique.

Proposition A.21 (Fonction de Green et mesure de Green)**.** *On suppose que $D = \mathbb{R}^d$ ou que D est un domaine lisse, régulier, et borné de $\mathbb{R}^d$, $d \geq 3$. Alors, on a*

$$\mathbf{G}_D(x, dy) = d g_D(x, y)\mathbb{1}_D(y)dy \tag{A.3.9}$$

où $g_{\mathbb{R}^d}$ est défini par (A.3.3) *et où g_D est défini par* (A.3.5) *si $D \neq \mathbb{R}^d$.*

Autrement dit, le temps moyen passé en y dans la vie d'un mouvement brownien tué sur ∂D issu de $x \in D$ est de densité $dg_D(x, y)$. La relation (A.3.9) reste valide pour les ensembles de Green (*greenian sets*) D lorsque $d \geq 2$, voir [34, Partie 2, Chapitre IX.17].

Exemple A.22 (Cas $D = \mathbb{R}^d, d \geq 3$.)**.** Si $D = \mathbb{R}^d$, établissons (A.3.9) par un calcul direct. On a en effet

$$\mathbf{G}_{\mathbb{R}^d}(x, dy) = \left(\int_0^{+\infty} p_t(x, y)dt\right) dy$$

où $p_t(x, y) = \frac{e^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{d/2}}$ la densité de transition de W_t. En effectuant le changement de variable $u = \frac{|x-y|^2}{2t}$, on obtient

$$\left(\int_0^{+\infty} p_t(x, y)dt\right) = \int_0^{+\infty} \frac{e^{-u}}{(2\pi)^{d/2}} \frac{(2u)^{d/2}}{|x-y|^d} \frac{|x-y|^2}{2u^2} du = \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}|x-y|^{d-2}} = d \times g_{\mathbb{R}^d}(x, y)$$

puisque $\alpha(d) = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$, ce qui implique directement (A.3.9) lorsque $D = \mathbb{R}^d$.

Preuve de la Proposition A.21. Soient $x \in D$ et A un Borélien de D. Comme le temps d'arrêt τ_{D^c} est fini $\mathbb{P}_x$-presque sûrement (car D est borné), on a en décomposant l'intervalle $[0, +\infty) = [0, \tau_{D^c}) \cup [\tau_{D^c}, +\infty)$:

$$\mathbf{G}_{\mathbb{R}^d}(x, A) = \mathbb{E}_x\left[\int_0^{\tau_{D^c}} \mathbb{1}_A(W_t)dt\right] dy + \mathbb{E}_x\left[\int_{\tau_{D^c}}^{+\infty} \mathbb{1}_A(W_t)dt\right] = \mathbf{G}_D(x, A) + \mathbb{E}_x[\mathbf{G}_{\mathbb{R}^d}(W_{\tau_{D^c}}, A)]$$

par la propriété de Markov forte. Grâce au cas $D = \mathbb{R}^d$ et à la relation précédente, on obtient

$$\mathbf{G}_D(x, A) = d \int_A (g_{\mathbb{R}^d}(x, y) - \mathbb{E}_x[g_{\mathbb{R}^d}(W_{\tau_{\partial D}}, y)])\, dy.$$

La fonction $x \longmapsto \mathbb{E}_x\left[g_{\mathbb{R}^d}(W_{\tau_{\partial D}}, y)\right]$ étant l'unique fonction harmonique sur D qui soit continue sur $\overline{D}$ et égale à $g_{\mathbb{R}^d}(\cdot, y) = g_{\mathbb{R}^d}(y, \cdot)$ sur ∂D (cf. Théorème A.7), elle est égale à $h_D^y(x) = h_D^x(y)$ par le Lemme A.20. On en déduit ainsi la relation (A.3.9) par la définition de g_D donnée par (A.3.5). ☐

A.3.3 Dualité et fonctions de Green probabilistes

Dans cette section, on définit les fonctions de Green probabilistes $g(x, y)$ comme les densités de la mesure de Green par rapport à une mesure m, par analogie avec la relation (A.3.9). Toutefois, cette définition de $g(x, \cdot)$ n'est valable que $m-$presque partout et ne peut donc pas être utilisée telle quelle. On introduit ainsi une hypothèse technique, nécessaire à la bonne définition de g et à la construction de la frontière de Martin (voir Kunita et Watanabe [78]). Cette hypothèse permet notamment d'assurer l'harmonicité de $g(\cdot, y)$ sur $E \setminus \{y\}$ pour tout $y \in E$. Formellement, on justifie la bonne définition de g en écrivant $g(x, y) = \widetilde{g}(y, x)$ où $\widetilde{g}$ désigne la fonction de Green d'un processus dual. En réalité, il suffit d'une hypothèse plus faible, portant sur l'existence d'une co-résolvante.

On présente les notions de dualité correspondantes en Section A.3.3.1 puis la bonne définition des fonctions de Green en Section A.3.3.2. Enfin, on démontre en Section A.3.3.3 l'harmonicité de $g(\cdot, y)$ sur $E\setminus\{y\}$ pour tout $y \in E$.

A.3.3.1 Co-résolvante et processus dual

Avant d'introduire le concept de processus dual (voir [78, Section 4], [109, Chapitre VII.4] ou [20, Chapitre 13]), on définit d'abord la notion de résolvante et de co-résolvante.

Définition A.23 (Résolvante)**.** *Une famille d'applications* $R_\alpha \ : E \times \mathcal{B} \longrightarrow [0, +\infty]$ *indexée par* $\alpha > 0$ *est appelée résolvante si*

- *Pour tout* $\alpha > 0$ *et* $x \in E$, $R_\alpha(x, \cdot)$ *est une mesure finie sur les compacts.*
- *Pour tout* $\alpha > 0$ *et* $f \ : E \longrightarrow \mathbb{R}_+$ *est mesurable bornée,* $R_\alpha f \ := \int_E f(y) R_\alpha(\cdot, dy)$ *est mesurable et bornée sur les compacts.*
- *Pour tout* $\alpha, \beta > 0$ *et* f *mesurable bornée, on a (équation de la résolvante)*

$$R_\alpha f - R_\beta f + (\alpha - \beta) R_\alpha R_\beta f = 0$$

- *Pour tout* $x \in E$ *et* f *mesurable bornée,* $R_\alpha f(x) \underset{\alpha \to +\infty}{\longrightarrow} 0$.

