Numerical methods for an optimal order execution problem

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Abstract

This paper deals with numerical solutions to an impulse control problem arising from optimal portfolio liquidation with bid-ask spread and market price impact penalizing speedy execution trades. The corresponding dynamic programming (DP) equation is a quasi-variational inequality (QVI) with solvency constraint satisfied by the value function in the sense of constrained viscosity solutions. By taking advantage of the lag variable tracking the time interval between trades, we can provide an explicit backward numerical scheme for the time discretization of the DPQVI. The convergence of this discrete-time scheme is shown by viscosity solutions arguments. An optimal quantization method is used for computing the (conditional) expectations arising in this scheme. Numerical results are presented by examining the behaviour of optimal liquidation strategies, and comparative performance analysis with respect to some benchmark execution strategies. We also illustrate our optimal liquidation algorithm on real data, and observe various interesting patterns of order execution strategies. Finally, we provide some numerical tests of sensitivity with respect to the bid/ask spread and market impact parameters.

Keywords: Optimal liquidation, Impulse control problem, Quasi-variational inequality, explicit backward scheme, quantization method, viscosity solutions.

JEL Classification : G11.


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1 Introduction

Portfolios managers define “implementation shortfall” as the difference in performance between a theoretical trading strategy and the implemented portfolio. In a theoretical strategy, the investor observes price displayed by the market and assumes that trades will actually be executed at this price. Implementation shortfall measures the distance between the realized transaction price and the pre-trade decision price. Indeed, the investor has to face several adverse effects when executing a trading strategy, usually referred to as trading costs. Let us describe the three main components of these illiquidity effects: the bid/ask spread, the broker’s fees and the market impact. The best bid (resp. best ask) price is the best offer to buy (resp. to sell) the asset, and the bid/ask spread is the difference (always positive in the continuous trading session) between the best ask price and best bid price. The broker’s fees are the amount paid to the broker for executing the order. The market impact refers to the following phenomenon: any buy or sell market order passed by an investor induces an adverse market reaction that will penalize quoted price from the investor point of view.

Market impact is a key factor when executing large orders. A famous worst case example is Jérôme Kerviel’s liquidation portfolio, operated by Société Générale in 2008. According to the report of Commission Bancaire, the book value of Kerviel’s portfolio was $-2.7G€$ when the Société Générale decided to unwind it on January 20, 2008. The liquidation was operated during 3 days and led to a supplementary loss of $3.6G€$. Even in regular operations, price impact may noticeably affect a trading strategy. On April 29, 2010, Reuters agency reports that Citadel Investment Group sold 170M shares of the E*Trade stock, and raised about 301M$: this operation led to a price fall of 7.1%. These examples explain why measurement and efficient management of market impact is a key issue for financial institutions, and the research of low-touch trading strategies has found a great interest among academics.

Most of market places and brokers offer several common tools to reduce market impact. We can cite as an example the simple time slicing (we will refer to this example later as the uniform strategy): a large order is split up in multiple children orders of the same size, and these children orders are sent to the market at regular time intervals. Brokers also propose more sophisticated tools as smart order routing (SOR) or volume weighted average price (VWAP) based algorithmic strategies. Indeed, one basic observation is that market impact can be reduced by splitting up a large order into several children orders. Then the investor has to face the following trade-off: if he chooses to trade immediately, he will penalize his performance due to market impact; if he trades gradually, he is exposed to price variation on the period of the operation. Our goal in this article is to provide a numerical method to find optimal schedule and associated quantities for the children orders.

Recently, there has been considerable interest for this problem in the academic literature. The seminal papers [5] and [2] first provided a framework for managing market impact in a discrete-time model. The optimality is determined according to a mean-variance criterion, and this leads to a static strategy, in the sense that it is independent of the stock price. Models of market impact based on stylized order book dynamics were proposed in [16], [20] and [9]. There also has been several optimal control approaches to the order
execution problem, using a penalizing function to model price impact: the papers [19] and [8] assume continuous-time trading, and use an Hamilton-Jacobi-Bellman approach for the mean-variance criterion, while [10], [14], and [11] consider real trading taking place in discrete-time by using an impulse control approach. This last approach combines the advantages of realistic modelling of portfolio liquidation and the tractability of continuous-time stochastic calculus. In these papers, the optimal liquidation strategies are price-dependent in contrast with static strategies.

In this article, we adopt the model investigated in [11]. Let us describe the main features of this model. The stock price process is assumed to follow a geometrical Brownian motion. The price impact is modelled via a nonlinear transaction costs function, that depends both on the quantity traded, and on a lag variable \( \theta \) tracking the time spent since the investor’s last trade. This lag variable will penalize rapid execution trades, and ensures in particular that trading times are strictly increasing, which is consistent with market practice in limit order books. In this context, we consider the problem of an investor seeking to unwind an initial position in stock shares over a finite horizon. Risk aversion of the investor is modelled through a utility function, and we use an impulse control approach for the optimal order execution problem, which consists in maximizing the expected utility from terminal liquidation wealth, under a natural economic solvency constraint involving the liquidation value of portfolio. The theoretical part of this impulse control problem is studied in [11], and the solution is characterized through dynamic programming by means of a quasi-variational inequality (QVI) satisfied by the value function in the (constrained) viscosity sense. The aim of this paper is to solve numerically this optimal order execution problem. There are actually few papers dealing with a complete numerical treatment of impulse control problems, see [6], [15], or [7]. In these papers, the domain has a simple shape, typically rectangular, and a finite-difference method is used. In contrast, our domain is rather complex due to the solvency constraint naturally imposed by the liquidation value under market impact, and we propose a suitable probabilistic numerical method for solving the associated impulse control problem. Our main contributions are the following:

- We provide a numerical scheme for the QVI associated to the impulse control problem and prove that this method is monotone, consistent and stable, hence converges to the viscosity solution of the QVI. For this purpose, we adapt a proof from [4].

- We take advantage of the lag variable \( \theta \) to provide an explicit backward scheme and then simplify the computation of the solution. This contrasts with the classical approach by iterative sequence of optimal stopping problems, see e.g. [6].

- We provide the detailed computational probabilistic algorithm with an optimal quantization method for the approximation of conditional expectations arising in the backward scheme.

- We provide several numerical tests and statistics, both on simulated and real data, and compare the optimal strategy to a benchmark of two other strategies: the uniform strategy and the naive one consisting in the liquidation of all shares in one block at the terminal date. We also provide some sensitivity numerical analysis with respect to the bid/ask spread and market impact parameters.
This paper is organized as follows: Section 2 recalls the problem formulation and main properties of the model, in particular the PDE characterization of the impulse control problem by means of constrained viscosity solutions to the QVI, as stated in [11]. Section 3 is devoted to the time discretization and the proof of convergence of the numerical scheme. Section 4 provides the numerical algorithm and numerical methods to solve the DPQVI. We also address the convergence of the numerical scheme when approximating the exact expectation by the quantized expectation, discuss the complexity of the algorithm, and compare with the finite-difference scheme methods. Section 5 presents the results obtained with our implementation, both on simulated and historical data.

2 Problem formulation

2.1 The model of portfolio liquidation

We consider a financial market where an investor has to liquidate an initial position of \( y > 0 \) shares of risky asset by time \( T \). He faces the following risk/cost tradeoff: if he trades rapidly, this results in higher costs due to market impact; if he splits the order into several smaller blocks, he is exposed to the risk of price depreciation during the trading horizon. We adopt the recent continuous-time framework of [11], who proposed a modeling where trading takes place at discrete random times through an impulse control formulation, and with a temporary price impact depending on the time interval between trades, and including a bid-ask spread.

Let us recall the details of the model. We set a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped with a filtration \(\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) supporting a one-dimensional Brownian motion \( W \) on a finite horizon \([0, T], T < \infty\). We denote by \( P_t \) the market price of the risky asset, by \( X_t \) the cash holdings, by \( Y_t \) the number of stock shares held by the investor at time \( t \) and by \( \Theta_t \) the time interval between \( t \) and the last trade before \( t \).

Trading strategies. We assume that the investor can only trade at discrete time on \([0, T]\). This is modelled through an impulse control strategy \( \alpha = (\tau_n, \zeta_n)_{n \geq 1} \) where \( \tau_1 \leq \ldots \tau_n \leq \ldots \leq T \) are stopping times representing the trading times and \( \zeta_n, n \geq 1 \), are \( \mathcal{F}_{\tau_n} \)-measurable random variables valued in \( \mathbb{R} \) and giving the quantity of stocks purchased if \( \zeta_n \geq 0 \) or sold if \( \zeta_n < 0 \) at these times. A priori, the sequence \( (\tau_n, \zeta_n) \) may be finite or infinite. We introduce the lag variable tracking the time interval between trades, which evolves according to

\[
\Theta_t = t - \tau_n, \quad \tau_n \leq t < \tau_{n+1}, \quad \Theta_{\tau_{n+1}} = 0, \quad n \geq 0.
\]

(2.1)

The dynamics of the number of stock shares \( Y \) is then given by:

\[
Y_t = Y_{\tau_n}, \quad \tau_n \leq t < \tau_{n+1}, \quad Y_{\tau_{n+1}} = Y_{\tau_n} + \zeta_{n+1}, \quad n \geq 0.
\]

(2.2)

Cost of illiquidity. The market price of the risky asset process follows a geometric Brownian motion:

\[
dP_t = P_t(bdt + \sigma dW_t),
\]

(2.3)
with constant $b$ and $\sigma > 0$. We focus here on the temporary price impact that penalizes the price at which an investor will trade the asset. Suppose now that the investor decides at time $t$ to trade the quantity $e$. If the current market price is $p$, and the time lag from the last order is $\theta$, then the price he actually gets for the order $e$ is:

$$Q(e, p, \theta) = pf(e, \theta),$$

(2.4)

where $f$ is a temporary price impact function from $\mathbb{R} \times [0, T]$ into $\mathbb{R}_+ \cup \{\infty\}$. Actually, in the rest of the paper, we consider a function $f$ in the form

$$f(e, \theta) = \exp\left(\lambda \frac{|e|}{\theta} \beta \text{sgn}(e)\right).$$

(2.5)

where $\beta > 0$ is the price impact exponent, $\lambda > 0$ is the temporary price impact factor, $\kappa_b < 1$, and $\kappa_a > 1$ are the bid and ask spread parameters. The impact of liquidity modelled in (2.4) is like a transaction cost combining nonlinearity and proportionality effects. The nonlinear costs come from the dependence of the function $f$ on $e$, but also on $\theta$. On the other hand, this transaction cost function $f$ can be determined implicitly from the impact of a market order placed by a large trader in a limit order book, as explained in [16], [20] or [19]. Moreover, the dependence of $f$ in $\theta$ in (2.5) means that rapid trading has a larger temporary price impact than slower trading. Such kind of assumption is also made in the seminal paper [2], and reflects stylized facts on limit order books. The form (2.5) was suggested in several empirical studies, see [13], [18], [3], and used also in [8], [11].

**Remark 2.1** We could consider a permanent price impact, i.e. the lasting effect of large trade, in our modelling by introducing a jump in the market price $P$ at a trading time (as in [10] or [14]), which depends on the order size and time lag from the last order size. Alternatively, one can introduce a permanent price impact in the spirit of [2], [8] or [1] by modelling the rate of return $b = (b_t)$ of the market price as a state variable process following the dynamics:

$$db_t = \rho(\theta_t)(\bar{b} - b_t)dt, \quad \tau_n \leq t < \tau_{n+1}, \quad b_{\tau_{n+1}} = b_{\tau_n} + g\left(\frac{\zeta_{\tau_{n+1}}}{\tau_{n+1} - \tau_n}\right),$$

where $g$ is the permanent price impact function, e.g. in the linear form $g(\eta) = \kappa_p \eta$, with a factor $\kappa_p > 0$, and $\rho$ is an increasing positive resilience function, e.g. in the linear form $\rho(\theta) = \kappa_r \theta$, $\kappa_r > 0$, measuring the reversion rate of the return process to a reference constant value $\bar{b}$.

