Reflected Brownian motion in a cone and Martin boundary.

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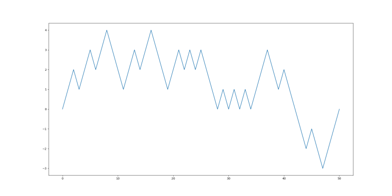
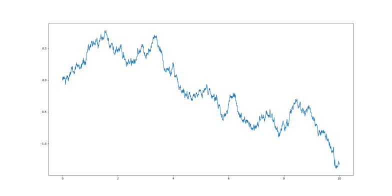
Recall : Donsker's Theorem

Brownian Motion is the scale limit of random walks.



(a) Centered random walk (L^2) (discrete time)



(b) Brownian motion (continuous time)

Motivation 2 of Brownian motion : Robert Brown the genius

Pollen grain on the water surface : does not have a C^1 trajectory !

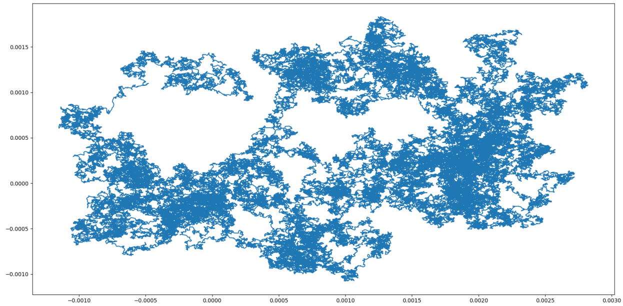


Figure 2 – Brownian motion dimension 2

Properties of Brownian motion

Proposition

A Brownian motion is (almost surely) everywhere **continuous and nowhere differentiable.**

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Proposition

If $(B_t)_{t\geqslant 0}$ is a Brownian motion, then it is a **Markov** process : it forgets the past.

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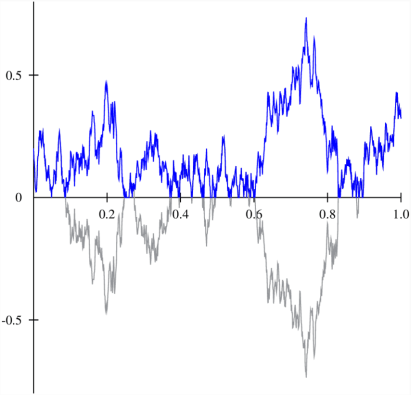
Reflecting a Brownian motion : naive approach

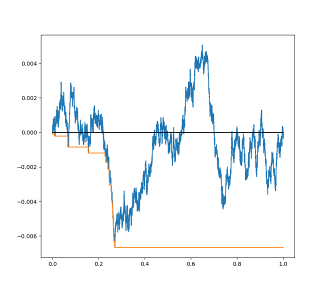
Naive approach : consider $(|B_t|)_{t \geqslant 0}$



(Source : Brownian Motions on Metric Graphs : Feller Brownian Motions on Intervals Revisited)

Reflecting a Brownian motion : local time approach

We define $L_t = -\min(B_s, \quad 0 \leqslant s \leqslant t)$



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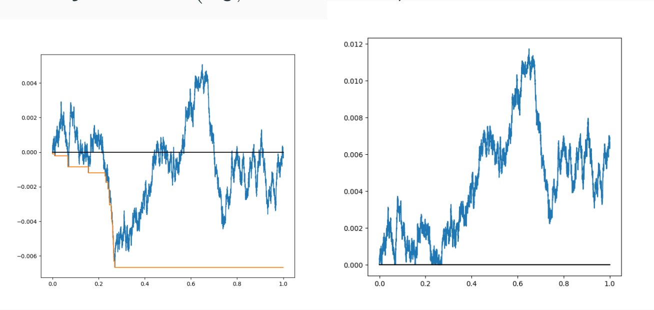

(a) $(B_t)_{t\geqslant 0}$, $(-L_t)_{t\geqslant 0}$ **(b)** $(B_t + L_t)_{t\geqslant 0}$ **is a RBM**

$(L_t)_{t\geqslant 0}$ is called the local time of the reflected Brownian motion at 0 : it is a **pushing process**.

