Complements and additional material for "Markov Chains" (2022/23)

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1. Branching process (Bienaymé-Galton-Watson process)

The BGW process (Z_n) is a MC on $E := \mathbb{N} \cup \{0\}$ defined starting from the IID family $\xi := (\xi_{n,j})_{(n,j)\in\mathbb{N}^2}$, with $\mathbb{P}(\xi_{1,1} \in E) = 1$. We use the notation $p_j := \mathbb{P}(\xi_{1,1} = j)$. The chain can be introduced by iteration once Z_0 independent of ξ is given (unless otherwise said, we choose $Z_0 = 1$) via

$$Z_{n+1} = \begin{cases} \xi_{n+1,1} + \xi_{n+1,2} + \ldots + \xi_{n+1,Z_n} & \text{if } Z_n > 0, \\ 0 & \text{if } Z_n = 0. \end{cases} \tag{0.1}$$

We assume hat $\mu = \mathbb{E}[\xi_{1,1}] = \sum_j jp_i = \mu \in (0, \infty)$ and that $p_1 < 1$ (to avoid trivialities).

It is useful to establish that (Z_n/μ^n) is a (non-negative) martingale with respect to the natural filtration of the MC (*Exercise*). Hence $\lim_n Z_n/\mu^n$ exists a.s. and we denote the limit (non-negative) random variable by H. Check that if $\mathbb{E}[\xi_{1,1}^2] < \infty$ then the martingale is UI (Uniformly Integrable), hence in this case $H \not\equiv 0$. For $s \in (0,1]$ we introduce also $\varphi(s) = \mathbb{E}[s^{\xi_{1,1}}]$. Note that (*Exercise*) $\varphi(\cdot)$ is convex, increasing and smooth. Since $\varphi(0) = \lim_{s\searrow 0}\varphi(s) = p_0$ and $\varphi(1) = 1$, there exists only one solution in $[0,1)$ to the the fixed point equation $s = \varphi(s)$. Call this solution ϱ

Proposition 1.1. 0 *is a recurrent state for the chain* (Z_n)*: note that* 0 *is accessible from any other state (i.e.,* $\rho_{n,0} > 0$*) if and only if* $p_0 > 0$*. All other states* n *are transient. Moreover*

(1) *if* $\mu \le 1$ *then* $\sum_n \mathbf{1}_{Z_n>0} < \infty$ *a.s. (hence* $H \equiv 0$*);*
(2) *if* $\mu > 1$ *then* $\mathbb{P}(H = 0) = \rho$*, hence* $\mathbf{1}_{H=0}\sum_n \mathbf{1}_{Z_n>0} < \infty$ *a.s., and if* $\mathbb{E}[\xi_{1,1}^2] < \infty$ *on the event* $\{H > 0\}$ *we have* $Z_n \sim H\mu^n$ *a.s. (here* $\sim$ *is aymptotic equivalence.*

The only (σ*-finite) invariant measure can be normalized and it is* δ_0*.*

Proof: *Exercise.*

2. Birth and death chain

$E = \mathbb{N} \cup \{0\}$ and Q is defined by

$$Q(j, j+1) = p_j\,, \quad Q(j, j-1) = q_j\,, \quad Q(j,j) = r_j, \tag{0.2}$$

with $p_j + q_j + r_j = 1$ for every j. We assume that $p_j > 0$ for every $j \in E$, $q_j > 0$ for every $j \in E \setminus \{0\}$ and $q_0 = 0$.

This MC is irreducible, moreover it is aperiodic if and only if there exists j such that $r_j > 0$ (*Exercise*).

We introduce the function $\varphi : E \mapsto [0, \infty)$ defined by $\varphi(0) = 0$, $\varphi(1) = 1$ and by imposing that

$$(Q\varphi)(k) = \varphi(k)\,, \tag{0.3}$$

for $k \in \mathbb{N}$. This yields $(\varphi(k+1) - \varphi(k)) = (q_k/p_k)(\varphi(k) - \varphi(k-1))$ for $k \geq 1$ and therefore for $n \geq 2$

$$\varphi(n) \ = \ 1 + \sum_{m=1}^{n-1} \prod_{j=1}^{m} \frac{q_j}{p_j} \, . \tag{0.4}$$

