Complements and additional material for "Markov Chains" (2022/23)

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1. Branching process (Bienaymé-Galton-Watson process)

The BGW process (Z_n) is a MC on $E := \mathbb{N} \cup \{0\}$ defined starting from the IID family $\xi := (\xi_{n,j})_{(n,j)\in\mathbb{N}^2}$, with $\mathbb{P}(\xi_{1,1} \in E) = 1$. We use the notation $p_j := \mathbb{P}(\xi_{1,1} = j)$. The chain can be introduced by iteration once Z_0 independent of ξ is given (unless otherwise said, we choose $Z_0 = 1$) via

$$Z_{n+1} = \begin{cases} \xi_{n+1,1} + \xi_{n+1,2} + \ldots + \xi_{n+1,Z_n} & \text{if } Z_n > 0, \\ 0 & \text{if } Z_n = 0. \end{cases} \tag{0.1}$$

We assume hat $\mu = \mathbb{E}[\xi_{1,1}] = \sum_j jp_i = \mu \in (0,\infty)$ and that $p_1 < 1$ (to avoid trivialities).

It is useful to establish that (Z_n/μ^n) is a (non-negative) martingale with respect to the natural filtration of the MC (*Exercise*). Hence $\lim_n Z_n/\mu^n$ exists a.s. and we denote the limit (non-negative) random variable by H. Check that if $\mathbb{E}[\xi_{1,1}^2] < \infty$ then the martingale is UI (Uniformly Integrable), hence in this case $H \not\equiv 0$. For $s \in (0,1]$ we introduce also $\varphi(s) = \mathbb{E}[s^{\xi_{1,1}}]$. Note that (*Exercise*) $\varphi(\cdot)$ is convex, increasing and smooth. Since $\varphi(0) = \lim_{s\searrow 0}\varphi(s) = p_0$ and $\varphi(1) = 1$, there exists only one solution in $[0,1)$ to the the fixed point equation $s = \varphi(s)$. Call this solution ϱ

PROPOSITION 1.1. *0 is a recurrent state for the chain (Z_n): note that 0 is accessible from any other state (i.e., $\rho_{n,0} > 0$) if and only if $p_0 > 0$. All other states n are transient. Moreover*

1. *if $\mu \le 1$ then $\sum_n \mathbf{1}_{Z_n>0} < \infty$ a.s. (hence $H \equiv 0$);*
2. *if $\mu > 1$ then $\mathbb{P}(H = 0) = \rho$, hence $\mathbf{1}_{H=0}\sum_n \mathbf{1}_{Z_n>0} < \infty$ a.s., and if $\mathbb{E}[\xi_{1,1}^2] < \infty$ on the event $\{H > 0\}$ we have $Z_n \sim H\mu^n$ a.s. (here $\sim$ is aymptotic equivalence.*

The only (σ-finite) invariant measure can be normalized and it is δ_0.

Proof: *Exercise.*

2. Birth and death chain

$E = \mathbb{N} \cup \{0\}$ and Q is defined by

$$Q(j,j+1) = p_j\,, \quad Q(j,j-1) = q_j\,, \quad Q(j,j) = r_j, \tag{0.2}$$

with $p_j + q_j + r_j = 1$ for every j. We assume that $p_j > 0$ for every $j \in E$, $q_j > 0$ for every $j \in E \setminus \{0\}$ and $q_0 = 0$.

This MC is irreducible, moreover it is aperiodic if and only if there exists j such that $r_j > 0$ (*Exercise*).

We introduce the function $\varphi : E \mapsto [0,\infty)$ defined by $\varphi(0) = 0$, $\varphi(1) = 1$ and by imposing that

$$(Q\varphi)(k) = \varphi(k)\,, \tag{0.3}$$

for $k \in \mathbb{N}$. This yields $(\varphi(k+1) - \varphi(k)) = (q_k/p_k)(\varphi(k) - \varphi(k-1))$ for $k \geq 1$ and therefore for $n \geq 2$

$$\varphi(n) \ = \ 1 + \sum_{m=1}^{n-1} \prod_{j=1}^{m} \frac{q_j}{p_j} \, . \tag{0.4}$$

Note that $\lim_{n\to\infty} \varphi(n) =: \varphi(\infty)$ exists and takes value in $(1, \infty]$. We set $T_a := \inf\{n = 0, 1, \ldots : \ X_n = a\}$. By the Optional Stopping Theorem we have that, for $a < x < b$, $\varphi(x) = \mathbb{E}_x[\varphi(X_{T_a \wedge T_b}]$ from which we readily extract

$$\mathbb{P}_x(T_b > T_a) \ = \ \frac{\varphi(x) - \varphi(a)}{\varphi(b) - \varphi(a)} \ \Longrightarrow \ \mathbb{P}_x(T_b > T_0) \ = \ \frac{\varphi(x)}{\varphi(b)} \, . \tag{0.5}$$

Note finally that $\nu(x) := \prod_{k=1}^{x} (p_{k-1}/q_k)$ is reversible for Q.

We can therefore conclude that (*Exercise*):

Proposition 2.1. *The birth and death MC is recurrent if and only if* $\varphi(\infty) = \infty$. *In this case* ν *is unique up to a multiplicative factor. Moreover it is finite recurrent if and only if* $\sum_x \prod_{k=1}^{x} (p_{k-1}/q_k) < \infty$.