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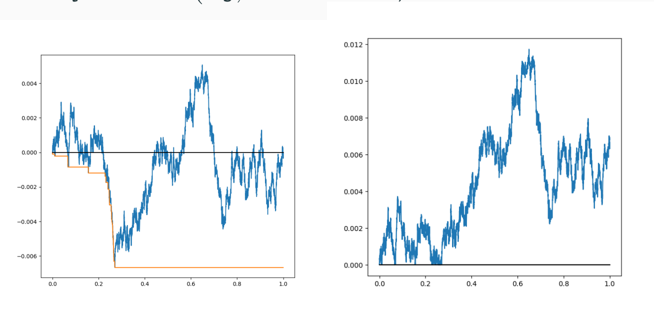

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Theorem : Tanaka formula

$$|W_t| = B_t + L_t$$

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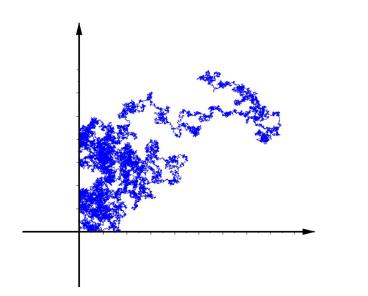
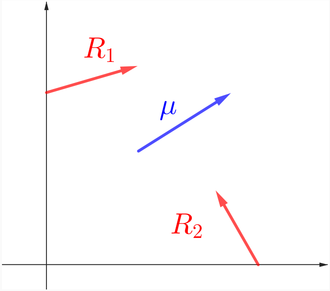
What is it ?

Reflected Brownian motion in $\mathbb{R}_+^2$: $Z_t = z_0 + B_t + \mu t + RL_t$



(a) Exemple de trajectoire



(b) Drift, Vecteurs de réflexion

- $(B_t)_{t\geqslant 0}$ is a Brownian motion, $\mu \in \mathbb{R}^2$ is the drift
- $R = (R_1, R_2)$ reflection matrix
- $(L_t)_{t\geqslant 0}$ local time on the axes $\text{supp}(dL^i) \subset \{t, Z_t^i = 0\}$.

Theorem : Weak existence and uniqueness

Existence and uniqueness in law at all positive times if and only if

$$r_{11} > 0,\ r_{22} > 0,\ \text{and}\ [\det(R) > 0 \quad \text{or} \quad r_{21},\ r_{12} > 0].$$

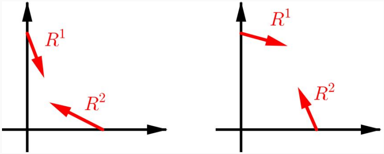


Figure 6 – Not defined   Defined for all time

Existence and uniqueness

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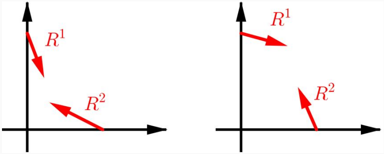

Figure 6 – Not defined   Defined for all time

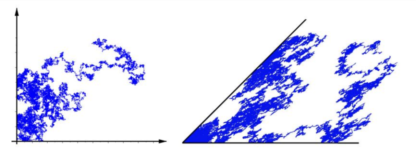

Figure 7 – From quadrant to cone : linear transformation

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Definition : Recurrent or Transient process

A Markov process is **recurrent** if it comes back almost surely in any open set. Otherwise, such a process is called **transient**.

Example :

- Drifted Brownian motion $(B_t + \mu t)_{t \geqslant 0}$ $(\mu \neq 0)$ is transient in $\mathbb{R}^d$
- Brownian motion in dimension $d \geqslant 3$ is transient in $\mathbb{R}^d$

Recurrence and transience : competition drift/reflection

Theorem : Conditions for transience (Hobson-Rogers, 1993)

$(Z_t)_{t\geqslant 0}$ is transient if and only if

$$r_{11}\mu_1 - r_{21}\mu_2^- \geqslant 0 \quad \text{or} \quad r_{12}\mu_1^- - r_{22}\mu_2 \leqslant 0.$$

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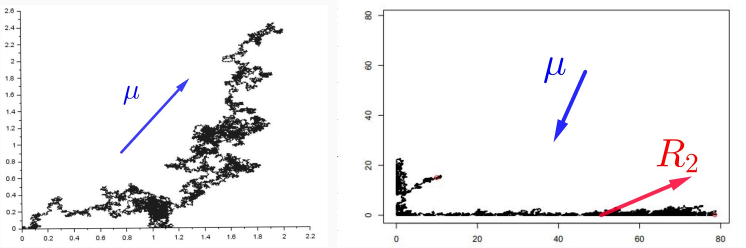


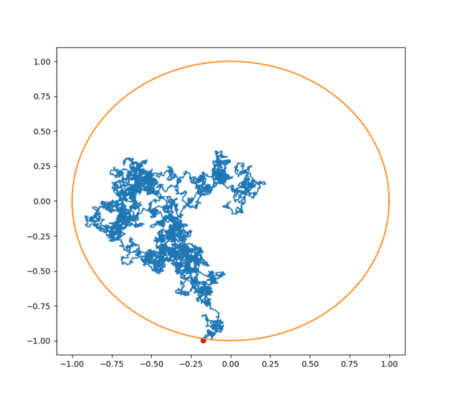
(a) Example where $\mu_1, \mu_2 > 0$.