En particulier, si X est un processus de Markov, la famille $(G_\alpha)_{\alpha>0}$ définie par

$$G_\alpha(x, A) = \int_0^{+\infty} e^{-\alpha t} \mathbb{P}_x(X_t \in A) dt, \quad A \in \mathcal{B} \tag{A.3.10}$$

est une famille de résolvantes. Avec l'équation de la résolvante, on voit que $R_\alpha(x, \cdot)$ est décroissant en α et que la limite croissante $R_0(x, \cdot) \ := \lim_{\alpha \to 0} R_\alpha(x, \cdot)$ est bien définie. On note également

$$G(x, \cdot) \ := G_0(x, \cdot) = \int_0^{+\infty} \mathbb{P}_x(X_t \in \cdot) dt, \tag{A.3.11}$$

cette mesure est appelée mesure de Green du processus X.

Définition A.24 (Co-résolvante)**.** *Soit* m *une mesure sur* E *et* $(G_\alpha(\cdot, \cdot))_{\alpha>0}$ *la résolvante définie par* (A.3.10)*. Une résolvante* $(\hat{G}_\alpha(\cdot, \cdot))_{\alpha>0}$ *est appelée co-résolvante de* $(G_\alpha(\cdot, \cdot))_{\alpha>0}$ *par rapport à la mesure* m *si pour tout* $\alpha > 0$ *et* $f, g \ : E \longrightarrow \mathbb{R}$ *continues à support compact,*

$$\langle f, G_\alpha g \rangle_m = \langle \hat{G}_\alpha f, g \rangle_m.$$

En particulier, si deux processus de Markov X et Y sur E sont tels que leurs semi-groupes respectifs $(P_t)_{t \geq 0}$ et $(Q_t)_{t \geq 0}$ satisfont la propriété suivante : pour toute fonction f continue à support compact et $t \geq 0$,

$$\int_E P_t f(x) g(x) m(dx) = \int_E f(x) Q_t g(x) m(dx),$$

alors les deux résolvantes canoniquement associées sont en dualité par rapport à la mesure m. Dans ce cas, on dit que *les processus sont en dualité* par rapport à la mesure m.

Exemple A.25. Le mouvement brownien dans un domaine D tué sur le bord de D (potentiellement $D = \mathbb{R}^d$), est auto-dual par rapport à la mesure de Lebesgue. Il s'agit d'une conséquence directe de la symétrie de la densité de transition.

On peut donner une interprétation trajectorielle au processus dual. Les trajectoires du processus dual s'interprètent en effet comme les trajectoires du processus X renversées en temps sur un intervalle $[0, L]$ [109, Section VII.4] où L est un temps dit *co-optionnel* (*co-optional time*). On renvoie le lecteur à [109, Chapitre VII.4] et [20, Chapitre 13] pour plus de détails.

Exemple A.26 (Dual du mouvement brownien réfléchi dans un cône). Soit X un mouvement brownien réfléchi (sans drift) dans un cône circulaire, avec covariance $\Sigma = Id$, champ de réflexion radial, et tué au sommet du cône. Deblassie [28] montre que le processus dual de X par rapport à la mesure de Lebesgue est un mouvement brownien réfléchi avec drift, dont le champ de vecteurs est réfléchi selon la normale au cône.

A.3.3.2 Densité des mesures de Green

On rappelle les définitions des résolvantes $(G_\alpha)_{\alpha>0}$ et de la fonction de Green données par (A.3.10) et (A.3.11). On suppose qu'il existe une mesure m telle que $G(x,\cdot)$ admette une densité $g(x,\cdot)$ sur E par rapport à m pour tout $x \in E$. Le choix de cette densité, a priori pas unique (car définie $m-$presque partout) est précisé par l'hypothèse de dualité.

Hypothèse A.27. Il existe une mesure m sur E et une co-résolvante $(\hat{G}_\alpha(\cdot,\cdot))_{\alpha>0}$ à $(G_\alpha(\cdot,\cdot))_{\alpha>0}$ par rapport à la mesure m telles que les propriétés suivantes soient satisfaites.

1. $G(x,\cdot)$ est dominé par m pour tout $x \in E$
2. Pour tout compact K de E, $G(\cdot, K)$ est borné
3. Pour toute fonction f continue sur E à support compact,
$$\alpha \hat{G}_\alpha f \underset{\alpha\to+\infty}{\longrightarrow} f$$
au sens de la convergence uniforme sur tout compact.
4. Pour toute fonction $f \ : E \longrightarrow \mathbb{R}$ borélienne bornée à support compact et $\alpha \geq 0$, $\hat{G}_\alpha f$ est continue et finie (partout).

Sous cette hypothèse, la mesure m charge tous les ouverts. En effet, si $x \in E$, $G(x,\cdot)$ charge tout voisinage de x par continuité à droite des trajectoires : il en est donc de même pour m par domination. Notons que, sous de bonnes hypothèses [59, Corollaire 2.3], le point 2 n'est rien d'autre que la transience de X. On se place à présent sous l'Hypothèse A.27.

Pour définir la fonction de Green $g(x,y)$ de manière rigoureuse, on introduit la notion de fonction co-excessive. On peut démontrer [78] qu'une fonction mesurable positive $f \ : E \longrightarrow \mathbb{R}_+$ est excessive si $G_\alpha f \ := \int_E f(z) G_\alpha(\cdot, dz)$ satisfait $\alpha G_\alpha f \leq f$ et $\alpha G_\alpha f \underset{\alpha\to+\infty}{\longrightarrow} f$. Par symétrie, on dit qu'une fonction f mesurable positive est *co-excessive* si $\hat{G}_\alpha f \ := \int_E f(z) \hat{G}_\alpha(\cdot, dz)$ satisfait $\alpha \hat{G}_\alpha f \leq f$ et $\alpha \hat{G}_\alpha f \underset{\alpha\to+\infty}{\longrightarrow} f$. On pourra se référer à Hunt [69] pour des propriétés usuelles des fonctions excessives et co-excessives. Par exemple, toute fonction excessive ou co-excessive est semi-continue inférieurement.

