A SIMPLE PROOF OF THE SUPPORT THEOREM FOR DIFFUSION PROCESSES

Annie Millet and Marta Sanz–Solé *

1. Introduction and Notations

Let W denote a d–dimensional standard Wiener process, $\sigma : \mathbf{R}^m \longrightarrow \mathbf{R}^m \otimes \mathbf{R}^d$ and $b : \mathbf{R}^m \longrightarrow \mathbf{R}^m$ satisfy the following condition

(H) σ is of class $\mathcal{C}^2$, bounded together with its partial derivative of order one and two, and b is globally Lipschitz and bounded.

For any $x \in \mathbf{R}^m$, let $(X_t, t \in [0,1])$ be the diffusion solution of the stochastic differential equation

$$X_t = x + \int_0^t \sigma(X_s)\, dW_s + \int_0^t b(X_s)\, ds \tag{1.1}$$

Let $\alpha > 0$, and denote by $\mathcal{C}^\alpha([0,1]; \mathbf{R}^m)$ the set of α–Hölder continuous functions, i.e., of continuous functions $f : [0,1] \longrightarrow \mathbf{R}^m$ such that

$$\| f \|_\alpha = \sup_t |\, f(t) \,| + \sup_{s \neq t} \frac{|\, f(t) - f(s) \,|}{|\, t - s \,|^\alpha} < \infty \, . \tag{1.2}$$

The norm $\| \cdot \|_\alpha$ is called the α–Hölder norm. It is well known that the trajectories of X are α–Hölder continuous for $\alpha \in \left[0, \frac{1}{2}\right[$.

Let $\mathcal{H}$ denote the Cameron–Martin space, and given $h \in \mathcal{H}$ let $S(h)$ be the solution of the differential equation

$$S(h)_t = x + \int_0^t \sigma\big[S(h)_s\big]\, \dot{h}_s\, ds + \int_0^t \Big[b(S(h)_s) - \frac{1}{2}\, (\nabla \sigma)\, \sigma(S(h)_s)\Big]\, ds \tag{1.3}$$

The aim of this paper is to give a simple proof of the characterization of the support of $P \circ X^{-1}$ as the closure $\mathcal{S}$ of the set $\{S(h)\,,\ h \in \mathcal{H}\}$ in $\mathcal{C}^\alpha([0,1]; \mathbf{R}^m)$. This

* Partially supported by a grant of the DGICYT n° PB 90–0452. This work was partially done while the author was visiting the "Laboratoire de Probabilités" at Paris VI.

characterization has been shown by Ben Arous, Gradinaru and Ledoux ([2], [3]) using the approximative continuity property - first introduced by Stroock and Varadhan [8] in the case of the norm of uniform convergence - and by Aida, Kusuoka and Stroock [1] by means of a sequence of non absolutely continuous transformations of Ω.

In the setting of stochastic differential equations driven by general semimartingales and for the uniform topology, Mackevičius [6], Gyöngy and Pröhle [4] have introduced families of probabilities $P^\delta << P$ connected with mollifiers, to conclude that the support $P\circ X^{-1}$ contains $\mathcal{S}$. Their approach relies on general convergence results of semimartingales under P^δ.

The idea of the method presented here consists in reducing both inclusions of the support to approximations of the diffusion using adapted linear interpolations ω^n of ω. Thus we check that $\| S(\omega^n) - X \|_\alpha$ and $\| X(\omega - \omega^n + h) - S(h) \|_\alpha$ converge to zero in L^2. Since the law of the transformation T_n of Ω defined by $T_n(\omega) = \omega - \omega^n + h$ is absolutely continuous with respect to P (by Girsanov's theorem), the second convergence yields that the support of $P\circ X^{-1}$ contains $\mathcal{S}$, while the first one provides the converse inclusion in the usual way.

The method is used in [7] for stochastic hyperbolic partial differential equations. In order to stress the method and avoid technical arguments, we suppose that the coefficients σ and b are bounded.

We will use the following notational convention: sums on repeated indices are omitted, and constants appearing in the proof are denoted by C, even though they may change from one line to the next one.

2. Preliminaries

In this section we state criteria of convergence in Hölder norms and a general theorem characterizing the support of the law of a Wiener functional. The following theorem is a consequence of the Garsia–Rodemich–Rumsey lemma (see e.g. [9], p. 60).