(b) Escape along an axis

Assuming μ points outward : Z is transient.

A last example

- Brownian motion in a ball killed when it reaches $\partial D(0,1)$



Motivation

(One) motivation for transient processes : is it possible to define a "limit at infinity" ?

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Boundary value problem and harmonic fonctions

$$\begin{cases} \Delta h = 0 \\ h|_{\partial D} = f \in C(\partial D) \end{cases}$$

(a) Analytic problem

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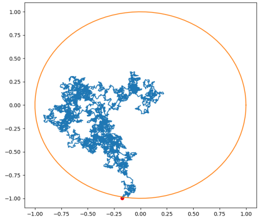

(b) Probabilistic interpretation

Theorem : Harmonic functions

If B is a Brownian motion and D is a smooth bounded domain,
$h(z) := \mathbb{E}_z[f(B_T)]$ is a solution where $T = \inf(t \geqslant 0, B_t \in \partial D)$.
In particular,
$$h(z) = \int_{\partial D} f(u)k(z,u)du$$
where $k(z,u)du = P_z(B_T \in du)$

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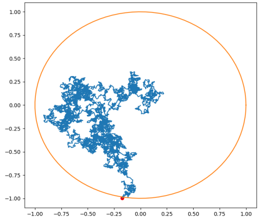

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Motivation : find harmonic functions with stochastic approach

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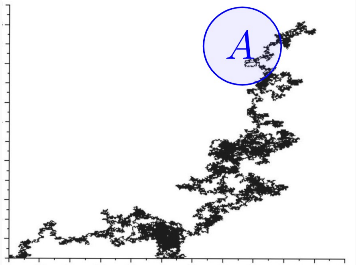
Green's measure

Let be $(B_t)_{t \geqslant 0}$ a transient Brownian motion (or transient process).

Definition : / Proposition : Green measure from z_0

The following measure has a density $g(z_0, \cdot)$ inside the cone :

$$G(z_0, A) := \mathbb{E}_{z_0}\left[\int_0^{\infty} \mathbf{1}_A(B_t)dt\right] = \iint_A g(z_0, z)dz$$



(a) Time passed in A.

Definition : Harmonic functions

A function h is harmonic for a Markov process $(Z_t)_{t\geqslant 0}$ if for all z,

$$\mathbb{E}_z[h(Z_t)] = h(z).$$

Example :
This is equivalent with $\Delta h = 0$ for $(Z_t)_{t\geqslant 0}$ Brownian motion.

Proposition

For $z_0 \in \mathbb{R}^d$, $g(\cdot, y)$ is harmonic on $\mathbb{R}^d \backslash \{y\}$

Martin Kernel and construction

Definition : Martin Kernel and metric

Fix z_0 in $\mathbb{R}^d$. For $x, y \in \mathbb{R}^d$, define

$$k(x,y) = \begin{cases} \frac{g(x,y)}{g(z_0,y)} & \text{if} \quad y \neq z_0 \\ 0 & \text{if} \quad y = z_0. \end{cases} \tag{1}$$

Then, the following expression defines a metric on $\mathbb{R}^d$:

$$\rho(z_1, z_2) = \int_{\mathbb{R}^d} \frac{|k(x,z_1) - k(x,z_2)|}{1 + |k(x,z_1) - k(x,z_2)|} e^{-|x|^2} dx. \tag{2}$$

Remark

We recognize a "pointwise convergence topology" for the family $(k(x,\cdot))_{x \in \mathbb{R}^d}$.

Definition : Martin Boundary

Define Γ as $\boldsymbol{\Gamma = \{(x_n)_{n\geqslant 0}, (k(\cdot, x_n))_n}$ **converges pointwise**$\}/\sim$ where $\sim$ is the equivalent relation caracterising the limit of $(k(\cdot, x_n))_n$

Theorem : Martin compactification

The metric ρ extends naturally to $\mathbb{R}^d \cup \Gamma$. Furthermore,

- $k(\cdot, \xi) = k(\cdot, \eta) \implies \eta = \xi$.
- $\mathbb{R}^d \cup \Gamma$ is **compact** under ρ.
- $y_n \underset{n\to\infty}{\longrightarrow} \eta \in \Gamma \iff k(\cdot, y_n) \underset{n\to\infty}{\longrightarrow} k(\cdot, \eta)$ pointwise.