**Cash holdings.** We assume a zero risk-free return, so that the cash holdings are constant between two trading times:

$$X_t = X_{\tau_n}, \quad \tau_n \leq t < \tau_{n+1}, \quad n \geq 0.$$  

(2.6)

When a discrete trading $\Delta Y_t = \zeta_{\tau_{n+1}}$ occurs at time $t = \tau_{n+1}$, this results in a variation of the cash amount given by $\Delta X_t := X_t - X_{\tau_n} = -\Delta Y_t Q(\Delta Y_t, P_t, \Theta_t)$ due to the illiquidity effects. Moreover, there is a fixed cost $\varepsilon \geq 0$ to be paid at each transaction. In other words, we have

$$X_{\tau_{n+1}} = X_{\tau_n} - \zeta_{\tau_{n+1}} P_{\tau_{n+1}} f(\zeta_{\tau_{n+1}}, \tau_{n+1} - \tau_n) - \varepsilon, \quad n \geq 0.$$  

(2.7)
Remark 2.2 Notice that since $f(e, 0) = 0$ if $e < 0$ and $f(e, 0) = \infty$ if $e > 0$, an immediate sale does not increase the cash holdings, i.e. $X_{\tau_{n+1}} = X_{\tau_n} = X_{\tau_n}$, while an immediate purchase leads to a bankruptcy i.e. $X_{\tau_{n+1}} = -\infty$.

Liquidation value and solvency constraint. The solvency constraint is a key issue in portfolio choice problem. The point is to define in an economically meaningful way what is the portfolio value of a position in cash and stocks. In our context, we first impose a no-short selling constraint on the trading strategies, i.e.

$$Y_t \geq 0, \quad 0 \leq t \leq T.$$ 

Next, we introduce the liquidation function $L_\varepsilon(x, y, p, \theta)$ representing the value that an investor would obtain by liquidating immediately his stock position $y$ by a single block trade, when the pre-trade price is $p$ and the time lag from the last order is $\theta$. It is defined on $\mathbb{R} \times \mathbb{R}_+ \times (0, \infty) \times [0, T]$ by

$$L_\varepsilon(x, y, p, \theta) = \max\{x, x + yp f(-y, \theta) - \varepsilon\}.$$ 

The interpretation of this liquidation function is the following. Due to the presence of the transaction fee at each trading, it may be advantageous for the investor not to liquidate his position in stock shares (which would give him $x + yp f(-y, \theta) - \varepsilon$), and rather bin his stock shares, by keeping only his cash amount (which would give him $x$). Hence, the investor chooses the best of these two possibilities, which induces a liquidation value $L_\varepsilon(z, \theta)$.

We thus constrain the portfolio’s liquidative value to satisfy the solvency criterion:

$$L_\varepsilon(X_t, Y_t, P_t, \Theta_t) \geq 0, \quad 0 \leq t \leq T.$$ 

We then naturally introduce the solvency region:

$$S_\varepsilon = \{(z, \theta) = (x, y, p, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times (0, \infty) \times [0, T] : L_\varepsilon(z, \theta) > 0\}.$$ 

and we denote its boundary and its closure by

$$\partial S_\varepsilon = \partial_y S_\varepsilon \cup \partial L S_\varepsilon \quad \text{and} \quad \bar{S}_\varepsilon = S_\varepsilon \cup \partial S_\varepsilon.$$ 

where

$$\partial_y S_\varepsilon = \{(z, \theta) = (x, y, p, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times (0, \infty) \times [0, T] : y = 0 \text{ and } x = L_\varepsilon(z, \theta) \geq 0\},$$

$$\partial L S_\varepsilon = \{(z, \theta) = (x, y, p, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times (0, \infty) \times [0, T] : L_\varepsilon(z, \theta) = 0\}.$$ 

In the sequel, we also introduce the corner lines in $\partial S_\varepsilon$:

$$D_0 = \{(0, 0) \times (0, \infty) \times [0, T] = \partial_y S_\varepsilon \cap \partial L S_\varepsilon.$$ 

Admissible trading strategies. Given $(t, z, \theta) \in [0, T] \times \bar{S}_\varepsilon$, we say that the impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n \geq 0}$ is admissible, denoted by $\alpha \in \mathcal{A}_\varepsilon(t, z, \theta)$, if $\tau_0 = t - \theta$, $\tau_n \geq t$, $n \geq 1$, and the process $\{(Z_s, \Theta_s) = (X_s, Y_s, P_s, \Theta_s), t \leq s \leq T\}$ solution to (2.1)-(2.2)-(2.3)-(2.6)-(2.7), with an initial state $(Z_{t-}, \Theta_{t-}) = (z, \theta)$ (and the convention that $(Z_t, \Theta_t)$
Immediately at time \( t \), satisfies \((Z_s, \Theta_s) \in [0,T] \times \bar{S} \) for all \( s \in [t,T] \). As usual, to alleviate notations, we omit the dependence of \((Z, \Theta)\) in \((t,z,\theta,\alpha)\), when there is no ambiguity.

**Portfolio liquidation problem.** We consider a utility function \( U \) from \( \mathbb{R}_+ \) into \( \mathbb{R} \), strictly increasing, concave and w.l.o.g. \( U(0) = 0 \), and s.t. there exists \( K \geq 0 \), \( \gamma \in [0,1) \):

\[
U(w) \leq Kw^\gamma, \quad \forall w \geq 0.
\]

The problem of optimal portfolio liquidation is formulated as

\[
v_\varepsilon(t,z,\theta) = \sup_{\alpha \in \mathcal{A}_\varepsilon(t,z,\theta)} \mathbb{E}[U_{L_\varepsilon}(Z_T, \Theta_T)], \quad (t,z,\theta) \in [0,T] \times \bar{S}_\varepsilon,
\]

where \( U_{L_\varepsilon}(z,\theta) = U(L_\varepsilon(z,\theta)) \) is the terminal liquidation utility function.

**Remark 2.3** The function \( z \to v_\varepsilon(t,z,0) \) is strictly increasing in the argument of cash holdings \( x \), for \((z = (x,y,p),0) \in \bar{S}_\varepsilon \), and fixed \( t \in [0,T] \). Indeed, for \( x < x' \), and \( z = (x,y,p), z' = (x',y,p) \), any strategy \( \alpha \in \mathcal{A}_\varepsilon(t,z,\theta) \) with corresponding state process \((Z_s = (X_s,Y_s,P_s), \Theta_s)_{s \geq t}\), is also in \( \mathcal{A}_\varepsilon(t,z',\theta) \), and leads to an associated state process \((Z'_s = (X_s + x' - x, Y_s, P_s), \Theta_s)_{s \geq t}\). Using the fact that the utility function is strictly increasing, we deduce that \( v_\varepsilon(t,x,y,p,0) < v_\varepsilon(t,x',y,p,0) \). Moreover, the function \( z \to v_\varepsilon(t,z,0) \) is nondecreasing in the argument of number of shares \( y \). Indeed, fix \( z = (x,y,p) \), and \( z' = (x,y',p) \) with \( y \leq y' \). Given any arbitrary \( \alpha = (\tau_n, \zeta_n)_{n \in \mathbb{N}} \in \mathcal{A}_\varepsilon(t,z,0) \), consider the strategy \( \alpha' = (\tau'_n, \zeta'_n) \), starting from \((x,y',p)\) at time \( t \), which consists in trading again immediately at time \( t \) by selling \( y' - y \) shares (which does not change the cash holdings, see Remark 2.2), and then follow the same strategy as \( \alpha \). The corresponding state process satisfies \((Z'_s, \Theta'_s) = (Z_s, \Theta_s) \) a.s. for \( s \geq t \), and in particular \( \alpha' \in \mathcal{A}_\varepsilon(t,z',0) \), together with \( \mathbb{E}[U_{L_\varepsilon}(Z'_T, \Theta'_T)] = \mathbb{E}[U_{L_\varepsilon}(Z_T, \Theta_T)] \leq v(t,z',\theta) \). Since \( \alpha \) is arbitrary in \( \mathcal{A}_\varepsilon(t,z,0) \), this shows that \( v(t,x,y,p,0) \leq v(t,x,y',p,0) \).

We recall from [11] that \( v_\varepsilon \) is in the set \( \mathcal{G}([0,T] \times \bar{S}_\varepsilon) \) of functions satisfying the growth condition:

\[
\mathcal{G}([0,T] \times \bar{S}_\varepsilon) = \left\{ \varphi : [0,T] \times \bar{S}_\varepsilon \rightarrow \mathbb{R} \ \text{s.t.} \ \sup_{[0,T] \times \bar{S}_\varepsilon} \frac{|\varphi(t,z,\theta)|}{1 + (x + yp)^\gamma} < \infty \right\}.
\]

In the sequel, we shall denote by \( \mathcal{G}_+([0,T] \times \bar{S}_\varepsilon) \) the set of functions \( \varphi \in \mathcal{G}([0,T] \times \bar{S}_\varepsilon) \) such that \( \varphi(t,x,y,p,0) \) is strictly increasing in \( x \) and nondecreasing in \( y \).

### 2.2 PDE characterization

The dynamic programming Hamilton-Jacobi-Bellman (HJB) equation corresponding to the stochastic control problem (2.8) is a quasi-variational inequality written as

\[
\min \left[-\frac{\partial v}{\partial t} - \mathcal{L}v, \ v - \mathcal{H}_x v\right] = 0, \quad \text{on} \quad [0,T] \times \bar{S}_\varepsilon,
\]

(2.9)

with the relaxed terminal condition

\[
\min \left[v - U_{L_\varepsilon}, v - \mathcal{H}_x v\right] = 0, \quad \text{on} \quad \{T\} \times \bar{S}_\varepsilon.
\]

(2.10)
Here, $\mathcal{L}$ is the infinitesimal generator associated to the process $(Z = (X,Y,P),\Theta)$ in a no-trading period:

$$\mathcal{L}\phi = \frac{\partial \phi}{\partial \theta} + b p \frac{\partial \phi}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 \phi}{\partial p^2},$$

$\mathcal{H}_\varepsilon$ is the impulse operator defined by

$$\mathcal{H}_\varepsilon \phi(t,z,\theta) = \sup_{e \in \mathcal{C}_\varepsilon(z,\theta)} \phi(t,\Gamma_\varepsilon(z,\theta,e),0), \quad (t,z,\theta) \in [0,T] \times \bar{S}_\varepsilon,$$

$\Gamma_\varepsilon$ is the impulse transaction function defined from $\bar{S}_\varepsilon \times \mathbb{R}$ into $\mathbb{R} \times \mathbb{R} \times (0,\infty)$:

$$\Gamma_\varepsilon(z,\theta,e) = (x - ep f(e,\theta) - \varepsilon, y + e, p), \quad z = (x,y,p) \in \bar{S}_\varepsilon, \quad e \in \mathbb{R},$$

and $\mathcal{C}_\varepsilon(z,\theta)$ the set of admissible transactions:

$$\mathcal{C}_\varepsilon(z,\theta) = \left\{ e \in \mathbb{R} : \left( \Gamma_\varepsilon(z,\theta,e),0 \right) \in \bar{S}_\varepsilon \right\}.$$

**Remark 2.4** Fix $t \in [0,T]$. For $\theta = 0$, and $z = (x,y,p)$ s.t. $(z,0) \in \bar{S}_\varepsilon$, the set of admissible transactions $\mathcal{C}_\varepsilon(z,0) = [-y,0]$ (and $\Gamma_\varepsilon(z,0,e) = (x - \varepsilon, y + e, p)$ for $e \in \mathcal{C}_\varepsilon(z,0)$) if $x \geq \varepsilon$, and is empty otherwise. Thus, $\mathcal{H}_\varepsilon w(t,z,0) = \sup_{e \in [-y,0]} w(t,x - \varepsilon, y + e, p,0)$ if $x \geq \varepsilon$, and is equal to $-\infty$ otherwise. This implies in particular that

$$\mathcal{H}_\varepsilon w(t,z,0) < w(t,z,0), \quad (2.11)$$

for any $w \in \mathcal{G}_+([0,T] \times \bar{S}_\varepsilon)$, which is the case of $v_\varepsilon$ (see Remark 2.3). Therefore, due to the market impact function $f$ in (2.5) penalizing rapid trades, it is not optimal to trade again immediately right after some trade, i.e. the optimal trading times are strictly increasing.

A main result in [11] is to provide a unique PDE characterization of the value functions $v_\varepsilon$, $\varepsilon > 0$, and to prove that the sequence $(v_\varepsilon)_\varepsilon$ converges, as $\varepsilon$ goes to zero, to the value function $v_0$ in the model without transaction fee, i.e. when $\varepsilon = 0$.

**Theorem 2.1** (1) The sequence $(v_\varepsilon)_\varepsilon$ is nonincreasing, and converges pointwise on $[0,T] \times (\bar{S}_0 \setminus \partial L_0 S_0)$ towards $v_0$ as $\varepsilon$ goes to zero, with $v_\varepsilon \leq v_0$.

