A model of optimal consumption under liquidity risk with random trading times

Huyën PHAM
Laboratoire de Probabilités et Modèles Aléatoires
CNRS, UMR 7599
Université Paris 7
e-mail: pham@math.jussieu.fr
and Institut Universitaire de France

Peter TANKOV
Laboratoire de Probabilités et Modèles Aléatoires
CNRS, UMR 7599
Université Paris 7
e-mail: tankov@math.jussieu.fr

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Abstract

We consider a portfolio/consumption choice problem in a market model with liquidity risk. The main feature is that the investor can trade and observe stock prices only at exogenous Poisson arrival times. He may also consume continuously from his cash holdings, and his goal is to maximize his expected utility from consumption. This is a mixed discrete/continuous stochastic control problem, nonstandard in the literature. The dynamic programming principle leads to a coupled system of Integro-Differential Equations (IDE), and we provide a convergent numerical algorithm for the resolution to this coupled system of IDE. Several numerical experiments illustrate the impact of the restricted liquidity trading opportunities, and we measure in particular the utility loss with respect to the classical Merton consumption problem.

Key words : liquidity, random trading times, portfolio/consumption problem, integro-differential equations, cost of liquidity.


JEL Classification : G11.

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1 Introduction

A fundamental assumption of the traditional portfolio/consumption choice paradigm of Merton [11] is that assets are liquid and readily continuously tradable by economic agents. In reality, there are some restrictions on securities trade, and investors cannot buy and sell them immediately; typical examples of assets in which trading is problematic include human capital, mutual funds, pension plans, inheritances, and residential real-estate. We then usually speak about liquidity risk meaning that one may have to wait some time before being able to unwind a position in some financial assets.

There are various approaches to model liquidity risk since it is in fact related to many factors. A familiar approach in the academic literature is to measure illiquidity in terms of bid-ask spread and transaction costs, see Davis and Norman [4], Jouini and Kallal [7] and many others. In this setting, potentially high cost is associated to frequent trading but the investors can trade whenever desired. On the other hand, there are some studies where illiquidity is represented by restrictions on trade times. For instance, Schwartz and Tebaldi [14] and Longstaff [9] assume in their model that illiquid assets can only be traded at the starting date and at a fixed terminal horizon. In a less extreme modelling, Rogers and Zane [13] and Matsumoto [10] consider random trade times by assuming that trade succeeds only at the jump times of a Poisson process, and study the impact on the portfolio choice problem. In these models, the price process is observed continuously, trading strategies are in continuous time, and the corresponding portfolio/consumption problem leads to a standard jump-diffusion control problem, see also Wang [15]. However, illiquidity is often viewed by practitioners as the situation where their ability to trade assets is limited or restricted to the times when a quote comes into the market.

In this paper, we consider a description of liquidity risk, which is consistent with the market microstructure oriented modelling of high frequency financial data such as tick-by-tick stock prices. We assume that stock prices can be observed and traded only at random times of a Poisson process corresponding to quotes in the market. This setup is inspired by recent papers of Frey and Runggaldier [5] and Cvitanic, Liptser and Rozovskii [2], who assume in addition that there is an unobservable stochastic volatility, and are interested in the estimation of this volatility. In our liquidity risk context, we suppose that the investor is also allowed to consume continuously from the bank account, and we study the Merton problem of maximizing the expected discounted utility of consumption.

From a mathematical viewpoint, the resulting optimization problem is a mixed discrete/continuous stochastic control problem, nonstandard in the literature. We show how it leads, via a dynamic programming principle, to a coupled system of nonlinear integro-differential equations (IPDE). We also provide a convergent numerical algorithm for solving this coupled system, and illustrate our results with some numerical experiments. In particular, we compare the value function and the optimal investment policy obtained in presence of liquidity risk with the ones in the classical Merton model, and we illustrate the impact of the restricted trading opportunities. Moreover, following the utility-indifference pricing approach, we measure the utility loss in monetary terms due to our liquidity constraints.

The plan of the paper is as follows. We formulate the liquidity risk model and the portfolio/consumption problem in Section 2. We show in Section 3 how it leads to a coupled system of IPDE. Section 4 describes a convergent numerical algorithm and we give several numerical illustrations in Section 5.
2 Model and problem formulation

Let us fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped with a filtration \(\mathbb{F} = (\mathbb{F}_t)_{t \geq 0}\) satisfying the usual conditions. All stochastic processes involved in this paper are defined on the stochastic basis \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\).

We consider a model of an illiquid market where the investor can observe the positive stock price process \(S\) and trade only at random times \(\{\tau_k\}_{k \geq 0}\) with \(\tau_0 = 0 < \tau_1 < \ldots < \tau_k < \ldots\). For simplicity, we assume that \(S_0\) is known, and we denote

\[ Z_k = \frac{S_{\tau_k} - S_{\tau_{k-1}}}{S_{\tau_{k-1}}}, \quad k \geq 1, \]

the observed return process valued in \((-1, \infty)\), where we set by convention \(Z_0\) to some fixed constant.

The investor may also consume continuously from the bank account (interest rate is assumed w.l.o.g to be zero) between two trading dates. We introduce the continuous observation filtration \(\mathbb{G}^c = (\mathbb{G}_t)_{t \geq 0}\) with:

\[ \mathbb{G}_t = \sigma\{ (\tau_k, Z_k) : \tau_k \leq t \}, \]

and the discrete observation filtration \(\mathbb{G}^d = (\mathbb{G}_{\tau_k})_{k \geq 0}\). Notice that \(\mathbb{G}_t\) is trivial for \(t < \tau_1\).

