Investment/consumption choice in illiquid markets with random trading times

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Abstract. We consider a portfolio/consumption selection problem in a liquidity risk model introduced in [11], and further investigated in [12] and [4]. This survey paper summarizes the main results in these works. In this illiquidity market modeling, stock prices are quoted and observed only at exogenous random times corresponding to the arrivals of buy/sell orders. The investor trades the stock at these random times, while consuming continuously from his cash holdings, and the goal is to maximize the expected utility from consumption. This mixed discrete/continuous stochastic control problem is solved by a dynamic programming approach, which leads to a coupled system of Integro-Partial Differential Equations (IPDE). Analytic characterization of the value functions and of the optimal strategies are derived, and we provide a convergent numerical algorithm for the resolution to this coupled system of IPDE. Several numerical experiments illustrate the impact of the restricted liquidity trading opportunities, and we measure in particular the utility loss with respect to the classical Merton consumption problem.

Key words. Liquidity, random trading times, portfolio/consumption problem, cost of liquidity, integro-partial differential equations, viscosity solutions.

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1 Introduction

Liquidity risk is one of the most significant risk factors in financial economy, yet a lot remains to be done at the theoretical level for designing appropriate measures of liquidity and understanding the mechanisms which underly it. In general, the market liquidity is the ability to quickly liquidate big volumes to low costs when assets have to be converted into cash. Therefore, the liquidity is a three-dimensional measure composed of: (i) volume: the size of traded position, (ii) price: the costs which are caused by trading the position, (iii) time: the point in time when one has to trade or execute the position. There have been numerous approaches to modeling liquidity over the years, mostly focusing on the volume and price measures of market liquidity. In this direction, the recent papers [1], [2], [3] or [14] studied the price impact, that is the correlation between an incoming order (to buy or sell) and the subsequent price change.

The temporal dimension of market liquidity is related to the restriction on asset price observation, trade or execution times, and is a crucial determinant for liquidity measure. It has been largely studied in the econometrics of high-frequency data, especially for volatility estimation. The next important issue in this modeling is to understand
the implications for pricing and risk management. In this perspective, Schwartz and Tebaldi [15] and Longstaff [8] assumed in their model that illiquid assets could only be traded at the starting date and at a fixed terminal horizon. In a less extreme modelling, Rogers and Zane [13] and Matsumoto [9] consider random trading dates with continuous-time observation, by assuming that trade succeed only at the jump times of a Poisson process, and study the impact on the portfolio choice problem.

In this paper, we consider a description of liquidity risk introduced in Pham and Tankov [11], which is consistent with the situation often viewed by practitioners where their ability to trade assets is limited or restricted to the times when a quote comes into the market. The price of the risky asset can be observed and trade orders can be passed only at random times of an exogenous Poisson process. These times model the arrival of buy/sell orders in an illiquid market, or the dates on which the results of a hedge fund are published. This setup was inspired by recent papers of Frey and Runggaldier [6] and Cvitanic, Liptser and Rozovskii [5], who assume in addition that there is an unobservable stochastic volatility, and are interested in the estimation of this volatility. In our liquidity risk context, we suppose that the investor is also allowed to consume (or distribute dividends) continuously from the bank account, and the objective is to maximize the expected discounted utility of consumption. The resulting optimization problem is a nonstandard mixed discrete/continuous time stochastic control problem, which leads via the dynamic programming principle to a coupled system of nonlinear integro-partial differential equations (IPDE). The aim of this paper is to summarize the main results recently obtained in [11], [12] and [4] about the IPDE characterization and regularity of the value functions, the existence and representation of the optimal strategies, and the numerical resolution of this investment/consumption problem in a liquidity risk context with computational illustrations compared to the classical Merton problem.

The rest of the paper is structured as follows. In Section 2, we describe the liquidity risk model, and we formulate in Section 3 the optimal investment/consumption problem. Section 4 contains the main results of the paper, by stating the IPDE viscosity characterization and regularity of the value function, which is then used for deriving the optimal portfolio and consumption policies. Finally, we describe in Section 5 a convergent numerical algorithm and give some numerical tests for measuring the impact of our liquidity trading constraints.

2 The liquidity risk model

We consider a market model in which the bids and offers on a risky asset are not available at any time. The arrivals of buy/sell orders occur at the jumps \((\tau_k)_k, \tau_0 = 0 < \tau_1 < \ldots < \tau_k\) of a Poisson process with constant intensity \(\lambda\), independent of the asset price process \(S\). For simplicity, we assume that the continuous time price process \(S\) follows a Black-Scholes dynamics:

\[
dS_t = S_t(bdt + \sigma dW_t),
\]  
(2.1)
where $W$ is a standard brownian motion on a probability space $(\Omega, \mathcal{G}, P)$, $b$, $\sigma > 0$ are positive constants, and we denote by $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ the natural filtration of $W$, which is also the filtration generated by the asset price $S$.

In this illiquid market, the investor can observe and trade $S$ only at the random times $(\tau_k)_{k \geq 0}$. We denote by $F = (F_t)_{t \geq 0}$ the natural filtration of $W$, which is also the filtration generated by the asset price $S$.

In this illiquid market, the investor can observe and trade $S$ only at the random times $(\tau_k)_{k \geq 0}$. We denote by $Z_{t,s} = S_s - S_t, 0 \leq t \leq s,$ and by $Z_k = Z_{\tau_k, \tau_k}, k \geq 1,$ the observed return process valued in $(-1, \infty)$. We set by convention $Z_0$ to some fixed constant. The investor may also consume continuously from the bank account (interest rate is assumed w.l.o.g. to be zero) between two trading dates. We introduce the continuous observation filtration $G^c = (G_t)_{t \geq 0}$ with :

$$G_t = \sigma \{ (\tau_k, Z_k) : \tau_k \leq t \} ,$$

and the discrete observation filtration $G^d = (G_{\tau_k})_{k \geq 0}$.

