Hedging and Optimization Problems in Continuous-Time Financial Models∗

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Abstract

This paper gives an overview of the results and developments in the area of hedging contingent claims in an incomplete market. We study three hedging criteria. We first present the superhedging approach. We then study the mean-variance criterion and finally, we describe the shortfall risk minimization problem. From a mathematical viewpoint, these optimization problems lead to nonstandard stochastic control problems in PDE and new variants of decomposition theorems in stochastic analysis.

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1 Introduction

Option hedging is a prominent problem in finance and has become an important field of theoretical and applied research in stochastic analysis and partial differential equations. We start the introduction with some general ideas and financial motivation before turning to more precise mathematical definitions.

We consider a financial market operating in continuous time and described by a probability space $(\Omega, \mathcal{F}, P)$, a time horizon $T$ and a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ representing the information available at time $t$. There are $d+1$ basic assets for trading. The 0-th asset is a bond with strictly positive price process and is used as numéraire. This means that we pass to discounted quantities so that asset 0 has discounted price equal to one at each time, and the $i$-th asset, $i = 1, \ldots, d$, has a (discounted) price process $S^i$.

One central problem in financial mathematics is the pricing and hedging of contingent claims by means of dynamic trading strategies based on $S$. An (European) contingent claim is an $\mathcal{F}_T$-measurable random variable describing the payoff at time $T$ of some derivative security. A dynamic portfolio strategy is pair $(V, \theta)$ where $V$ is a real-valued $\mathcal{F}$-adapted process and $\theta$ is an $\mathbb{R}^d$-valued predictable process. In such a strategy, $\theta^i_t$ represents the number of shares of asset $i$ held at time $t$ and $V_t$ is the portfolio wealth. Notice that $V_t - \theta^i_t S^i_t$ is the amount invested in asset 0 at time $t$. A strategy $(V, \theta)$ is called self-financing if:

$$V_t = V_0 + \int_0^t \theta_u dS_u,$$

which means that gain or losses on the portfolio wealth are uniquely due to price fluctuations on $S$. Notice that a self-financing strategy is completely determined by the initial capital $V_0$ and $\theta$.

Suppose now that, given a contingent claim $H$, there exists a self-financing strategy $(V, \theta)$ whose terminal value $V_T$ equals $H$ almost surely. Under the basic no-arbitrage assumption on the financial market, the price of $H$ must be equal to $V_0$ and $\theta$ provides a hedging strategy against $H$. This is the basic idea behind the seminal paper of Black and Scholes (1973) and it is mathematically clarified in the martingale theory by Harrison and Pliska (1981): A contingent claim is attainable if there exists a self-financing strategy with
$V_T = H$, a.s., or in other words, if $H$ can be written as the sum of a constant and a stochastic integral with respect to $S$:

$$H = H_0 + \int_0^T \theta_t^H S_t dt.$$ (1.1)

We say that the market is complete if every contingent claim is attainable. However, completeness of the market is a property largely denied by empirical studies on financial market, focusing attention of researchers on extensions of the Black-Scholes model: stochastic volatility models, jump-diffusion models. In this case, given an arbitrary contingent claim $H$, representation (1.1) is no more possible and we say that the market is incomplete. We mention that there are many others modifications of the Black-Scholes model destroying the completeness property: These are portfolio constraints, transaction costs on trading, etc ... More generally, we speak of imperfect markets. In this paper, we focus mainly on the incompleteness situation. For a nonattainable contingent claim, it is then impossible to find a self-financing strategy with final value equal to $H$. The problem of pricing and hedging can then be formulated as follows: Approximate a contingent claim $H$ by the family of terminal wealth of self-financing strategies. Of course, the approximation depends on the choice of the risk measure and leads to various stochastic optimization problems. A first approach is to looking for trading strategies with terminal value $V_T$ larger than $H$: In this case, one (super)hedges the contingent claim. This idea, introduced by Bensaid, Lesne, Pagès and Scheinkman (1992), is behind the concept of superreplication. An alternative approach is to introduce subjective criteria according to which strategies are chosen. The measure of riskiness by a mean-variance criterion was first proposed by Föllmer and Sondermann (1986), and consists in minimizing the expected square of the replication error between the contingent claim and the terminal portfolio wealth. One drawback of this approach is the fact that one penalizes both situations where the terminal wealth is larger or smaller than $H$. Finally, a third measure of riskiness that circumvents this last point, is proposed by Föllmer and Leukert (1998). It consists in minimizing the expected shortfall $(H - V_T)_+$ weighted by some loss function.

The paper is organized as follows. Section 2 introduces the model and formulates the hedging problems. Section 3 describes in detail the super-
hedging approach. In Section 4, we study the mean-variance hedging criterion and the final Section 5 is devoted to the shortfall risk minimization problem.

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2 Hedging problems

2.1 The model

We consider a financial market that operates in uncertain conditions described by a probability space \((\Omega, \mathcal{F}, P)\) equipped with a filtration \(\mathcal{F}^T = \{\mathcal{F}_t\}_{0 \leq t \leq T}\) representing the flow of information on \([0, T]\). For simplicity, we assume that \(\mathcal{F}_0\) is trivial and \(\mathcal{F}_T = \mathcal{F}\). There are \(d + 1\) assets in the market. The 0-th asset is riskless, equal to 1 at any time. The last \(d\) assets could be risky (typically stocks), and their price process is given by an \(\mathbb{R}^d\)-valued semimartingale \(S\).

An important notion in mathematical finance is the set of equivalent martingale measures: We let

\[ \mathcal{P} = \{ Q \sim P : S \text{ is a } Q - \text{local martingale}\} \]

denote the set of probability measures \(Q\) on \(\mathcal{F}\) equivalent to \(P\) and such that \(S\) is a local martingale under \(Q\). We shall assume that:

\[ \mathcal{P} \neq \emptyset. \quad (2.1) \]

This standing assumption is related to some kind of no-arbitrage condition and we refer to Delbaen and Schachermayer (1994) for a general version of this first fundamental theorem of asset pricing.

A trading portfolio strategy is an \(\mathbb{R}^d\)-valued predictable process \(\theta\), integrable with respect to \(S\). We denote \(\theta \in L(S)\) and we refer to Jacod (1979) for vector stochastic integration with respect to a semimartingale. Here \(\theta_t\) represents the number of shares invested in the assets of price \(S\). Given an initial capital \(x \in \mathbb{R}\) and a trading strategy \(\theta \in L(S)\), the self-financed
wealth process is governed by:

\[ V_t^{x,\theta} = x + \int_0^t \theta_u dS_u, \quad 0 \leq t \leq T. \]

2.2 Formulation of the problem

An (European) contingent claim is an \( \mathcal{F}_T \)-measurable random variable. Typical examples are call option of exercise price \( \kappa \) on the \( i \)-th asset \( S^i \), \( H = (S_T^i - \kappa)^+ \) and put option of exercise price \( \kappa \) on the \( i \)-th asset \( S^i \), \( H = (\kappa - S_T^i)^+ \). More generally, \( H \) may depend on the whole history of \( S \) up to time \( T \).