Note that $\lim_{n\to\infty} \varphi(n) =: \varphi(\infty)$ exists and takes value in $(1, \infty]$. We set $T_a := \inf\{n = 0, 1, \ldots : \, X_n = a\}$. By the Optional Stopping Theorem we have that, for $a < x < b$, $\varphi(x) = \mathbb{E}_x[\varphi(X_{T_a \wedge T_b}]$ from which we readily extract

$$\mathbb{P}_x(T_b > T_a) \ = \ \frac{\varphi(x) - \varphi(a)}{\varphi(b) - \varphi(a)} \ \Longrightarrow \ \mathbb{P}_x(T_b > T_0) \ = \ \frac{\varphi(x)}{\varphi(b)} \, . \tag{0.5}$$

Note finally that $\nu(x) := \prod_{k=1}^{x} (p_{k-1}/q_k)$ is reversible for Q.

We can therefore conclude that (*Exercise*):

> PROPOSITION 2.1. *The birth and death MC is recurrent if and only if $\varphi(\infty) = \infty$. In this case ν is unique up to a multiplicative factor. Moreover it is finite recurrent if and only if $\sum_x \prod_{k=1}^{x} (p_{k-1}/q_k) < \infty$.*

Note that this result implies in particular that the symmetric simple random walk is null recurrent, as well as the well known fact that an asymmetric simple random walk is transient.

3. Discrete renewal processes

The basic object is $\tau = \{\tau_j\}_{j=0,1,\ldots}$ with $\tau_0 = 0$ and $\tau_j - \tau_{j-1} =: \eta_j$ and $(\eta_j)j = 1, 2, \ldots$ are IID variables taking values in $\{1, 2, \ldots, \infty\} = \mathbb{N} \cup \{\infty\}$. We set $K(n) := \mathbb{P}(\eta_1 = n)$ so in general $\sum_{n\in\mathbb{N}} K(n) \leq 1$, while $\sum_{n\in\mathbb{N}} K(n) + K(\infty) = 1$. We can view τ as a random subset of $\mathbb{N} \cup \{\infty\}$. In fact either $K(\infty) = 0$ and $|\tau| = \infty$ a.s. or $K(\infty) > 0$ and τ is a.s. a finite set (containing ∞). We set

$$u(n) \ := \ \mathbb{P}(n \in \tau) \ = \ \mathbb{P}\left(\exists j \text{ such that } \tau_j = n\right) \, , \tag{0.6}$$

and $u(\cdot)$ is called *renewal function*. We say that τ is aperiodic if there exists no integer $p > 1$ such that $\{n \in \mathbb{N} : \, K(n) > 0\} \subset p\mathbb{N}$. We write η for η_1

> THEOREM 3.1 (Renewal Theorem). *For an aperiodic renewal*
>
> $$\lim_{n\to\infty} u(n) \ = \ \frac{1}{\mathbb{E}[\eta]} \, . \tag{0.7}$$

Of course the right-hand side of (0.7) is zero if $\mathbb{E}[\eta_1] = \infty$ and the statement holds without aperiodicity condition in this case. The proof is a direct application (*Exercise*) of the Ergodic Theorem for MC to one of the two MC's that we are going to build (and study) now.

3.1. The backward recurrence time. We set $A_n := n - \sup(\tau \cap [0, n]) \in E := \mathbb{N} \cup \{0\}$. Looking from time n, this is the time elapsed since the last renewal.

PROPOSITION 3.2. *A is a Q-MC with*

$$Q(j, j+1) = \mathbb{P}(\eta > j+1 | \eta > j) \quad \text{and} \quad Q(j, 0) = 1 - \mathbb{P}(\eta > j+1 | \eta > j)\,,$$

for $j = 0, 1, \ldots$ with the convention that $\mathbb{P}(\eta > j+1 | \eta > j) = 0$ if $\mathbb{P}(\eta > j) = 0$. Various facts:

- *A is transient if and only if $\mathbb{P}(\eta = \infty) > 0$;*
- *A is positive recurrent if and only if $\mathbb{E}[\eta] < \infty$;*
- *A is aperiodic if and only if τ is;*
- *the unique (up to a factor) invariant measure μ satisfies*

$$\mu(n) = \mu(0)\mathbb{P}(\eta > n)\,;$$

- *A has no invariant measure if (and only if) it is transient;*
- *A is positive recurrent if and only if $\mathbb{E}[\eta] < \infty$ and we remark, once μ is normalized, that $\mu(0) = 1/\mathbb{E}[\eta]$.*

Proof: *Exercise.*

3.2. The forward recurrence time. We set $B_n = \inf(\tau \cap [n, \infty]) - n \in E = \mathbb{N} \cup \{0, \infty\}$. Looking from time n, this is the time that is missing till the next renewal (unless at time n there is a renewal, so the next renewal is considered to be the one at n and $B_n = 0$).