A control policy is a mixed discrete-continuous process \((\alpha, c)\), where \(\alpha = (\alpha_k)_{k \geq 1}\) is real-valued \(\mathbb{G}^d\)-predictable, i.e. \(\alpha_k\) is \(\mathbb{G}_{\tau_{k-1}}\)-measurable, and \(c = (c_t)_{t \geq 0}\) is a nonnegative \(\mathbb{G}^c\)-predictable process. \(\alpha_k\) represents the amount of stock invested for the period \((\tau_{k-1}, \tau_k]\) after observing the stock price at time \(\tau_{k-1}\), and \(c_t\) is the consumption rate at time \(t\) based on the available information. Starting from an initial capital \(x \geq 0\), and given a control policy \((\alpha, c)\), we denote \(X^x_k\) the wealth of the investor at time \(\tau_k\) defined by:

\[ X^x_k = x - \int_0^{\tau_k} c_t dt + \sum_{i=1}^{k} \alpha_i Z_i, \quad k \geq 1, \quad X^x_0 = x. \]  

(2.1)

Given \(x \geq 0\), we say that a control policy \((\alpha, c)\) is admissible, and we denote \((\alpha, c) \in \mathcal{A}(x)\) if:

\[ X^x_k \geq 0, \quad a.s. \quad \forall \ k \geq 1. \]  

(2.2)

We are interested in the optimal portfolio/consumption problem:

\[ v(x) = \sup_{(\alpha, c) \in \mathcal{A}(x)} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right], \quad x \geq 0, \]  

(2.3)

where \(\rho > 0\) is a positive discount factor, and \(U\) is an utility function defined on \(\mathbb{R}_+\), with w.l.o.g. \(U(0) = 0\), nondecreasing, concave and \(C^1\) on \((0, \infty)\) satisfying the Inada conditions \(U'(0^+) = \infty\) and \(U''(\infty) = 0\). We shall assume the following growth condition on \(U\) : there exists \(\gamma \in (0, 1)\) s.t.

\[ U(x) \leq K_1 x^\gamma, \quad x \geq 0, \]  

(2.4)

for some positive constant \(K_1\). We denote \(\hat{U}\) the convex conjugate of \(U\) i.e.:

\[ \hat{U}(y) = \sup_{x > 0} [U(x) - xy], \quad y \geq 0. \]  

(2.5)
Notice that $\tilde{U}$ is nonincreasing, $\tilde{U}(\infty) = U(0)$, and under (2.4) we have

$$\tilde{U}(y) \leq \tilde{K}_1 y^{-\tilde{\gamma}}, \quad y \geq 0, \quad \text{with } \tilde{\gamma} = \frac{\gamma}{1-\gamma} > 0,$$

for some positive constant $\tilde{K}_1$ (actually $\tilde{K}_1 = (\frac{K_1}{\gamma})^{\frac{1-\gamma}{\gamma}}$).

**Remark 2.1.** Denote by $\mu(dt, dz) = \sum_{k=1}^{\infty} \delta_{(\tau_k, Z_k)} dt dz$ the integer-valued random measure associated to the multivariate point process $(\tau_k, Z_k)_{k \geq 1}$. Let us then consider the piecewise deterministic controlled jump process:

$$\bar{X}_t^x = x - \int_0^t c_i dt + \int_0^t \tilde{\alpha}_t z \mu(dt, dz), \quad (2.7)$$

where $\tilde{\alpha} = (\tilde{\alpha}_t)_{t \geq 0}$ is a $\mathcal{G}^c$-predictable control process, $c = (c_t)_{t \geq 0}$ is a nonnegative $\mathcal{G}^c$-predictable control process, and define the related standard continuous control problem:

$$\bar{v}(x) = \sup_{(\bar{\alpha}, c) \in \bar{A}(x)} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right], \quad x \geq 0, \quad (2.8)$$

where $\bar{A}(x)$ is the set of control processes $(\bar{\alpha}, c)$ s.t. $X_t^x \geq 0$, for all $t \geq 0$. This control problem is interpreted as a consumption/investment problem where the investor may consume and trade continuously in a stock price whose return process $\bar{Z}$ is modelled as a pure jump process of dynamics $d\bar{Z}_t = \int z \mu(dt, dz)$. Problems of type (2.7)-(2.8) belong to the class of piecewise deterministic control problems, see e.g. the book by Davis [3], and lead to integrodifferential equations for the corresponding value functions. The link with our original control problem (2.3) is the following. Given $(\alpha, c) \in A(x)$, if we define the predictable process $\bar{\alpha}$ by $\bar{\alpha}_t = \sum_k \alpha_k I_{(\tau_{k-1}, \tau_k)}(t)$, for $t \geq 0$, then it is easy to see that $(\bar{\alpha}, c) \in \bar{A}(x)$, so that $v(x) \leq \bar{v}(x)$. Problem (2.3) is a mixed discrete/continuous-time stochastic control problem: this is a nonstandard control problem, which was not yet studied in the literature. In particular, we cannot derive as usual the Backward or Bellman equation associated to (2.3). Our paper is a first attempt to study such a control problem.

In the rest of the paper, the following conditions on $(\tau_k, Z_k)$ stand in force.

**H1** \{$(\tau_k)_{k \geq 1}$ is the sequence of jump times of a Poisson process with intensity $\lambda$.\}

**H2**

(i) For all $k \geq 1$, conditionally on the interarrival time $\tau_k - \tau_{k-1} = t \in \mathbb{R}_+$, $Z_k$ is independent from $(\tau_i, Z_i)_{i < k}$ and has a distribution denoted $p(t, dz)$.

(ii) For all $t \geq 0$, the support of $p(t, dz)$ is

- either an interval with interior equal to $(-\varepsilon, \varepsilon)$, $\varepsilon \in (0, 1]$ and $\varepsilon \in (0, \infty]$,

- or is finite equal to $(-\varepsilon, \ldots, \varepsilon)$, $\varepsilon \in (0, 1]$ and $\varepsilon \in (0, \infty]$.