A portfolio/consumption policy is a mixed discrete-continuous process $(\alpha, c)$, where $\alpha = (\alpha_k)_{k \geq 0}$ is real-valued $G^d$-adapted, and $c = (c_t)_{t \geq 0}$ is a nonnegative $G^c$-adapted process: $\alpha_k$ represents the amount of stock invested for the period $[\tau_k, \tau_{k+1}]$ after observing the stock price at time $\tau_k$, and $c_t$ is the consumption rate at time $t$ based on the available information. Starting from an initial capital $x \geq 0$, and given a control policy $(\alpha, c)$, we denote by $X^{x}_{\tau_k}$ the wealth of the investor at time $\tau_k$ given by:

$$X^{x}_{\tau_k} = x - \int_0^{\tau_k} c_t dt + \sum_{i=0}^{k} \alpha_i Z_{i+1}, \quad k \geq 0. \tag{2.2}$$

Given $x \geq 0$, we say that a control policy $(\alpha, c)$ is admissible, and we denote it by $(\alpha, c) \in A(x)$ if :

$$X^{x}_{\tau_k} \geq 0, \quad a.s. \quad \forall \ k \geq 0. \tag{2.3}$$

**Remark 2.1** For all $k \geq 0$, conditionally on the interarrival times $\tau_{k+1} - \tau_k = t \geq 0$, we see from (2.1) that $Z_{k+1}^\tau$ is independent of $\mathcal{G}_{\tau_k}$, and has distribution $p(t, dz)$ of support $(-1, \infty)$, with

$$p(t, dz) = P \left[ e^{(b-\frac{\sigma^2}{2})t + \sigma W_t} - 1 \in dz \right].$$

Notice that

$$\int zp(t, dz) = \mathbb{E} \left[ Z_{k+1} \mathbb{1}_{\mathcal{G}_{\tau_k}, \tau_{k+1} - \tau_k = t} \right] = e^{bt} - 1 \geq 0, \quad k \geq 0, t \geq 0. \tag{2.4}$$
Remark 2.2

Constrained policies. Since $X_{\tau_{k+1}}^x = X_{\tau_k}^x - \int_{\tau_k}^{\tau_{k+1}} c_u \, du + \alpha_k Z_{k+1}$, and the support of $Z_{k+1}$ is $(-1, \infty)$, we see that the admissibility condition (2.3) for $(\alpha, c) \in A(x)$ is written as:

$$X_{\tau_k}^x - \int_{\tau_k}^{s} c_u \, du + \alpha_k z \geq 0, \quad \forall k \geq 0, \forall s \geq \tau_k, \forall z \in (-1, \infty).$$

almost surely. This means that we have a no-short sale constraint (both on the risky asset and bank account):

$$0 \leq \alpha_k \leq X_{\tau_k}^x, \quad \forall k \geq 0,$$

together with the consumption constraint:

$$\int_{\tau_k}^{\infty} c_u \, du \leq X_{\tau_k}^x - \alpha_k, \quad \forall k \geq 0.$$ (2.6)

Remark 2.3

Embedding in a continuous-time wealth process. Let us introduce the continuous time filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0} = \mathbb{F} \vee \mathbb{G}$. In other words, $\mathcal{H}_t$ corresponds to the path observation of the asset price and of the random times up to time $t$. Notice that $W$ is still a Brownian motion under $\mathbb{H}$, and the dynamics of $S$ under $(\mathbb{P}, \mathbb{H})$ is still governed by (2.1). Given $x \geq 0$, and $(\alpha, c) \in A(x)$ with corresponding discrete time wealth process $(X_{\tau_k}^x)_{k \geq 0}$, let us define the continuous time process $(X_t^x)_{t \geq 0}$ by

$$X_t^x = X_{\tau_k}^x - \int_{\tau_k}^{t} c_u \, du + \alpha_k Z_{\tau_k}, \quad \tau_k < t \leq \tau_{k+1} \quad k \geq 0,$$

$$= x - \int_{0}^{t} c_u \, du + \int_{0}^{t} H_u \, dS_u, \quad t \geq 0,$$ (2.7)

where $H$ is the simple integrand process

$$H_t = \sum_{k=0}^{\infty} \frac{\alpha_k}{S_{\tau_k}} 1_{\tau_k < t \leq \tau_{k+1}}, \quad t \geq 0,$$

representing the number of shares invested in the risky asset. Notice that $H$ is $\mathbb{H}$-predictable (hence adapted), and $c$ is $\mathbb{H}$-adapted. Moreover, from (2.5) and (2.6), we easily see that $H$ satisfies the no-short sale constraint:

$$0 \leq H_t S_t \leq X_t^x, \quad t \geq 0.$$ (2.8)

The continuous time process $X$ has the meaning of a shadow wealth process: it is not observed except for at times $\tau_k$.

3 The optimal investment/consumption problem

We investigate an optimal investment/consumption problem in the illiquid market described in the previous section. We are given a utility function defined on $\mathbb{R}_+$, with
$U(0) = 0$, strictly increasing, strictly concave and $C^1$ on $(0, \infty)$, satisfying the Inada conditions $U''(0^+) = \infty$ and $U''(\infty) = 0$. We also assume the following growth condition on $U$: there exists $\gamma \in (0, 1)$ s.t.

$$U(x) \leq K_1 x^\gamma, \; x \geq 0,$$

for some positive constant $K_1$.