(2) For any $\varepsilon > 0$, the value function $v_\varepsilon$ is continuous on $[0,T] \times \bar{S}_\varepsilon$, and is the unique (in $[0,T] \times \bar{S}_\varepsilon$) constrained viscosity solution to (2.9)-(2.10), satisfying the growth condition in $\mathcal{G}([0,T] \times \bar{S}_\varepsilon)$, and the boundary condition:

$$\lim_{(t',z',\theta') \to (t,z,\theta)} v_\varepsilon(t',z',\theta') = v_\varepsilon(t,z,\theta) = U(0), \quad \forall (t,z = (0,0,p),\theta) \in [0,T] \times D_0. \quad (2.12)$$

The rest of this paper is devoted to the numerical analysis and resolution of the QVI (2.9)-(2.10) characterizing the optimal portfolio liquidation problem with fixed transaction fee. On the other hand, this also provide an $\varepsilon$-approximation of the optimal portfolio liquidation problem without fixed transaction fee.
3 Time discretization and convergence analysis

In this section, we fix \( \varepsilon > 0 \), and we study time discretization of the QVI (2.9)-(2.10) characterizing the value function \( v_z \). For a time discretization step \( h > 0 \) on the interval \([0, T]\), let us consider the following approximation scheme:

\[
S^h(t, z, \theta, v^h(t, z, \theta), v^h) = 0, \quad (t, z, \theta) \in [0, T] \times \mathcal{S}_\varepsilon,
\]

where \( S^h : [0, T] \times \mathcal{S}_\varepsilon \times \mathbb{R} \times \mathcal{G}_+(0, T] \times \mathcal{S}_\varepsilon \rightarrow \mathbb{R} \) is defined by

\[
S^h(t, z, \theta, r, \varphi) = \begin{cases} 
\min & r - \mathbb{E}[\varphi(t + h, Z_{t+h}^{0,t,z}, \Theta_{t+h}^{0,t,\theta})] - r - \mathcal{H}_e \varphi(t, z, \theta) \quad & \text{if } t \in [0, T - h] \\
\min & r - \mathbb{E}[\varphi(T, Z_T^{0,t,z}, \Theta_T^{0,t,\theta})] - r - \mathcal{H}_e \varphi(t, z, \theta) \quad & \text{if } t \in (T - h, T) \\
\min & r - U_{L_e}(z, \theta) - r - \mathcal{H}_e \varphi(t, z, \theta) \quad & \text{if } t = T.
\end{cases}
\]

Here, \((Z_{s,t}^{0,t,z}, \Theta_{s,t}^{0,t,\theta})\) denotes the state process starting from \((z, \theta)\) at time \( t \), and without any impulse control strategy: it is given by

\[
\left(Z_{s,t}^{0,t,z}, \Theta_{s,t}^{0,t,\theta}\right) = (x, y, P_{s,t}^{t,y}, \theta + s - t), \quad s \geq t,
\]

with \( P_{s,t}^{t,y} \) the solution to (2.3) starting from \( p \) at time \( t \). Notice that (3.1) is formulated as a backward scheme for the solution \( v^h \) through:

\[
v^h(T, z, \theta) = \max \left[U_{L_e}(z, \theta), \mathcal{H}_e v^h(T, z, \theta)\right],
\]

(3.3)

\[
v^h(t, z, \theta) = \max \left[\mathbb{E}[v^h(t + h, Z_{t+h}^{0,t,z}, \theta + h)] - \mathcal{H}_e v^h(t, z, \theta)\right], \quad 0 \leq t \leq T - h,
\]

(3.4)

and \( v^h(t, z, \theta) = v^h(T - h, z, \theta) \) for \( T - h < t < T \). This approximation scheme seems a priori implicit due to the nonlocal obstacle term \( \mathcal{H}_e \). This is typically the case in impulse control problems, and the usual way (see e.g. [6], [15]) to circumvent this problem is to iterate the scheme by considering a sequence of optimal stopping problems:

\[
v^{h,n+1}(T, z, \theta) = \max \left[U_{L_e}(z, \theta), \mathcal{H}_e v^{h,n}(T, z, \theta)\right],
\]

\[
v^{h,n+1}(t, z, \theta) = \max \left[\mathbb{E}[v^{h,n+1}(t + h, Z_{t+h}^{0,t,z}, \theta + h)] - \mathcal{H}_e v^{h,n}(t, z, \theta)\right],
\]

starting from \( v^{h,0} = \mathbb{E}[U_{L_e}(Z_T^{0,t,z}, \Theta_T^{0,t,\theta})] \). Here, we shall make the numerical scheme (3.1) explicit, i.e. without iteration, by taking effect of the state variable \( \theta \) in our model. Recall indeed from Remark 2.4 that it is not optimal to trade again immediately right after some trade. Thus, for \( v^h \in \mathcal{G}_+(0, T] \times \mathcal{S}_\varepsilon \), and any \((z', 0) \in \mathcal{S}_\varepsilon\), we have from (2.11) and (3.3)-(3.4):

\[
v^h(T, z', 0) = U_{L_e}(z', 0)
\]

\[
v^h(t, z', 0) = \mathbb{E}[v^h(t + h, Z_{t+h}^{0,t,z'}, \theta + h)]
\]

Therefore, by using again the definition of \( \mathcal{H}_e \) in the relations (3.3)-(3.4), we see that the scheme (3.1) is written equivalently as an explicit backward scheme:

\[
v^h(T, z, \theta) = \max \left[U_{L_e}(z, \theta), \mathcal{H}_e U_{L_e}(z, \theta)\right],
\]

(3.5)

\[
v^h(t, z, \theta) = \max \left[\mathbb{E}[v^h(t + h, Z_{t+h}^{0,t,z}, \theta + h)] - \mathcal{H}_e v^h(t, z, \theta)\right], \quad \sup_{e \in \mathcal{C}_e(z, \theta)} \mathbb{E}[v^h(t + h, Z_{t+h}^{0,t,z}, \theta + h)],
\]

(3.6)
for $0 \leq t \leq T - h$, and $v^h(t, z, \theta) = v^h(T - h, z, \theta)$ for $T - h < t < T$, where we denote $z^e = \Gamma(z, \theta, e)$ in (3.6) to alleviate notations. Notice that at this stage, this approximation scheme is not yet fully implementable since it requires an approximation method for the expectations arising in (3.6). This is the concern of the next section.

We focus now on the convergence (when $h$ goes to zero) of the solution $v^h$ to (3.1) towards the value function $v_\epsilon$ solution to (2.9)-(2.10). Following [4], we have to show that the scheme $S^h$ in (3.2) satisfies monotonicity, stability and consistency properties. As usual, the monotonicity property follows directly from the definition (3.2) of the scheme.

**Proposition 3.1 (Monotonicity)**

For all $h > 0$, $(t, z, \theta) \in [0, T] \times \tilde{S}_\epsilon$, $r \in \mathbb{R}$, and $\varphi, \psi \in \mathcal{G}_+([0, T] \times \tilde{S}_\epsilon)$ s.t. $\varphi \leq \psi$, we have

$$S^h(t, z, \theta, r, \varphi) \geq S^h(t, z, \theta, r, \psi).$$

We next prove the stability property.

**Proposition 3.2 (Stability)**

For all $h > 0$, there exists a unique solution $v^h \in \mathcal{G}_+([0, T] \times \tilde{S}_\epsilon)$ to (3.1), and the sequence $(v^h)_h$ is uniformly bounded in $\mathcal{G}([0, T] \times \tilde{S}_\epsilon)$: there exists $w \in \mathcal{G}([0, T] \times \tilde{S}_\epsilon)$ s.t. $|v^h| \leq |w|$ for all $h > 0$.

**Proof.** The uniqueness of a solution $\in \mathcal{G}_+([0, T] \times \tilde{S}_\epsilon)$ to (3.1) follows from the explicit backward scheme (3.5)-(3.6). For $t \in [0, T]$, denote by $N_{t,h}$ the integer part of $(T - t)/h$, and $T_{t,h} = \{t_k = t + kh, k = 0, \ldots, N_{t,h}\}$ the partition of the interval $[t, T]$ with time step $h$. For $(t, z, \theta) \in [0, T] \times \tilde{S}_\epsilon$, we denote by $A^h_t(t, z, \theta)$ the subset of elements $\alpha = (\tau_n, \zeta_n)_n$ in $\mathcal{A}_\epsilon(t, z, \theta)$ such that the trading times $\tau_n$ are valued in $T_{t,h}$. Let us then consider the impulse control problem

$$v^h(t, z, \theta) = \sup_{\alpha \in A^h_t(t, z, \theta)} \mathbb{E}[U_{\alpha}(Z^\epsilon_T, \Theta_T)], \quad (t, z, \theta) \in [0, T] \times \tilde{S}_\epsilon. \quad \text{(3.7)}$$

It is clear from the representation (3.7) that for all $h > 0$, $0 \leq v^h \leq v_\epsilon$, which shows that the sequence $(v^h)_h$ is uniformly bounded in $\mathcal{G}([0, T] \times \tilde{S}_\epsilon)$. Moreover, similarly as for $v_\epsilon$, and by the same arguments as in Remark 2.3, we see that $v^h(t, z, 0)$ is strictly increasing in $x$ and nondecreasing in $y$ for $(z, 0) = (x, y, p, 0) \in \tilde{S}_\epsilon$. Finally, we observe that the numerical scheme (3.1) is the dynamic programming equation satisfied by the value function $v^h$. This proves the required stability result. \hfill \Box

We now move on the consistency property.

**Proposition 3.3 (Consistency)**

(i) For all $(t, z, \theta) \in [0, T] \times \tilde{S}_\epsilon$ and $\phi \in C^{1,2}([0, T] \times \tilde{S}_\epsilon)$, we have

$$\limsup_{(t', z', \theta') \to (t, z, \theta), (t', z', \theta') \in [0, T] \times \tilde{S}_\epsilon} \min \left\{ \frac{\phi(t', z', \theta') - \mathbb{E}\left[ \phi(t' + h, Z^\epsilon_{t'+h}, \Theta^0_{t'+h}) \right]}{h}, \left( \phi - \mathcal{H}_\epsilon \phi \right)(t', z', \theta') \right\}$$

$$\leq \min \left\{ \left( -\partial \phi \over \partial t \right)(t, z, \theta), \left( \phi - \mathcal{H}_\epsilon \phi \right)(t, z, \theta) \right\} \quad \text{(3.8)}$$
and
\[
\lim \inf_{(h,t',z',\theta') \to (0, t, z, \theta)} \limsup_{(t',z',\theta') \in [0, T] \times \mathcal{S}_c} \min \left\{ \frac{\phi(t', z', \theta') - \mathbb{E} \left[ \phi(t' + h, Z_{t'+h}^{0,t',z'}, \Theta_{t'+h}^{0,t',z'}) \right]}{h}, \left( \phi - \mathcal{H}_c \phi \right)(t', z', \theta') \right\}
\]
\[
\geq \min \left\{ \left( - \frac{\partial \phi}{\partial t} - \mathcal{L} \phi \right)(t, z, \theta), \left( \phi - \mathcal{H}_c \phi \right)(t, z, \theta) \right\}
\]
(3.9)

(ii) For all \((z, \theta) \in \mathcal{S}_c\) and \(\phi \in C^{1,2}([0, T] \times \mathcal{S}_c)\), we have
\[
\lim \sup_{(t',z',\theta') \to (T, z, \theta)} \min \left\{ \phi(t', z', \theta') - U_{L_c}(z', \theta'), \left( \phi - \mathcal{H}_c \phi \right)(t', z', \theta') \right\}
\]
\[
\leq \min \left\{ \phi(T, z, \theta) - U_{L_c}(z, \theta), \left( \phi - \mathcal{H}_c \phi \right)(T, z, \theta) \right\}
\]
(3.10)

and
\[
\lim \inf_{(t',z',\theta') \to (T, z, \theta)} \min \left\{ \phi(t', z', \theta') - U_{L_c}(z', \theta'), \left( \phi - \mathcal{H}_c \phi \right)(t', z', \theta') \right\}
\]
\[
\geq \min \left\{ \left( \phi(T, z, \theta) - U_{L_c}(z, \theta) \right), \left( \phi - \mathcal{H}_c \phi \right)(T, z, \theta) \right\}
\]
(3.11)