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Proposition

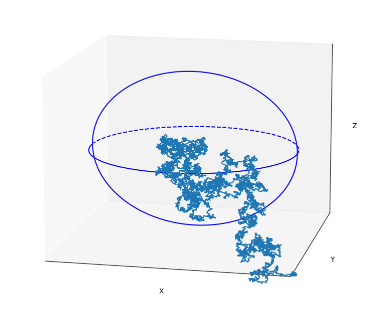
For $\eta \in \Gamma$, $\quad k(\cdot,\eta)$ is harmonic on $\mathbb{R}^d$.

Theorem : Limit at infinity

Almost surely, $B_t \underset{t\to\infty}{\longrightarrow} B_\infty \in \Gamma$

Example :

Martin Boundary of **drifted** Brownian Motion : $\Gamma \sim \mathbb{S}^{d-1}$



Remark

The Martin Boundary see more than the limit at infinity

Representation of nonnegative harmonic functions

Theorem : Representation of nonnegative harmonic function

Let $h \geqslant 0$ be a harmonic function. Then, there is a Radon mesure μ_h on Γ satisfying

$$h(z) = \int_\Gamma k(z,\eta) d\mu_h(\eta). \tag{3}$$

Furthermore, every function defined by (3) is harmonic.

Example :

For the killed Brownian motion at $\partial D(0,1)$,

- $\Gamma = \partial D(0,1)$
- μ_h is the boundary condition

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Remark

It works only for **nonnegative** harmonic function

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Harmonic functions

Theorem : Doob Transform

If h is nonnegative harmonic, then $P^h(A) = \mathbb{E}\left[\frac{h(Z_t)}{h(Z_0)}\mathbf{1}_A\right]$ defines a probability under which $(Z_t)_{t\geqslant 0}$ remains a Markov process.

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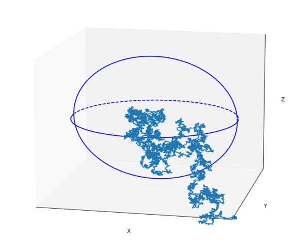
Proposition

If Γ is minimal, for all $\eta \in \Gamma$, $P_x^{k(\cdot,\eta)}(X_\infty = \eta) = 1$.

Example :

For $u \in \mathbb{S}^{d-1}$,[Condition by $k(\cdot, u)$] = [Converge in direction u]



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Reflected Brownian motion in a cone

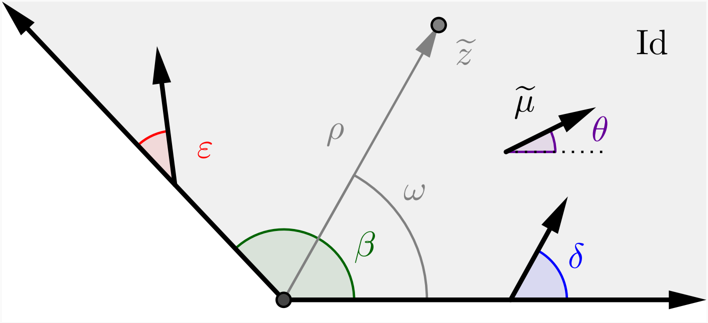
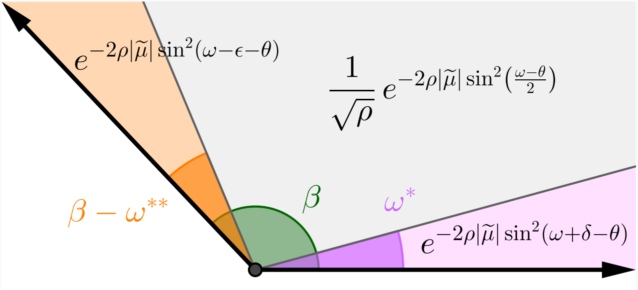


Figure 10 – Notations in the cone

Theorem : Asymptotics of the Green's Density

Let $\omega^* = \theta - 2\delta$ and $\omega^{**} = \theta + 2\varepsilon$. We have the asymptotics for $g^{(z_0)}(\rho\cos(\omega), \rho\sin(\omega))$, $\rho \to \infty$, $\omega \to \omega_0 \notin \{0, \omega^*, \omega^{**}, \beta\}$:

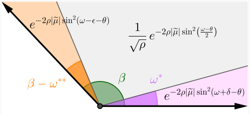
$$\mathbf{g}^{(z_0)}(\rho\cos(\omega), \rho\sin(\omega)) \sim \ldots$$



(a) $0 < \omega^* < \omega^{**} < \beta$

Remarks

- According to the angles of the parameters, we have 4 configurations



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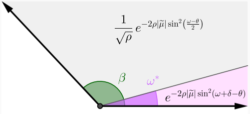
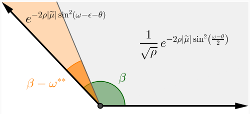
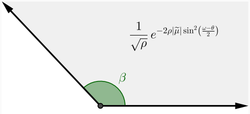


(b) $0 < \omega^* < \beta < \omega^{**}$



(c) $\omega^* < 0 < \omega^{**} < \beta$



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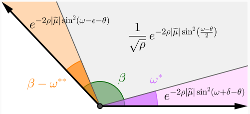

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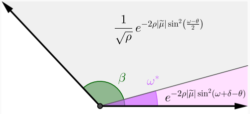

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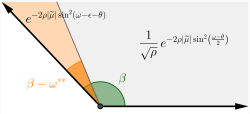

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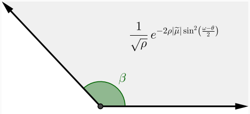


(d) $\omega^* < 0 < \beta < \omega^{**}$

- Asymptotic expansion (middle zone :)

$$g(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to\omega_0}}{\sim} e^{-2\rho|\mu|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{1}{\sqrt{\rho}} \sum_{k=0}^{n} \frac{c_k(\omega)}{\rho^k}$$

Proof Strategy

We consider the $\mathbb{R}_+^2$ quadrant.

- Definition of Green measures on the boundaries
- Laplace transforms, **functional equation** and extension of Laplace transforms
- **Laplace inversion**
- Residue theorem
- **Saddle point method**

Corresponding Martin Boundary

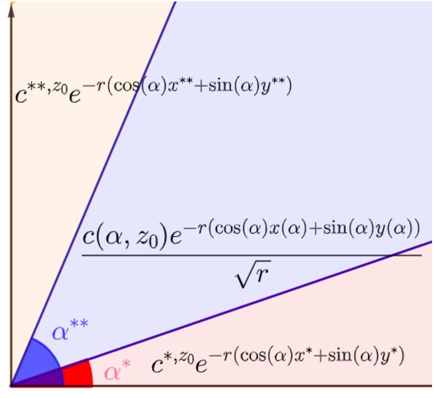


Figure 13 – Asymptotics of $g^{(z_0)}(r\cos(\alpha), r\sin(\alpha))$ as $r \to \infty$

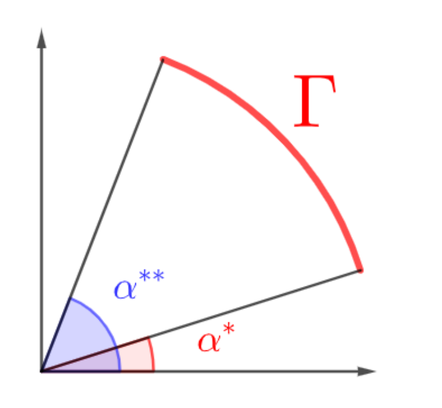


Figure 14 – Shape of Martin Boundary of the process : $\Gamma = [\alpha^*, \alpha^{**}]$

Remark

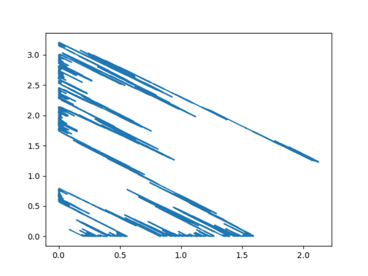
Constants $c(\alpha, z_0)$ are the harmonic functions $k(z_0, \alpha)$ relative to Martin kernel

Degenerate Model - Link to particles ?

Three particles $X_1 \leqslant X_2 \leqslant X_3$ colliding with each other, and the middle one is a Brownian motion : **gapping process**

$(X_3 - X_2, X_2 - X_1)$ is a reflected Brownian motion.

(a) System of 3 particles (Karatzas Ichiba paper)



(b) Degenerate reflected Brownian motion

Thank you for listening !

References and existence

- Survey : SRBM in the orthant - Williams (1995)
- Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant - Taylor and Williams (1992)
- Other...
- Tomoyuki Ichiba, Ioannis Karatzas - Degenerate Competing Three-Particle Systems (2021)