**H3** \[ \int z p(t, dz) \geq 0, \text{ for all } t \geq 0, \text{ and there exist some } \kappa \in \mathbb{R}_+ \text{ and } b \in \mathbb{R}_+ \text{ s.t.} \]

$$\int (1 + z) p(t, dz) \leq \kappa e^{bt}, \quad \forall t \geq 0.$$  

The last condition (H3) means that for all $k \geq 1$,

$$1 \leq \mathbb{E}\left[ \frac{S_{\tau_k}}{S_{\tau_{k-1}}} | \tau_k - \tau_{k-1} = t \right] \leq \kappa e^{bt}, \quad \forall t \geq 0.$$
Remark 2.2. The assumption (H1) that random trading times occur via a Poisson process is a simplified story for liquidity constraints, and could be extended by considering for instance Cox processes. Here, the Poisson process simplifies the explicit derivation of the equations arising below from the dynamic programming principle, and in the limit as the intensity of the Poisson process increases to infinity, provides a valuable comparison with the original Merton problem. Assumption (H2)(i) means that the return process has stationary and independent increments, and is satisfied typically when it is extracted from a Lévy model for price process. The condition (H2)(ii) is not restrictive and is imposed here simply for making explicit the a-posteriori bounds on the controls (see Remark 2.3). It is easy to see that if the support of $Z_k$ is included in $(0, \infty)$, i.e. the sequence $(S_k)_k$ is increasing, or is included in $(-1, 0)$, i.e. $(S_k)_k$ is decreasing, then the value function $v$ is infinite. Indeed, suppose that $\bar{z} > 0$. Then, one can consume as much as wanted, by buying enough stock in order to satisfy the admissibility condition, so that $v$ is infinite. A similar argument is valid (by selling stock) when $\bar{z} < 0$. The condition $\int z p(t, dz) \geq 0$ in (H3) is simply put for financial interpretation, but could be relaxed. The other condition in (H3) is a more crucial technical one. Intuitively, with an exponential discount factor $e^{-\rho t}$, we cannot hope to obtain a finite value function if the expected return grows faster than exponential.

Remark 2.3. Since $X_{k+1}^x = X_k^x - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_{k+1} Z_{k+1}$, and by the condition (H2) on the support of $Z_{k+1}$, we see that the admissibility condition (2.2) is written as:

$$X_k^x - \int_{\tau_k}^{s} c_u du + \alpha_{k+1} z \geq 0, \quad \forall k \geq 0, \forall s \geq \tau_k, \forall z \in \{-\bar{z}, \bar{z}\}.$$  

almost surely. This may be also formulated directly in terms of $(\alpha, c) \in A(x)$ as:

$$-\frac{X_k^x}{\bar{z}} \leq \alpha_{k+1} \leq \frac{X_k^x}{\bar{z}}, \quad \forall k \geq 0,$$

$$\int_{\tau_k}^{s} c_u du \leq X_k^x - \ell(\alpha_{k+1}), \quad \forall k \geq 0, \forall s \geq \tau_k,$$  

(2.9) (2.10)

where we set for all $a \in \mathbb{R}$:

$$\ell(a) = \max(a_{\bar{z}}, -a \bar{z}),$$

with the convention that $\max(a_{\bar{z}}, -a \bar{z}) = a_{\bar{z}}$ when $\bar{z} = \infty$. In particular, we see that for $x = 0$, $A(0) = \{0, 0\}$ and so $\nu(0) = 0$. Notice that in the usual case of stock price with distribution support $(0, \infty)$, i.e. $\bar{z} = 1$ and $\bar{z} = \infty$, as in the Example 2.1 below, we have $\ell(a) = a$, and the bounds in (2.9) is written as $\alpha_{k+1} \in [0, X_k^x]$; no short selling and no borrowing.

The following simple but important examples illustrate these assumptions (H1)-(H2)-(H3).

Example 2.1. $S$ is extracted from a Black-Scholes model: $dS_t = bS_t dt + \sigma S_t dW_t$, with $b \geq 0, \sigma > 0$. Then $p(t, dz)$ is the distribution of

$$Z(t) = \exp \left[ \left( b - \frac{\sigma^2}{2} \right) t + \sigma W_t \right] - 1,$$

with support $(-1, \infty)$, and (H3) is clearly satisfied, since in this case $\int (1 + z)p(t, dz) = \mathbb{E} \left[ \exp \left( (b - \sigma^2/2)t + \sigma W_t \right) \right] = e^{bt}$.  

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Example 2.2. \( Z_k \) is independent of the waiting times \( \tau_k - \tau_{k-1} \), in which case its distribution \( p(dz) \) does not depend on \( t \). In particular, \( p(dz) \) may be a discrete distribution with support \( \{z_0, \ldots, z_d\} \) s.t. \( z = -z_0 \in (0, 1] \) and \( z_d = \bar{z} \in (0, \infty) \).

### 3 A first-order coupled system of nonlinear IPDE

In this section, we derive the coupled system of Integro Partial Differential Equations (IPDE) that will be satisfied by the value function of our control problem. The starting point is the following version of the dynamic programming principle (DPP) adapted to our context, and proved rigorously in [12] :

\[
v(x) = \sup_{(a,c) \in A_d(x)} \mathbb{E} \left[ \int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} v(X_t^x) \right].
\] (3.1)

From the expression (2.1) of the wealth, and the measurability conditions on the control, the above dynamic programming relation is written as

\[
v(x) = \sup_{(a,c) \in A_d(x)} \mathbb{E} \left[ \int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} v(x - \int_0^{\tau_1} c_t dt + aZ_1) \right],
\] (3.2)

where \( A_d(x) \) is the set of pairs \((a,c)\) with a deterministic constant, and \( c \) a deterministic nonnegative process s.t. (see Remark 2.3) \( a \in [-x/\bar{z}, x/\bar{z}] \) and

\[
\int_0^t c_u du \leq x - \ell(a) \quad \text{i.e.} \quad x - \int_0^t c_u du + az \geq 0, \; \forall t \geq 0, \; \forall z \in (-\bar{z}, \bar{z}).
\] (3.3)