We consider the optimal investment/consumption problem:

$$v(x) = \sup_{(\alpha, c) \in \mathcal{A}(x)} E \left[ \int_0^\tau e^{-\rho t} U(c_t) dt \right], \; x \geq 0, \tag{3.1}$$

where $\rho > 0$ is a positive discount factor. Actually, it is proved in [12] that for $\rho$ large enough, namely

$$\rho > b \gamma,$$

(which we shall assume in the sequel), then the nonnegative value function $v$ is finite, and satisfies the growth condition

$$v(x) \leq K x^\gamma, \; x \geq 0, \tag{3.2}$$

for some positive constant $K$.

**Remark 3.1** Given $x \geq 0$, denote by $\mathcal{A}_H(x)$ (resp. $\mathcal{A}_F(x)$) the set of pairs of $\mathbb{H}$-adapted (resp. $\mathbb{F}$-adapted) processes $(H, c)$ with $c$ nonnegative, and corresponding wealth process given in (2.7), such that the no-short sale constraint (2.8) holds. Consider the associated continuous time optimal investment/consumption problems

$$v_H(x) = \sup_{(H, c) \in \mathcal{A}_H(x)} E \left[ \int_0^\tau e^{-\rho t} U(c_t) dt \right], \; x \geq 0,$$

and

$$v_M(x) = \sup_{(H, c) \in \mathcal{A}_F(x)} E \left[ \int_0^\tau e^{-\rho t} U(c_t) dt \right], \; x \geq 0. \tag{3.3}$$

Problem (3.3) is the classical Merton portfolio/consumption choice problem under no-short sale constraints, and based on the continuous time observation of the stock price. It is not hard to check that independent information provided by the random times $\tau_k$ does not increase the maximal expected utility of consumption; in other words, the value functions $v_H$ and $v_M$ coincide, and in view of Remark 2.3, we have

$$v \leq v_M (= v_H). \tag{3.4}$$

We use a dynamic programming approach for solving the control problem (3.1). The starting point is the following version of the dynamic programming principle (DPP) adapted to our context, and proved rigorously in [11]:

$$v(x) = \sup_{(\alpha, c) \in \mathcal{A}(x)} E \left[ \int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} v(X^{x}_{\tau_1}) \right]. \tag{3.5}$$
From the expression (2.2) of the wealth, and the measurability conditions on the control, the above dynamic programming relation is written as

\[
v(x) = \sup_{(a,c) \in A_d(x)} \mathbb{E} \left[ \int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} v(x - \int_0^{\tau_1} c_t dt + aZ_1) \right],
\]

where \(A_d(x)\) is the set of pairs \((a,c)\) with \(a\) deterministic constant, and \(c\) a deterministic nonnegative process s.t. (see Remark 2.2) \(a \in [0,x]\) and

\[
\int_0^{\infty} c_u du \leq x - a
\]

Given \(a \in [0,x]\), we denote by \(C_a(0,x)\) the set of deterministic nonnegative processes satisfying (3.7). The r.h.s. of (3.6) is then written explicitly in:

\[
v(x) = \sup_{a \in [0,x]} \int_0^{\infty} e^{-(\rho + \lambda) t} \left[ U(c_t) + \lambda \int v(x - \int_t^s c_u du + az) p(s,dz) dt \right].
\]

Denote by

\[
\mathcal{D} = \mathbb{R}_+ \times \mathcal{X} \quad \text{with} \quad \mathcal{X} = \{ (x,a) \in \mathbb{R}_+ \times \mathbb{R}_+ : a \leq x \},
\]

and let us introduce the dynamic auxiliary control problem: for \((t,x,a) \in \mathcal{D},\)

\[
\hat{v}(t,x,a) = \sup_{c \in C_a(t,x)} \int_t^{\infty} e^{-(\rho + \lambda)(s-t)} \left[ U(c_s) + \lambda \int v(Y^{t,x}_s + az) p(s,dz) ds \right],
\]

where \(C_a(t,x)\) is the set of deterministic nonnegative processes \(c = (c_s)_{s \geq t}\) s.t.

\[
\int_t^{\infty} c_u du \leq x - a
\]

and \(Y^{t,x}\) is the deterministic controlled process by \(c \in C_a(t,x):\)

\[
Y^{t,x}_s = x - \int_t^s c_u du, \quad s \geq t.
\]

From (3.8)-(3.9), the original value function is then related to this auxiliary optimization problem by:

\[
v(x) = \sup_{a \in [0,x]} \hat{v}(0,x,a).
\]
The Hamilton-Jacobi (in short HJ) equation associated to the deterministic control problem (3.9) is the following Integro Partial Differential Equation (in short IPDE):

\[(\rho + \lambda)\hat{v} - \frac{\partial \hat{v}}{\partial t} - \hat{U}\left(\frac{\partial \hat{v}}{\partial x}\right) - \lambda \int v(x + az)p(t, dz) = 0, (t, x, a) \in \mathcal{D}, (3.13)\]

where \(\hat{U}\) is the Legendre transform of \(U\), i.e. \(\hat{U}(y) = \sup_{x \geq 0} [U(x) - xy]\).