Hedging of the contingent claim \( H \) in the financial market described in the previous paragraph consists in approximating \( H \) by the terminal wealth values \( V_T^{x,\theta} \). We say that the market is complete if every contingent claim \( H \) can be represented as:

\[ H = V_T^{H_0,\theta_H} = H_0 + \int_0^T \theta_H u dS_u, \quad (2.2) \]

for some \( H_0 \in \mathbb{R} \) and \( \theta \in L(S) \). Completeness property depends on the class of contingent claims and trading strategies considered and one has to precise the integrability conditions on \( H \) and \( \theta \). Under the no-arbitrage assumption (2.1), a sufficient condition ensuring the completeness of the market is that the set of equivalent martingale measures \( \mathcal{P} \) is reduced to a singleton. This result follows from general results on martingale representation due to Jacod (1979) and is applied for the purpose of finance theory in Harrison and Pliska (1981). In this case, one can perfectly hedge (approximate) \( H \) by \( V_T^{H_0,\theta_H} \). The initial capital \( H_0 \) is a fair price of \( H \) and \( \theta_H \) is called perfect replicating strategy of \( H \). Typical example of complete market is the classical Black-Scholes model.

In the general semimartingale model of Section 1, given an arbitrary \( H \), representation (2.2) is no more possible. We say that the market is incomplete and in this case the set of equivalent martingale measures \( \mathcal{P} \) is infinite. Typical examples of incomplete markets are stochastic volatility models and jump-diffusion models. In such a context, one can no more perfectly hedge (approximate) \( H \) by \( V_T^{x,\theta} \) and one has to choose a hedging criteria. From a mathematical viewpoint, this leads to various stochastic
optimization problems. In the sequel, we shall study the following three hedging criteria:
- Superhedging approach
- Mean-variance hedging criterion
- Shortfall risk minimization

3 The superhedging approach

We are given a nonnegative contingent claim of the form $H = g(S_T)$, where $g$ is a continuous function from $\mathbb{R}^d$ into $\mathbb{R}_+$, with linear growth condition.

The integrability conditions on the trading strategies are defined as follows.

Given $x \geq 0$, we say that a trading strategy $\theta \in \mathcal{L}(S)$ is admissible, and we note $\theta \in \mathcal{A}(x)$, if $V_t^{x,\theta} \geq 0$, for all $t$ in $[0,T]$. The superhedging approach consists in looking for an initial capital $x \geq 0$ and an admissible trading strategy $\theta \in \mathcal{A}(x)$, such that:

$$V_T^{x,\theta} = x + \int_0^T \theta_t dS_t \geq H = g(S_T), \text{ a.s.}$$

In this case, we say that the contingent claim $H$ is superhedged (dominated).

We define then the superreplication (superhedging) cost of $H$ as the least initial capital that allows the superhedging of $H$:

$$U_0 = \inf \{ x \geq 0 : \exists \theta \in \mathcal{A}(x), V_T^{x,\theta} \geq g(S_T) \}.$$ 

This is a nonstandard stochastic control problem and we describe different methods for solving this problem, i.e. calculate $U_0$ and the associated optimal control.

3.1 Dual approach

The starting point of the dual approach is to notice that for any $Q \in \mathcal{P}$, $x \geq 0$ and $\theta \in \mathcal{A}(x)$, the wealth process $V_t^{x,\theta}$ is a nonnegative local martingale, hence a supermartingale, under $Q$. It follows that:

$$E^Q[V_T^{x,\theta}] \leq x. \quad (3.1)$$
Now, let $x \geq 0$ such that there exists $\theta \in A(x) : X_{T}^{x,\theta} \geq g(S_T)$. By (3.1), we then have:

$$E^Q[g(S_T)] \leq x, \quad \forall \ Q \in \mathcal{P},$$

and so by definition of the superreplication cost:

$$V_0 := \sup_{Q \in \mathcal{P}} E^Q[g(S_T)] \leq U_0.$$

The dual approach consists in studying and calculating $V_0$ and then $U_0$.

**Remark 3.1** Actually, we have the equality $V_0 = U_0$. The converse inequality $V_0 \geq U_0$ is proved by using optional decomposition for supermartingales, first proved by El Karoui and Quenez (1995) and then extended by Kramkov (1996). This theorem states that if $V$ is a supermartingale under any $Q \in \mathcal{P}$, then $V$ admits a decomposition of the form:

$$V_t = V_0 + \int_0^t \theta_u dS_u - C_t, \quad 0 \leq t \leq T, \quad (3.2)$$

where $\theta \in L(S)$ and $C$ is an optional nondecreasing process. Applying this theorem to the RCLL version of the process $V_t = \text{esssup}_Q E^Q[g(S_T)|\mathcal{F}_t]$, we deduce from (3.2) for $t = T$, that $g(S_T) \leq V_T^{V_0,\theta}$. By definition of the superreplication cost, this shows that $V_0 \geq U_0$. We mention that we don’t need the equality in the dual approach but only the easy inequality $V_0 \leq U_0$.

**Application: Stochastic volatility models**

We use the dual approach to the computation of $V_0$ and of the superreplication cost $U_0$ in the context of stochastic volatility models. This application is developed in Cvitanić, Pham and Touzi (1999a), see also Frey and Sin (1999).

We consider a standard stochastic volatility diffusion model:

$$dS_t = S_t \left( \mu(t, S_t, Y_t) dt + \sigma(t, S_t, Y_t) dW_t^1 \right), \quad (3.3)$$

$$dY_t = \eta(t, S_t, Y_t) dt + \gamma(t, S_t, Y_t) dW_t^2. \quad (3.4)$$
Here $Y$ is an exogeneous factor influencing the volatility of the stock price $S$ and $\mathcal{F}$ is the augmented filtration generated by the two-dimensional Brownian motion $(W^1, W^2)$. In this model, we have a parametrization of equivalent martingale measures: Consider the process

$$Z^\nu_t = \exp\left(-\int_0^t \frac{\mu}{\sigma}(u, S_u, Y_u) dW^1_u - \frac{1}{2} \int_0^t \left(\frac{\mu}{\sigma}\right)^2 (u, S_u, Y_u) du\right).$$

$$\exp\left(-\int_0^t \nu_u dW^2_u - \frac{1}{2} \int_0^t \nu_u^2 du\right),$$

where $\nu$ is an $\mathcal{F}$-adapted process satisfying $\int_0^T \nu_u^2 du < \infty$. Denote by $\mathcal{D}$ the set of processes $\nu$ such that $E[|Z^\nu_T|] = 1$. Then one can define a probability measure $P^\nu$ equivalent to $P$, by $dP^\nu / dP = Z^\nu_T$, and we have $\mathcal{P} = \{P^\nu, \nu \in \mathcal{D}\}$. It follows that:

$$V_0 = V(0, S_0, Y_0) := \sup_{\nu \in \mathcal{D}} E^{P^\nu} [g(S_T)] \ (\leq U_0).$$

We are then led to a more standard stochastic control problem, by studying the value function $V(t, s, y)$. Indeed, by usual dynamic programming principle, one shows that the value function $V$ is a supersolution (in the viscosity sense) of the Bellman equation:

$$-\frac{\partial V}{\partial t} + \inf_{\nu \in \mathbb{R}} \left\{-\eta + \nu \gamma \frac{\partial V}{\partial y} - \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} - \frac{1}{2} \gamma^2 \frac{\partial^2 V}{\partial y^2}\right\} = 0,$$

for all $(t, s, y) \in [0, T) \times (0, \infty) \times \mathbb{R}$, and satisfies the terminal condition $V(T^-, s, y) \geq g(s)$.