**Proof.** The arguments are standard, and can be adapted e.g. from [6] or [7]. We sketch the proof, and only show the inequality (3.8) since the other ones are derived similarly. Fix \(t \in [0, T)\). Since the minimum of two upper-semicontinuous (usc) functions is also usc and using the characterization of usc functions, we have
\[
\lim \sup_{(h,t',z',\theta') \to (0, t, z, \theta)} \min \left\{ \phi(t', z', \theta'), \frac{\phi(t', z', \theta') - \mathbb{E} \left[ \phi(t' + h, Z_{t'+h}^{0,t',z'}, \Theta_{t'+h}^{0,t',z'}) \right]}{h} \right\}
\]
\[
\leq \lim \sup_{(h,t',z',\theta') \to (0, t, z, \theta)} \lim \sup_{(t'',z'',\theta'') \to (0, t', z', \theta')} \min \left\{ \phi(t'', z'', \theta''), \frac{\phi(t'', z'', \theta'') - \mathbb{E} \left[ \phi(t'' + h, Z_{t''+h}^{0,t'',z''}, \Theta_{t''+h}^{0,t'',z''}) \right]}{h} \right\}
\]
\[
\leq \min \left\{ \phi(t', z', \theta'), \frac{\phi(t', z', \theta') - \mathbb{E} \left[ \phi(t' + h, Z_{t'+h}^{0,t',z'}, \Theta_{t'+h}^{0,t',z'}) \right]}{h} \right\}
\]
\[
\leq \min \left\{ \phi(t, z, \theta) - \mathcal{H}_c \phi(t, z, \theta) \right\}
\]
(3.12)
where the last inequality follows from the continuity of $\phi$ and the lower semicontinuity of $\mathcal{H}_\varepsilon$. Moreover, by Itô’s formula applied to $\phi(s, Z^0_{s+t'}, Z^{0, t'}_{s}, \Theta^0_{s+t'}, \Theta^0_{t'})$, and standard arguments of localization to remove in expectation the stochastic integral, we get

$$
\limsup_{(h, t', z', \theta') \to (0, t, z, \theta)} \frac{\phi(t', z', \theta') - \mathbb{E} \left[ \phi(t' + h, Z^0_{t'+h}, \Theta^0_{t'+h}) \right]}{h} = - \left( \frac{\partial \phi}{\partial t} + \mathcal{L} \phi \right)(t, z, \theta)
$$

Substituting into (3.12), we obtain the desired inequality (3.8).

Since the numerical scheme (3.1) is monotone, stable and consistent, we can follow the viscosity solutions arguments as in [4] to prove the convergence of $v^h$ to $v_\varepsilon$, by relying on the PDE characterization of $v_\varepsilon$ in Theorem 2.1 (2), and the strong comparison principle for (2.9)-(2.10) proven in [11].

**Theorem 3.1 (Convergence)** The solution $v^h$ of the numerical scheme (3.1) converges locally uniformly to $v_\varepsilon$ on $[0, T) \times S_\varepsilon$.

**Proof.** Let $v^\leq_\varepsilon$ and $v^\geq_\varepsilon$ be defined on $[0, T] \times S_\varepsilon$ by

$$
v^\leq_\varepsilon(t, z, \theta) = \limsup_{(h, t', z', \theta') \to (t, z, \theta)} v^h(t', z', \theta')
$$

$$
v^\geq_\varepsilon(t, z, \theta) = \liminf_{(h, t', z', \theta') \to (t, z, \theta)} v^h(t', z', \theta')
$$

We first see that $v^\leq_\varepsilon$ and $v^\geq_\varepsilon$ are respectively viscosity subsolution and supersolution of (2.9)-(2.10). These viscosity properties follow indeed, by standard arguments as in [4] (see also [6] or [7] for impulse control problems), from the monotonicity, stability and consistency properties. Details can be obtained upon request to the authors. Moreover, from (3.7), we have the inequality: $U(0) \leq v^h \leq v_\varepsilon$, which implies by (2.12):

$$
\liminf_{(t', z', \theta') \to (t, z, \theta)} v^\geq_\varepsilon(t', z', \theta') = U(0) = v^\leq_\varepsilon(t, z, \theta), \quad \forall (t, z, \theta) \in [0, T] \times D_0
$$

Thus, by using the strong comparison principle for (2.9)-(2.10) stated in Theorem 5.2 [11], we deduce that $v^\leq_\varepsilon \leq v^\geq_\varepsilon$ on $[0, T] \times S_\varepsilon$ and so $v^\leq_\varepsilon = v^\geq_\varepsilon = v_\varepsilon$ on $[0, T] \times S_\varepsilon$. This proves the required convergence result. $\square$

**4 Numerical Algorithm**

Let us consider a time step $h = T/m$, $m \in \mathbb{N} \setminus \{0\}$, and denote by $T_m = \{t_i = ih, i = 0, \ldots, m\}$ the regular grid over the interval $[0, T]$. We recall from the previous section that the time discretization of step $h$ for the QVI (2.9)-(2.10) leads to the convergent explicit
backward scheme:

\[
\begin{align*}
\v^h(t_m, z, \theta) &= \begin{cases} 
U_L(\v, \theta), & \text{if } \theta = 0 \\
\max \left[ U_L(\v, \theta), \sup_{e \in C(z, \theta)} \v(t_m, \Gamma_\v(z, \theta, e), 0) \right], & \text{if } \theta > 0,
\end{cases} \\
\v^h(t_i, z, \theta) &= \begin{cases} 
\mathbb{E}[\v(t_{i+1}, Z^{0, t_{i+1}}, \theta + h)] \\
\max \left[ \mathbb{E}[\v(t_{i+1}, Z^{0, t_{i+1}}, \theta + h)], \sup_{e \in C(z, \theta)} \v(t_i, \Gamma_\v(z, \theta, e), 0) \right], & \text{if } \theta > 0.
\end{cases}
\end{align*}
\]

for \(i = 0, \ldots, m - 1\), \((z = (x, y, p), \theta) \in \bar{\mathcal{S}}_\v\). Recall that the variable \(\theta\) represents the time lag between the current time \(t\) and the last trade. Thus, it suffices to consider at each time step \(t_i\) of \(T_m\), a discretization for \(\theta\) valued in the time grid

\[T_i = \{ \theta_j = jh, \ j = 0, \ldots, i \}, \ i = 0, \ldots, m.\]

On the other hand, the above scheme involves nonlocal terms in the variable \(z\) for the solution \(\v^h\) in relation with the supremum over \(e \in C_\v(z, \theta)\) and the expectations in (4.1)-(4.2), and thus the practical implementation requires a discretization of the set of admissible transactions \(C_\v(z, \theta)\) and a computational approximation for the above expectations. Moreover, since the state space \(\bar{\mathcal{S}}_\v\) is unbounded, we also need to localize the domain on which computations are done. For any \(\theta_j \in T_i\), let us denote by

\[
\begin{align*}
\mathcal{Z}^j &= \{ z = (x, y, p) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ : (z, \theta_j) \in \bar{\mathcal{S}}_\v \}, \\
\mathcal{Z}^j_{\text{loc}} &= \mathcal{Z}^j \cap ([x_{\text{min}}, x_{\text{max}}] \times [0, y_{\text{max}}] \times [0, p_{\text{max}}]),
\end{align*}
\]

where \(x_{\text{min}} < x_{\text{max}}\) in \(\mathbb{R}\), \(0 < y_{\text{max}}\), \(0 < p_{\text{max}}\) are fixed constants.

Let us first discretize the set of admissible transactions \(C_\v(z, \theta_j)\) over which the supremum in (4.2) is taken, for any \(\theta_j \in T_i\), \(z \in \mathcal{Z}^j_{\text{loc}}\). Recall from [11] that \(C_\v(z, \theta_j)\) is compact in the form \([c(z, \theta_j), e(z, \theta_j)]\). We then consider the discrete set of admissible transactions of size \(M\):

\[
C^M(z, \theta_j) = \{ e = c(z, \theta_j) + \frac{i}{M} (e(z, \theta_j) - c(z, \theta_j)), i = 0, \ldots, M : \Gamma_\v(z, \theta_j) \in \mathcal{Z}^0_{\text{loc}} \},
\]

and define the associated discrete impulse operator:

\[
\mathcal{H}^M_{\v}(v^h)(t_i, z, \theta_j) = \sup_{e \in C^M(z, \theta_j)} v^h(t_i, \Gamma_\v(z, \theta_j, e), 0).
\]

**Optimal quantization method and truncation.** Let us now describe the numerical procedure for computing the expectations arising in (4.2). Recalling that \(Z^{0, t, z} = (x, y, P^\v, p)\), this involves only the expectation with respect to the price process, assumed here to follow a Black-Scholes model (2.3). We shall then use an optimal quantization for the standard normal random variable \(\mathcal{U}\), which consists in approximating the distribution of \(\mathcal{U}\) by the

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discrete law of a random variable $\hat{U}$ of support $(u_k)_{1 \leq k \leq N} \in \mathbb{R}^N$, and defined as the projection of $U$ on the grid $(u_k)_{1 \leq k \leq N}$ according to the closest neighbour. The grid $(u_k)_{1 \leq k \leq N}$ is optimized in order to minimize the distortion error, i.e. the quadratic norm between $U$ and $\hat{U}$. This optimal grid and the associated weights $(\pi_k)_{1 \leq k \leq N}$ are downloaded from the website: “http://www.quantize.maths-fi.com/downloads”. We refer to the survey article [17] for more details on the theoretical and computational aspects of optimal quantization methods. From (4.2), we have to compute at any time step $t_i \in T_m$, and for any $\theta_j \in T_i$, $z = (x, y, p) \in Z^j_{loc}$, expectations in the form:

$$
E[v^h(t_i + h, Z^0_{t_i+h}, \theta_j + h)] = E[v^h(t_i + h, x, y, p \exp(bh + \sigma \sqrt{h} U), \theta_j + h)],
$$

where we set $\hat{b} = (b - \sigma^2 / 2)$. The optimal quantization method consists then in approximating the above exact expectation by the discrete expectation operator:

$$
E^N[h(t_i + h, Z^0_{t_i+h}, \theta_j + h)] := \sum_{k=1}^{N} \pi_k v^h(t_i + h, x, y, p \exp(bh + \sigma \sqrt{h} u_k), \theta_j + h)
$$

$$
= E[v^h(t_i + h, \hat{Z}^0_{t_i+h}, \theta_j + h)],
$$

where $\hat{Z}^0_{t_i+h} = (x, y, \hat{P}_{t_i+h}^z)$, and $\hat{P}_{t_i+h}^z = p \exp(bh + \sigma \sqrt{h} U)$ is the discrete random variable valued in $P_{t_i+h}^z = p \exp(bh + \sigma \sqrt{h} u_k)$, with weights $\pi_k$, $k = 1, \ldots, N$. Actually, since for $0 \leq p \leq p_{max}$ the discrete positive random variable $\hat{P}_{t_i+h}^z$ can take values above $p_{max}$, we truncate to the nearest neighbour of $p_{max}$, and consider the approximate expectation operator:

$$
E^N_{loc}[v^h(t_i + h, Z^0_{t_i+h}, \theta_j + h)] := \sum_{k=1}^{N} \pi_k v^h(t_i + h, x, y, \text{Proj}_{0,p_{max}}(P_{t_i+h}^z), \theta_j + h)
$$

$$
= E[v^h(t_i + h, x, y, \hat{P}_{t_i+h}^{loc,t_i+h}, \theta_j + h)],
$$

where $\text{Proj}_{0,p_{max}}(p) = p 1_{p \leq p_{max}} + p_{max} 1_{p > p_{max}}$ for $p \geq 0$, and $\hat{P}_{t_i+h}^{loc,t_i+h} = \text{Proj}_{0,p_{max}}(\hat{P}_{t_i+h}^z)$. We may then rewrite the actual numerical scheme used as:

$$
S^h_{loc,M,N}(t_i, z, \theta_j, v^h(t, z, \theta_j), v^h) = 0, \quad t_i \in T_m, \theta_j \in T_i, z \in Z^j_{loc},
$$

for $i = 0, \ldots, m$, $j = 0, \ldots, i$, where $S^h_{loc,M,N}$ is defined by

$$
S^h_{loc,M,N}(t_i, z, \theta_j, r, \varphi)
:= \begin{cases} 
\min \left[ r - E^N[\varphi(t_i + h, Z^0_{t_i+h}, \theta_j + h)] , r - H^M_{loc,\epsilon}(t_i, z, \theta) \right] & \text{for } i = 0, \ldots, m - 1 \\
\min \left[ r - U_{loc}(z, \theta_j) , r - H^M_{loc,\epsilon}(T, z, \theta_j) \right] & \text{for } i = m.
\end{cases}
$$