Proposition 2.1. (i) Let $(Y_n(t),\ t\in[0,1])$ be a sequence of $\mathbf{R}^m$–valued processes such that

(A1) For every $p\in[1,\infty)$ there exists C such that

$$\sup_n E\Big(|\,Y_n(t)-Y_n(s)\,|^{2p}\Big) \le C\,|\,t-s\,|^p$$

for every $s,t\in[0,1]$.

Then, for any $\lambda>0$ and $\beta<\frac{p-1}{2p}$, there exists $C>0$ such that

$$\sup_n P\left(\sup_{s\neq t}\frac{|\,Y_n(t)-Y_n(s)\,|}{|\,t-s\,|^\beta}>\lambda\right)\le C\,\lambda^{-2p} \tag{2.1}$$

(ii) Let $(Y_n(t),\ t\in[0,1])$ be a sequence of $\mathbf{R}^m$–valued processes such that (A1) and the following assumption (A2) is satisfied:
(A2) For any $\varepsilon>0$,

$$\lim_{n\to\infty} P\left(\sup_{0\le i\le 2^n} |\,Y_n(i\,2^{-n})\,|>\varepsilon\right)=0\ .$$

Then, for any $\alpha\in\left[0,\frac{1}{2}\right[$ one has that

$$\lim_n P\Big(\|\,Y_n\,\|_\alpha>\varepsilon\Big)=0$$

Sketch of proof

Part (i) is a simple consequence of the Garsia–Rodemich–Rumsey lemma
(ii) Approximating Y_t^n by $Y_{\underset{\sim}{t}_n}^n$, where $\underset{\sim}{t}_n$ denotes the greatest dyadic point less than t, and using (2.1) we obtain that

$$P\Big(\sup_t |\,Y_n(t)\,|>\varepsilon\Big)\ \le\ P\Big(\sup_i |\,Y_n(i2^{-n})\,|>\frac{\varepsilon}{2}\Big)+C\,\varepsilon^{-2p}\cdot 2^{-2p(n\,\beta-1)}\ .$$

Let $\alpha\in\left[0,\frac{1}{2}\right[$, $p>1$, $\delta>0$ be such that $\beta=\alpha+\delta<\frac{p-1}{2p}$. Then, considering the cases $|\,s-t\,|\le 2^{-m_0}$ and $|\,s-t\,|>2^{-m_0}$, we obtain that for every n,

$$\begin{aligned}P\left(\sup_{t\neq s}\frac{|\,Y_n(t)-Y_n(s)\,|}{|\,t-s\,|^\alpha}>\varepsilon\right)&\le P\left(\sup_{|\,t-s\,|>2^{-m_0}}\frac{|\,Y_n(t)-Y_n(s)\,|}{|\,t-s\,|^\alpha}>\varepsilon\right)\\&\quad+\ P\left(\sup_{t\neq s}\frac{|\,Y_n(t)-Y_n(s)\,|}{|\,t-s\,|^\beta}>\varepsilon\,2^{m_0\delta}\right)\\&\le 2\,P\big(\sup_t\ |\,Y_n(t)\,|>\varepsilon\,2^{-m_0-1}\big)+C\ \varepsilon^{-2p}\ 2^{-2m_0\delta\,p}\end{aligned}$$

Thus choosing m_0 large enough we conclude that

$$\lim_n P\left(\sup_{s\neq t}\frac{|\,Y_n(t)-Y_n(s)\,|}{|\,t-s\,|^\alpha}>\varepsilon\right)=0\ . \qquad \diamond$$

We now state sufficient conditions for inclusions on the support of the law of a measurable map $F:\Omega\longrightarrow B$, where $(B,\|\ \|)$ is a Banach space; the proof is straightforward.

Proposition 2.2. Let $F:\Omega\longrightarrow B$ be measurable

(i) Let $\zeta_1 : \mathcal{H} \longrightarrow E$ be a measurable map, and let $H_n : \Omega \longrightarrow \mathcal{H}$ be a sequence of random variables such that for any $\varepsilon > 0$,

$$\lim_n P\Big(\| F(\omega) - \zeta_1(H_n(\omega)) \| > \varepsilon \Big) = 0 . \tag{2.2}$$

Then

$$\text{support } \left(P \circ F^{-1} \right) \subset \overline{\zeta_1(H)} \tag{2.3}$$