We show (formally by sending $\nu$ to $\pm \infty$ in Bellman equation) that:

$$V \text{ does not depend on } y : \quad V(t, s, y) = V(t, s)$$

so that $V$ is supersolution of:

$$\inf_{y \in \mathbb{R}} \left\{-\frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 (t, s, y) s^2 \frac{\partial^2 V}{\partial s^2}\right\} = 0,$$

(3.5)
At this point, the explicit calculation of $V$ depends on the interval of variation of the volatility:

$$\left[ \inf_y \sigma(t, s, y), \sup_y \sigma(t, s, y) \right] = [\underline{\sigma}(t, s), \bar{\sigma}(t, s)].$$

We shall distinguish two cases.

**Case 1 : unbounded volatility model**

$$\bar{\sigma}(t, s) = \infty \quad \text{and} \quad \underline{\sigma}(t, s) = 0. \quad (3.6)$$

From the Bellman equation and the conditions (3.6), we show that:

$V$ is concave in $s$ and $V$ is nonincreasing in $t$

Using also the terminal condition $V(T^{-}, s, y) \geq g(s)$, this shows that $V(0, S_0) \geq \hat{g}(S_0)$, where $\hat{g}$ is the concave envelope of $g$, i.e. the least concave majorant function of $g$. Moreover, since $\hat{g}$ is concave and is a majorant of $g$, we have:

$$\hat{g}(S_0) + \hat{g}'(S_0)(S_T - S_0) \geq \hat{g}(S_T) \geq g(S_T),$$

where $\hat{g}'$ is the left derivative of $\hat{g}$. This shows that $g(S_T)$ can be super-hedged from an initial capital $\hat{g}(S_0)$ and following the constant strategy $\hat{g}'(S_0)$. We deduce that $\hat{g}(S_0) \geq U_0 \geq V(0, S_0)$. In conclusion, we obtain:

$$U_0 = V(0, S_0) = \hat{g}(S_0)$$
$$\theta^* = \hat{g}'(S_0) \quad \text{(constant : trivial buy-and-hold strategy)}.$$

**Case 2 : Model with bounded volatility (Avellaneda, Lévy and Paras)**

$$\bar{\sigma}(t, s) < \infty \quad \text{and} \quad \underline{\sigma}(t, s) \geq \varepsilon > 0.$$

Then there exists an unique smooth solution $W$ to the nonlinear PDE, called Black-Scholes-Barenblatt (BSB in short) PDE:
\[- \frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2(t, s) s^2 \left( \frac{\partial^2 W}{\partial s^2} \right) + \frac{1}{2} \sigma^2(t, s) s^2 \left( \frac{\partial^2 W}{\partial s^2} \right) - 0, \quad (3.7)\]
\[W(T, s) = g(s), (3.8)\]

Actually, this Black-Scholes-Barenblatt PDE is the Bellman equation:

\[\inf_{y \in \mathbb{R}} \left\{ - \frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2(t, s, y) s^2 \frac{\partial^2 V}{\partial s^2} \right\} = 0.\]

By the maximum principle, we deduce that \( W \leq V \). Moreover, by Itô’s formula and using the fact that \( W \) solves the above Bellman equation, we have:

\[g(S_T) = W(T, S_T) \leq W(0, S_0) + \int_0^T \frac{\partial W}{\partial s}(t, S_t) dS_t,\]

and then by definition of the superreplication cost, \( W(0, S_0) \geq U_0 \geq V(0, S_0) \).

In conclusion, we have that \( U_0 = W(0, S_0) (= V(0, S_0)) \) is the unique smooth solution of the BSB equation (3.7)-(3.8), and the optimal control is given by:

\[\theta^*_t = \frac{\partial W}{\partial s}(t, S_t).\]

**Other applications**

By the dual approach, one can also calculate the superreplication cost in:
- models with jumps: see Eberlein and Jacod (1997) and Bellamy and Jeanblanc (1998),
- models with transaction costs: see Cvitanić, Pham and Touzi (1999b).

### 3.2 Direct approach

Soner and Touzi (1998) developed a dynamic programming principle directly on the primal problem:

\[U_0 = \inf\{ x \geq 0 : \exists \theta \in \mathcal{A}(x), V_T^{x, \theta} \geq g(S_T) \},\]
and derived then an associated Bellman equation for $U_0$. This Bellman equation is actually the PDE (3.5) in the case of the stochastic volatility model (3.3)-(3.4). Such a direct approach allows to study superreplication in models with gamma constraints, i.e. constraints on the sensibility of $\theta$ with respect to stock price.

### 3.3 BSDE approach

The superhedging problem can also be viewed as a problem of finding a triple $(V, \theta, C)$ of adapted processes, with $C$ nonincreasing, solution of the backward stochastic differential equation:

$$
\begin{align*}
\frac{dV_t}{dt} &= \theta_t dS_t - dC_t \\
V_T &= g(S_T).
\end{align*}
$$

Such a method is studied in El Karoui, Peng and Quenez (1997), see also Yong (1999).

### 4 Mean-variance hedging

We are looking for a strategy $\theta$ which minimizes the quadratic error of replication between the contingent claim $H \in L^2(P)$ and the terminal wealth $V_{T}^{x,\theta} = x + \int_0^T \theta_t dS_t$:

$$
\text{minimize over } \theta \quad E \left[ H - x - \int_0^T \theta_t dS_t \right]^2. \quad (4.1)
$$

This is problem of $L^2$-projection of a random variable on a space of stochastic integrals. In order to ensure the existence of a solution to this quadratic optimization problem, we need to precise the class of admissible trading strategies $\theta$ so that the space of stochastic integrals is closed in $L^2(P)$. Following Delbaen and Schachermayer (1996), we introduce the subset $\mathcal{P}_2$ of probability measure $Q$ in $\mathcal{P}$ with square-integrable density : $dQ/dP \in L^2(P)$, and we assume that $\mathcal{P}_2$ is nonempty. We then define the integrability conditions on the trading strategies :

$$
\Theta_2 = \left\{ \theta \in L(S) : \int_0^T \theta_t dS_t \in L^2(P) \text{ and} \right\}
$$

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\[ \int \theta dS \text{ is a } Q \text{-martingale, } \forall Q \in \mathcal{P}_2 \].

It is showed in Delbaen and Schachermayer (1996) that the set \( G_T(\Theta_2) = \{ \int_0^T \theta_t dS_t : \theta \in \Theta_2 \} \) is closed in \( L^2(P) \) so that for any \( H \in L^2(P) \), the problem

\[
J_2(x) = \min_{\theta \in \Theta_2} E \left[ H - x - \int_0^T \theta_t dS_t \right]^2,
\]

admits a solution. We now focus on the characterization of the solution.