Let us now address the convergence proofs of this computational scheme by adapting the arguments in Section 3. The monotonicity in the sense of Proposition 3.1 easily follows since the weights $(\pi_k)_{1 \leq k \leq N}$ appearing in the definition of $E^N_{loc}$ are nonnegative. In order to get the stability, we notice that the numerical scheme (4.4) is actually the dynamic programming equation for the following discrete impulse control problem:

$$
v^h_{loc,M}(t_i, z, \theta_j) = \sup_{\alpha \in A^h_{loc}(t_i, z, \theta_j)} E[U_{loc}(\hat{Z}_{t_i+m}^{loc}, \hat{\Theta}_{t_i+m})].
$$
where $\mathcal{A}_{e}^{h,M,\text{loc}}(t_i, z, \theta_0)$ is the set of elements $\alpha = (\tau_n, \zeta_n)_n$ s.t. the trading times $\tau_n$ are $\hat{\mathcal{F}}_{t_\ell}^{e} = (\hat{\mathcal{F}}_{t_\ell}^{e})$-stopping times, valued in $\mathbb{T}_{i,m} = \{t_\ell = \ell h, \ell = i, \ldots, m\}$, and $\zeta_n$ is $\hat{\mathcal{F}}_{\tau_n}$-measurable, valued in the discrete set of admissible transactions $\mathcal{C}_{e}^{M,\text{loc}}(\mathcal{\hat{Z}}_{\tau_n}^{\text{loc}}; \tau_n - \tau_{n-1})$, where the discrete time controlled process $\{\hat{Z}_{t_\ell}^{\text{loc}} = (\hat{X}_{t_\ell}, \hat{Y}_{t_\ell}, \hat{P}_{t_\ell}^{\text{loc}}), \hat{\Theta}_{t_\ell}, \ell = i, \ldots, m\}$ is governed by $\hat{Z}_{t_\ell}^{\text{loc}} = z$, and

\[
\hat{X}_{t_\ell} = \hat{X}_{t_{\ell}}, \quad \hat{Y}_{t_\ell} = \hat{Y}_{t_{\ell}}, \quad \hat{\Theta}_{t_\ell} = t_{\ell} - \tau_{n}, \quad \tau_{n} \leq t_{\ell} < \tau_{n+1},
\]

\[
\hat{P}_{t_\ell}^{\text{loc}} = \text{Proj}_{[\min, \max]} (\hat{P}_{t_{\ell-1}}^{\text{loc}} \exp (\frac{\|\|}{2} h + \sigma \sqrt{h} U_{t_{\ell}}));
\]

\[
(\mathcal{\hat{Z}}_{\tau_{n+1}}^{\text{loc}}, \hat{\Theta}_{\tau_{n+1}}) = \left(\mathcal{\Gamma}_{\varepsilon}(\mathcal{\hat{Z}}_{\tau_{n}}^{\text{loc}}, \tau_{n+1} - \tau_{n}, \zeta_{n+1}), 0\right),
\]

where $U_{t_{\ell}}, \ell = i + 1, \ldots, m$ are i.i.d. discrete random variables with support $(u_{k})_{k=1,\ldots,N}$ and weights $(\pi_{k})_{k=1,\ldots,N}$, and $\hat{\mathcal{F}}_{t_{\ell}}$ is the $\sigma$-algebra generated by $U_j, j \leq \ell$. Assuming for simplicity that the utility function $U$ is bounded, we then see that the solution $u_{h,N,M}^{\text{loc}}$ to the numerical scheme is pointwise bounded uniformly in $(h, N, M)$ and the localization parameters $(x_{\min}, x_{\max}, y_{\max}, p_{\max})$. For proving the (pointwise) consistency in the line of Proposition 3.3, we have to estimate, for any fixed $t_i \in \mathbb{T}_m$, $\theta_j \in \mathbb{T}_i$, $z \in \mathbb{Z}_i$, any smooth test function $\phi$, the accuracy of the approximate expectation $\mathbb{E}^{N}_t[\phi(t_i + h, Z_{t_i+h}^{0,\theta_j}, \theta_j + h)]$ with respect to the exact expectation $\mathbb{E}[\phi(t_i + h, Z_{t_i+h}^{0,\theta_j}, \theta_j + h)]$, when $h$ goes to zero, $N$ goes to infinity, and $R := \min\{x_{\min}, x_{\max}, y_{\max}, p_{\max}\}$ goes to infinity. Assuming that the smooth test function is uniformly Lipschitz in $p$, we have:

\[
\mathbb{E} \left[ \phi(t_i + h, Z_{t_i+h}^{0,\theta_j}, \theta_j + h) \right] - \mathbb{E}^{N}_t \left[ \phi(t_i + h, Z_{t_i+h}^{0,\theta_j}, \theta_j + h) \right] = \mathbb{E} \left[ \phi(t_i + h, x, y, \mathcal{P}_{t_i+h}^{\text{loc}}, \theta_j + h) \right] - \mathbb{E} \left[ \phi(t_i + h, x, y, \mathcal{P}_{t_i+h}^{\text{loc}}, \theta_j + h) \right]
\]

\[
= \mathbb{E} \left[ \phi(t_i + h, x, y, \mathcal{P}_{t_i+h}^{\text{loc}}, \theta_j + h) \right] - \mathbb{E} \left[ \phi(t_i + h, x, y, \mathcal{P}_{t_i+h}^{\text{loc}}, \theta_j + h) \right]
\]

\[
- \mathbb{E} \left[ \phi(t_i + h, x, y, \text{Proj}_{[0,p_{\max}]}(\mathcal{P}_{t_i+h}^{\text{loc}}, \theta_j + h) \right]
\]

\[
\leq C \mathbb{E} \left[ \mathcal{P}_{t_i+h}^{\text{loc}} - \text{Proj}_{[0,p_{\max}]}(\mathcal{P}_{t_i+h}^{\text{loc}}, \theta_j + h) \right]
\]

Now, since $\mathcal{U}$ is an optimal quantization of $\mathcal{U}$, we have the stationary property, meaning that $\mathbb{E}[\mathcal{U} \mathcal{U}] = \mathcal{U}$ (see [17]), which implies from Jensen’s inequality applied to the convex
function \( u \to e^{2bh+2\sigma\sqrt{Nh}} \), and the law of iterated conditional expectations:

\[
E \left[ e^{2bh+2\sigma\sqrt{Nh}} \hat{U} \right] \leq E \left[ e^{2bh+2\sigma\sqrt{Nh}} U \right] = e^{(2b+\sigma^2)h}.
\]

Denoting by \( \Phi \) the distribution function of \( U \), we then have:

\[
\left| E \left[ \phi(t_i + h, Z_{t_{i+1}}^{0,t_i}, \theta_j + h) - \mathcal{E}_N^N \left[ \phi(t_i + h, Z_{t_{i+1}}^{0,t_i}, \theta_j + h) \right] \right] \right| \leq Cpe^{(b+\frac{\sigma^2}{2})h} \left\{ \sqrt{1 - \Phi \left( \frac{p_{\text{max}}}{pe^{bh}} \right)} + \sqrt{h} \sqrt{E[U - \hat{U}]^2} \right\}.
\]

From Zador's theorem (see [17]), the asymptotic distortion error for the optimal quantization satisfies:

\[
\lim_{N \to \infty} N \sqrt{E[U - \hat{U}]^2} = O(1/N).
\]

Recalling the well known estimate: \( 1 - \Phi(d) \sim \varphi(d)/d \), as \( d \) goes to infinity, where \( \varphi = \Phi' \) is the density of \( U \), we obtain by taking \( N \) s.t. \( N\sqrt{h} \to \infty \), e.g. \( N = O(1/h^{\frac{1}{2}+\varepsilon}) \), with \( \varepsilon > 0 \), and \( p_{\text{max}} > p \), the pointwise estimation:

\[
\mathcal{E}_N^N \left[ \phi(t_i + h, Z_{t_{i+1}}^{0,t_i}, \theta_j + h) \right] = E \left[ \phi(t_i + h, Z_{t_{i+1}}^{0,t_i}, \theta_j + h) \right] + o(h),
\]

where the notation \( o(h) \) means that \( o(h)/h \) goes to zero as \( h \) goes to zero. This yields

\[
\lim_{h \to 0} \frac{\phi(t_i, z, \theta_j) - \mathcal{E}_N^N \left[ \phi(t_i + h, Z_{t_{i+1}}^{0,t_i}, \theta_j + h) \right]}{h} = \lim_{h \to 0} \frac{\phi(t_i, z, \theta_j) - \mathcal{E}_N^N \left[ \phi(t_i + h, Z_{t_{i+1}}^{0,t_i}, \theta_j + h) \right]}{h}.
\]

On the other hand, for fixed \( t_i \in T_m, \theta_j \in T_i, z \in Z \), we notice that \( \bigcup_{M,R} \mathcal{C}^M_{\varepsilon}(z, \theta_j) \) is dense in \( C_\varepsilon(z, \theta_j) \). Hence, by continuity of \( \phi, \Gamma, \) and compacity of \( C_\varepsilon(z, \theta_j) \), we deduce that

\[
\lim_{M,R \to \infty} H_\varepsilon^M_{\varepsilon}(t_i, z, \theta_j) = H_\varepsilon \phi(t_i, z, \theta_j).
\]

Together with (4.6), we then obtain similarly as in Proposition 3.3:

\[
\lim_{h \to 0} \min_{N \sqrt{h}, M, R \to \infty} \left\{ \frac{\phi(t_i, z, \theta_j) - \mathcal{E}_N^N \left[ \phi(t_i + h, Z_{t_{i+1}}^{0,t_i}, \theta_j + h) \right]}{h}, (\phi - \mathcal{H}_\varepsilon^M_{\varepsilon}(\phi)(t_i, z, \theta_j)) \right\} = \min \left\{ -\left( \frac{\partial \phi}{\partial t} + L\phi \right)(t_i, z, \theta_j), (\phi - \mathcal{H}_\varepsilon \phi)(t_i, z, \theta_j) \right\},
\]

which then proves the convergence of the numerical scheme \( S_{\varepsilon}^{N,M,N} \).

**Algorithm description.** In summary, our numerical scheme provides an algorithm for computing approximations \( \nu^h \) of the value function, and \( \zeta^h \) of the optimal trading strategy at each time step \( t_i \in T_m \), and each point \((z, \theta)\) of the grid \((X_n \times Y_n \times P_n \times T) \cap \mathcal{S}_\varepsilon \), where \( X_n \) is the uniform grid with \( n \) nodes on \([x_{\min}, x_{\max}]\), i.e. of step \((x_{\max} - x_{\min})/n\), and similarly for \( Y_n, P_n \). Let us also denote by \( Z_n^j = \{ z \in Z_n : (z, \theta_j) \in \mathcal{S}_\varepsilon \} \). The parameters in the algorithm are:
- $T$ the maturity
- $b$ and $\sigma$ the Black and Scholes parameters of the stock price
- $\lambda$ the impact parameter, $\beta$ the impact exponent in the market impact function (2.5)
- $\kappa_a, \kappa_b$ the spread parameters in percent, $\varepsilon$ the transactions costs fee
- We take by default a CRRA utility function: $U(x) = x^\gamma$
- $x_{\text{min}}, x_{\text{max}} \in \mathbb{R}$, $0 \leq y_{\text{min}} < y_{\text{max}}$, $0 \leq p_{\text{min}} < p_{\text{max}}$, the boundaries of the localized domain
- $m$ number of steps in time discretization, $n$ the number of steps in space discretization
- $N$ number of points for optimal quantization of the normal law, $M$ number of points used in the static supremum in $e$

The algorithm is described explicitly in backward induction as follows:

- **Initialization step at time $t_m = T$**:
  - $(s:0)$ For $j = 0$, set $v^h(t_m, z, 0) = U_{L_e}(z, 0)$, $\zeta^h(t_m, z, 0) = 0$ on $Z^0_{\text{loc}}$, and interpolate $v^h(t_m, z, 0)$ on $Z^0_{\text{loc}}$.
  - $(s:j)$ For $j = 1, \ldots, m$,
    - for $z \in Z^j_h$, compute $v := \sup_{\varepsilon \in C^M_{\text{loc}}(z, \theta_j)} U_{L_e}(\Gamma_{\varepsilon}(z, \theta_j, e), 0)$ and denote by $\hat{v}$ the argument maximum:
      - if $v > U_{L_e}(z, \theta_j)$, then set $v^h(t_m, z, \theta_j) = v$ and $\zeta^h(t_m, z, \theta_j) = \hat{v}$,
      - else set $v^h(t_m, z, \theta_j) = U_{L_e}(z, \theta_j)$, and $\zeta^h(t_m, z, \theta_j) = 0$.
    - Interpolate $z \rightarrow v^h(t_m, z, \theta_j)$ on $Z^j_{\text{loc}}$.