Proposition A.28 (Définition de $g(x,y)$ [78]). *On se place sous l'Hypothèse A.27; en particulier $G(x,\cdot)$ est dominé par la mesure m pour tout $x \in E$. Alors, il en est de même pour $\hat{G}_\alpha(dx, y)$ pour tout $y \in E$. De plus, pour tout $\alpha \geq 0$ (0 inclus) il existe une unique fonction $g_\alpha \ : E \times E \longrightarrow [0, +\infty]$ telle que*

- $G_\alpha(x, dy) = g_\alpha(x, y) m(dy)$
- $\hat{G}_\alpha(y, dx) = g_\alpha(x, y) m(dx)$

- $g_\alpha(\cdot, y)$ *est* α*-excessive pour tout* y
- $g_\alpha(x, \cdot)$ *est* α *co-excessive pour tout* x.

On appelle alors *fonction de Green du processus* X *pour la mesure* m la fonction

$$g \;: (x,y) \in E \times E \longmapsto g_0(x,y) =: g(x,y). \tag{A.3.12}$$

La fonction g est ainsi bien définie sur $E \times E$, est semi-continue inférieurement en chaque variable et satisfait $G(x,dy) = g(x,y)m(dy)$. Le lemme suivant [93, Lemme T5, Chapitre III.1] justifie l'unicité de la fonction de Green et nous sera utile dans la Section A.3.3.3.

Lemme A.29 (Fonctions excessives égales m-presque partout)**.** *On suppose l'Hypothèse A.27.*

(i) Soient u *et* v *deux fonctions excessives égales* m*-presque partout. Alors,* $u = v$ *partout.*

(ii) Soient u *et* v *deux fonctions co-excessives égales* m*-presque partout sur un ouvert* U *de* E*. Alors,* $u = v$ *partout sur* U*.*

Démonstration. Prouvons tout d'abord (i), considérons u et v deux fonctions excessives égales m-p.p. sur E. Si $\alpha \geq 0$ et $x \in E$,

$$\alpha G_\alpha u(x) = \alpha \int_E u(z) g_\alpha(x,z) m(dz) = \alpha \int_E v(z) g_\alpha(x,z) m(dz) = \alpha G_\alpha v(x).$$

On obtient le résultat en faisant tendre α vers $+\infty$. Pour (ii), considérons deux fonctions u et v co-excessives égales m-p.p. sur U. Par l'Hypothèse A.27, $\alpha \hat{G}_\alpha(x,\cdot)$ converge vaguement vers le Dirac δ_x pour tout $x \in E$. Par le théorème du portemanteau, étant donné que $u\mathbb{1}_U$ est semi-continue inférieurement, pour tout $x \in U$ on a

$$u(x) \leq \liminf_{\alpha \to +\infty} \alpha \hat{G}_\alpha(u\mathbb{1}_U)(x).$$

La fonction u étant co-excessive, on a

$$\liminf_{\alpha \to +\infty} \alpha \hat{G}_\alpha(u\mathbb{1}_U)(x) \leq \lim_{\alpha \to +\infty} \alpha \hat{G}_\alpha u(x) = u(x)$$

donc les inégalités sont des égalités. La quantité $\alpha \hat{G}_\alpha(u\mathbb{1}_U)(x)$ ne dépendant que de la classe de $u\mathbb{1}_U$ $m-$presque partout (même preuve que le cas (i)), on en déduit que $u\mathbb{1}_U = v\mathbb{1}_U$ partout, ce qui conclut la preuve. □

A.3.3.3 Harmonicité de $g(\cdot, y)$ sur $E\backslash\{y\}$.

On présente le résultat clef suivant [78, Proposition 6.2] sur les fonctions de Green, définies par (A.3.12).

Proposition A.30 (Harmonicité des fonctions de Green)**.** *La fonction excessive* $g(\cdot, y)$ *est harmonique sur* $E\backslash\{y\}$*.*

Pour démontrer la Proposition A.30, on établit le lemme suivant [78, Proposition 6.2].

Lemme A.31 (Propriété de la moyenne pour $g(\cdot, z)$)**.** *Si* $u \;: E \longrightarrow \mathbb{R}_+$ *est mesurable positive et si* U *est un ouvert de* E*, on note* $H_U u(x) = \mathbb{E}_x[\mathbb{1}_{\tau_U < +\infty} u(X_{\tau_U})]$ *pour* $x \in E$ *et* $H_U(x, dy)$ *la mesure canoniquement associée. Alors, pour tout* $x \in E$ *et* $z \in U$*,*

$$g(x,z) = \int_{y \in E} g(y,z) H_U(x,dy). \tag{A.3.13}$$

Démonstration. Fixons $x \in E$. Soit $f : E \longrightarrow \mathbb{R}_+$ une fonction mesurable positive à support dans U. Alors, d'une part, par le théorème de Fubini,

$$H_U[Gf](x) = \int_E Gf(y)H_U(x,dy) = \int_E \left(\int_E g(y,z)H_U(x,dy) \right) f(z)m(dz).$$

D'autre part, par la propriété de Markov forte,

$$\begin{aligned} H_F[Gf](x) &= \mathbb{E}_x[Gf(X_{\tau_U})\mathbb{1}_{\tau_U<+\infty}] = \mathbb{E}_x \left[\mathbb{E}_{X_{\tau_U}} \left[\int_0^{+\infty} f(X_t)dt \right] \mathbb{1}_{\tau_U<+\infty} \right] \\ &= \mathbb{E}_x \left[\int_{X_{\tau_U}}^{+\infty} f(X_t)dt\mathbb{1}_{\tau_U<+\infty} \right] = \mathbb{E}_x \left[\int_0^{+\infty} f(X_t)dt\mathbb{1}_{\tau_U<+\infty} \right] \\ &= Gf(x) = \int_E g(x,z)f(z)m(dz) \end{aligned}$$

où la quatrième égalité est justifiée par le fait que f est à support dans U. Comme f est prise de manière arbitraire (à support dans U), l'égalité (A.3.13) est valide pour $m-$presque tout $z \in U$. Les deux fonctions étant co-excessives sur l'ouvert U (voir Proposition A.28 et [78]), elles sont égales sur U par le Lemme A.29. □

Preuve de la Proposition A.30. Soit $y \in E$ et A un compact ne contenant pas x. En posant $A = U^c$, l'égalité (A.3.13) se réécrit $g(x,y) = \mathbb{E}_x[g(X_{\tau_{A^c}},y)\mathbb{1}_{\tau_{A^c}<+\infty}]$, ce qui implique l'harmonicité de $g(\cdot,y)$ sur $E\backslash\{y\}$. □

A.4 Frontière de Martin

En Section A.4.1, on présente la construction théorique de la frontière de Martin, fondée sur les fonctions de Green probabilistes introduites en Section A.3.3. On décrit ensuite en Section A.4.2 la méthode pratique permettant de déterminer cette frontière à partir de l'asymptotique des fonctions de Green, en l'illustrant par l'exemple du mouvement brownien avec drift. Enfin, on énonce en Section A.4.3 le théorème central de la théorie, à savoir le théorème de représentation des fonctions excessives.