PROPOSITION 3.3. *B is a Q-MC on E with*

$$Q(j, j-1) = 1 \text{ for } j \in \mathbb{N} \text{ and } Q(0, j) = \mathbb{P}(\eta = j+1) \text{ for } j \in \mathbb{N} \cup \{0, \infty\}\,,$$

and $Q(\infty, \infty) = 1$. B is not irreducible: it is impossible to go from ∞ to $n = 0, 1, \ldots$. Various facts:

- *$E \setminus \{\infty\}$ is transient (in the obvious sense) if and only if $\mathbb{P}(\eta = \infty) > 0$;*
- *$E \setminus \{\infty\}$ is positive recurrent if and only if $\mathbb{E}[\eta] < \infty$;*
- *B is aperiodic if and only if τ is;*
- *the unique (up to a factor) invariant measure μ supported on $E \setminus \{\infty\}$ satisfies*

$$\mu(n) = \mu(0)\mathbb{P}(\eta > n) \quad \text{for every } n \in E \setminus \{\infty\}\,;$$

- *B has δ_∞ as unique invariant measure if (and only if) $E \setminus \{\infty\}$ is transient;*
- *Assume $B_0 \neq \infty$ a.s.. Then B is positive recurrent if and only if $\mathbb{E}[\eta] < \infty$ and we remark, once μ is normalized, that $\mu(0) = 1/\mathbb{E}[\eta]$.*

Proof: *Exercise.*

4. Lindley process

The Lindley process is the MC on $E = [0, \infty)$ ($\mathcal{E} := \mathcal{B}(E)$) defined by the iteration map $f_\xi(x) := (x + \xi)_+$, $x \geq 0$ and $\xi \in \mathbb{R}$. That means that, given $W_0 \in E$ and an IID sequence $(\xi_j)_{j=1,2,\ldots}$, with W_0 and $(\xi_j)_{j=1,2,\ldots}$ independent, we define for $n\ in \mathbb{N}$

$$W_n \,:=\, f_{\xi_n}(W_{n-1})\,. \tag{0.8}$$

Hence (W_n) is a $p-MC$ with $p(x, [0, y]) = \mathbb{P}(\xi_1 \leq y - x)$. In particular, $p(x, \{0\}) = \mathbb{P}(\xi_1 \leq -x)$. This process has immediate interpretations in terms of a storage processes or of basic queueing systems [**1**, Ch. III, Sec. 6]. In order to avoid trivialities, we assume that $\mathbb{P}(\xi_1 < 0) > 0$ as well as $\mathbb{P}(\xi_1 > 0) > 0$.

There is an interesting direct link between the Lindley process with $W_0 = 0$ and the random walk (S_n) with $S_0 = 0$ and increments $(\xi_j)_{j=1,2,\ldots}$. In fact, it is straightforward to see that the hitting times $(T_0^{(j)})_{j=0,1,\ldots}$ of 0 by (W_n) are the descending ladder times of (S_n): the descending ladder tiles are defined by setting $\tau_0 = 0$ and, for $k \geq 0$, $\tau_{k+1} := \inf\{n > \tau_k : S_n \leq S_{\tau_k}\}$. So the ladder times are the times in which the walk hits a new minimum. The identity we just claimed is actually *pathwise*: that is $T_0^{(j)}(\omega) = \tau_j(\omega)$ for every ω.

Since it is immediate to check that 0 is accessible for every $x \in E$, i.e. $\mathbb{P}_x(T_0 < \infty) > 0$, showing recurrence or transience of 0 is crucial. One can actually show that (S_n) has only three possible behaviors [**1**, p. 224, Th. 2.4]:

(1) either $S_n \to \infty$ a.s.;
(2) or $S_n \to -\infty$ a.s.;
(3) or $\limsup_n S_n = +\infty$ and $\liminf_n S_n = -\infty$ a.s.;

and $\mathbb{P}(\tau_1 < \infty) = 1$ if and only if we are in cases (2) or (3). Therefore this is a necessary and sufficient condition for recurrence of the Lindley process. One can actually show that (W_n) is null recurrent if and only if (S_n) is in case (3), and this happens if and only if $\xi \in \mathbb{L}^1$ and $\mathbb{E}\xi_1 = 0$ or $\xi \sim -\xi$.