### 4.1 Case \( S \) martingale under \( P \)

In this paragraph, we assume that \( S \) is a continuous local martingale under \( P \). This case was first considered in Föllmer and Sondermann (1986). Then, given \( H \in L^2(P) \), the Kunita-Watanabe projection theorem provides:

\[
H = E[H] + \int_0^T \theta_t^H dS_t + R_t,
\]

where \( \theta^H \in \Theta_2 \) and \( R \) is a square-integrable martingale orthogonal to \( S \). It follows immediately that the solution to \( J_2(x) \) is given by:

\[
\theta^* = \theta^H.
\]

**Example**

Consider the stochastic volatility model:

\[
\begin{align*}
\text{d}S_t &= S_t \sigma(t, S_t, Y_t) \text{d}W^1_t, \\
\text{d}Y_t &= \eta(t, S_t, Y_t) dt + \gamma(t, S_t, Y_t) \text{d}W^2_t.
\end{align*}
\]

Then, the integrand in the Kunita-Watanabe projection of \( H = g(S_T) \) is:

\[
\theta_t^H = \frac{\partial V}{\partial S}(t, S_t),
\]

where \( V(t, s, y) = E[g(S_T) | S_t = s, Y_t = y] \). More general applications and examples are studied in Bouleau and Lamberton (1989).
4.2 Case $S$ semimartingale

We now turn to the general situation where $S$ is a continuous semimartingale under $P$. Recall that there exists a solution $\theta^*(x) \in \Theta_2$ to the problem $J_2(x)$. Characterization of the solution has been obtained by Duffie and Richardson (1991), Schweizer (1994), Hipp (1993) and Pham, Rheinländer and Schweizer (1998) under more and less restrictive assumptions. The most general results are obtained for the case where $S$ is a continuous semimartingale, independently by Gouriéroux, Laurent and Pham (1998) (GLP in short) and Rheinländer and Schweizer (1997) (RS in short). We present here the approach of the former authors. Their basic idea is to state an invariance property of the space of stochastic integrals by a change of numéraire, and to combine this change of coordinates with an appropriate change of probability measure in order to transform $J_2(x)$ into an equivalent $L^2$-projection problem corresponding to the martingale case.

The suitable change of probability measure and change of coordinates use the so-called variance-optimal martingale measure and hedging numéraire. Under the standing assumption (2.1) and the condition that $S$ is continuous, Delbaen and Schachermayer (1996) prove that there exists an unique solution $\tilde{P}$, called variance-optimal martingale measure, to the problem

$$\min_{Q \in \mathcal{P}_2} E \left[ \frac{dQ}{dP} \right]^2. \quad (4.2)$$

Moreover, there exists $\tilde{\theta} \in \Theta_2$ such that

$$\tilde{Z}_t := \frac{E^P \left[ \frac{d\tilde{P}}{dP} \bigg| \mathcal{F}_t \right]}{E \left[ \frac{d\tilde{P}}{dP} \right]^2} = V_t^{1,\tilde{\theta}}, \quad P \text{ a.s., } 0 \leq t \leq T, \quad (4.3)$$

and $\tilde{\theta}$, called hedging numéraire, is solution of the optimization problem:

$$\min_{\theta \in \Theta_2} E \left[ V_T^{1,\theta} \right]^2. \quad (4.4)$$

It follows that the process $\tilde{Z}$ is a strictly positive $Q$-martingale for any $Q \in \mathcal{P}_2$, with initial value 1. We can then associate to each $Q \in \mathcal{P}_2$ the probability measure $\tilde{Q} \sim Q$ defined by:

$$\frac{d\tilde{Q}}{dQ} \bigg|_{\mathcal{F}_t} = \tilde{Z}_t, \quad 0 \leq t \leq T. \quad (4.5)$$
We denote then by \( \tilde{\mathcal{P}}_2 \) the set of all elements \( \tilde{Q} \sim Q \) defined by (4.5) when \( Q \) varies in \( \mathcal{P}_2 \). In particular, we associate to \( \tilde{Q} \in \tilde{\mathcal{P}}_2 \) the probability measure \( \tilde{Q} \sim Q \) defined by (4.5). Notice that by definition of \( \tilde{Z} \) in (4.3), the Radon-Nikodym density of \( \tilde{P} \) with respect to \( P \) can be written as:

\[
\frac{d\tilde{P}}{dP} = \mathbb{E} \left[ \frac{d\tilde{P}}{dP} \right]^2 \tilde{Z}_T^2. \tag{4.6}
\]

We consider the \( \mathbb{R}^{d+1} \)-valued continuous process \( \tilde{X} \) with \( \tilde{X}_0 = 1/\tilde{Z} \) and \( \tilde{X}_i = S^i/\tilde{Z}, \ i = 1, \ldots, d \). As a direct consequence of the fact that \( S \) is a continuous local martingale under any \( Q \in \mathcal{P}_2 \) and Bayes formula, we deduce that the process \( \tilde{X} \) is a continuous local martingale under any \( \tilde{Q} \in \tilde{\mathcal{P}}_2 \). We denote by \( \tilde{\Phi}_2 \) the set of all \( \mathbb{R}^{d+1} \)-valued predictable processes \( \phi \tilde{X} \)-integrable, such that \( \int_0^T \phi_t d\tilde{X}_t \in L^2(\tilde{P}) \) and \( \int \phi d\tilde{X} \) is a \( \tilde{Q} \)-martingale under any \( \tilde{Q} \in \tilde{\mathcal{P}}_2 \).

The following result is crucial in the method of resolution in GLP.

**Theorem 4.1** Assume that \( S \) is continuous. Let \( x \in \mathbb{R} \). Then we have:

\[
\{ V^x_\theta : \theta \in \Theta_2 \} = \left\{ \tilde{Z}_T \left( x + \int_0^T \phi_t d\tilde{X}_t \right) : \phi \in \tilde{\Phi}_2 \right\}. \tag{4.7}
\]

Moreover, the relation between \( \theta = (\theta^1, \ldots, \theta^d)' \in \Theta_2 \) and \( \phi = (\phi^0, \ldots, \phi^d)' \in \tilde{\Phi}_2 \) is given by:

\[
\phi^0 = V^x_\theta - \theta'S \quad \text{and} \quad \phi^i = \theta^i, \ i = 1, \ldots, d, \tag{4.8}
\]

and

\[
\theta^i = \phi^i + \tilde{\theta} \left( x + \int \phi d\tilde{X} - \phi' \tilde{X} \right), \ i = 1, \ldots, d. \tag{4.9}
\]

**Proof.** We follow arguments of GLP and RS. The proof is mainly based on Itô’s product rule and for simplicity we omit the integrability questions.

1. By Itô’s product rule, we have:

\[
d \left( \frac{S}{\tilde{Z}} \right) = S d \left( \frac{1}{\tilde{Z}} \right) + \frac{1}{\tilde{Z}} dS + d < S, \frac{1}{\tilde{Z}} >. \tag{4.10}
\]
Let $\theta \in \Theta_2$. Then we have :

$$\int \theta d \left( \frac{S}{Z} \right) = \int \theta S d \left( \frac{1}{Z} \right) + \int \frac{1}{Z} dS + \int \theta d < S, \frac{1}{Z} > . \quad (4.11)$$

By Itô’s formula, the self-financing condition $dV^{x,\theta} = \theta dS$, and by (4.11) we obtain :

$$d \left( \frac{V^{x,\theta}}{Z} \right) = V^{x,\theta} \left( \frac{1}{Z} \right) + \frac{1}{Z} \theta dS + \theta' d < S, \frac{1}{Z} > = \left( V^{x,\theta} - \theta' S \right) d \left( \frac{1}{Z} \right) + \theta d \left( \frac{S}{Z} \right) = \phi d \tilde{X}, \quad (4.12)$$

with an $\tilde{X}$-integrable process $\phi$ given by (4.8). Then relation (4.12) shows that

$$V^{x,\theta} = x + \int \theta dS = \tilde{Z} \left( x + \int \phi d\tilde{X} \right). \quad (4.13)$$

Since $\tilde{Z}_T$ and $\int_0^T \theta_t dS_t \in L^2(P)$, we have $\tilde{Z}_T \int_0^T \phi_t d\tilde{X}_t \in L^2(P)$ or equivalently by (4.6) $\int_0^T \phi_t d\tilde{X}_t \in L^2(\tilde{P})$. Since $\int \theta dS$ is a $Q$-martingale under any $Q \in \mathcal{P}_2$, we deduce by definition of $\mathcal{P}_2$ and (4.13) that $\int \phi d\tilde{X}$ is a $\tilde{Q}$-martingale under any $\tilde{Q} \in \tilde{\mathcal{P}}_2$ and so $\phi \in \tilde{\Phi}_2$. Therefore the inclusion $\subseteq$ in (4.7) is proved.