Given \( a \in [-x/\bar{z}, x/\bar{z}] \), we denote by \( C_a(x) \) the set of deterministic nonnegative processes satisfying (3.3). Moreover, under conditions (H1) and (H2), the r.h.s. of (3.2) is written explicitly in :

\[
v(x) = \sup_{a \in [-x/\bar{z}, x/\bar{z}], c \in C_a(x)} \int_0^{\infty} e^{-(\rho + \lambda)t} \left[ U(c_t) + \lambda \int v(x - \int_0^t c_s ds + az) p(t, dz) \right] dt.
\] (3.4)

Let

\[
\mathcal{D} = \mathbb{R}_+ \times \mathcal{X} \quad \text{with} \quad \mathcal{X} = \left\{ (x, a) \in \mathbb{R}_+ \times \mathbb{R} : -\frac{x}{\bar{z}} \leq a \leq \frac{x}{\bar{z}} \right\}.
\] (3.5)

By setting \( A = \mathbb{R} \) if \( \bar{z} < \infty \), and \( A = \mathbb{R}_+ \) if \( \bar{z} = \infty \), notice that \( \mathcal{X} \) is written also as

\[
\mathcal{X} = \left\{ (x, a) \in \mathbb{R}_+ \times A : x \geq \ell(a) \right\}.
\]

Now, we introduce the dynamic auxiliary control problem : for \((t, x, a) \in \mathcal{D}\),

\[
\hat{v}(t, x, a) = \sup_{c \in C_a(t, x)} \int_t^{\infty} e^{-(\rho + \lambda)(s-t)} \left[ U(c_s) + \lambda \int v(Y_{s,x}^t + az) p(s, dz) \right] ds,
\] (3.6)

where \( C_a(t, x) \) is the set of deterministic nonnegative processes \( c = (c_s)_{s \geq t} \) s.t.

\[
\int_t^s c_u du \leq x - \ell(a) \quad \text{i.e.} \quad Y_{s,x}^t + az \geq 0, \; \forall s \geq t, \; \forall z \in (-\bar{z}, \bar{z}),
\] (3.7)
and $Y_{t,x}$ is the deterministic controlled process by $c \in C(a(t,x))$:

$$Y_{t,x} = x - \int_t^s c_u \, du, \quad s \geq t. \quad (3.8)$$

One can check (see details in [12]) that $\hat{v}$ lies in $C_+(D)$, the set of nonnegative continuous functions on $D$. From (3.4)-(3.6), the original value function is then related to this auxiliary optimization problem by:

$$v = \mathcal{H}\hat{v} \quad (3.9)$$

where $\mathcal{H}$ is the operator mapping $C_+(D)$ into the set $B_+(\mathbb{R}_+)$ of nonnegative measurable functions on $\mathbb{R}_+$ by:

$$\mathcal{H}\hat{w}(x) = \sup_{a \in [-x/z,x/z]} \hat{w}(0,x,a). \quad (3.10)$$

**Remark 3.1.** For a given $a \in A$, $\hat{v}$ is the value function of an optimal consumption problem over an infinite horizon in a certain environment:

$$\hat{v}(t,x,a) = \sup_{c \in C_a(t,x)} \int_t^\infty e^{-(\rho + \lambda)(s-t)} V_a(s,Y_{s,x},c_s) \, ds,$$

where $V_a$ is a modified utility function depending not only on the current consumption rate $c_s$, but also on the cumulated consumption $\int c_s \, ds$. The optimal trading policy for the initial portfolio/investment problem $v(x)$ is given by

$$\alpha^*_k \in \arg \max_{-\frac{x_k}{z} \leq a \leq \frac{x_k}{z}} \hat{v}(0,X_k,a), \quad k \geq 0.$$

At this stage, we may study the deterministic control problem (3.6) by standard dynamic programming methods: the associated Hamilton-Jacobi equation is

$$\sup_{c \geq 0} \left[ -(\rho + \lambda)\hat{v} + \frac{\partial \hat{v}}{\partial t} - c \frac{\partial \hat{v}}{\partial x} + U(c) + \lambda \int v(x + az)p(t,dz) \right] = 0, \quad (t,x,a) \in D,$$

which may be rewritten as a first order Integro Partial Differential Equation (IPDE)

$$(\rho + \lambda)\hat{v} - \frac{\partial \hat{v}}{\partial t} - \hat{U} \left( \frac{\partial \hat{v}}{\partial x} \right) - \lambda \int v(x + az)p(t,dz) = 0, \quad (t,x,a) \in D. \quad (3.11)$$

**Remark 3.2.** In the particular case where the distribution $p(t,dz) = p(dz)$ does not depend on $t$, the above IPDE reduces to the integro ordinary differential equation for $\hat{v}(x,a)$:

$$(\rho + \lambda)\hat{v} - \hat{U} \left( \frac{\partial \hat{v}}{\partial x} \right) - \lambda \int v(x + az)p(dz) = 0, \quad (t,x,a) \in D,$$

with $v(x) = \sup_{a \in [-x/z,x/z]} \hat{v}(x,a)$

We have then split our original stochastic optimization problem into two coupled tractable deterministic optimization problems: Problem (3.6) is a family over $a \in A$ of standard deterministic control problems on infinite horizon, which is stationary (i.e. $\hat{v}$ does not depend on $t$), whenever the distribution $p(t,dz)$ does not depend on $t$, and problem (3.9) is
a classical one-dimensional extremum problem over \( a \). Notice that these two optimization problems are coupled since the reward function appearing in the definition of problem (3.6) or in its IPDE (3.11) depends on the value function of problem (3.9) and vice-versa. However, this suggests a fixed point algorithm for solving our original optimization problem: this argument will be developed in the next section.