- **From time step $t_{i+1}$ to $t_i$, $i = m-1, \ldots, 0$**:
  - $(s:0)$ For $j = 0$, compute $\mathcal{E}^N_{\text{loc}}[v^h(t_i + h, Z^{0, t_{i+1}}_{t_i + h}, \theta_j + h)]$ from (4.3) and $(s:1)$ of time step $t_{i+1}$, and set $v^h(t_i, z, 0) = \mathcal{E}^N_{\text{loc}}[v^h(t_i + h, Z^{0, t_{i+1}}_{t_i + h}, \theta_j + h)]$, $\zeta^h(t_i, z, 0) = 0$ on $Z^0_{\text{loc}}$; interpolate $v^h(t_i, z, 0)$ on $Z^0_{\text{loc}}$.
  - $(s:j)$ For $j = 1, \ldots, i$,
    - for $z \in Z^j_h$, compute $\mathcal{E}^N_{\text{loc}}[v^h(t_i + h, Z^{0, t_{i+1}}_{t_i + h}, \theta_j + h)]$ from (4.3) and $(s:j+1)$ of time step $t_{i+1}$, $v := \sup_{\varepsilon \in C^M_{\text{loc}}(z, \theta_j)} v^h(t_i, \Gamma_{\varepsilon}(z, \theta_j, e), 0)$ from $(s:0)$, and denote by $\hat{v}$ the argument maximum:
      - if $v > \mathcal{E}^N_{\text{loc}}[v^h(t_i + h, Z^{0, t_{i+1}}_{t_i + h}, \theta_j + h)]$, then set $v^h(t_i, z, \theta_j) = v$, $\zeta^h(t_i, z, \theta_j) = \hat{v}$,
      - else set $v^h(t_i, z, \theta_j) = \mathcal{E}^N_{\text{loc}}[v^h(t_i + h, Z^{0, t_{i+1}}_{t_i + h}, \theta_j + h)]$, and $\zeta^h(t_i, z, \theta_j) = 0$.
    - Interpolate $z \rightarrow v^h(t_i, z, \theta_j)$ on $Z^j_{\text{loc}}$.

**Complexity of the algorithm.** Due to the high dimension of the grid

$$ S = T_m \times \bigcup_{i=1 \ldots m} \left( \mathcal{X}_n \times \mathcal{Y}_n \times \mathcal{P}_n \times T_i \right) \cap \mathcal{S}_e, $$

17
the computation of the optimal policy on the entire grid has an expensive computational cost. Indeed, this grid contains $O(m^2 n^3)$ points, and at each point $(t_i, z, \theta_j) \in S$, one has to compute:

- The approximation of conditional expectation $E_{\text{loc}}^N [v^h(t_i + h, Z_{t_i+h}^0, \theta_j + h)]$ that costs $O(N)$ unitary operations.
- The approximation of the static supremum $\sup_{e \in C_{M,\text{loc}}^Z(z,\theta_j)} v^h(t_i, \Gamma_{\epsilon}(z, \theta_j, e), 0)$, together with its argument maximum, that costs $O(M)$ unitary operations when using linear search\(^1\).
- The localization procedure and the interpolation procedure has constant computational cost $O(1)$.

Therefore, we obtain a complexity of:

\[
\text{Complexity} = O(m^2 n^3 \max(N, M)).
\]

Actually, denoting by $K = \max(n, m, N, M)$, the complexity of the algorithm is $O(K^6)$. Yet, practical implementation of the algorithm can achieve quite better performance. First, in the optimal quantization for the computation of the expectations in the numerical algorithm, we can choose $N = O(m^{1/2} n^{1/2} \epsilon)$ for all $\epsilon > 0$. Assuming that we are able to use a dichotomy-based method for computing the static supremum, which has logarithmic complexity, the main computational costs are due to the computation of the approximate conditional expectation, and we can neglect the cost of computing the static supremum. In this case, the complexity is reduced to:

\[
\text{Complexity} = O(m^{5/2} n^3), \quad \forall \epsilon > 0,
\]

which is satisfactory when considering that there is $O(m^2 n^3)$ points to compute in the grid. Second, the grid computation algorithm can be parallelized easily, which is a very desirable property when targeting an industrial application. Indeed, at each date $t_i$ the computation of $E_{\text{loc}}^N [v^h(t_i + h, x, y, P^0, t_i, p_{t_i+h}, \theta_j + h)]$ and $\sup_{e \in C_{M,\text{loc}}^Z(z,\theta_j)} v^h(t_i, \Gamma_{\epsilon}(x, y, p, \theta_j, e), 0)$ can be done independently for each quadruplet $(x, y, p, \theta_j)$ provided that $\theta_j > 0$.

Finally, the complexity displayed above represents the amount of computations needed to build up the optimal policy. When targeting a live trading application, one can compute off-line and store optimal policies for a given set of market parameters, and when actually trading, one does only need to read (with constant cost) the optimal policy corresponding to current market state.

**Comparison with finite difference scheme.** In order to motivate our numerical scheme proposal, let us compare it with usual finite difference scheme. Let us briefly introduce the

\(^1\)Note that the supremum computation can be improved by the use of dichotomy-based search instead of linear search if we are able to use a concavity argument on $e \mapsto v(t, \Gamma(x, y, p, \theta, e), 0)$ which would lead to a complexity of $O(\ln(M))$. From numerical experiments, this dichotomy search method leads to acceptable results.
class of theta-schemes. We refer to [12] for complete discussion about this class of schemes.
We will assume that the value function is sufficiently smooth, and we focus in this paragraph on the diffusive part of the QVI, so that our target equation to solve is:

\[
\frac{\partial}{\partial t} + \mathcal{L}v = 0 \text{ on } \bar{S}_\varepsilon \times [0, T),
\]

together with a terminal condition on \( \bar{S}_\varepsilon \times \{T\} \). To solve numerically this Kolmogorov parabolic equation with finite time horizon, we can discretize it using a theta-scheme of parameter \( a \) according to [12]. This approximation consists in the following:

\[
\left( \frac{\partial}{\partial t} v + \mathcal{L}v \right)(t, z, \theta) \simeq \mathcal{P}_{h, \delta}^a v(t, z, \theta)
\]

where

\[
\mathcal{P}_{h, \delta}^a v(t, z, \theta) = \frac{v(t + h, z, \theta + h) - v(t, z, \theta)}{h} + aL_\delta v(t, z, \theta) + (1 - a)L_\delta v(t + h, z, \theta + h)
\]

and \( L_\delta \) is the finite difference approximation of \( \tilde{\mathcal{L}} := bp \frac{\partial}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2}{\partial p^2} \) of (space) step \( \delta \) and \( a \in [0, 1] \). The discretized equation is:

\[
\mathcal{P}_{h, \delta}^a v(t, z, \theta) = 0 \text{ on } \mathcal{O}_\delta \cap \bar{S} \times [0, T),
\]

where \( \mathcal{O}_\delta \) is a suitable regular grid of (space) step \( \delta \). From the finite differences approximation, we have the following precision:

\[
\left( \frac{\partial}{\partial t} v + \mathcal{L}v \right)(t, z, \theta) = \mathcal{P}_{h, \delta}^a v(t, z, \theta) + o(h^p + \delta^q),
\]

where \( p \) and \( q \) depends on the choice of \( a \): if \( a \neq 1/2 \) we obtain that \( p = 1 \), and if \( a = 1/2 \) we obtain that \( p = 2 \), which corresponds to the Crank-Nicholson scheme. Due to the second order derivative in \( \mathcal{L} \), and by using standard finite difference approximation, the rate of convergence for the spatial approximation is \( q = 1 \), \( \forall a \in [0, 1] \). Therefore, in our case, we see that theta-schemes have order 1 in time and order 1 in space, except for the Crank-Nicholson scheme, which gives an order 2 in time and order 1 in space. For comparison purpose, the optimally quantized scheme that we use has order 1 in time provided that \( N = O(h^{-1/2 + \varepsilon}) \) where \( N \) is the number of points in the optimal quantization grid:

\[
\left( \frac{\partial}{\partial t} v + \mathcal{L}v \right)(t, z, \theta) = 0, \quad N = O(h^{-1/2 + \varepsilon}) \implies v^h(t, z, \theta) = \mathcal{E}_{\text{loc}}^N [v^h(t+h, Z_{t+h}^{0,t,z}, \theta+h)] + o(h).
\]

This raises two comments. First, we see that in contrast with finite difference scheme, the precision of the optimally quantized scheme is controlled by the number of points \( N \) of the optimal quantization grid, and not by the space step \( \delta \), provided that interpolation procedure is sufficiently efficient. Therefore, one can improve the precision by increasing \( N \) and without increasing the size of the grid, which is very interesting when dealing with high-dimension state space. Second, the above result allows us to choose \( n = O(m^{1/2}) \), while keeping a precision of \( o(1/m) \), whereas if using a finite-difference scheme, the precision would be \( o(1/m^{1/2}) \) due to spatial approximation. Therefore, by using an optimally
quantized scheme, we can obtain a satisfactory precision, while managing efficiently the size of the grid, and subsequently the memory needed to achieve computation, which is quite relevant when dealing with high-dimensional state space.

Yet, two other theta-schemes may be good candidates for solving numerically our QVI, the Crank-Nicholson scheme due to its higher order in time, and the fully-implicit scheme, corresponding to \( a = 1 \) because it has the property of being stable without restriction on the choice of time step versus space step.

5 Numerical Results

5.1 Procedure

For each of the numerical tests, we used the same procedure consisting in the following steps:

1. Set the parameters according to the parameter table described in the first subsection of each test
2. Compute and save the grids representing value function and optimal policy according to the optimal liquidation algorithm
3. Generate \( Q \) paths for the stock price process following a geometrical Brownian motion: we choose parameters \( b \) and \( \sigma \) that allows us to observe several empirical facts on the performance and the behavior of optimal liquidation strategy. These parameters can also be estimated from historical observations on real data by standard statistical methods.
4. Consider the portfolio made of \( X_0 \) dollars and \( Y_0 \) shares of risky asset
5. For each price path realization, update the portfolio along time and price path accordingly to the policy computed in the second step
6. Save each optimal liquidation realization
7. Compute statistics

In the sequel, we shall use the following quantities as descriptive statistics:

- The performance of the \( i \)-th realization of the optimal strategy is defined by

\[
L_{\text{opt}}^{(i)} = \frac{L_{\epsilon}(Z_T^{(i),\alpha_{\text{opt}}}, \Theta_T^{(i),\alpha_{\text{opt}}})}{X_0 + Y_0 P_0}
\]

where \((Z_T^{(i),\alpha_{\text{opt}}}, \Theta_T^{(i),\alpha_{\text{opt}}})\) is the state process, starting at date 0 at \((X_0, Y_0, P_0, 0)\), evolving under the \( i \)-th price realization and the optimal control \( \alpha_{\text{opt}} \). This quantity can be interpreted as the ratio between the cash obtained from the optimal liquidation strategy and the ideal Merton liquidation. We define in the same way the quantities \( L_{\text{naive}}^{(i)} \) and \( L_{\text{uniform}}^{(i)} \) respectively associated with the controls \( \alpha_{\text{naive}} \) and \( \alpha_{\text{uniform}} \) of the naïve and uniform strategy, referred to as benchmark strategies. Recall that the naïve strategy consists in liquidating the whole portfolio in one block at the last date, and the uniform strategy consists in liquidating the same quantity of asset at each predefined date until the last date.
Notice that the score 1 corresponds to the strategy, which consists in liquidating the whole portfolio immediately in an ideal Merton market.

When denoting by $Q$ the number of paths of our simulation, we define:

- The mean utility $\hat{V} = \frac{1}{Q} \sum_{i=1}^{Q} U(L^{(i)})$

- The mean performance $\hat{L} = \frac{1}{Q} \sum_{i=1}^{Q} L^{(i)}$

- The standard deviation of the strategy $\hat{\sigma} = \sqrt{\frac{1}{Q} \sum_{i=1}^{Q} (L^{(i)})^2 - \hat{L}^2}$

Here the dot . stands for opt, naive or uniform. We will also compute the third and fourth standardized moments for the series $(L^{(i)})_i$.