To sum up, the dynamic programming principle for our original stochastic optimization problem (3.1) leads to a first-order coupled IPDE (3.12)-(3.13): Problem (3.9) is a family over \(a \in \mathbb{R}_+\) of standard deterministic control problems on infinite horizon, associated to the HJ equation (3.13), and the coupling comes from the fact that the reward function appearing in the definition of problem (3.9) or in its IPDE (3.13) depends on the value function of problem (3.12) and vice-versa. The next section provides a rigorous analytic characterization of the value functions through their dynamic programming (in short DP) equations (3.12)-(3.13), by showing the regularity properties of the value functions, and then as a byproduct the existence (and uniqueness) of the optimal control feedback.

4 Analytic characterization of the value functions and optimal strategies

We first recall from [12] some basic properties on the value functions \((v, \hat{v})\) defined in the previous section. The value function \(v\) is strictly increasing, concave on \(\mathbb{R}_+\), and lies in \(C_+^{\mathbb{R}_+}\), the set of nonnegative continuous functions on \(\mathbb{R}_+\). The value function \(\hat{v}\) lies in \(C_+^{\mathcal{D}}\), the set of nonnegative continuous functions on \(\mathcal{D}\), and satisfies the boundary condition

\[
\lim_{\bar{x} \to a} \hat{v}(\bar{t}, \bar{x}, a) = \lambda \int_{t}^{\infty} e^{-(\rho + \lambda)(s-t)} \int v(a + az)p(s, dz)ds, \forall t \geq 0. \quad (4.1)
\]

It satisfies the growth estimate

\[
\hat{v}(t, x, a) \leq Ke^{bt}x, \quad (t, x, a) \in \mathcal{D}, \quad (4.2)
\]

for some positive constant \(K\). Moreover, \(\hat{v}\) is strictly increasing in \(x \geq a\), given \((t, a) \in \mathbb{R}_+ \times \mathbb{R}_+,\) and is concave in \((x, a) \in \mathcal{X}\), given \(t \in \mathbb{R}_+\).

We now provide a characterization of the value functions to the DP equation (3.12)-(3.13) by means of the notion of viscosity solution adapted to our context.

**Definition 4.1** A pair of functions \((w, \tilde{w}) \in C_+^{\mathbb{R}_+} \times C_+^{\mathcal{D}}\) is a viscosity solution to (3.12)-(3.13) if the two following properties hold simultaneously:

(i) **viscosity supersolution property**: \(w(x) \geq \sup_{a \in [0, x]} \tilde{w}(0, x, a)\), for all \(x \geq 0\), and

\[
(\rho + \lambda)\tilde{w}(\bar{t}, \bar{x}, a) - \frac{\partial \tilde{w}}{\partial t}(\bar{t}, \bar{x}) - \tilde{U}\left(\frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})\right) - \lambda \int w(\bar{x} + az)p(\bar{t}, dz) \geq 0,
\]
for all $a \in \mathbb{R}_+$, for any test function $\varphi \in C^1(\mathbb{R}_+ \times (a, \infty))$, and $\left(\bar{t}, \bar{x}\right) \in \mathbb{R}_+ \times (a, \infty)$, which is a local minimum of $(\hat{w}(\cdot, \cdot, a) - \varphi)$.

(ii) viscosity subsolution property: $w(x) \leq \sup_{a \in [0, x]} \hat{w}(0, x, a)$, for all $x \geq 0$, and

$$\left(\rho + \lambda\right)\hat{w}(\bar{t}, \bar{x}, a) - \frac{\partial \varphi}{\partial \bar{t}}(\bar{t}, \bar{x}) - \bar{U} \left(\frac{\partial \varphi}{\partial \bar{x}}(\bar{t}, \bar{x})\right) - \lambda \int w(\bar{x} + az)p(\bar{t}, dz) \leq 0,$$

for all $a \in \mathbb{R}_+$, for any test function $\varphi \in C^1(\mathbb{R}_+ \times (a, \infty))$, and $\left(\bar{t}, \bar{x}\right) \in \mathbb{R}_+ \times (a, \infty)$, which is a local maximum of $(\hat{w}(\cdot, \cdot, a) - \varphi)$.

We reformulate the viscosity characterization result proved in [12].

**Theorem 4.2** The pair of value functions $(v, \hat{v})$ defined in (3.1), (3.9) is the unique viscosity solution to (3.12)-(3.13) in the sense of Definition 4.1, satisfying the growth conditions (3.2), (4.2), and the boundary condition (4.1).

The above characterization makes the computation of the value functions possible (see the next section) but does not yield the optimal policies in explicit form. We need to go beyond the viscosity property, and focus on the regularity of the value functions. By using arguments of (semi)conavity and the strict convexity of the Hamiltonian for the IPDE in connection with viscosity solutions, it is proved in [4] that the value functions are continuously differentiable.

**Theorem 4.3** (1) The value function $v$ lies in $C^1(0, \infty)$, and any maximum point in (3.12) is interior for every $x > 0$. Moreover, $v'(0^+) = \infty$.

(2) For all $a \in \mathbb{R}_+$, we have $\hat{v}(\cdot, a) \in C^1([0, \infty) \times (a, \infty))$, and

$$\lim_{x \downarrow a} \frac{\partial \hat{v}}{\partial x}(t, x, a) = \infty, \quad t \geq 0.$$  

In particular, $\hat{v}$ satisfies the IPDE (3.13) in the classical sense.