The rigorous characterization of the value function for the original control problem (2.3) by means of viscosity solutions to the coupled IPDE

\[
(\rho + \lambda)\dot{v} - \frac{\partial \dot{v}}{\partial t} - \tilde{U} \left( \frac{\partial \dot{v}}{\partial x} \right) - \lambda \int v(x + az)p(t, dz) = 0, \quad (t, x, a) \in \mathbb{D}, \quad (3.12)
\]

\[
v = \mathcal{H} \dot{v} := \sup_{a \in [-x/\bar{z}, x/\bar{z}]} \dot{v}(0, x, a), \quad (3.13)
\]

is proved in [12]. In particular, it is shown that under conditions (2.4) and (H1)-(H2)-(H3), and provided that \( \rho \) satisfies

\[
\rho > b\gamma + \lambda \left( \frac{\kappa\gamma}{2\gamma} - 1 \right), \quad (3.14)
\]

we have for all \( x \geq 0, (\alpha, c) \in \mathcal{A}(x), \)

\[
\mathbb{E} \left[ e^{-\rho T_n} (X_n^x)^\gamma \right] \leq x^\gamma \delta^n, \quad (3.15)
\]

where

\[
\delta = \frac{\lambda}{\rho - b\gamma + \lambda \frac{\kappa\gamma}{2\gamma}} < 1. \quad (3.16)
\]

Moreover, the value functions satisfy the growth conditions

\[
\dot{v}(t, x, a) \leq K(e^{bt} x)^\gamma, \quad \forall (t, x, a) \in \mathcal{D}, \quad (3.17)
\]

\[
v(x) \leq Kx^\gamma, \quad \forall x \geq 0, \quad (3.18)
\]

for some positive constant \( K \). The case where \( U(0^+) = -\infty \), e.g. for power utility with negative exponent and logarithmic utility, does not change the form of the coupled IPDE (3.12)-(3.13), but leads to a different type of growth condition on the value functions, which does not allow to deduce the convergence of our algorithm as in Theorem 4.1.

Remark 3.3. In the case of Example 2.1, we have \( \bar{z} = 1 \) and \( \kappa = 1 \). Hence, the condition (3.14) is written as: \( \rho > b\gamma \), which is independent of \( \lambda \). This will enable us to study the limiting behavior of this model when \( \lambda \) goes to infinity, see Section 5.

In this paper, we focus now on the numerical resolution of the coupled IPDE, and then on simulations illustrating the impact of the liquidity restricted trading strategies.

4 A numerical decoupling algorithm

The main difficulty in the numerical resolution of the IPDE (3.12) for \( \dot{v} \) comes from the integrodifferential term involving \( \mathcal{H} \dot{v} \). To overcome this problem, we suggest the following iterative procedure. We start from an initial function \( v_0 \) defined on \( \mathbb{R}_+ \), as the value function of the consumption problem without trading:

\[
v_0(x) = \sup_{c \in \mathcal{C}(x)} \int_0^\infty e^{-\rho t} U(c_t) dt,
\]
where $\mathcal{C}(x)$ is the set of nonnegative (deterministic) processes $c = (c_t)_t$ s.t. $x - \int_0^t c_s ds \geq 0$ for all $t \geq 0$. $v_0$ is the unique solution with linear growth condition to the first-order differential equation

$$\rho v_0 - \hat{U} \left( \frac{\partial v_0}{\partial x} \right) = 0, \quad x > 0,$$

together with the boundary condition $v_0(0^+) = 0$. We then construct a sequence of functions $(\hat{v}_n(t, x, a))_{n \geq 1}$ defined on $\mathcal{D}$ and $(v_n(x))_{n \geq 0}$ defined on $\mathbb{R}_+$ by:

$$\hat{v}_{n+1}(t, x, a) = \sup_{c \in \mathcal{C}_n(t, x)} \int_t^\infty e^{-(\rho + \lambda)(s-t)} \left[ U(c_s) + \lambda \int v_n(Y_s^t x + az)p(s, dz) \right] ds \quad (4.1)$$

$$v_{n+1} = \mathcal{H}\hat{v}_{n+1}, \quad n \geq 0.$$  

By similar arguments as in the previous section (actually simpler since here there is no more coupling system), we see that $\hat{v}_{n+1}$ and $v_{n+1}$ are solutions to the recursive system:

$$-(\rho + \lambda)\hat{v}_{n+1} + \frac{\partial \hat{v}_{n+1}}{\partial t} + \hat{U} \left( \frac{\partial \hat{v}_{n+1}}{\partial x} \right) + \lambda \int_{-1}^z v_n(x + az)p(t, dz) = 0, \quad (t, x, a) \in \mathcal{D},$$

$$v_{n+1} = \mathcal{H}\hat{v}_{n+1},$$

and we have an approximate trading policy by taking:

$$\alpha_{k+1}^{(n)} \in \arg \max_{-\bar{X}_k^t \leq a \leq \bar{X}_k^t} \hat{v}_n(0, X_k^t, a), \quad k \geq 0.$$  

Notice also that the determination of $v_{n+1}$ is easily obtained from the operator $\mathcal{H}$, involving simply a standard maximization procedure.

### 4.1 Numerical solution of the decoupled control problem

At step $n$ of the iterative algorithm, to solve the deterministic control problem (4.1) we need to compute $\hat{v}_{n+1}(0, x, a)$ for different values of $a$ (on a discrete grid) and then find the maximum of $\hat{v}_{n+1}(0, x, a)$ to compute $v_{n+1}(x)$. We now explain how the optimization problem (4.1) is solved for each given value of $a$. Let us fix $a \in A$, and consider the following function:

$$f_n(t, x, a) = \lambda \int v_n(x + az)p(t, dz).$$

The deterministic control problem to be solved is therefore

$$\hat{v}_{n+1}(t, x, a) = \sup_{c \in \mathcal{C}_n(t, x)} \int_t^\infty e^{-(\rho + \lambda)(s-t)} [U(c_s) + f_n(s, Y_s^t x, a)] ds. \quad (4.2)$$