A.4.1 Compactification de Martin

On expose à présent la construction de la frontière de Martin comme elle est décrite dans l'article [78]. Pour cela, on part des fonctions de Green et de l'heuristique suivante, faisant suite à la Proposition A.30 : les fonctions $g(\cdot,y)$ étant harmoniques sur $E\backslash\{y\}$, on peut espérer obtenir – après renormalisation – des fonctions harmoniques en envoyant y “vers l'infini” (au sens de sortir de tout compact). Cela conduit à la définition du noyau de Martin.

Définition A.32 (Mesure de référence et noyau de Martin)**.** *Soit r une mesure borélienne sur E. On note $r\hat{G} := \int_E g(x,\cdot)r(dx) : E \longrightarrow [0,+\infty]$. On dit que r est une mesure de référence lorsque $r\hat{G}$ est continue (pour la topologie de $[0,+\infty]$ à l'arrivée) et strictement positive sur E. On appelle noyau de Martin pour la mesure de référence r la quantité suivante :*

$$K(x,y) = \begin{cases} \frac{g(x,y)}{r\hat{G}(y)} & \text{si} \quad r\hat{G}(y) < +\infty \\ 0 & \text{si} \quad r\hat{G}(y) = +\infty. \end{cases} \tag{A.4.1}$$

En pratique, on prendra $r = \delta_{x_0}$ pour un x_0 fixé dans E, c'est à dire $r\hat{G}(y) = g(x_0,y)$. On discute dans la Remarque A.39 de l'existence et du choix d'une telle mesure de référence.

L'idée de la frontière de Martin est alors de compactifier l'espace d'états E pour construire les "points à l'infini", correspondant chacun à une limite du noyau de Martin K. Pour cela, définissons une métrique sur E avec le noyau de Martin.

Désignons par $C_c(E)$ l'ensemble des fonctions continues à support compact de E dans $\mathbb{R}$. Pour $f \in C_c(E)$, on note f_K la fonction définie par $f_K := \int_E f(x)K(x,\cdot)m(dx)$. Notons que $f_K = \frac{\hat{G}_0 f}{r\hat{G}}$ où r est une mesure de référence, donc f_K est continue par le point 3 de l'Hypothèse A.27.

Soit $(f^n)_{n\geq 0}$ une suite de fonctions de $C_c(E)$ qui soit dense dans $C_c(E)$ pour la norme uniforme et soit ρ la *distance de Martin* définie par

$$\rho(y,y') = \rho_1(y,y') + \sum_{n=0}^{+\infty} \frac{\min(1, |f_K^n(y) - f_K^n(y')|)}{2^n} \tag{A.4.2}$$

où ρ_1 est la distance de la compactification d'Alexandroff (ou *one point compactification*). On définit la *compactification de Martin M de E* par la complétion de E pour la métrique ρ. Autrement dit, $M := F/\sim$ où F désigne l'ensemble des suites de Cauchy de E pour la distance ρ et où $\sim$ est la relation d'équivalence définie par :

$$(y_n)_{n\geq 0} \sim (y'_n)_{n\geq 0} \iff \rho(y_n, y'_n) \underset{n\to+\infty}{\longrightarrow} 0.$$

Ainsi, une suite $(y_n)_{n\geq 0}$ de E converge dans M si elle converge dans E ou si elle tend vers l'infini et que pour tout $f \in C_c(E)$, $(f_K(y_n))_{n\geq 0}$ converge. Par [78, Théorème 4], on a le théorème suivant.

Théorème A.33 (Frontière de Martin)**.** *L'injection (continue) $E \hookrightarrow M$ construite satisfait les propriétés suivantes*

1. *M est compact.*
2. *E est dense dans M pour la métrique ρ.*
3. *A tout point $\eta \in M\backslash E$ correspond une fonction excessive $K(\cdot,\eta) : E \longrightarrow [0,+\infty]$. De plus, si η, η' sont distincts dans $M\backslash E$, il existe $x \in E$ tel que $K(x,\eta) \neq K(x,\eta')$.*
4. *Si $\eta \in M\backslash E$ et si $(y_n)_{n\geq 0}$ est une suite de E convergente vers η, alors $K(x,y_n) \underset{n\to+\infty}{\longrightarrow} K(x,\eta)$ pour tout $x \in E$.*

De plus, la compactification M est caractérisée par les propriétés 1, 2, 3 et (A.4.3) donnée par

$$\forall f \in C_c(E),\, \forall \eta \in M, \qquad \int_E K(x,y)f(x)m(dx) \underset{\substack{y\in E \\ y \underset{\rho}{\to} \eta}}{\longrightarrow} \int_E K(x,\eta)f(x)m(dx). \tag{A.4.3}$$

On appelle M la compactification de Martin et $\partial M := M\backslash E$ la frontière de Martin. Les fonctions $K(\cdot,\eta)$ pour $\eta \in \partial M$ sont appelées fonctions de Martin.

Remarque A.34 (Harmonicité des noyaux de Martin)*.* En général, les fonctions $K(\cdot,\eta)$ pour $\eta \in \partial M$ ne sont pas nécessairement toutes harmoniques. Cependant, si l'on montre que pour tout A compact de E, $K(x,y)$ est uniformément borné en $x \in A$ lorsque $y \to \eta$ (voir par exemple l'inégalité de Harnack [102, Théorèmes 4.0.1 et 4.3.1]), alors $K(\cdot,\eta)$ est harmonique sur E. En effet, soit U un ouvert relativement compact dans E. Par la Proposition A.30, si $y \in E\backslash U$ et si $x \in E$ est distinct de y, alors l'harmonicité de $K(\cdot,y)$ sur $E\backslash\{y\}$ s'écrit

$$K(x,y) = \mathbb{E}_x\left[K(X_{\tau_{U^c}}, y)\right].$$

On peut alors appliquer le théorème de convergence dominée avec $y \to \eta$ dans l'équation précédente pour obtenir l'harmonicité de $K(\cdot,\eta)$.