(2) The proof of the converse is very similar. By Itô’s product rule, we have :

$$d(\tilde{Z}X) = \tilde{Z} d\tilde{X} + \tilde{X} d\tilde{Z} + d < \tilde{Z}, \tilde{X} > . \quad (4.14)$$

Let $\phi \in \tilde{\Phi}_2$. By Itô’s formula, (4.14), (4.3) and definition of $\tilde{X}$, we have :

$$d \left( \tilde{Z} \left( x + \int \phi d\tilde{X} \right) \right) = \left( x + \int \phi d\tilde{X} \right) d\tilde{Z} + \tilde{Z} \phi d\tilde{X} + \phi' d < \tilde{Z}, \tilde{X} > = \left( x + \int \phi d\tilde{X} \right) d\tilde{Z} + \phi d(\tilde{Z} \tilde{X}) - \phi \tilde{X} d\tilde{Z} = \theta dS,$$

with the $S$-integrable process $\theta$ given by (4.9). We then obtain :

$$\tilde{Z} \left( x + \int \phi d\tilde{X} \right) = x + \int \theta dS. \quad (4.15)$$
Since $\int_0^T \phi_t d\tilde{X}_t \in L^2(\tilde{P})$, we see from (4.6) and (4.15) that $\int_0^T \theta_t dS_t \in L^2(P)$. Moreover, $\int \phi d\tilde{X}$ is a $\tilde{Q}$-martingale for any $\tilde{Q} \in \tilde{P}_2$ and so $\int \theta dS$ is a $Q$-martingale for all $Q \in P_2$. This shows that $\theta \in \Theta_2$ and so the inclusion $\supseteq$ in (4.7) is proved.

By (4.6), we have $H/\tilde{Z}_T \in L^2(\tilde{P})$ since $H \in L^2(P)$. Moreover, the process $\tilde{X}$ is a continuous local martingale under $\tilde{P} \in \tilde{P}_2$. We can then apply the Kunita-Watanabe projection and obtain:

$$\frac{H}{Z_T} = E^{\tilde{P}}\left[\frac{H}{Z_T}\right] + \int_0^T \tilde{\phi}_t^H d\tilde{X}_t + \tilde{L}_T^H, \quad P \text{ a.s.}$$

where $\tilde{\phi}^H \in \tilde{\Phi}_2$ and $\tilde{L}$ is a square integrable martingale under $\tilde{P}$ orthogonal to $\tilde{X}$. We have then the following characterization result of a solution to the mean-variance hedging problem.

**Theorem 4.2** Assume that $S$ is continuous. Then for all $x \in \mathbb{R}$, there exists a unique solution $\theta^*(x) \in \Theta_2$ to problem $J_2(x)$ given by:

$$(\theta^*(x))^i = (\tilde{\phi}^H)^i + \bar{\theta}^i \left(x + \int \tilde{\phi}^H d\tilde{X} - \tilde{\phi}^{H'} \tilde{X}\right), \quad i = 1, \ldots, d.$$  (4.16)

The associated value function is given by:

$$J_2(x) = \frac{\left(E^{\tilde{P}}[H] - x\right)^2 + E^{\tilde{P}}\left[\tilde{L}_T\right]^2}{E\left[\frac{d\tilde{P}}{dP}\right]^2}, \quad x \in \mathbb{R}. \quad (4.17)$$

**Proof.** In view of Theorem 4.1 and (4.6), we have:

$$J_2(x) = \frac{1}{E\left[\frac{d\tilde{P}}{dP}\right]^2} \inf_{\phi \in \tilde{\Phi}_2} E^{\tilde{P}}\left[H/\tilde{Z}_T - x - \int_0^T \phi_t d\tilde{X}_t\right]^2. \quad (4.18)$$

This is an optimization problem as in the martingale case (see Paragraph 4.1), and therefore the unique solution of (4.18) is given by $\tilde{\phi}^H$. The unique solution $\theta^*(x)$ to $J_2(x)$ is then obtained via (4.9) from $\tilde{\phi}^H$. Moreover, we have:

$$J_2(x) = \frac{\left(E^{\tilde{P}}\left[H/\tilde{Z}_T\right] - x\right)^2 + E^{\tilde{P}}\left[\tilde{L}_T\right]^2}{E\left[\frac{d\tilde{P}}{dP}\right]^2}, \quad x \in \mathbb{R}. \quad (4.19)$$

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By definition of $\tilde{P}$ in function of $\hat{P}$, we then obtain (4.17).

**Remark 4.1** RS prove that $H$ can be decomposed into:

$$ H = E^{\tilde{P}}[H] + \int_0^T \tilde{\theta}_t^H dS_t + \tilde{L}_T^H, $$

where $\tilde{\theta}^H \in \Theta_2$ and $\tilde{L}^H$ is a martingale under $\tilde{P}$ orthogonal to $S$. They obtain then a description of the solution to $J_2(x)$ in feedback form:

$$ \theta_t^\ast(x) = \tilde{\theta}_t^H - \tilde{\theta}_t \left( \frac{1}{Z_t} \left( x + E^\tilde{P}[H|\mathcal{F}_t] - \int_0^t \theta_s^\ast(x) dS_s \right) \right). \quad (4.20) $$

It is also checked in RS that the expression given in (4.20) coincide with the one given in (4.16).

Description of the optimal hedging strategy requires finding the variance-optimal martingale measure and the hedging numéraire. Hipp (1993), Pham, Rheinländer and Schweizer (1998) studied the special case where the variance-optimal martingale measure coincide with the minimal martingale measure of Föllmer and Schweizer (1991). More general results have been obtained by Laurent and Pham (1999) in a multidimensional diffusion model by dynamic programming arguments, with applications to stochastic volatility models; see also in this direction the recent works of Heath, Platen and Schweizer (1998) and Biagini, Guasoni, Pratelli (1999).

## 5 Shortfall risk minimization

This is an alternative criterion to the extrem approach of superreplication and to the symmetrical approach of the mean-variance hedging criterion. Here one penalizes only situations when:

$$ V_T^{x,\theta} \leq H, $$

and we want to minimize the expected shortfall $(H - V_T^{x,\theta})_+ = \max(H - V_T^{x,\theta}, 0)$ weighted by some loss function $l$, i.e. $l(0) = 0$, $l$ is nondecreasing and convex on $\mathbb{R}_+$.  

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Given an initial capital $x \geq 0$ and a nonnegative contingent claim $H$, we consider the stochastic optimization problem:

$$\min_{\theta \in \mathcal{A}(x)} E \left[ l(H - V_{T}^{x,\theta})_{+} \right] \quad (\mathcal{P}(x))$$

Such a problem has been first studied by Föllmer and Leukert (1998a,b) in the context of incomplete semimartingale model. It is studied in the context of diffusion models by Cvitanić and Karatzas (1998), by Cvitanić (1998) for diffusion models with portfolio constraints. Pham (1999) extended this shortfall risk minimization problem to a general framework including semimartingale models with constrained portfolios, large investor models, labor income, reinsurance models.