The dynamic programming principle for this control problem implies for any $0 \leq t < T$, $x \geq \ell(a)$:

$$\hat{v}_{n+1}(t, x, a) = \sup_{c \in \mathcal{C}_n(t, x)} \left\{ \int_t^T e^{-(\rho + \lambda)(s-t)} [U(c_s) + f_n(s, Y_s^t x, a)] ds + e^{-(\rho + \lambda)(T-t)} \hat{v}_{n+1}(T, Y_T^t x, a) \right\}. \quad (4.3)$$
Let us then introduce a finite-horizon deterministic control problem with terminal cost

$$
\hat{v}^n_{n+1}(t, x, a) = \sup_{c \in C_n(t, x)} \left\{ \int_t^T e^{-(\rho + \lambda)(s-t)}[U(c_s) + f_n(s, Y^t_{n+1}, a)] ds + e^{-(\rho + \lambda)(T-t)}G(T, Y^t_{n+1}, a) \right\},
$$

where

$$
G(t, x, a) = \int_t^\infty e^{-(\rho + \lambda)(s-t)} f_n(s, x, a) ds.
$$

This problem is obtained from (4.2) by restricting the set of consumption policies to those for which there is no consumption after date $T$.

Similarly as in (3.17)-(3.18), one can derive an uniform bound on $(\hat{v}_n, v_n)_n$:

$$
\hat{v}_n(t, x, a) \leq K(e^{bt}x)^\gamma, \quad \forall (t, x, a) \in D,
$$

$$
v_n(x) \leq K x^\gamma, \quad \forall x \geq 0,
$$

for some constant $K$ independent of $n$. Hence, under the condition (3.14), which implies $\rho + \lambda > b\gamma$, we have

$$
e^{-(\rho + \lambda)(T-t)}\hat{v}_{n+1}(T, Y^t_{n+1}, a) \leq e^{-(\rho + \lambda)(T-t)}\hat{v}_{n+1}(t, x, a) \leq K x^\gamma e^{(b\gamma - \rho - \lambda)T + (\rho + \gamma)t}.
$$

Therefore, $e^{-(\rho + \lambda)(T-t)}\hat{v}_{n+1}(T, Y^t_{n+1}, a)$ converges to zero exponentially fast and uniformly on $(t, x, a)$ on compacts as $T \to \infty$. This shows from (4.3) that we can approximate $\hat{v}_{n+1}(t, x, a)$ by $\hat{v}^T_{n+1}(t, x, a)$ with any desired precision. On the other hand, (4.4) is a finite-horizon deterministic control problem well studied in the literature. For fixed $a \in A$, the value function $\hat{v}^T_{n+1}(t, x, a)$ is the unique viscosity solution of the HJB equation

$$
-(\rho + \lambda)\hat{v}^T_{n+1} + \frac{\partial \hat{v}^T_{n+1}}{\partial t} + \hat{U} \left( \frac{\partial \hat{v}^T_{n+1}}{\partial x} \right) + f_n(t, x, a) = 0
$$

in the domain $(t, x) \in [0, T] \times [\ell(a), \infty)$, with boundary condition

$$
\hat{v}^T_{n+1}(t, \ell(a), a) = G(t, \ell(a), a), \quad t \in [0, T],
$$

and terminal condition $\hat{v}^T_{n+1}(T, x, a) = G(T, x, a)$ for all $x \in [\ell(a), \infty)$. This equation can be approximated numerically by a standard backward discretization scheme as discussed for example in [1].

### 4.2 Convergence of the iterative decoupling algorithm

We now focus on the convergence of the sequence of functions $(\hat{v}_n, v_n)_n$ as $n$ goes to infinity. Although we have a uniform bound (4.6) on $(\hat{v}_n, v_n)_n$, the equicontinuity of $(\hat{v}_n, v_n)_n$ seems much more difficult to establish in order to apply Ascoli-Arzelà theorem and thus to get the convergence of the sequence $(\hat{v}_n, v_n)_n$. Instead, by means of dynamic programming arguments, we provide an autonomous probabilistic representation of $\hat{v}_n$ and $v_n$. Given $x \in \mathbb{R}_+$, we denote by $A_n(x)$ the subset of controls $(\alpha, \rho) = ((\phi_k)_{k \in \mathbb{R}}) \in A(x)$ s.t. $\alpha_k =$
0 for \( k \geq n + 1 \). In other words, \( A_n(x) \) is the set of admissible controls with at most \( n \) trading interventions and we have

\[
A_n(x) \subset A_{n+1}(x) \subset A(x).
\]

We then have the following representation of \( v_n \):

**Proposition 4.1.** For all \( n \geq 0 \), we have

\[
v_n(x) = \sup_{(\alpha, c) \in A_n(x)} \mathbb{E} \left[ \int_0^{\infty} e^{-\rho t} U(c_t) dt \right], \quad x \geq 0,
\]

**Proof.** We set for all \( n \geq 0 \),

\[
w_n(x) = \sup_{(\alpha, c) \in A_n(x)} \mathbb{E} \left[ \int_0^{\infty} e^{-\rho t} U(c_t) dt \right], \quad x \geq 0.
\]

From the dynamic programming principle for this control problem, we have

\[
w_{n+1}(x) = \sup_{(a, c) \in A_d(x)} \mathbb{E} \left[ \int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} w_n(X_{\tau_1}^x) \right], \quad x \geq 0.
\]

Then, by the same arguments as in the derivation of relations (3.6), (3.9), the sequence of functions \((w_n)_n\) is given in inductive form by:

\[
w_{n+1} = H \hat{w}_{n+1}, \quad \forall n \geq 0,
\]

\[
\hat{w}_{n+1}(t, x, a) = \sup_{c \in C_a(t, x)} \int_t^{\infty} e^{-(\rho + \lambda)(s-t)} \left[ U(c_s) + \lambda \int w_n(Y_t^{s, x} + az)p(s, dz) \right] ds.
\]

From the definition of \((\hat{v}_n, v_n)\) and by induction starting from \(w_0 = v_0\), we deduce that \(\hat{w}_n = \hat{v}_n, w_n = v_n\), for all \( n \geq 1 \). This ends the proof. \(\square\)

As a consequence, we can prove the convergence of the sequence of value functions.