A.4.2 Calcul de la frontière de Martin

Dans cette thèse, la description de la frontière de Martin s'obtient en analysant l'asymptotique des fonctions de Green, et donc celle du noyau de Martin K. Résumons la méthode pratique.

(i) Vérifier l'hypothèse technique A.27 de la Section A.3.3.2. Dans le cadre de notre étude, la mesure m sera toujours la mesure de Lebesgue (ou une transformation absolument continue dans le cas de la barrière perméable lorsque $q \neq q_0$, cf. Proposition 1.9).

(ii) Poser une mesure de référence r au sens de la Définition A.32 ($r = \delta_{x_0}$ dans le cadre de cette thèse, avec $x_0 \in E$) .

(iii) Étudier l'asymptotique des fonctions de Green le long de tous les chemins qui sortent de tout compact.

(iv) En déduire via l'asymptotique $y \to \infty$ du noyau de Martin $K(x,y)$ un ensemble Γ et ensemble de fonctions limites $(K(\cdot,\eta))_{\eta\in\Gamma}$ (avec $\eta \in \Gamma \longmapsto K(\cdot,\eta)$ injective). Montrer que pour tout $x \in E$, $K(x,y) \longrightarrow K(x,\eta)$ lorsque y tend vers l'infini dans une direction $\alpha = \alpha(\eta)$ (on note $y \to \eta$). Justifier que, pour toute fonction $f \in C_c(E)$, on a

$$\int_E K(x,y)f(x)m(dx) \underset{\substack{y\in E\\ y\to\eta}}{\longrightarrow} \int_E K(x,\eta)f(x)m(dx).$$

Pour cela, il suffit de montrer que, pour tout compact A de E et tout $\eta \in \Gamma$,

$$\sup_{x\in A} |K(x,y) - K(x,\eta)| \underset{\substack{y\in E\\ y\to\eta}}{\longrightarrow} 0. \tag{A.4.4}$$

Une autre approche consiste à établir que pour tout compact A et toute suite $(y_n)_{n\geq 0}$ de E telle que $y_n \underset{n\to+\infty}{\longrightarrow} \eta$, la famille $(K(x,y_n))_{x\in A, n\geq n_0}$ est bornée.

Avec la construction ci-dessus, les fonctions $(K(\cdot,\eta))_{\eta\in\Gamma}$ forment l'ensemble des fonctions de Martin. De plus, l'espace Γ, muni de la topologie induite par $\eta \in \Gamma \longmapsto K(\cdot,\eta)$ pour la topologie de la convergence simple des fonctions, est la frontière de Martin ∂M.

Exemple A.35 (Mouvement brownien drifté sur $\mathbb{R}^d$). Soient $d \geq 1$, $(W_t)_{t\geq 0}$ un mouvement brownien standard $d-$dimensionnel et $\mu \in \mathbb{R}^d\backslash\{0\}$. On considère le mouvement brownien de drift μ donné par $(X_t)_{t\geq 0} = (W_t + \mu t)_{t\geq 0}$.

(i) Le processus X satisfait les hypothèses techniques, on peut remarquer par exemple que le processus dual de X est un mouvement brownien de drift $-\mu$.

(ii)-(iii) La densité de transition étant explicite, la fonction de Green s'écrit :

$$\begin{aligned} g(x,y) &= \int_0^\infty e^{-\frac{||y-x-\mu t||^2}{2t}} \frac{dt}{2\pi t^{d/2}} = e^{\langle \mu, y-x\rangle} \int_0^\infty e^{-\frac{1}{2}\left(||\mu||^2 t + \frac{||y-x||^2}{t}\right)} \frac{dt}{(2\pi t)^{d/2}} \\ &= \frac{e^{\langle \mu, y-x\rangle}}{(2\pi)^{d/2}(||\mu||.||y-x||)^{\frac{d}{2}-1}} \int_0^{+\infty} \frac{e^{-\frac{1}{2}||\mu||.||y-x||(u+u^{-1})}}{u^{d/2}} du \end{aligned}$$

via le changement de variables $u = \frac{||\mu||}{||y-x||}t$. On choisit $x_0 = (0,0)$ et $r = \delta_{x_0}$, on peut vérifier directement qu'il s'agit d'une mesure de référence. Par une méthode de Laplace (voir [32, Chapitre IV] ou Section 1.2.3.6), on obtient

$$g(x,y) \underset{||y||\to+\infty}{\sim} \frac{e^{-||\mu||.||x-y|| + \langle \mu, y-x\rangle}}{(2\pi||\mu||.||x-y||)^{\frac{d-1}{2}}} \tag{A.4.5}$$

(iv) Par (A.4.5), on a l'asymptotique $K(x,y) \sim_{||y|| \to +\infty} e^{-\langle \mu + ||\mu|| \frac{y}{||y||}, x \rangle}$ et la propriété (A.4.4) peut être facilement vérifiée. L'application

$$u \in \mathbb{S}^{d-1} \longmapsto [h_u \ : x \mapsto e^{-\langle \mu + ||\mu|| u, x \rangle}] \tag{A.4.6}$$

étant continue (pour la topologie de la convergence simple à l'arrivée), on en déduit que la frontière de Martin est homéomorphe à la sphère unité $\mathbb{S}^{d-1}$ de $\mathbb{R}^d$.

A.4.3 Représentation des fonctions excessives et harmoniques

Nous avons précédemment introduit les fonctions excessives $K(\cdot, y)$, $y \in M$ (voir Propositions A.30 et Théorème A.33). L'idée centrale de la théorie de la frontière de Martin est que toute fonction excessive peut s'exprimer de manière unique comme l'intégrale de ces fonctions sur un sous-ensemble de $E \cup \partial M$. Pour garantir l'unicité de cette représentation, nous introduisons la notion de fonction minimale.