From the regularity of the value functions, we derive the existence of an optimal control through a verification theorem, and the optimal consumption strategy is produced in feedback form in terms of the classical derivatives of the value functions. We denote by $I = (U')^{-1} : (0, \infty) \rightarrow (0, \infty)$ the inverse function of the derivative $U'$, and we consider for each $a \in \mathbb{R}_+$ the nonnegative measurable function

$$\hat{c}(t, x, a) = \arg \max_{c \geq 0} \left[U(c) - c \frac{\partial \hat{v}}{\partial x}(t, x, a)\right] = I \left(\frac{\partial \hat{v}}{\partial x}(t, x, a)\right), \quad (t, x, a) \in \mathcal{D}.$$  

**Theorem 4.4** (1) Let $(x, a) \in \mathcal{X}$, i.e. $x \geq a \geq 0$. There exists a unique solution, denoted by $\hat{Y}(x, a)$, to the equation

$$Y_t = x - \int_0^t \hat{c}(s, Y_s, a) ds, \quad t \geq 0,$$

(4.3)
and the pair \((\hat{Y}(x,a), a)\) lies in \(X\), i.e. \(\hat{Y}_t(x,a) \geq a\) for all \(t \geq 0\). The feedback control 
\[ \{ \hat{c}(t, \hat{Y}_t(x,a), a), t \geq 0 \} \text{ is optimal for } \hat{v}(0, x, a). \]
(2) For any \(x \geq 0\), there exists an optimal control policy \((\alpha^*, \epsilon^*) \in A(x)\) for \(v(x)\), given by
\[ \alpha^*_k = \arg \max_{a \in [0, \hat{X}_{\tau_k}]} \hat{v}(0, X^x_{\tau_k}, a), \quad k \geq 0, \]  
\[ \epsilon^*_k = \hat{c}(t - \tau_k, Y^{(k)}_t, \alpha^*_k), \quad \tau_k \leq t < \tau_{k+1}, \quad k \geq 0, \]  
where \(X^x_{\tau_k}\) is the wealth of the investor at time \(\tau_k\) given in (2.2) with the feedback control \((\alpha^*, \epsilon^*)\), and \(Y^{(k)}_t = \hat{Y}_{t-\tau_k}(X^x_{\tau_k}, \alpha^*_k), t \geq \tau_k\), solution to
\[ Y^{(k)}_t = X^x_{\tau_k} - \int_{\tau_k}^t \epsilon^*_s ds, \quad t \geq \tau_k, \]  
represents the wealth between two trading dates \(\tau_k\) and \(\tau_{k+1}\).

**Proof.** (1) Let \(c \in C_u(0, x)\) and \(Y^x = x - \int_0^T c(t) dt\) the corresponding wealth process. By applying standard differential calculus to \(e^{-(\rho + \lambda)t} \hat{v}(t, Y^x_t, a)\) between \(t = 0\) and \(t = T\), we have
\[ e^{-(\rho + \lambda)T} \hat{v}(T, Y^x_T, a) - \hat{v}(0, x, a) = \int_0^T e^{-(\rho + \lambda)t} \left[ - (\rho + \lambda) \hat{v} + \frac{\partial \hat{v}}{\partial t} + \frac{\partial}{\partial x} \frac{\partial \hat{v}}{\partial x} \right] (t, Y^x_t, a) dt \]
\[ = - \int_0^T e^{-(\rho + \lambda)t} \left[ U(c_t) + \lambda \int v(Y^x_t + az) \right] p(t, dz) + \int_0^T e^{-(\rho + \lambda)t} \left[ U(c_t) - c_t \frac{\partial}{\partial x} (t, Y^x_t, a) - \tilde{U} \left( \frac{\partial}{\partial x} (t, Y^x_t, a) \right) \right] dt, \]
where we used in the last equality the property that \(\hat{v}\) satisfies the IPDE (3.13). From the growth estimate (4.2) for \(\hat{v}\), the increasing monotonicity of \(\hat{v}(T, ., a)\), and since \(\rho > b_\gamma\), we see that \(\lim_{T \to \infty} e^{-(\rho + \lambda)T} \hat{v}(T, Y^x_T, a) = 0\), and thus by sending \(T\) to infinity into (4.6):
\[ \hat{v}(0, x, a) = \int_0^\infty e^{-(\rho + \lambda)t} \left[ U(c_t) + \lambda \int v(Y^x_t + az) \right] p(t, dz) + \int_0^\infty e^{-(\rho + \lambda)t} \left[ U(c_t) - c_t \frac{\partial}{\partial x} (t, Y^x_t, a) - \tilde{U} \left( \frac{\partial}{\partial x} (t, Y^x_t, a) \right) \right] dt. \]

The existence and uniqueness of a solution \(\hat{Y}(x,a)\) to (4.3), which satisfies \(\hat{Y}_t(x,a) \geq a\) for all \(t \geq 0\) is proved in [4]. The wealth process \(Y(x,a)\) is associated to the admissible control \(\hat{c}_t = \hat{c}(t, \hat{Y}_t(x,a), a), t \geq 0\), and by definition of the function \(\hat{c}\), we have
\[ \hat{U} \left( \frac{\partial}{\partial x} (t, \hat{Y}_t(x,a), a) \right) = U(\hat{c}_t) - \hat{c}_t \frac{\partial}{\partial x} (t, \hat{Y}_t(x,a), a), \quad t \geq 0, \]
so that from (4.7)
\[
\dot{v}(0, x, a) = \int_0^\infty e^{-(\rho + \lambda)t} \left[ U(\dot{z}) + \lambda \int v(\dot{Y}_t(x, a) + az) \right] p(t, dz), \quad (4.8)
\]
which shows the optimality of the control \(\hat{c}\).