These results are rather advanced, but it is for example immediate to see that $W_n \geq S_n$, hence $W_n \to \infty$ if $S_n \to \infty$. Therefore, this happens in particular if $\xi \in \mathbb{L}^1$ and $\mathbb{E}[\xi_1] > 0$ and in this case one obtains without difficulty also that $W_n/n \to \mathbb{E}[\xi_1]$ a.s.. Therefore (W_n) is transient for $\mathbb{E}[\xi_1] > 0$ (in the sense that 0 is visited only a finite number of times, but of course this result says that also $[0, x]$ is visited only finitely many times).

Another case that we can treat in detail with relatively elementary methods is the case in which $S_n \to -\infty$, hence in particular if $\xi \in \mathbb{L}^1$ and $\mathbb{E}[\xi_1] < 0$. In analyzing this case the following magic identity (that involves no probability!) will the of help:

LEMMA 4.1. *If $\xi_1, \xi_2, \ldots$ are real numbers, then for every $n = 1, 2, \ldots$ and $x \geq 0$ we have*

$$f_{\xi_1} \circ f_{\xi_2} \circ \ldots \circ f_{\xi_2}(x) = \max\left(0, \xi_1, \xi_1 + \xi_2, \ldots, \xi_1 + \ldots + \xi_{n-1}, \xi_1 + \ldots + \xi_{n-1} + \xi_n + x\right) . \quad (0.9)$$

PROOF. Of course it suffices to consider the case $x = 0$. We can proceed by induction: the case $n = 1$ follows from the definition of $f_\xi(\cdot)$. To prove the induction step it suffices to exploit that $(x+y_+)_+ = \max(0, x, x+y)$ (this identity is established by considering separately the cases $y \geq 0$ and $y < 0$). □

The consequences of Lemma 4.1 are immediate: if $\lim_n \xi_1 + \xi_2 + \ldots \xi_n = -\infty$ we have that for every x

$$\lim_n f_{\xi_1} \circ f_{\xi_2} \circ \ldots \circ f_{\xi_2}(x) = \max\left(0, \xi_1, \xi_1 + \xi_2, \xi_1 + \xi_2 + \xi_3, \ldots\right) \in [0, \infty). \quad (0.10)$$

This is still just a deterministic result, but of course (ξ_j) are IID and $S_n \to -\infty$ a.s. we have that, with $Y_n := f_{\xi_1} \circ f_{\xi_2} \circ \ldots \circ f_{\xi_2}(x)$, $\lim_n Y_n$ converges almost surely to a limit that we call Y_∞ (see the right-hand side of (0.10) for an explicit expression).

Note however that we are not interested in (Y_n), but in (W_n)! This is not a minor change:

- (Y_n) is not (at all) a MC and it the trajectories of (Y_n) have very little to do with those of (W_n). In fact, Lemma 4.1 is telling us that it is a non decreasing process (while (W_n) oscillates a lot!).
- nevertheless $Y_n \sim W_n$ for every n, just because the ξ_j variables are exchangeable. We stress that this holds only for the 1-marginal: we do not claim at all for example that $(Y_1, Y_2) \sim (W_1, W_2)$.

From these considerations we can extract

PROPOSITION 4.2. *If $S_n \to -\infty$ a.s., in particular if $\mathbb{E}[(\xi_1)_+] < \infty$ and $\mathbb{E}[\xi_1] \in [-\infty, 0)$, we have that (W_n) converges in law, for every choice of $W_0 = x$, to W_∞. The law of W_∞ is the unique invariant probability of the Markov chain and $\mathbb{P}(W\infty = 0) \in (0, 1)$.*

PROOF. *Exercise* □

Proposition 4.2 collects the results that one obtains directly from Lemma 4.1 and the trick of inversing the order of the ξ variables. But once we know that there is an invariant probability for which 0 has positive probability, we can extract that 0 is recurrent and exploit this to obtain (*Exercise*)

PROPOSITION 4.3. *If $S_n \to -\infty$ a.s. we have that the law of W_n converges in total variation, for every choice of $W_0 = x$, to the law of W_∞.*

Bibliography

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