Définition A.36 (Fonction minimale)**.** *Une fonction excessive $h \ : E \longrightarrow [0, +\infty]$ est dite minimale si pour toute décomposition $h = h_1 + h_2$ avec h_1, h_2 excessives, il existe $\alpha, \beta \geq 0$ tels que $h_1 = \alpha h$ et $h_2 = \beta h$.*

On note $\partial_{min} M$ le sous-ensemble des points de la frontière de Martin tels que la fonction $K(\cdot, \eta)$ est harmonique et minimale. On note également E' le sous-ensemble des points $y \in E$ tels que $g(\cdot, y)$ est excessive mais non harmonique – ces fonctions sont par ailleurs minimales (voir [78, Section 11]). Par une approche directe [78, Théorème 4] ou en utilisant la théorie de Choquet [93], on a le théorème de représentation suivant.

Théorème A.37. *Soit h une fonction excessive sur E telle que $h \in L^1(r)$. Alors, il existe une unique mesure borélienne finie ν^h sur $\partial_{min} M \cup E'$ telle que*

$$\forall x \in E, \quad h(x) = \int_{\partial_{min} M \cup E'} K(x, \eta) d\nu^h(\eta). \tag{A.4.7}$$

De plus, toute fonction de la forme (A.4.7) est excessive. Enfin, toute fonction de la forme (A.4.7) est harmonique si et seulement si $\nu^h(E') = 0$.

Remarque A.38 (Points de $E \backslash E'$). En réalité, les fonctions $g(\cdot, y)$ avec $y \in E \backslash E'$ coïncident avec une fonction $K(\cdot, \eta)$ où $\eta \in \partial M_{min}$ (cf. [78, Section 12]). Si la co-résolvante est la résolvante d'un processus standard, alors $E' = E$ [78, Proposition 13.1].

Remarque A.39 (Existence et dépendance de la mesure r). Si h est une fonction excessive, il existe toujours une mesure de référence r telle que h soit r-intégrable (voir [93, Théorème 8, Chapitre II.1]). Soient r et w deux mesures de références et K_r et K_w les noyaux de Martin associés. Justifions que les frontières de Martin Γ_r, Γ_w correspondantes sont égales sous de bonnes hypothèses. Par un calcul direct, on a

$$K_r(x, y) = K_w(x, y) \int_E K_r(u, y) w(du). \tag{A.4.8}$$

Soit $(y_n)_{n \geq 0}$ une suite de E telle que $y_n \underset{n \to +\infty}{\longrightarrow} \eta \in \Gamma_r$ pour la topologie induite par la compactification par K_r. Alors, au moins formellement, (A.4.8) donne, pour tout $x \in E$,

$$K_w(x, y_n) \underset{n \to +\infty}{\longrightarrow} K_r(x, \eta) \left(\int_E K_r(u, \eta) w(du) \right)^{-1}. \tag{A.4.9}$$

La convergence précédente est en particulier vraie si r et w sont des Diracs, ce qui est le cas dans cette thèse. Ainsi, $(y_n)_{n \geq 0}$ converge pour la topologie induite par la compactification par K_w, et la fonction excessive limite est la même que pour $\eta \in \Gamma_r$ à une constante multiplicative près. Sous réserve de (A.4.9), les compactifications associées à r et w sont donc les mêmes.

Reprenons l'Exemple A.35 du mouvement brownien drifté et montrons que la frontière de Martin est minimale.

Exemple A.40 (Minimalité de la frontière de Martin du brownien drifté). On rappelle l'expression (A.4.6) des fonctions harmoniques h_u issues de la compactification de Martin. Pour montrer la minimalité de ∂M, on passe par le fait que si $u \neq v$ dans $\mathbb{S}^{d-1}$, alors $h_u(rv)$ est asymptotiquement négligeable devant $h_v(rv)$ lorsque $r \to +\infty$. Soit $u \in \mathbb{S}^{d-1}$. On va montrer que s'il existe une mesure borélienne ν sur $\mathbb{S}^{d-1}$ telle que $h_u = \int_{\mathbb{S}^{d-1}} h_v d\nu(v)$, alors $\nu = \delta_u$: cela implique directement la minimalité de ∂M. Soit ν une telle mesure. Soit V un ouvert de $\mathbb{S}^{d-1}$ tel que $d(u, V) > 0$. On a

$$h_u(ru) \geq \int_V h_v(ru) d\nu(v).$$

Or, par propriété du produit scalaire, il existe $\epsilon > 0$ tel que $\langle u, v\rangle \leq 1 - \epsilon$ pour tout $v \in V$. Ainsi, par (A.4.6),

$$e^{-r||\mu||} \geq \int_V e^{-r||\mu||\langle v,u\rangle} d\nu(v) \geq e^{-r||\mu||(1-\epsilon)}\nu(V).$$

On a donc $\nu(V) \leq e^{-r\epsilon||\mu||}$ pour tout $r \geq 0$, c'est à dire $\nu(V) = 0$. Ainsi, *supp*$(\nu) = \{u\}$ et donc $\nu = c\delta_u$ avec $c = 1$ par une simple vérification. La frontière de Martin est donc minimale.

A.5 Transformée de Doob et point terminal

La notion de fonction excessive est étroitement liée à celle de conditionnement et plus précisément à la h-transformée de Doob. Le théorème de représentation des fonctions excessives (Théorème A.37) permet ainsi d'identifier, dans un certain sens, l'ensemble des conditionnements possibles du processus de Markov considéré. Ce théorème admet en outre une interprétation probabiliste en termes de point terminal du processus. Pour cette raison, nous présentons en Section A.5.1 la notion de h-conditionnement de Doob et en Section A.5.2 l'interprétation probabiliste en question.

A.5.1 h-transformée de Doob

Si h est une fonction excessive, elle peut être utilisée pour conditionner le processus de Markov et obtenir un semi-groupe sous-markovien. On rappelle ici les éléments fondamentaux de la h-transformée de Doob, une description plus complète y est détaillée dans Chung et Walsh [20, Section 11]. On présente également le cas des diffusions, pour lesquelles la h-transformée s'interprète comme l'ajout d'un drift dans l'EDS correspondante.

Définition A.41 ($h-$transformée de Doob). *Soit h une fonction excessive pour un semi-groupe sous-markovien $(P_t)_{t\geq 0}$, et soit $E_h := \{x \in E \mid 0 < h(x) < +\infty\}$. On appelle $h-$transformée de Doob du noyau $(P_t)_{t\geq 0}$ le noyau $(P_t^h)_{t\geq 0}$ défini par :*

$$P_t^h(x, dy) = \begin{cases} \dfrac{h(y)}{h(x)} P_t(x, dy) & si\ x \in E_h \\ \delta_x(dy)e^{-t} & si\ x \in E - E_h. \end{cases} \tag{A.5.1}$$

On a alors le résultat suivant [78, Section 12].