(2) We first show that for any \((\alpha, c) \in A(x)\), and \(k \geq 0\),
\[
E \left[ \int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)} U(c_t) dt + e^{-\rho(\tau_{k+1}-\tau_k)} v(X_{\tau_{k+1}}^x) \big| G_{\tau_k} \right]
\]
\[
= \int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)} \left[ U(c_t) + \lambda \int v \left( X_{\tau_k}^x - \int_{\tau_k}^{t} c_u du + \alpha_k z \right) p(t-\tau_k, dz) \right] dt.
\]
Indeed, since \(X_{\tau_{k+1}}^x = X_{\tau_k}^x - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_k Z_{k+1}\), we have by the law of conditional toy expectations:
\[
E \left[ \int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)} U(c_t) dt + e^{-\rho(\tau_{k+1}-\tau_k)} w(X_{\tau_{k+1}}^x) \big| G_{\tau_k} \right]
\]
\[
= E \left[ \int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)} U(c_t) dt + e^{-\rho(\tau_{k+1}-\tau_k)} w \left( X_{\tau_k}^x - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_k Z_{k+1} \right) \big| G_{\tau_k}, \tau_{k+1} - \tau_k \right] \big| G_{\tau_k}
\]
\[
= \int_{\tau_k}^{\tau_{k+1}} E \left[ e^{-\rho(t-\tau_k)} U(c_t) dt + e^{-\rho(\tau_{k+1}-\tau_k)} \int w \left( X_{\tau_k}^x - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_k Z_{k+1} \right) p(\tau_{k+1} - \tau_k, dz) \big| G_{\tau_k} \right]
\]
\[
= \int_{0}^{\infty} \int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)} U(c_t) dt + e^{-\rho s} \int w \left( X_{\tau_k}^x - \int_{\tau_k}^{s} c_u du + \alpha_k Z_{k+1} \right) p(s, dz) \big| G_{\tau_k} \right]
\]
where we used Remark 2.1 in the second equality and the fact that \(\tau_{k+1} - \tau_k\) follows an exponential law of parameter \(\lambda\), in the last one. We obtain (4.9) with Fubini’s theorem and the change of variable \(s \rightarrow s + \tau_k\).

We next prove that for any \((\alpha, c) \in A(x)\), and \(n \geq 0\),
\[
E \left[ e^{-\rho \tau_n} (X_{\tau_n}^x)^\gamma \right] \leq x^\gamma \delta^n, \quad \text{with} \quad \delta = \frac{\lambda}{\rho - b\gamma + \lambda} < 1 \quad (4.10)
\]
Indeed, from Jensen’s inequality
\[
E \left[ (X_{\tau_{n-1}}^x + \alpha_{n-1} Z_n)^\gamma \big| G_{\tau_{n-1}}, \tau_n - \tau_{n-1} \right] \leq \left( X_{\tau_{n-1}}^x + \alpha_{n} \int \rho \left( \tau_n - \tau_{n-1}, dz \right) \right)^\gamma
\]
\[
= \left( X_{\tau_{n-1}}^x + X_{\tau_{n-1}}^x \left( e^{b(\tau_n - \tau_{n-1})} - 1 \right) \right)^\gamma
\]
\[
= (X_{\tau_{n-1}}^x)^\gamma e^{b\gamma(\tau_n - \tau_{n-1})},
\]
where we used (2.4) and (2.5). Thus, by writing that $X^x_{\tau_n} \leq X^x_{\tau_{n-1}} + \alpha_{n-1}Z_n$, and by the law of iterated conditional expectations, we get:

\[
E[e^{-\rho\tau_n}(X^x_{\tau_n})^\gamma] \leq E[e^{-(\rho-b\gamma)(\tau_n-\tau_{n-1})}e^{-\rho\tau_{n-1}}(X^x_{\tau_{n-1}})^\gamma] = E[E[e^{-(\rho-b\gamma)(\tau_n-\tau_{n-1})}\int_0^{\tau_n} e^{-(\rho-b\gamma+\lambda)t} dt | \mathcal{G}_{\tau_{n-1}}]] = \delta E[e^{-\rho\tau_{n-1}}(X^x_{\tau_{n-1}})^\gamma]
\]

We obtain the required inequality (4.10) by induction on $n$.

Consider the control policy in (4.4)-(4.5). By definition of $\hat{Y}$ in (4.3), the associated wealth process satisfies for all $k \geq 0$

\[
X^x_{\tau_{k+1}} = X^x_{\tau_k} + \int_{\tau_k}^{\tau_{k+1}} c_t^* dt + \alpha^*_k Z_{k+1} + \alpha^*_k Z_{k+1} \geq 0, \ a.s.,
\]

and thus $(\alpha^*, c^*) \in \mathcal{A}(x)$. From (3.12), definition of $\alpha^*$ and (4.8), we have

\[
v(X^x_{\tau_k}) = \hat{v}(0, X^x_{\tau_k}, \alpha^*_k) = \int_0^{\tau_k} e^{-(\rho+\lambda)(t-\tau_k)} U(c_t^*) + \lambda \int_{0}^{\tau_k} v(Y_t^{(k)} + \alpha^*_k Z_{k+1}) \rho(t-\tau_k, dz) dt + e^{-\rho(\tau_{k+1}-\tau_k)}v(X^x_{\tau_{k+1}} | \mathcal{G}_{\tau_k}),
\]

where we used (4.9) in the last equality. By iterating these relations for all $k$, and using the law of iterated conditional expectations, we obtain

\[
v(x) = E\left[ \int_0^{\tau_n} e^{-\rho t} U(c_t^*) dt + e^{-\rho \tau_n}v(X^x_{\tau_n}) \right].
\]

From the growth estimate (3.2), relation (4.10), and sending $n$ to infinity, we conclude that

\[
v(x) = E\left[ \int_0^{\infty} e^{-\rho t} U(c_t^*) dt \right].
\]

Furthermore, by using maximum principle, additional properties on the consumption policy between two trading dates are derived in [4], as solution of an Euler-Lagrange ordinary differential equation.