### 5.2 Test 0: Convergence of the numerical scheme

In order to experiment numerically the convergence of the scheme, we performed two series of convergence tests. First, we computed a reference value function with a fine discretization grid, and computed for various sizes of grids the difference to this reference result. Second, we backtested the optimal policy obtained with various discretization grid sizes, using the procedure described in 5.1, and compared the results.

Due to the high dimension of the problem, we restricted our convergence analysis to reasonably sized discretization grids, except for the reference computation, and therefore missing values in tables 3 and 4 correspond either to grids that required too much memory space or too much time for computations. When targeting industrial applications, one can avoid these restrictions by using a suitable parallel algorithm, as we did for computing the reference value function. Yet, with a reasonable size of grid, for example $(m = 64, n = 32)$ one can achieve satisfactory results (see table 3).

**Convergence of the value function** First, we computed a reference value function denoted by $v^\infty$, and using a parallelized version of our algorithm with parameters shown in table 1. We ran the computations on two SGI Altix ICE 8200EX supercomputers made of 256 computing cores 64-bit at 2.83 GHz with 512 GB of distributed RAM. The complete computations took 11 hours and 36 minutes, and the size of computer representation of $v^\infty$ was 0.991 TB.

Second, we computed value functions $v^{n,m}$ for different values of $n$ (number of space steps) and $m$ (number of time steps), and we chosed $N$ (number of quantization points) equal to $\sqrt{m}$ to match the consistency requirement in Section 4, see parameters in table 2. The relative error compared to $v^\infty$, i.e. $\frac{\|v^\infty - v^{n,m}\|_2}{\|v^\infty\|_2}$, is reported on table 3, which shows clearly the convergence of our algorithm when $n, m$ increase.

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<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>1 day</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>0.02</td>
</tr>
<tr>
<td>(\beta)</td>
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</tr>
<tr>
<td>(\gamma)</td>
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<tr>
<td>(\kappa_A)</td>
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<td>(\kappa_B)</td>
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<tr>
<td>(\epsilon)</td>
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<tr>
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</tr>
<tr>
<td>(N)</td>
<td>30</td>
</tr>
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Table 1: Test 0: parameters for the reference computation \(v^\infty\)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>Maturity</td>
<td>1 day</td>
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<tr>
<td>(X_0)</td>
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</tr>
<tr>
<td>(Y_0)</td>
<td>250</td>
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<tr>
<td>(N)</td>
<td>(\sqrt{m})</td>
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<td>(Q)</td>
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Table 2: Test 0: parameters

<table>
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<tr>
<th>(m)</th>
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<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
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<tr>
<td>16</td>
<td>0.2251</td>
<td>0.1043</td>
<td>0.0668</td>
<td>0.0582</td>
<td>0.0563</td>
</tr>
<tr>
<td>32</td>
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<td></td>
</tr>
</tbody>
</table>

Table 3: Test 0: convergence of the value function. Quantity displayed is \(\frac{\|v^\infty - v^{n,m}\|_2}{\|v^\infty\|_2}\).

As a consequence of this convergence test, we will use in the following tests the following values: \(m \in [30...60]\) and \(n \in [20...60]\). Indeed these sizes of grid are a good compromise between computational complexity and precision.

**Backtesting the optimal strategy** We compared the fully implicit scheme to our optimally quantized scheme, following the procedure described in section 5.1. The fully implicit scheme corresponds to a theta-scheme with parameter \(a = 1\), and has the property of inducing no restriction on the choice of timestep. Therefore we use it as a benchmark for our optimally quantized scheme. Parameters are reported in table 2 and results in table 4.
<table>
<thead>
<tr>
<th>scheme</th>
<th>m</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
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</thead>
<tbody>
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<td>Quantized</td>
<td>16</td>
<td>0.5238</td>
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<td>0.8752</td>
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<tr>
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<td>0.8747</td>
<td>0.8743</td>
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<tr>
<td>Implicit</td>
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<td>0.8458</td>
<td>0.8589</td>
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</tr>
<tr>
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<td>0.8465</td>
<td>0.8578</td>
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<tr>
<td>Implicit</td>
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<td>0.8465</td>
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<tr>
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<td>0.8417</td>
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<tr>
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<td>0.8456</td>
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<tr>
<td>Quantized</td>
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<td>0.5278</td>
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</tr>
</tbody>
</table>

Table 4: Test 0: Convergence of the numerical algorithm: table of value function estimated by Monte-Carlo simulation $\hat{V}$ with initial portfolio $(X_0, Y_0, P_0)$ when varying grid size ($m$ is number of time steps, $n$ the number of space steps, with boundaries fixed). We display results for the optimally quantized scheme (referred to as "Quantized" scheme in the table) against the benchmark made of the theta-scheme of parameter $a = 1$ and usual finite difference approximation (referred to as "Implicit" scheme in the table).

In table 5 we display the same convergence test measured in terms of the statistics $\frac{\hat{L}_{\text{Quantized}} - \hat{L}_{\text{Implicit}}}{\hat{\sigma}_{\text{Quantized}}}$ where $\hat{L}_{\text{Quantized}}$ (resp. $\hat{L}_{\text{Implicit}}$) is the estimate of performance for the initial portfolio $(X_0, Y_0, P_0)$ using the optimally quantized scheme (resp. the fully implicit scheme) for computing the optimal policy and $\hat{\sigma}_{\text{Quantized}}$ its standard deviation. This quantity is more intuitive from the financial point of view, and can be interpreted as the gain in mean performance when using the optimally quantized scheme compared to using the fully implicit scheme, measured with the standard deviation as unit. We remark that the optimally quantized scheme performs better for most values of $(m, n)$, especially for small-sized time grids. When increasing the size of the time grid, the difference of performance for these two scheme seems to vanish, in terms of the above statistics, but more precise tests are needed to conclude.

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td></td>
<td>0.1191</td>
<td>0.1179</td>
<td>0.1085</td>
<td>0.1662</td>
<td>0.1482</td>
</tr>
<tr>
<td>32</td>
<td></td>
<td>0.0890</td>
<td>0.0612</td>
<td>0.0902</td>
<td>0.1175</td>
<td>0.1065</td>
</tr>
<tr>
<td>64</td>
<td></td>
<td>0.0109</td>
<td>0.0367</td>
<td>0.0328</td>
<td>0.0521</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td></td>
<td>-0.0439</td>
<td>0.0127</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>256</td>
<td></td>
<td>-0.3437</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Test 0: Convergence of the numerical algorithm: table for the statistics $(\hat{L}_{\text{Quantized}} - \hat{L}_{\text{Implicit}})/\hat{\sigma}_{\text{Quantized}}$ when varying grid size.
5.3 Test 1: A toy example

The goal of this test is to show the main characteristics of our results. We choose a set of parameters that is unrealistic but that has the advantage of emphasizing the typical behavior of the optimal liquidation strategy.

Parameters  We choose the set of parameters shown in table 6.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>1 year</td>
<td>$X_0$</td>
<td>2000</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>5.00E-07</td>
<td>$Y_0$</td>
<td>2500</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.5</td>
<td>$P_0$</td>
<td>5.0</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.5</td>
<td>$x_{min}$</td>
<td>-30000</td>
</tr>
<tr>
<td>$\kappa_A$</td>
<td>1.01</td>
<td>$x_{max}$</td>
<td>80000</td>
</tr>
<tr>
<td>$\kappa_B$</td>
<td>0.99</td>
<td>$y_{min}$</td>
<td>0</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>0.001</td>
<td>$y_{max}$</td>
<td>5000</td>
</tr>
<tr>
<td>$b$</td>
<td>0.1</td>
<td>$p_{min}$</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.5</td>
<td>$p_{max}$</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$m$</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$n$</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$N$</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Q$</td>
<td>$10^5$</td>
</tr>
</tbody>
</table>

Table 6: Test 1: parameters

Execution statistics  The results were computed using Intel® Core 2 Duo at 2.93Ghz CPU with 2.98 Go of RAM. Statistics are shown in table 7.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Elapsed for grid computation in seconds</td>
<td>7520</td>
</tr>
<tr>
<td>Number Of Available Processors</td>
<td>2</td>
</tr>
<tr>
<td>Estimated Memory Used (Upper bound)</td>
<td>953MB</td>
</tr>
<tr>
<td>Time Elapsed for statistics Computation in seconds</td>
<td>21</td>
</tr>
</tbody>
</table>

Table 7: Test 1: Execution statistics

Shape of policy  In this paragraph we plotted the shape of the policy sliced in the plane $(x,y)$, i.e. the (cash, shares) plane, for a fixed $(t,\theta,p)$ (figure 1). The color of the map at $(x_0,y_0)$ on the graph represents the action one has to take when reaching the state $(t,\theta,x_0,y_0,p)$. We can see three zones: a buy zone (denoted BUY on the graph), a sell zone (denoted SELL on the graph) and a no trade zone (denoted NT on the graph). Note that the bottom left zone on the graph is outside the domain $\bar{S}$. These results have the
intuitive financial interpretation: when $x$ is big and $y$ is small, the investor has enough cash to buy shares of the risky asset and tries to profit from an increased exposure. When $y$ is large and $x$ is small, the investor has to reduce exposure to match the terminal liquidation constraint.

We also plotted the shape of the policy sliced in the plane $(y,p)$, i.e. the (shares,price) plane, for a fixed $(t,\theta,x)$ (figure 2). As before, the color of the map at $(y_0,p_0)$ on the graph represents the action one has to take when reaching the state $(t,\theta,x,y_0,p_0)$. Again, we can distinguish the three zones: buy, sell and no trade.

**Remark 5.1** In our modelling, we allow buying to occur during liquidation. This may be a priori undesirable in practice, and one could easily enforce a no-buying constraint in our model by requiring that the strategies $(\zeta_n)$ should be nonpositive, so that the shape policy is reduced to two zones instead of three zones as above: a no-trade and a sell zone. However, by giving more flexibility to the investor, we allow him to take advantage of a drop of the asset price, as illustrated in Figure 7, and so to realize a better performance.

**Shape of value function** Figure 3 shows the value function sliced in the $(x,y)$ plane. This figure is a typical pattern of the value function. Recall from Proposition 3.1 in [11] the following Merton theoretical bound for the value function:

$$v(t,z,\theta) \leq v_M(t,x,y,p) = e^{\rho(T-t)}(x + yp)^\gamma, \quad \text{with} \quad \rho = \frac{\gamma}{1 - \gamma} \frac{b^2}{2\sigma^2}.$$
In the figure 4 we plotted the difference between the value function and this theoretical bound. We observe that this difference is increasing with the number of shares, and decreasing with the cash. This result is interpreted as follows: the price impact increases with the number of shares, but this can be reduced by the liquidation strategy whose efficiency is greater if the investor can sustain bigger cash variations.

5.4 Test 2: Short term liquidation

The goal of this test is to show the behavior of the algorithm on a realistic set of parameters and real data. We used Reuters™ data fed by OneTick™ TimeSeries Database. We used the spot prices (Best Bid and Best Ask) for the week starting 04/19/2010 on BNP.PA. We computed mid-price that is the middle between best bid and best ask price. We choose the impact parameter $\lambda$ in order to penalize by approximately 1% the immediate liquidation of the whole portfolio compared to Merton liquidation. In other words, we take $\lambda$ so that:

$$\lambda |\frac{Y_0}{T}|^{\beta} \simeq 0.01.$$

**Parameters**

We computed the strategy with parameters shown in table 8.