Proposition A.42 (Propriété de Markov de la $h-$transformée de Doob). *Si X est un processus standard et si h est une fonction excessive pour X, alors le noyau $(P_t^h)_{t\geq 0}$ forme un semi-groupe sous-markovien, et le processus X^h induit est un processus standard.*

Formellement, le générateur $\mathcal{G}^h$ associé au semi-groupe $(P_t^h)_{t\geq 0}$ s'écrit :

$$\mathcal{G}^h f(x) = \lim_{t\to 0} \frac{h(x)^{-1}\mathbb{E}_x[h(X_t)f(X_t)] - f(x)}{t} = \frac{1}{h(x)} \lim_{t\to 0} \frac{\mathbb{E}_x[h(X_t)f(X_t)] - h(x)f(x)}{t} = \frac{1}{h(x)}\mathcal{G}(hf)(x).$$

Ce dernier calcul est rendu légitime pour les diffusions [109, Section VIII, Proposition 3.9].

Exemple des diffusions sur $\mathbb{R}^d$

Pour les diffusions sur $\mathbb{R}^d$, la h-transformée de Doob par une fonction harmonique peut être interprétée comme l'ajout d'un drift. En effet, soit $\mathcal{G} = \frac{1}{2}\sum_{1\leq i,j\leq d} a_{i,j}\partial_i\partial_j + \sum_{i=1}^d b_i\partial_i$ le générateur d'une diffusion sur $\mathbb{R}^d$ donnée par

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt \tag{A.5.2}$$

où W est un mouvement brownien et où σ, b sont continus bornés et où σ est uniformément elliptique. Soit h une fonction harmonique pour X, i.e. $\mathcal{G}h = 0$. On a :

$$\begin{aligned}\mathcal{G}^h f &= \frac{1}{2h}\sum_{i,j=1}^d a_{i,j}\left[h\partial^2_{ij}f + \partial_i h\partial_j f + \partial_i f\partial_j h + f\partial^2_{ij}h\right] + \frac{1}{h}b(x)h\nabla f + \frac{1}{h}b(x)f\nabla h \\ &= \mathcal{G}f + \frac{\sigma\sigma^T\nabla h}{h}\cdot\nabla f + \underbrace{\frac{f\mathcal{G}h}{h}}_{\substack{=0 \text{ car} \\ h \text{ est harmonique}}} .\end{aligned}$$

Ainsi, la h transformée de Doob revient à l'ajout du drift $\frac{\sigma\sigma^T\nabla h}{h}$ (voir [109, Section VIII] pour plus de précisions). Plus précisément, on peut montrer avec une version multidimensionnelle du théorème de Girsanov [75, Section 3.5] le résultat suivant : sur tout intervalle $[0,T]$ $(0 < T < +\infty)$, on a

$$dX_t = \sigma(X_t)d\widetilde{W}_t + b(X_t)dt + \sigma(X_t)\frac{\nabla h(X_t)}{h(X_t)}dt$$

sous la probabilité $\mathbb{Q}_T := \frac{h(X_T)}{h(X_0)}\mathbb{P}$ où $\widetilde{W}$ est un mouvement brownien sous $\mathbb{Q}_T$.

Exemple A.43 (Mouvement brownien drifté dans $\mathbb{R}^d$). Soit X un mouvement brownien dans $\mathbb{R}^d$ de covariance Σ non dégénérée et de drift μ. Son générateur s'écrit $\mathcal{L} = \frac{1}{2}\nabla\cdot\Sigma\nabla + \mu\cdot\nabla$. On vérifie que tout vecteur $v \in \mathbb{R}^d$ satisfaisant $\frac{1}{2}v^T\Sigma v + \mu\cdot v = 0$ (équation d'une ellipsoïde) rend la fonction $h_v : x \longmapsto e^{v\cdot x}$ harmonique pour X. De plus, la h_v-transformée de Doob de X est un mouvement brownien de covariance Σ et de drift $\mu + \Sigma v$.

Exemple A.44 (Pont brownien). Soit W un mouvement brownien libre sur $\mathbb{R}^d$ et $X = (t, W_t)_{t\in[0,1]}$ le mouvement brownien espace-temps associé. Le générateur de X étant $\partial_t + \frac{1}{2}\Delta_x$, on vérifie par un calcul direct que si $y \in \mathbb{R}^d$, la fonction $h : (t,x) \longmapsto p_{1-t}(x,y)$ est harmonique pour X. Formellement, l'analyse faite précedemment donne l'EDS suivante pour le pont brownien Z^y allant de 0 en y en un temps $t = 1$:

$$dZ^y_t = d\widetilde{W}_t - \frac{Z^y_t - y}{1-t}dt.$$

Pour plus de détails, on pourra consulter [109, Chapitre IX.2, (2.12)] ou [75, Section 5.6.B].

A.5.2 Point terminal

Maintenant que la notion de $h-$transformée de Doob est définie, nous sommes en mesure de donner l'interprétation probabiliste en termes de point terminal de la représentation (A.4.7) des fonctions excessives. Fixons X un processus transient satisfaisant l'Hypothèse A.27.

$h-$transformée de Doob et distribution terminale

Le théorème suivant donne l'interprétation trajectorielle de la $h-$transformée de Doob. On rappelle les définitions de E' et $\partial_{min}M$ données après la Définition A.36.

Théorème A.45 (Conditionnement et point terminal)**.** *Supposons $E = E'$. Soit $\eta \in E \cup \partial M_{min}$.*

(i) Pour tout $x \in E^{K(\cdot,\eta)}$, $\mathbb{P}_x^{K(\cdot,\eta)}-$ presque sûrement, $X_t \underset{\substack{t\to\zeta \\ t<\zeta}}{\longrightarrow} \eta$ dans la topologie de M.

(ii) De plus, si $\eta \in E$ et $x \in E^{K(\cdot,\eta)}$, on a $\mathbb{P}_x^{K(\cdot,\eta)}(\zeta < +\infty) = 1$.