**Proposition 4.5** Suppose that $U \in C^2(0, \infty)$. Given an investment $a \in \mathbb{R}_+$ at time $t$ in the stock, and starting from an initial capital $Y(t, x, a) = x \geq a$, the optimal wealth process $Y(l, x, a)$ between two trading dates is twice differentiable, satisfies the
second-order ordinary differential equation
\[
\frac{d^2 \hat{Y}_s(t, x, a)}{ds^2} = \lambda \int v'(\hat{Y}_s(t, x, a) + az)p(s, dz) - (\rho + \lambda)U'(c_s), \quad s \geq t,
\]
\[
c_s = -\frac{d\hat{Y}_s(t, x, a)}{ds},
\]
and we have \( \lim_{s \to \infty} \hat{Y}_s(t, x, a) = a \).

5 Numerical solution and illustrations

In this section, we focus on the resolution of the DP equation (3.12)-(3.13), and we give some numerical tests for illustrating the impact of liquidity risk induced by the random trading times.

5.1 A numerical decoupling algorithm

The main difficulty in the numerical resolution of the IPDE (3.13) for \( \hat{v} \) comes from the coupling in the integral term involving \( \hat{v} \) via \( v \). To overcome this problem, we suggest the following iterative procedure. We start from an initial function \( v_0 \) defined on \( \mathbb{R}_+ \), as the value function of the consumption problem without trading:
\[
v_0(x) = \sup_{c \in C(x)} \int_0^\infty e^{-\rho t} U(c_t) dt,
\]
where \( C(x) \) is the set of nonnegative (deterministic) processes \( c = (c_t) \), s.t. \( x - \int_0^t c_s ds \geq 0 \) for all \( t \geq 0 \). \( v_0 \) is the unique solution with linear growth condition to the first-order differential equation
\[
\rho v_0 - \hat{U} \left( \frac{\partial v_0}{\partial x} \right) = 0, \quad x > 0,
\]
together with the boundary condition \( v_0(0^+) = 0 \). We then construct by induction a sequence of functions \((\hat{v}_n)_{n \geq 1}\) defined on \( \mathcal{D} \) and \((v_n)_{n \geq 0}\) defined on \( \mathbb{R}_+ \) by:
\[
\hat{v}_{n+1}(t, x, a) = \sup_{c \in C_n(t, x)} \int_t^\infty e^{-(\rho+\lambda)(s-t)} \left[ U(c_s) + \lambda \int v_{n}(\hat{Y}_s+x+az)p(s, dz) \right] ds
\]
\[
v_{n+1}(x) = \sup_{a \in [0, x]} \hat{v}_{n+1}(0, x, a), \quad n \geq 0.
\]
By the dynamic programming principle, the function \( \hat{v}_{n+1} \) satisfies the first-order PDE
\[
-(\rho + \lambda)\hat{v}_{n+1} + \frac{\partial \hat{v}_{n+1}}{\partial t} + \hat{U} \left( \frac{\partial \hat{v}_{n+1}}{\partial x} \right) + \lambda \int v_{n}(x + az)p(t, dz) = 0, \quad (t, x, a) \in \mathcal{D},
\]
and we have an approximate trading policy by taking:

$$\alpha_k^{(n)} \in \arg \max_{a \in [0,X_{2k}^x]} \hat{v}_n (0,X_{2k}^x,a), \quad k \geq 0.$$ 

The convergence of this iterative decoupling algorithm was studied in [11], where it is proved that the sequence of functions $\langle v_n, \hat{v}_n \rangle_n$ converges uniformly on any compact subset of $D$ and $\mathbb{R}_+$ to $(v, \hat{v})$. More precisely, for any compact subset $F$ and $G$ of $D$ and $\mathbb{R}_+$, there exist some positive constants $C_F$ and $C_G$ s.t.

$$0 \leq \sup_F (\hat{v} - \hat{v}_n) \leq C_F \delta^n, \quad \text{and} \quad 0 \leq \sup_G (v - v_n) \leq C_G \delta^n,$$

where $0 < \delta < 1$ is defined in (4.10).

### 5.2 Numerical illustrations

We now provide simulations for illustrating the impact of liquidity constraints on the attainable utility level and on the investment strategy. We shall compare our numerical experiments with the original Merton problem with no-short sale constraints, and defined in (3.3). We consider the case of power utility functions $U(x) = x^\gamma / \gamma$, and we recall that the value function and the optimal trading strategy (in amount) are explicitly given by

$$v_M(x) = K_M x^\gamma, \quad \bar{\alpha}_M = \min \left[ \frac{b}{(1-\gamma)\sigma^2}, 1 \right] X_t^x,$$

with

$$K_M = \frac{1}{\gamma} \left( \frac{1-\gamma}{\rho - \eta} \right)^{1-\gamma}, \quad \eta = \gamma \max_{\pi \in [0,1]} \left[ \pi b - \frac{1}{2} \pi^2 (1-\gamma) \sigma^2 \right].$$

We know from (3.4) that $v \leq v_M$. On the other hand, the value function $v$ is always bounded from below by the value function of the consumption problem without trading $v_0$, given in our present setting by $v_0(x) = K_0 x^\gamma$, with $K_0 = \frac{1}{\gamma} \left( \frac{1-\gamma}{\rho} \right)^{1-\gamma}$.