**Theorem 4.1.** Under \((H1)-(H2)-(H3)\), (2.4), and (3.14), the sequence of functions \((\hat{v}_n, v_n)_{n \geq 0}\) converges uniformly on any compact subset of \(D\) and \(\mathbb{R}_+\) to \((\hat{v}, v)\). More precisely, for any compact subset \(F\) and \(G\) of \(D\) and \(\mathbb{R}_+\), there exist some positive constants \(C_F\) and \(C_G\) s.t.

\[
0 \leq \sup_F (\hat{v} - \hat{v}_n) \leq C_F \delta^n, \quad (4.12)
\]

\[
0 \leq \sup_G (v - v_n) \leq C_G \delta^n, \quad (4.13)
\]

where \(\delta\) is defined in (3.16).

**Proof.** 1) From the dynamic programming principle (3.1), for all \( \varepsilon > 0, n \geq 1, x \in \mathbb{R}_+ \), one can find \((\hat{\alpha}, \hat{c}) \in A(x)\) s.t.

\[
v(x) - \varepsilon \leq \mathbb{E} \left[ \int_0^{\tau_1} e^{-\rho t} U(\hat{c}_t) dt + e^{-\rho \tau_1} v(Y_{\tau_1}^x) \right]. \quad (4.14)
\]
Now, observe that the “truncated” control \((\alpha^{(n)}, c^{(n)})\) defined by \(\alpha_k^{(n)} = \hat{\alpha}_k 1_{k \leq n}, k \in \mathbb{N}^*,\)
\(\hat{c}_t^{(n)} = \hat{c}_t 1_{t \leq n}, t \geq 0,\) lies in \(A_n(x)\). Then, from the representation (4.9) of \(v_n\), we have
\[
v_n(x) \geq \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(\hat{c}_t) dt \right].
\] (4.15)

Moreover, from (3.15) and (3.18), we have
\[
\mathbb{E} \left[ e^{-\rho t} v(X_t^x) \right] \leq K \mathbb{E} \left[ e^{-\rho t} (X_t^n)^\gamma \right] \leq K x^\gamma \delta^n.
\] (4.16)

Therefore, by noting also from (4.9) that \((v_n)_n\) is nondecreasing with \(v_n \leq v\), and plugging (4.15)-(4.16) into (4.14), we obtain:
\[
v(x) - \varepsilon - K x^\gamma \delta^n \leq v_n(x) \leq v(x).
\] (4.17)

This proves the uniform convergence of \(v_n\) to \(v\) on any compact subset of \(\mathbb{R}_+\), and the estimate (4.13).

2) From the definition of \(\hat{v}_n\) and since \(v_n\) is a nondecreasing sequence converging to \(v\), we clearly have \(\hat{v}_n \leq \hat{v}_{n+1} \leq \hat{v}\). On the other hand, by definition of \(\hat{v}\), for all \(\varepsilon > 0\), \((t, x, a) \in \mathcal{D}\), one can find \(c \in \mathcal{C}_n(t, x)\) s.t.
\[
\hat{v}(t, x, a) - \varepsilon \leq \int_t^\infty e^{-(\rho + \lambda)(s-t)} \left[ U(c_s) + \lambda \int v(Y^{t,x} + az)p(s, dz) \right] ds.
\]

By using (4.17) and observing also that \(Y^{t,x}_s \leq x\), we get :
\[
\hat{v}(t, x, a) - \varepsilon \leq \int_t^\infty e^{-(\rho + \lambda)(s-t)} \left[ U(c_s) + \lambda \int v_n(Y^{t,x} + az)p(s, dz) \right] ds
\]
\[
+ \int_t^\infty e^{-(\rho + \lambda)(s-t)} \lambda \int \varepsilon + K \delta^n (x + az)^\gamma p(s, dz) ds
\]
\[
\leq \hat{v}_{n+1}(x) + \frac{\varepsilon \lambda}{\rho + \lambda} + \lambda K \delta^n \int_t^\infty e^{-(\rho + \lambda)(s-t)} x^\gamma \frac{K^\gamma}{\varepsilon\gamma} e^{b\gamma s} ds
\]
\[
\leq \hat{v}_{n+1}(x) + \frac{\varepsilon \lambda}{\rho + \lambda} + \lambda K \delta^n x^\gamma \frac{1}{\rho + \lambda - b\gamma} \frac{K^\gamma e^{b\gamma t}}{\varepsilon\gamma},
\]
where we used in the second inequality, Jensen’s inequality and conditions (H2)-(H3). This proves the uniform convergence of \(\hat{v}_n\) to \(\hat{v}\) on any compact subset of \(\mathcal{D}\), and the estimate (4.12). \(\square\)

5 Numerical illustrations

In this section, we provide simulations for illustrating the impact of liquidity constraints on the attainable utility level and on the investment strategy. We consider the liquidity risk model (2.1), (2.3) where \((Z_k)\) is extracted from the Black-Scholes model \(BS(b, \sigma)\) with drift \(b\) and volatility \(\sigma\) for stock price \(S\), and zero interest rate (see Example 2.1), and we shall compare our numerical experiments with the original Merton problem without liquidity constraints. We recall that this problem is defined by :
\[
v_M(x) = \sup_{(\hat{a}, \hat{c}) \in \mathcal{A}(x)} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(\hat{c}_t) dt \right].
\]
Here $\mathcal{A}(x)$ is the set $(\alpha, c)$ of predictable processes (with respect to the filtration generated by the stock price process $S$ in the Black-Scholes model $BS(b, \sigma)$) s.t. the corresponding wealth, with an amount $\alpha_t$ invested in stock price $S_t$, a consumption rate $c_t$, and given by $X_t = x - \int_0^t c_u\,du + \int_0^t \alpha_u\,dS_u$ is nonnegative. It is well-known (see e.g. the book [8]) that in the case of power utility functions $U(x) = x^{\gamma}/\gamma$, the value function and the optimal investment strategy are explicitly given by

$$v_M(x) = K_M x^\gamma, \quad \tilde{\alpha}_t^M = \frac{b}{(1-\gamma)\sigma^2} X_t^{\gamma},$$

with

$$K_M = \frac{1}{\gamma} \left( \frac{1-\gamma}{\rho - \eta} \right)^{1-\gamma}, \quad \eta = \frac{b^2\gamma}{2\sigma^2(1-\gamma)}.$$