(iii) Plus généralement, si $h \in L^1(r)$ est une fonction excessive, alors X^h est transient et pour tout $x \in E^h$, $\lim\limits_{\substack{t\to\zeta \\ t<\zeta}} X_t^h$ existe $\mathbb{P}_x^h-$p.s. dans M et

$$\forall A \in \mathcal{B}(M), \quad \mathbb{P}_x^h(X_{\zeta^-} \in A) = \frac{1}{h(x)} \int_A K(x,\eta)\mu^h(d\eta) \tag{A.5.3}$$

où μ^h désigne l'unique mesure dans la représentation (A.4.7).

Exemple A.46 (Mouvement brownien tué sur le bord d'un domaine borné lisse régulier)*.* Soit $E = B(0,1)$ la boule ouverte centrée en 0 de rayon 1 et X le mouvement brownien dans E tué sur le bord du domaine $\partial B(0,1)$. On connaît via l'Exemple A.18 l'expression des fonctions de Green, et donc du noyau de Martin. La frontière de Martin est alors donnée par $\partial B(0,1)$. Comme X est autodual, on a $E' = E$. L'expression (A.3.7) n'est alors rien d'autre que le théorème de représentation (A.4.7) pour les fonctions harmoniques. En particulier, si $d \geq 3$ et $y \in \partial B(0,1)$, conditionner par rapport à la fonction harmonique

$$x \longmapsto \frac{1-|x|^2}{|y-x|^d}$$

revient à conditionner le processus pour qu'il converge vers y.

Plus généralement, si D est un domaine borné Lipschitzien, alors la frontière de Martin est minimale et coïncide avec la frontière topologique [70]. On a par ailleurs la même interprétation en termes de point terminal. Il en est de même pour des diffusions uniformément elliptiques [102, Théorème 1.5, Section 8.1] sous la même hypothèse sur D. Il existe d'autres critères suffisants sur le domaine [5, 114] pour avoir $\partial M = \partial D$.

On considère à présent un exemple dans lequel la frontière de Martin ne coïncide pas avec la frontière topologique.

Exemple A.47 (Pacman)*.* Soit $E = B(0,1)\backslash([0,1] \times \{0\}) \subset \mathbb{R}^2$, à savoir le disque unité ouvert bidimensionnel privé d'un rayon. Soit X un mouvement brownien dans E tué sur le bord. Le processus X, s'il est tué sur le segment, peut être tué en arrivant d'en haut ou d'en bas du segment. Cela crée alors deux points distincts sur la frontière de Martin placés au même endroit sur la frontière topologique. On peut alors montrer [20, Exemple 14.16] que la frontière de Martin est homéomorphe au "pacman", voir Figure A.1.

Remarque A.48 (Cas où $E \neq E'$)*.* Si $E' \neq E$ et si $x \in E\backslash E'$, alors il existe un unique point $\pi(x) \in \partial M_{min}$ tel que $K(\cdot,x) = K(\cdot,\pi(x))$ (cf. [78, Section 12]). On peut alors montrer que pour un tel x, les points d'accumulation de $X_t^{K(\cdot,x)}$ quand $t \to \zeta$ sont exactement x et $\pi(x)$, et donc que $X^{K(\cdot,x)}$ ne converge pas $\mathbb{P}^{K(\cdot,x)}-$presque sûrement dans M. Cependant, si h est une fonction excessive, $\lim\limits_{\substack{t\to\zeta \\ t<\zeta}} X_t^h$ existe $\mathbb{P}^h-$presque sûrement, si et seulement si, $\pi(E\backslash E')$ est μ^h-négligeable. Dans ce cas, la loi de $X_{\zeta^-}^h$ est donnée par (A.5.3).

Retour sur le problème de Dirichlet

Grâce à la théorie de la frontière de Martin, on peut préciser le sens de la condition frontière du problème de Dirichlet lorsque celle-ci est donnée par une fonction mesurable. La solution (A.2.4) du problème de Dirichlet (A.2.2) associé à (D,f) pour le mouvement brownien X tué au bord du domaine borné D peut se réécrire

$$h(x) = \mathbb{E}_x\left[f(X_{\zeta^-})\right]. \tag{A.5.4}$$

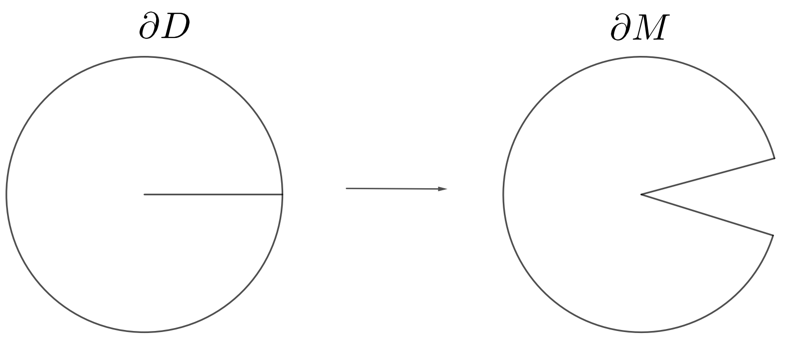


FIGURE A.1 : Frontière topologique et frontière de Martin relatives à l'Exemple A.47

Par la propriété de Markov forte, cette dernière quantité est harmonique en général lorsque f est mesurable positive. Pour une fonction $h : E \longrightarrow \mathbb{R}_+$, on dit que $h(x)$ converge finement vers $L \geq 0$ quand $x \to \eta \in \partial M$ si "$h(X_t)$ converge presque sûrement vers L lorsque $X_t \to \eta$", ou plus rigoureusement si :

$$\mathbb{P}^{K(\cdot,\eta)}\left(h(X_t) \underset{t\to\zeta^-}{\longrightarrow} L\right) = 1.$$

Notons $H_D(x, A) = \mathbb{P}_x(X_{\zeta^-} \in A)$ la distribution terminale de X sur ∂M. On a alors le résultat suivant [20, Théorème 14.18].

Proposition A.49 (Condition au bord du problème de Dirichlet sur M)**.** *Soit f une fonction mesurable positive sur ∂M, h la fonction harmonique définie par* (A.5.4) *et $x_0 \in D$. Alors, pour $H_D(x_0, \cdot)$-presque tout $\eta \in \partial M$, $h(x)$ converge finement vers $f(\eta)$.*

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