In a first step, notice that by the nondecreasing property of the loss function $l$, we can transform the original dynamic control problem into a static one:

$$\min_{\theta \in \mathcal{A}(x)} E \left[ l(H - V_{T}^{x,\theta})_{+} \right] \quad (\mathcal{P}(x))$$

$$= \min_{X \in \mathcal{C}(x)} E \left[ l(H - X) \right] := J(x)$$

where

\[
\mathcal{C}(x) = \{X \mathcal{F}_{T} - \text{measurable} : 0 \leq X \leq H, \text{and } \exists \theta \in \mathcal{A}(x), X \leq V_{T}^{x,\theta}\}.
\]

Now, from the optional decomposition theorem for supermartingales which gives a dual characterization of the superreplication cost (see Remark 3.1), we have:

$$\exists \theta \in \mathcal{A}(x), X \leq V_{T}^{x,\theta} \iff \sup_{Q \in \mathcal{P}} E^{Q}[X] \leq x.$$  \hspace{1cm} (5.2)

Therefore the set of constraints $\mathcal{C}(x)$ can be written as:

$$\mathcal{C}(x) = \{X \mathcal{F}_{T} - \text{measurable} : 0 \leq X \leq H, \text{and } \sup_{Q \in \mathcal{P}} E^{Q}[X] \leq x\}.$$  \hspace{1cm} (5.3)

We are then amounted to a static convex optimization $J(x)$ with linear constraints $\mathcal{C}(x)$ given by (5.3).
The following result proves the existence of a solution to the dynamic problem \((\mathcal{P}(x))\) and relates it to the solution of the static problem \(J(x)\). It also provides some qualitative properties of the associated value function. In the sequel, given a nonnegative contingent claim \(X\), we denote by \(v_0(X) = \sup_{Q \in \mathcal{P}} E^Q[X]\) its superreplication cost.

**Theorem 5.1** Assume that \(l(H) \in L^1(P)\).

1. For any \(x \geq 0\), there exists \(X^*(x) \in \mathcal{C}(x)\) solution of \(J(x)\) and \(H\) is solution of \(J(x)\) for \(x \geq v_0(H)\). Moreover, if \(l\) is strictly convex, any two such solutions coincide \(P\) a.s.

2. The function \(J\) is nonincreasing and convex on \([0, \infty)\), strictly decreasing on \([0, v_0(H)]\) and equal to zero on \([v_0(H), \infty)\). For any \(x \in [0, v_0(H)]\), we have:

\[
\sup_{Q \in \mathcal{P}} E^Q [X^*(x)] = x. \tag{5.4}
\]

Moreover, if \(l\) is strictly convex, then \(J\) is strictly convex on \([0, v_0(H)]\).

3. For any \(x \geq 0\), there exists \(\theta^*(x) \in \mathcal{A}(x)\) such that \(X^*(x) \leq V^x_{T, \theta^*(x)}\), \(P\) a.s., and \(\theta^*(x)\) is solution to the dynamic problem \((\mathcal{P}(x))\).

**Proof.** (1) Let \(x \geq 0\) and \((X^n)_n \in \mathcal{C}(x)\) be a minimizing sequence for the problem \(J(x)\), i.e.

\[
\lim_{n \to \infty} E[l(H - X^n)] = \inf_{X \in \mathcal{C}(x)} E[l(H - X)].
\]

Since \(X^n \geq 0\), then by Lemma A.1.1 of Delbaen and Schachermayer (1994), there exists a sequence of \(\mathcal{F}_T\)-measurable random variables \(\hat{X}^n \in \text{conv}(X^n, X^{n+1}, \ldots)\) such that \(\hat{X}^n\) converges almost surely to \(X^*(x)\) \(\mathcal{F}_T\)-measurable. Since \(0 \leq \hat{X}^n \leq H\), we deduce that \(0 \leq X^*(x) \leq H\). By Fatou’s lemma, we have for all \(Q \in \mathcal{P}\):

\[
E^Q [X^*(x)] \leq \liminf_{n \to \infty} E^Q [\hat{X}^n] \leq x,
\]

hence \(X^*(x) \in \mathcal{C}(x)\). Now, since \(l\) is convex and \(l(H) \in L^1(P)\), we have by the dominated convergence theorem:

\[
\inf_{X \in \mathcal{C}(x)} E[l(H - X)] = \lim_{n \to \infty} E[l(H - X^n)] \\
\geq \lim_{n \to \infty} E[l(H - \hat{X}^n)] \\
= E[l(H - X^*(x))],
\]

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which proves that $X^*(x)$ solves $J(x)$. Now, suppose that $x \geq v_0(H)$. Then $H \in \mathcal{C}(x)$ and is obviously solution to $J(x)$, and in this case $J(x) = 0$.

Let $X^1$ and $X^2$ be two solutions of $J(x)$ and $\varepsilon \in (0, 1)$. Set $X^\varepsilon = (1 - \varepsilon)X^1 + \varepsilon X^2 \in \mathcal{C}(x)$. By convexity of function $l$, we have:

$$E \left[ l(H - X^\varepsilon) \right] \leq (1 - \varepsilon)E \left[ l(H - X^1) \right] + \varepsilon E \left[ l(H - X^2) \right] \quad (5.5)$$

$$= \inf_{X \in \mathcal{C}(x)} E[l(H - X)]. \quad (5.6)$$

Suppose that $P[X^1 \neq X^2] > 0$. Then by the strict convexity of $l$, we should have strict inequality in (5.5), which is a contradiction with (5.6).

(2) Let $0 \leq x_1 \leq x_2$. Since $\mathcal{C}(x_1) \subset \mathcal{C}(x_2)$, we deduce that $J(x_2) \leq J(x_1)$ and so $J$ is nonincreasing on $[0, \infty)$. Notice also that $(X^*(x_1) + X^*(x_2))/2 \in \mathcal{C}((x_1 + x_2)/2)$. Then, by convexity of function $l$, we have:

$$J\left(\frac{x_1 + x_2}{2}\right) \leq E \left[ l\left( H - \frac{X^*(x_1) + X^*(x_2)}{2} \right) \right]$$

$$\leq \frac{1}{2} E \left[ l(H - X^*(x_1)) \right] + \frac{1}{2} E \left[ l(H - X^*(x_2)) \right]$$

$$= \frac{1}{2} J(x_1) + \frac{1}{2} J(x_2),$$

which proves the convexity of $J$ on $[0, \infty)$. We have already seen that $J(x) = 0$ for $x \geq v_0(H)$. To end the proof of assertion (2), we now suppose that $v_0(H) > 0$ (otherwise there is nothing else to prove). First, notice that since $l$ is a nonnegative function, cancelling only on 0, it follows that $J(x) = 0$ if and only if $X^*(x) = H$ which implies that $x \geq v_0(H)$. Therefore, for all $0 \leq x < v_0(H)$, we have $J(x) > 0$. Let us check that $J$ is strictly decreasing on $[0, v_0(H)]$. On the contrary, there would exist $0 \leq x_1 < x_2 < v_0(H)$ such that $J(x_1) = J(x_2)$. Then, there exists $\alpha \in (0, 1)$ such that $x_2 = \alpha x_1 + (1 - \alpha)v_0(H)$. By convexity of function $l$, we should have:

$$J(x_2) \leq \alpha J(x_1) + (1 - \alpha)J(v_0(H)) = \alpha J(x_1).$$

Since $J(x_1) = J(x_2) > 0$, this would imply that $\alpha > 1$, a contradiction. Let us now prove (5.4). On the contrary, we should have $0 \leq \tilde{x} := \sup_{Q \in \mathcal{P}} E^Q[X^*(x)] < x$. Then $X^*(x) \in \mathcal{C}(\tilde{x})$ and so $J(\tilde{x}) \leq E[l(H - X^*(x))] = J(x)$, a contradiction with the fact that $J$ is strictly decreasing on $[0, v_0(H)]$. Let
0 \leq x_1 < x_2 \leq v_0(H)$. We have $(X^*(x_1) + X^*(x_2))/2 \in C((x_1 + x_2)/2)$. Moreover, since $0 < J(x_2) < J(x_1)$, we have $X^*(x_1) \neq X^*(x_2)$. Then, by the strict convexity of function $l$, we obtain:

\[
J\left(\frac{x_1 + x_2}{2}\right) \leq E\left[l\left(H - \frac{X^*(x_1) + X^*(x_2)}{2}\right)\right] < \frac{1}{2}E\left[l\left(H - X^*(x_1)\right)\right] + \frac{1}{2}E\left[l\left(H - X^*(x_2)\right)\right] = \frac{1}{2}J(x_1) + \frac{1}{2}J(x_2),
\]

which proves the strict convexity of $J$ on $[0, v_0(H)]$.