**Execution statistics**

We obtained the results using Intel® Core 2 Duo at 2.93Ghz CPU with 2.98 Go of RAM, the computations statistics are gathered in table 9.
Performance Analysis  We computed the mean utility and the first four moments of the optimal strategy and the two benchmark strategies in table 10 and plotted the empirical distribution of performance in figure 5. It is remarkable that the optimal strategy gives
### Table 8: Test 2: Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>1 Day</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>5.00E-04</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\kappa_A$</td>
<td>1.0001</td>
</tr>
<tr>
<td>$\kappa_B$</td>
<td>0.9999</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>0.001</td>
</tr>
<tr>
<td>$b$</td>
<td>0.005</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.25</td>
</tr>
<tr>
<td>$X_0$</td>
<td>20000</td>
</tr>
<tr>
<td>$Y_0$</td>
<td>2500</td>
</tr>
<tr>
<td>$P_0$</td>
<td>52.0</td>
</tr>
<tr>
<td>$x_{\min}$</td>
<td>-30000</td>
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<tr>
<td>$x_{\max}$</td>
<td>200000</td>
</tr>
<tr>
<td>$y_{\min}$</td>
<td>0</td>
</tr>
<tr>
<td>$y_{\max}$</td>
<td>5000</td>
</tr>
<tr>
<td>$p_{\min}$</td>
<td>50.0</td>
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<tr>
<td>$p_{\max}$</td>
<td>54.0</td>
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<tr>
<td>$m$</td>
<td>30</td>
</tr>
<tr>
<td>$n$</td>
<td>40</td>
</tr>
<tr>
<td>$N$</td>
<td>100</td>
</tr>
<tr>
<td>$Q$</td>
<td>$10^5$</td>
</tr>
</tbody>
</table>

### Table 9: Test 2: Execution statistics

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Elapsed for grid computation in seconds</td>
<td>8123</td>
</tr>
<tr>
<td>Number Of Available Processors</td>
<td>2</td>
</tr>
<tr>
<td>Estimated Memory Used (Upper bound)</td>
<td>573MB</td>
</tr>
</tbody>
</table>

an empirical performance that is above the immediate liquidation at date 0 in the Merton ideal market (represented by performance $\hat{L} = 1$, and usually referred to as reference price benchmark). This is due to the fact that the optimal strategy has an ”opportunistic” behavior: indeed, an optimal trading strategy is embedded with the liquidation constraint: in this example, this feature not only compensates the trading costs, but also provides an extra performance compared to an ideal immediate liquidation at date 0. Still, the Merton case is a theoretical upper bound in the following sense: the optimal value function with trading costs is below the optimal value function without trading costs, recall the figure 4. As expected, the empirical distribution is between the distributions of the two other benchmark strategies. We also notice that the optimal strategy outperforms the two others by approximatively 0.25% in utility and in performance. We also computed other statistics in table 11.

**Behavior Analysis** In this paragraph, we analyze the behaviour of the strategy as follows: first, we plotted in figure 6 the empirical distribution of the number of trades for one trading session. Secondly, we plotted trades realizations for three days of the BNPP.PA stock for the week starting on 04/19/2010.

The three following graphs represent three days of market data for which we computed
<table>
<thead>
<tr>
<th>Strategy</th>
<th>Utility $V$</th>
<th>Mean $L$</th>
<th>Standard Dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Naive</td>
<td>0.99993</td>
<td>0.99986</td>
<td>0.00429</td>
<td>0.94584</td>
<td>4.68592</td>
</tr>
<tr>
<td>Uniform</td>
<td>0.99994</td>
<td>0.99988</td>
<td>0.00240</td>
<td>0.42788</td>
<td>3.34397</td>
</tr>
<tr>
<td>Optimal</td>
<td>1.00116</td>
<td>1.00233</td>
<td>0.00436</td>
<td>1.03892</td>
<td>4.89161</td>
</tr>
</tbody>
</table>

Table 10: Test 2: Utility and first four moments for the optimal strategy and the two benchmark strategies

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Formula</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winning percentage</td>
<td>$\frac{1}{Q} \sum_{i=1}^{Q} 1{L^{(i)}<em>{\text{opt}} &gt; \max(L^{(i)}</em>{\text{naive}}, L^{(i)}_{\text{uniform}})}$</td>
<td>58.8%</td>
</tr>
<tr>
<td>Relative Optimal Utility</td>
<td>$\frac{\hat{V}<em>{\text{opt}} - \max(\hat{V}</em>{\text{naive}}, \hat{V}<em>{\text{uniform}})}{\hat{V}</em>{\text{opt}}}$</td>
<td>0.00238</td>
</tr>
<tr>
<td>Relative Optimal Performance</td>
<td>$\frac{\hat{L}<em>{\text{opt}} - \max(\hat{L}</em>{\text{naive}}, \hat{L}<em>{\text{uniform}})}{\hat{L}</em>{\text{opt}}}$</td>
<td>0.00244</td>
</tr>
<tr>
<td>Utility Sharpe Ratio</td>
<td>$\frac{\hat{V}<em>{\text{opt}} - \max(\hat{V}</em>{\text{naive}}, \hat{V}<em>{\text{uniform}})}{\hat{\sigma}</em>{\text{opt}}}$</td>
<td>0.28017</td>
</tr>
<tr>
<td>Performance Sharpe Ratio</td>
<td>$\frac{\hat{L}<em>{\text{opt}} - \max(\hat{L}</em>{\text{naive}}, \hat{L}<em>{\text{uniform}})}{\hat{\sigma}</em>{\text{opt}}}$</td>
<td>0.56140</td>
</tr>
<tr>
<td>VaR 95% Naive Strategy</td>
<td>$\sup \left{ x \mid \frac{1}{Q} \sum_{i=1}^{Q} 1{L^{(i)}_{\text{naive}} &gt; x} \geq 0.95 \right}$</td>
<td>0.994</td>
</tr>
<tr>
<td>VaR 95% Uniform Strategy</td>
<td>$\sup \left{ x \mid \frac{1}{Q} \sum_{i=1}^{Q} 1{L^{(i)}_{\text{uniform}} &gt; x} \geq 0.95 \right}$</td>
<td>0.996</td>
</tr>
<tr>
<td>VaR 95% Optimal Strategy</td>
<td>$\sup \left{ x \mid \frac{1}{Q} \sum_{i=1}^{Q} 1{L^{(i)}_{\text{opt}} &gt; x} \geq 0.95 \right}$</td>
<td>0.997</td>
</tr>
<tr>
<td>VaR 90% Naive Strategy</td>
<td>$\sup \left{ x \mid \frac{1}{Q} \sum_{i=1}^{Q} 1{L^{(i)}_{\text{naive}} &gt; x} \geq 0.90 \right}$</td>
<td>0.995</td>
</tr>
<tr>
<td>VaR 90% Uniform Strategy</td>
<td>$\sup \left{ x \mid \frac{1}{Q} \sum_{i=1}^{Q} 1{L^{(i)}_{\text{uniform}} &gt; x} \geq 0.90 \right}$</td>
<td>0.997</td>
</tr>
<tr>
<td>VaR 90% Optimal Strategy</td>
<td>$\sup \left{ x \mid \frac{1}{Q} \sum_{i=1}^{Q} 1{L^{(i)}_{\text{opt}} &gt; x} \geq 0.90 \right}$</td>
<td>0.998</td>
</tr>
</tbody>
</table>

Table 11: Test 2: Other statistics on performance of optimal strategy

the mid-price (lines) with associated trades realizations for the optimal strategy (vertical bars). A positive quantity for the vertical bar means a buying operation, while a negative quantity means a selling operation.

Figure 7 shows the trade realizations of the optimal strategy for the day 04/19/2010 on the BNPP.PA stock. The interesting feature in this first graph is that we see two buying decisions when the price goes down through the 54.5 € barrier, and which corresponds
Figure 5: Test 2: Strategy empirical distribution

Figure 6: Test 2: Empirical distribution of the number of trades
roughly to a daily minimum. The following selling decision can be viewed as a failure. On the contrary, the two last selling decisions correspond quite precisely to local maxima.

Figure 8 (resp. 9) shows the trade realizations of the optimal strategy for the day 04/22/2010 (resp. 04/23/2010) on the BNPP.PA stock. Note that in figure 9, the naive strategy was overperforming the optimal strategy, due to an unexpected price increase. Despite this, it is satisfactory to see that there are only three trades, which is less than on April 19 and 22, 2010, and that trading occurs when price conditions are favourable.

5.5 Test 3: Sensitivity to Bid/Ask spread

In this last section, we are interested in the sensitivity of the results to the bid/ask spread, determined here by the two parameters $\kappa_a$ and $\kappa_b$. More precisely, we look at the dominant effect between the spread and the multiplicative price impact through the parameter $\lambda$.

We proceeded to two tests here: one without bid/ask spread, i.e. $\kappa_a = \kappa_b = 1$ and with $\lambda = 5.10^{-4}$ as before, and one with a spread of 0.2% and a price impact parameter $\lambda = 0$.

**Parameters** The table 12 shows the parameters of the two tests. We only changed the impact and spread parameters and let the others be identical.

**Performance Analysis** In table 13 we computed several statistics on the results. In figure 10 we plotted the empirical distribution of performance in the two tests, with the test 2 distribution (Cf. figure 5) serving as a reference. In figure 11 we plotted the empirical
Figure 8: Test 2: Strategy realization on the BNP.PA stock the 04/22/2010.

Figure 9: Test 2: Strategy realization on the BNP.PA stock the 04/23/2010.
Table 12: Test 3: Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>No spread test</th>
<th>No impact test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>1 Day</td>
<td>1 Day</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>5.00E-04</td>
<td>0</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$\kappa_a$</td>
<td>1</td>
<td>1.001</td>
</tr>
<tr>
<td>$\kappa_b$</td>
<td>1</td>
<td>0.999</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>$b$</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>$X_0$</td>
<td>20000</td>
<td>20000</td>
</tr>
<tr>
<td>$Y_0$</td>
<td>2500</td>
<td>2500</td>
</tr>
<tr>
<td>$P_0$</td>
<td>51</td>
<td>51</td>
</tr>
<tr>
<td>$x_{\min}$</td>
<td>-20000</td>
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<td>$x_{\max}$</td>
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<td>$y_{\min}$</td>
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<td>$p_{\max}$</td>
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<tr>
<td>$m$</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>$N$</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>$Q$</td>
<td>$10^5$</td>
<td>$10^5$</td>
</tr>
</tbody>
</table>

distribution of the number of trades in the two tests, which is helpful for interpreting the results. Indeed, we observe from figure 11 that increasing the spread reduces the number of trades of the optimal strategy. Intuitively, the more frequently a strategy trades, the smaller its standard deviation: for example, the limiting case of the uniform strategy achieve the smallest standard deviation in our benchmark, and the naïve strategy, that trades only once, the biggest. Qualitatively speaking, the standard deviation increases when the number of trades decreases: this help us explain qualitatively why the standard deviation is higher in the case of a large spread (we used $\kappa_a - \kappa_b = 20$ bps, which is much larger than usually observed in equity markets). Now, to provide an interpretation of why the optimal strategy trades less frequently when the spread is large, we can note two facts. First, in the large spread test, we considered that $\lambda = 0$, in other words that there is no market impact. Therefore, any trading rate $\xi/\theta$ will lead to same transaction price: this explain the clustering effect: the optimal strategy tends to trade a bigger quantity of assets at the same time to match terminal liquidation constraint. Second, a large spread will penalize strategies that can both buy and sell, and in particular the optimal strategy. Indeed, let us consider the typical scale of quantities involved in our optimization: we expect the optimal strategy to profit from price variation at the scale of 1€ in our example; if the spread is about 0.1€, like in our last example, and if we usually do about 10 trades on the liquidation period, the effect of the spread ($10 \times 0.1 \text{ €} = 1€$) is at the same scale as the price fluctuation. Therefore, the larger the spread, the more the optimal strategy tends to be one-sided, i.e. trading quantities ($\xi_n$) tends to be negative. Due to this phenomenon, the profit from optimal trading reduces with the spread, and the optimization becomes less efficient in this one-sided setup. This is consistent with the financial viewpoint: an investor that can both buy and sell have opportunities to profit from price fluctuations, whereas an investor that can only sell may only have opportunities to sell at high price; therefore the number of
trades decreases as the optimal strategy tends to be one-sided. Finally, we observe that
both spread and non-linear impact influence the trading schedule. We also expect that
the optimal quantity $\xi_n$ to trade at date $\tau_n$ is influenced directly by the non-linear impact
parameter $\lambda$.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>No spread test</th>
<th>No impact test</th>
<th>No spread vs. T2</th>
<th>No impact vs. T2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Utility</td>
<td>1.00113</td>
<td>1.00025</td>
<td>$-3.00.10^{-5}$</td>
<td>$-9.08.10^{-4}$</td>
</tr>
<tr>
<td>Mean Performance</td>
<td>1.00227</td>
<td>1.00053</td>
<td>$-5.98.10^{-5}$</td>
<td>$-1.80.10^{-3}$</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.00432</td>
<td>0.00906</td>
<td>$-9.17.10^{-3}$</td>
<td>1.078</td>
</tr>
</tbody>
</table>

Table 13: Test 3: Statistics. In the two last columns "No spread vs. T2" (resp."No impact vs. T2")
are shown the relative values of "No spread" test (resp. "No impact" test) against the values of test
2 of the preceding section.
Figure 11: Test 3: Empirical distributions of number of trades

References