In the sequel, for the numerical experiments, we choose a power utility function with $\gamma = 0.5$. The choice of other parameters is a more subtle matter. It is clear that the value function of an investor in an illiquid market will always be strictly smaller than the value function of the Merton problem. An important question that we would like to address is whether it converges to $v_M$ when the trading frequency $\lambda$ tends to infinity. This convergence cannot take place when Merton’s optimal investment proportion $\frac{b}{(1-\gamma)\sigma^2}$ does not belong to the interval $[0, 1]$ since in the illiquid market, the investment policy always belongs to this interval (cf Remark 2.3). We thus must choose parameters for which $\frac{b}{(1-\gamma)\sigma^2} < 1$. On the other hand, the value function is always bounded from below by the value function of the consumption problem without trading $v_0(x)$, given in our present setting by $v_0(x) = K_0 x^\gamma$, $K_0 = \frac{1}{\gamma} \left( \frac{1-\gamma}{\rho} \right)^{1-\gamma}$. The parameters must therefore be such that $K_M$ is substantially different from $K_0$. These two requirements on the model parameters correspond to a high-risk return market, where the economic agent can considerably increase her utility with relatively little investment. In addition, the discount factor $\rho$ must satisfy $\rho > b\gamma$ for the convergence of the iterative decoupling scheme (equation (3.14)). To satisfy all these conditions, we take $b = 0.4$, $\sigma = 1$ and $\rho = 0.2$, yielding $K_0 = 3.16$, $K_M = 4.08$ and $\frac{b}{(1-\gamma)\sigma^2} = 0.8$. The intensity $\lambda$ is a free parameter that can be changed to adjust the “illiquidity” of the market.

For numerical solution, equation (4.8) was discretized using a fully implicit finite difference scheme. The computation domain was $\{t, x\} \in [0, 10] \times [0, 5]$ with 50 discretization points in $x$ coordinate (non-uniformly spaced) and 200 discretization points in $t$ (uniformly spaced). The computation time per iteration was about 1 second on an Intel P-IV PC. The number of iterations needed to achieve convergence is increasing with $\lambda$, since the convergence parameter $\delta$ approaches 1 from below as $\lambda \to \infty$ (cf. equation (3.16)). For example, for $\lambda = 1$, 20 iterations are sufficient to achieve a relative precision of $10^{-3}$, and for $\lambda = 5$, 100 iterations may be required.

In the first series of tests, we study the performance of our decoupling algorithm in a strongly illiquid market ($\lambda = 1$). In figure 1, the left graph shows the form of the value function and the right graph that of the optimal investment strategy obtained at different iterations of the numerical decoupling algorithm. As expected, the limiting value function lies between the solution corresponding to the model without trading $v_0$ and the value function of the Merton problem $v_M$. Moreover, we observe the same qualitative behavior of the results as in Merton’s model (the value function resembles a power law and the optimal
investment is a constant fraction of the total wealth), however due to the cost of liquidity the value function in an illiquid market is smaller than that of the Merton portfolio problem.

In the second experiment, we vary the Poisson parameter $\lambda$ governing the trading frequency, to study the convergence of the illiquid market to the Merton portfolio problem. Figure 2 presents the behavior of the value functions $v(x)$ and the associated optimal trading strategies. From these graphs we observe, empirically, that

(i) for a fixed value of $x$, both the value function and the optimal investment policy are increasing in $\lambda$ and

(ii) as $\lambda \to \infty$, the value function and the optimal investment policy seem to converge to the corresponding functions in the Merton portfolio problem.

Next, we would like to study the utility loss due to liquidity constraints. Similarly to the utility-indifference pricing approach introduced in [6] for transaction costs and then extended in numerous papers for incomplete markets, we define the utility loss in monetary terms (which can also be called cost of liquidity) as the extra amount of initial wealth $\pi(x)$ needed to reach the same level of expected utility as an investor without trading restrictions and initial capital $x$. This cost of liquidity is then computed as the solution to

$$v(x + \pi(x)) = v_M(x).$$

In our setting (power utility), the cost of liquidity $\pi(x)$ is roughly proportional to $x$. We therefore study the cost of liquidity per unit of initial wealth $\pi(1)$. Table 1 reproduces the values $\pi(1)$ for different values of the Poisson parameter $\lambda$. As expected, the cost of liquidity decreases to zero as $\lambda \to \infty$.

![Figure 1: Left : Convergence of the iterative algorithm for computing the value function in an illiquid market with $\lambda = 1$. The limiting value is smaller than that of the classical Merton problem due to the cost of liquidity. Right : Convergence of the iterative algorithm for computing the optimal investment policy (the amount to invest in stock as a function of the total wealth at the trading date).](image-url)
Figure 2: Behavior of the value function in an illiquid market (left) and of the optimal investment policy (right) for different values of the Poisson parameter $\lambda$.

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<th>$\lambda$</th>
<th>0 (No trading)</th>
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<th>5</th>
<th>40</th>
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<td>0.6671</td>
<td>0.2749</td>
<td>0.1214</td>
<td>0.0539</td>
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Table 1: Cost of liquidity $\pi(1)$ as a function of the parameter $\lambda$. 
References