Given $(t, x, a) \in D$, notice that for any $\beta > 0$, we have $c \in C_\beta(t,x)$ if and only if $\beta c \in C_{\beta \bar{a}}(t,\beta x)$. We then easily deduce from (3.9) and (3.12) a scaling relation for the value function $v$ and the auxiliary value function $\hat{v}$:

$$\hat{v}(t, \beta x, \beta a) = \beta^\gamma \hat{v}(t, x, a), \quad v(\beta x) = \beta^\gamma v(x), \quad \forall \beta > 0.$$

The scaling relation for $v$ shows that it is of power type: $v(x) = v(1)x^\gamma$, hence of the same form as in the Merton model (see figure 5.2). The scaling relation for $\hat{v}$ implies that for all $\beta > 0$,

$$\hat{a} \in \arg \max_{a \in [0,x]} \hat{v}(0,x,a) \quad \text{if and only if} \quad \beta a \in \arg \max_{a \in [0,\beta x]} \hat{v}(0,\beta x,a).$$
From the feedback form (4.4), this shows that $\alpha_k^* \text{ is linear in } X_x^\tau_k$, or in other words the optimal investment strategy consists in investing a fixed proportion of the wealth into the risky asset. Moreover, we can reduce the dimension of the problem and denote by

$$v(x) = \vartheta_1 x^\gamma, \quad \hat{v}(t, x, a) = a^\gamma \tilde{v}(t, \xi), \quad \xi = x/a,$$

where $\vartheta_1$ and $\tilde{v}$ are solution to

$$(\rho + \lambda)\tilde{v} - \frac{\partial \tilde{v}}{\partial t} - \tilde{U} \left( \frac{\partial \tilde{v}}{\partial \xi} \right) - \lambda \vartheta_1 \int (\xi + z)^\gamma \rho(t, dz) = 0,$$

$$\vartheta_1 = \sup_{\xi \geq 1} \xi^{-\gamma} \tilde{v}(0, \xi).$$

In the sequel, for the numerical experiments, we consider a power utility function with $\gamma = 0.5$. We choose parameters for which $b(1 - \gamma)\sigma^2 < 1$, and such that $K_M$ is substantially different from $K_0$. These two requirements on the model parameters correspond to a high-risk return market, where the economic agent can considerably increase her utility with relatively little investment. In addition, the discount factor $\rho$ must satisfy $\rho > b\gamma$. To satisfy all these conditions, we take $b = 0.4, \sigma = 1$ and $\rho = 0.2$, yielding $K_0 = 3.16, K_M = 4.08$ and $\frac{b}{1 - \gamma}\sigma^2 = 0.8$. The intensity $\lambda$ is a free parameter that can be changed to adjust the “illiquidity” of the market.

A first series of tests computed in [11] studied the performance of the decoupling algorithm in a strongly illiquid market ($\lambda = 1$). In figure 5.1, the left graph shows the form of the value function and the right graph that of the optimal investment strategy obtained at different iterations of the numerical decoupling algorithm. As expected, the limiting value function lies between the solution corresponding to the model without trading $v_0$ and the value function of the Merton problem $v_M$.

In the second experiment, we vary the Poisson parameter $\lambda$ governing the trading frequency, to study the convergence of the illiquid market to the Merton portfolio problem. Figure 5.2 presents the behavior of the value functions $v(x)$ and the associated optimal trading strategies. From these graphs we observe, empirically, that

(i) for a fixed value of $x$, both the value function and the optimal investment policy are increasing in $\lambda$ and

(ii) as $\lambda \to \infty$, the value function and the optimal investment policy seem to converge to the corresponding functions in the Merton portfolio problem.

Next, we would like to study the utility loss due to liquidity constraints. Following the utility-indifference pricing approach introduced in [7], we define the utility loss in monetary terms (which can also be called cost of liquidity) as the extra amount of initial wealth $\pi(x)$ needed to reach the same level of expected utility as an investor without trading restrictions and initial capital $x$. This cost of liquidity is then computed as the solution to $v(x + \pi(x)) = v_M(x)$. In our setting (power utility), the cost of liquidity $\pi(x)$ is roughly proportional to $x$. We therefore study the cost of liquidity per unit of initial wealth $\pi(1)$. Table 5.1 reproduces the values $\pi(1)$ for different values of the Poisson parameter $\lambda$. As expected, the cost of liquidity decreases to zero as $\lambda \to \infty$. 
Figure 5.1 Left: Convergence of the iterative algorithm for computing the value function in an illiquid market with $\lambda = 1$. Right: Convergence of the iterative algorithm for computing the optimal investment policy (the amount to invest in stock as a function of the total wealth at the trading date).

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0 (No trading)</th>
<th>1</th>
<th>5</th>
<th>40</th>
</tr>
</thead>
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<tr>
<td>$\pi(1)$</td>
<td>0.6671</td>
<td>0.2749</td>
<td>0.1214</td>
<td>0.0539</td>
</tr>
</tbody>
</table>

Table 5.1 Cost of liquidity $\pi(1)$ as a function of the parameter $\lambda$.

Figure 5.2 Behavior of the value function in an illiquid market (left) and of the optimal investment policy (right) for different values of the Poisson parameter $\lambda$. 
Bibliography


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