Note that this result implies in particular that the symmetric simple random walk is null recurrent, as well as the well known fact that an asymmetric simple random walk is transient.

3. Discrete renewal processes

The basic object is $\tau = \{\tau_j\}_{j=0,1,\ldots}$ with $\tau_0 = 0$ and $\tau_j - \tau_{j-1} =: \eta_j$ and $(\eta_j)j = 1, 2, \ldots$ are IID variables taking values in $\{1, 2, \ldots, \infty\} = \mathbb{N} \cup \{\infty\}$. We set $K(n) := \mathbb{P}(\eta_1 = n)$ so in general $\sum_{n\in\mathbb{N}} K(n) \leq 1$, while $\sum_{n\in\mathbb{N}} K(n) + K(\infty) = 1$. We can view τ as a random subset of $\mathbb{N} \cup \{\infty\}$. In fact either $K(\infty) = 0$ and $|\tau| = \infty$ a.s. or $K(\infty) > 0$ and τ is a.s. a finite set (containing ∞). We set

$$u(n) \ := \ \mathbb{P}(n \in \tau) \ = \ \mathbb{P}\left(\exists j \text{ such that } \tau_j = n\right) \, , \tag{0.6}$$

and $u(\cdot)$ is called *renewal function*. We say that τ is aperiodic if there exists no integer $p > 1$ such that $\{n \in \mathbb{N} : \ K(n) > 0\} \subset p\mathbb{N}$. We write η for η_1

Theorem 3.1 (Renewal Theorem). *For an aperiodic renewal*

$$\lim_{n\to\infty} u(n) \ = \ \frac{1}{\mathbb{E}[\eta]} \, . \tag{0.7}$$

Of course the right-hand side of (0.7) is zero if $\mathbb{E}[\eta_1] = \infty$ and the statement holds without aperiodicity condition in this case. The proof is a direct application (*Exercise*) of the Ergodic Theorem for MC to one of the two MC's that we are going to build (and study) now.

3.1. The backward recurrence time. We set $A_n := n - \sup(\tau \cap [0,n]) \in E := \mathbb{N} \cup \{0\}$. Looking from time n, this is the time elapsed since the last renewal.

PROPOSITION 3.2. *A is a Q-MC with*

$$Q(j, j+1) = \mathbb{P}(\eta > j+1 | \eta > j) \quad \text{and} \quad Q(j,0) = 1 - \mathbb{P}(\eta > j+1 | \eta > j)\,,$$

for $j = 0, 1, \ldots$ with the convention that $\mathbb{P}(\eta > j+1|\eta > j) = 0$ if $\mathbb{P}(\eta > j) = 0$. Various facts:

- *A is transient if and only if $\mathbb{P}(\eta = \infty) > 0$;*
- *A is positive recurrent if and only if $\mathbb{E}[\eta] < \infty$;*
- *A is aperiodic if and only if τ is;*
- *the unique (up to a factor) invariant measure μ satisfies*

$$\mu(n) = \mu(0)\mathbb{P}(\eta > n)\,;$$

- *A has no invariant measure if (and only if) it is transient;*
- *A is positive recurrent if and only if $\mathbb{E}[\eta] < \infty$ and we remark, once μ is normalized, that $\mu(0) = 1/\mathbb{E}[\eta]$.*

Proof: *Exercise.*

3.2. The forward recurrence time. We set $B_n = \inf(\tau \cap [n, \infty]) - n \in E = \mathbb{N} \cup \{0, \infty\}$. Looking from time n, this is the time that is missing till the next renewal (unless at time n there is a renewal, so the next renewal is considered to be the one at n and $B_n = 0$).

PROPOSITION 3.3. *B is a Q-MC on E with*

$$Q(j, j-1) = 1 \text{ for } j \in \mathbb{N} \text{ and } Q(0, j) = \mathbb{P}(\eta = j+1) \text{ for } j \in \mathbb{N} \cup \{0, \infty\}\,,$$

and $Q(\infty, \infty) = 1$. B is not irreducible: it is impossible to go from ∞ to $n = 0, 1, \ldots$. Various facts:

- *$E \setminus \{\infty\}$ is transient (in the obvious sense) if and only if $\mathbb{P}(\eta = \infty) > 0$;*
- *$E \setminus \{\infty\}$ is positive recurrent if and only if $\mathbb{E}[\eta] < \infty$;*
- *B is aperiodic if and only if τ is;*
- *the unique (up to a factor) invariant measure μ supported on $E \setminus \{\infty\}$ satisfies*

$$\mu(n) = \mu(0)\mathbb{P}(\eta > n) \quad \text{for every } n \in E \setminus \{\infty\}\,;$$

- *B has δ_∞ as unique invariant measure if (and only if) $E \setminus \{\infty\}$ is transient;*
- *Assume $B_0 \neq \infty$ a.s.. Then B is positive recurrent if and only if $\mathbb{E}[\eta] < \infty$ and we remark, once μ is normalized, that $\mu(0) = 1/\mathbb{E}[\eta]$.*

Proof: *Exercise.*