(3) The third assertion follows from (5.1) giving the relation between the dynamic problem $(P(x))$ and the static problem $J(x)$. \hfill \Box

We provide a quantitative description of $X^*(x)$ and of $\theta^*(x)$ solutions of $J(x)$ and of $(P(x))$ by adopting a convex duality approach, which is now a standard tool in financial mathematics, see e.g. Karatzas (1998).

We assume that the function $l \in C^1(0, \infty)$, the derivative $l'$ is strictly increasing with $l'(0^+) = 0$ and $l'(\infty) = \infty$. We denote by $I = (l')^{-1}$ the inverse function of $l'$. Starting from the state-dependent convex function $0 \leq x \leq H \mapsto l(H - x)$, we consider its stochastic Fenchel-Legendre transform:

\[
\tilde{L}(y, \omega) = \max_{0 \leq x \leq H} [-l(H - x) - xy] = -l(H \wedge I(y)) - y(H - I(y))_+ , \quad y \geq 0.
\]

We now consider the dual control problem:

\[
(D(y)) \quad \tilde{J}(y) = \inf_{Q \in \mathcal{P}} E\left[\tilde{L}\left(\frac{dQ}{dP}, \omega\right)\right], \quad y \geq 0.
\]

It is straightforward to see that $\tilde{J}$ is convex on $[0, \infty)$.

Our object is to provide a description of the solution to the problem $(P(x))$ in function of a solution to the problem $(D(y))$ when it exists. This can be viewed as a verification theorem expressed in terms of the dual control problem. In a Markovian context, this is an alternative to the usual verification theorem of stochastic control problems expressed in terms of the value function of the primal problem. Notice that, due to the state constraints, the Bellman equation associated to the dynamic primal problem
will involve non “classical” boundary conditions, which are delicate from a theoretical and numerical viewpoint (see e.g. Fleming and Soner 1993).

In the sequel, we shall assume that the nonnegative contingent claim $H$ is not equal to zero a.s. and that its superreplication cost is finite. We then assume that $0 < v_0(H) < \infty$.

**Theorem 5.2** Assume that $l(H) \in L^1(P)$ and $0 < v_0(H) < \infty$. Suppose that for all $y > 0$, there exists a solution $Q^*(y) \in \mathcal{P}$ to problem $(D(y))$. Then:

1. $\tilde{J}$ is differentiable on $(0, \infty)$ with derivative:

$$
\tilde{J}'(y) = -E^{Q^*(y)} \left[ \left( H - I \left( y \frac{dQ^*(y)}{dP} \right) \right)_+ \right],
$$

for all $y > 0$.

2. Let $0 < x < v_0(H)$. Then, there exists $\hat{y} > 0$ that attains the infimum in $\inf_{y>0} \{ \tilde{J}(y) + xy \}$, and we have:

$$
\tilde{J}'(\hat{y}) = -x.
$$

The unique solution of $J(x)$ is then given by:

$$
X^*(x) = \left( H - I \left( \hat{y} \frac{dQ^*(\hat{y})}{dP} \right) \right)_+.
$$

There exists $\theta^*(x) \in A(x)$ such that $X^*(x) = V_T^{x,\theta^*(x)}$, $P$ a.s., and $\theta^*(x)$ is solution to $(P(x))$. Moreover, we have:

$$
V_t^{x,\theta^*(x)} = E^{Q^*(\hat{y})} \left[ X^*(x)| \mathcal{F}_t \right], 0 \leq t \leq T.
$$

(3) We have the duality relation:

$$
J(x) = \max_{y \geq 0} \left[ -\tilde{J}(y) - xy \right], \ \forall x > 0.
$$

**Proof.** First notice that the maximum in (5.7) is attained for:

$$
\chi(y, \omega) = (H - I(y))_+, \ y \geq 0.
$$
The function $\tilde{L}(.,\omega)$ is convex, differentiable on $(0,\infty)$ with derivative:

$$\tilde{L}'(y,\omega) = -\chi(y,\omega), \ y \geq 0. \quad (5.11)$$

(1) Let $y > 0$. Then for all $\delta > 0$, we have:

$$\frac{\tilde{J}(y+\delta) - \tilde{J}(y)}{\delta} \leq \frac{1}{\delta} E \left[ \tilde{L} \left( (y+\delta) \frac{dQ^*(y)}{dP},\omega \right) - \tilde{L} \left( y \frac{dQ^*(y)}{dP},\omega \right) \right]$$

$$\leq - E Q^*(y) \left[ \chi \left( (y+\delta) \frac{dQ^*(y)}{dP},\omega \right) \right]$$

where we used (5.11) and convexity of $\tilde{L}(.,\omega)$. By Fatou’s lemma, we deduce that:

$$\lim_{\delta \downarrow 0^+} \frac{\tilde{J}(y+\delta) - \tilde{J}(y)}{\delta} \leq - E Q^*(y) \left[ \chi \left( y \frac{dQ^*(y)}{dP},\omega \right) \right]. \quad (5.12)$$

Similarly, for all $\delta < 0$, $y+\delta > 0$, we have:

$$\frac{\tilde{J}(y+\delta) - \tilde{J}(y)}{\delta} \geq - E Q^*(y) \left[ \chi \left( (y+\delta) \frac{dQ^*(y)}{dP},\omega \right) \right].$$

Since $|\chi|$ is bounded by $H$ and under the assumption that $v_0(H) < \infty$, one can apply the dominated convergence theorem to deduce that:

$$\lim_{\delta \downarrow 0^-} \frac{\tilde{J}(y+\delta) - \tilde{J}(y)}{\delta} \geq - E Q^*(y) \left[ \chi \left( y \frac{dQ^*(y)}{dP},\omega \right) \right]. \quad (5.13)$$

Relations (5.12)-(5.13) and convexity of the function $\tilde{J}$ imply the differentiability of $\tilde{J}$ and provide the expression (5.8) of $\tilde{J}'$.

(2) The function $y \mapsto f_x(y) = \tilde{J}(y) + xy$ is convex on $(0,\infty)$. Let us check that:

$$\lim_{y \to \infty} f_x(y) = \infty, \ \forall x > 0. \quad (5.14)$$

Indeed, by noting that $\tilde{L}(y,\omega) \geq -l(H)$, we have $\tilde{J}(y) \geq -E[l(H)]$. We deduce that $f_x(y) \geq -E[l(H)] + xy$, which proves (5.14). We now check that for all $0 < x < v_0(H)$, there exists $y_0 > 0$ such that $f_x(y_0) < 0$. On the contrary, we should have:

$$E \left[ \tilde{L} \left( y \frac{dQ}{dP},\omega \right) \right] + xy > 0, \ \forall y > 0, \forall Q \in \mathcal{P},$$

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and then
\[
x > E \left[ -\frac{1}{y} \tilde{L} \left( y \frac{dQ}{dP}, \omega \right) \right], \quad \forall y > 0, \forall Q \in \mathcal{P}.
\]

Since \(-\tilde{L}(yk/dP, \omega)/y \geq 0\) and \(-\tilde{L}(yk/dQ, \omega)/y\) converges to \(HdQ/dP\) as \(y\) goes to infinity, we deduce by Fatou’s lemma that :
\[
x \geq E^Q[H], \quad \forall Q \in \mathcal{P},
\]
and then \(x \geq v_0(H)\), a contradiction. We can then deduce that for all \(0 < x < v_0(H)\), the function \(f_x(.)\) attains an infimum for \(\hat{y} > 0\) and since \(\hat{J}\) and so \(f_x\), is differentiable on \((0, \infty)\), we have \(f'_x(\hat{y}) = 0\), i.e. \(\hat{J}'(\hat{y}) = -x\).

Fix some \(y > 0\) and let \(Q\) be an arbitrary element of \(\mathcal{P}\). Denote :
\[
Q^\varepsilon = (1 - \varepsilon)Q^*(y) + \varepsilon Q, \quad \varepsilon \in (0, 1).
\]

Obviously, \(Q^\varepsilon \in \mathcal{P}\) so that by definition of \(\hat{J}\), we have :
\[
0 \leq \frac{1}{\varepsilon} E \left[ \frac{\tilde{L} \left( y \frac{dQ^\varepsilon}{dP}, \omega \right) - \tilde{L} \left( y \frac{dQ^*(y)}{dP}, \omega \right) \right] y
\]
\[
\leq \frac{1}{\varepsilon} E \left[ \frac{\tilde{L} \left( y \frac{dQ^\varepsilon}{dP}, \omega \right) - \tilde{L} \left( y \frac{dQ^*(y)}{dP}, \omega \right) \right] y
\]
\[
\leq E \left[ - \left( \frac{dQ^\varepsilon}{dP} - \frac{dQ^*(y)}{dP} \right) \chi \left( y \frac{dQ^\varepsilon}{dP}, \omega \right) \right]
\]
where the third inequality from (5.11) and the convexity of \(\tilde{L}\). We obtain then :
\[
E^Q \left[ \chi \left( y \frac{dQ^\varepsilon}{dP}, \omega \right) \right] \leq E^{Q^*(y)} \left[ \chi \left( y \frac{dQ^*(y)}{dP}, \omega \right) \right]. \tag{5.15}
\]

By the dominated convergence theorem and Fatou’s lemma applied respectively to the R.H.S. and the L.H.S. of (5.15), we have :
\[
E^Q \left[ \chi \left( y \frac{dQ^*(y)}{dP}, \omega \right) \right] \leq E^{Q^*(y)} \left[ \chi \left( y \frac{dQ^*(y)}{dP}, \omega \right) \right].
\]

From (5.8) and (5.10), this can be written also as :
\[
\sup_{Q \in \mathcal{P}} E^Q \left[ \chi \left( y \frac{dQ^*(y)}{dP}, \omega \right) \right] \leq -\hat{J}'(y),
\]
for all $y > 0$. By choosing $y = \hat{y}$ defined above, we get:

$$\sup_{Q \in \mathcal{P}} E^Q \left[ \chi \left( \frac{dQ^*(\hat{y})}{dP}, \omega \right) \right] \leq x, \quad (5.16)$$

which proves that $X^*(x) = \chi \left( \frac{dQ^*(\hat{y})}{dP}, \omega \right)$ lies in $\mathcal{C}(x)$.

Moreover, by definition (5.7) of $\tilde{L}$ and by definition of $X^*(x)$, we have for all $x \in \mathcal{C}(x)$:

$$\tilde{L} \left( \hat{y} \frac{dQ^*(\hat{y})}{dP}, \omega \right) = -l(H - X^*(x)) - \hat{y} \frac{dQ^*(\hat{y})}{dP} X^*(x) \quad (5.17)$$

Taking expectation under $P$ in (5.17)-(5.18) and using the facts that:

$$E^Q \left[ X^*(x) \right] = -J'(\hat{y}) = x, \quad (5.19)$$

$$E^Q \left[ V \right] \leq x, \quad (5.20)$$

we obtain that:

$$E \left[ l(H - X^*(x)) \right] \leq E \left[ l(H - X) \right],$$

which proves that $X^*(x)$ is solution to problem $(\mathcal{S}(x))$. Relations (5.16) and (5.19) show that $Q^*(\hat{y})$ attains the supremum in $\sup_{Q \in \mathcal{P}} E^Q[X^*(x)]$, and by Theorem 5.1, this proves that there exists $\theta^*(x) \in \mathcal{A}(x)$ such that $X^*(x) \leq V^{x,\theta^*(x)}_T$, a.s., and $\theta^*(x)$ is solution of the dynamic problem $(\mathcal{P}(x))$. Moreover, since the associated wealth process $V^{x,\theta^*(x)}$ is a supermartingale under $Q^*(\hat{y})$, we have from (5.8)-(5.9):

$$x = E^Q \left[ X^*(x) \right] \leq E^Q \left[ V^{x,\theta^*(x)}_T \right] \leq x,$$

which shows that $X^*(x) = V^{x,\theta^*(x)}_T$, a.s., and that the wealth process $V^{x,\theta^*(x)}$ is a martingale under $Q^*(\hat{y})$. The assertion (2) of Theorem 5.2 is then proved.

(3) By definition (5.7) of the function $\tilde{L}$, we have for all $x \geq 0$, $y \geq 0$, $X \in \mathcal{C}(x)$, $Q \in \mathcal{P}$:

$$-l(H - X) - y \frac{dQ}{dP} X \leq \tilde{L} \left( y \frac{dQ}{dP}, \omega \right),$$
hence by taking expectation under $P$:

$$-E[l(H - X)] - yx \leq E\left[\tilde{L}\left(y \frac{dQ}{dP}, \omega\right)\right],$$

and therefore,

$$\sup_{y \geq 0} \left[ -\tilde{J}(y) - xy \right] \leq J(x), \quad \forall x \geq 0. \quad (5.21)$$

For $x \geq v_0(H)$, we have $J(x) = 0 = -\tilde{J}(0)$. Fix now $0 < x < v_0(H)$. From relations (5.17) and (5.19), we have:

$$\tilde{J}(\hat{y}) = -E[l(H - X^*(x))] - \hat{y} E\left[\frac{dQ^*(\hat{y})}{dP} X^*(x)\right] = -J(x) - x\hat{y},$$

which proves that $J(x) = -\tilde{J}(\hat{y}) - x\hat{y}$. The proof is ended. \hfill \Box

The description of the optimal hedging strategy is proceeded in two steps. In the first step, one has to solve a dual problem. Notice that in a markovian framework, such as the stochastic volatility model described in the previous sections, one has a parametrization of the set $\mathcal{P}$ and so the dual problem leads to a classical stochastic control problem. The optimal hedging strategy is then obtained as the (super)replicating strategy of a modified contingent claim, and can be computed via the superhedging approach described in Section 3.

References