(ii) Let $\zeta_2 : \mathcal{H} \longrightarrow E$ be a map, and for fixed h let $T_n^h : \Omega \longrightarrow \Omega$ be a sequence of measurable transformations such that $P \circ \left(T_n^h\right)^{-1} << P$, and for any $\varepsilon > 0$,

$$\limsup_n P\Big(\| F\left(T_n^h(\omega)\right) - \zeta_2(h) \| < \varepsilon \Big) > 0 . \tag{2.4}$$

Then $\text{support}(P \circ F^{-1}) \supset \overline{\zeta_2(\mathcal{H})}$. (2.5)

Given a positive integer n, let D_n denote the set of n–dyadic points, $D_n = \{ i\, 2^{-n};\ 0 \le i \le 2^m \}$. For $t \in [0,1]$, $\frac{k}{2^n} \le t < \frac{k+1}{2^n}$, set

$$\underset{\sim}{t}_n = \frac{k}{2^n}, \quad \underline{t}_n = \frac{k-1}{2^n} \vee 0 , \tag{2.6}$$

and let W^n be the adapted linear interpolation of ω defined by

$$W_t^n = W_{\underline{t}_n} + 2^n (t - \underset{\sim}{t}_n) \left[W_{\underset{\sim}{t}_n} - W_{\underline{t}_n} \right] . \tag{2.7}$$

We consider the map $\zeta_1 = \zeta_2 = S(\cdot)$, $H_n(\omega) = \omega^n$, and $T_n^h(\omega) = \omega - \omega^n + h$. Then Girsanov's theorem implies that $P \circ \left(T_n^h\right)^{-1}$ is absolutely continuous with respect to P.

Thus, by Proposition 2.2 the equality $\text{supp } P \circ X^{-1} = \mathcal{S}$ will follow from the following convergence results for every $\varepsilon > 0$:

$$\lim_n P\Big(\| X(\omega) - S(\omega^n) \|_\alpha > \varepsilon \Big) = 0 \tag{2.8}$$

$$\lim_n P\Big(\| X(\omega - \omega^n + h) - S(h) \|_\alpha > \varepsilon \Big) = 0 . \tag{2.9}$$

Approximations of stochastic integrals by Riemann sums imply that $X^n(\omega) := X(\omega - \omega^n + h)$ is solution of the stochastic differential equation

$$\begin{aligned} X_t^n = x &+ \int_0^t \sigma(X_s^n)\, dW_s - \int_0^t \sigma(X_s^n)\, \dot{\omega}_s^n\, ds + \int_0^t \sigma(X_s^n)\, \dot{h}_s\, ds \\ &+ \int_0^t b(X_s^n)\, ds , \end{aligned} \tag{2.10}$$

while $S(\omega^n)$ satisfies

$$S(\omega^n)_t = x + \int_0^t \sigma\big(S(\omega^n)_s\big)\ \dot\omega_s^n\ ds + \int_0^t \Big[b - \frac{1}{2}\,(\nabla\,\sigma)\,\sigma\Big]\ \big(S(\omega^n)\big)_s\ ds\ . \tag{2.11}$$

Thus both processes $(X_\cdot^n)$ and $\big(S(\omega^n)_\cdot\big)$ are particular cases of a diffusion $(Y_\cdot^n)$ solution of the stochastic differential equation

$$Y_t^n = x \int_0^t F(Y_s^n)\ dW_s + \int_0^t G(Y_s^n)\,\dot\omega_s^n\ ds + \int_0^t H(Y_s^n)\,\dot h_s\,ds + \int_0^t B(Y_s^n)\ ds\ , \tag{2.12}$$

where the coefficients $F,\ G,\ H$ and B satisfy the condition:

(C) $F,\, G,\, H\ \ \mathbf{R}^m \longrightarrow \mathbf{R}^m \otimes \mathbf{R}^d$ are globally Lipschitz functions, G is of class $\mathcal{C}^2$ with bounded partial derivatives, $B : \mathbf{R}^m \longrightarrow \mathbf{R}^m$ is globally Lipschitz.

Given the coefficients $F,\, G,\, H$ and B, let (Z_s) be solution of the stochastic differential equation

$$\begin{aligned} Z_t = x + &\int_0^t \big[F(Z_s) + G(Z_s)\big]\ dW_s + \int_0^t H(Z_s)\ \dot h_s\ ds + \int_0^t B(Z_s)\ ds \\ + &\int_0^t \nabla\,G(Z_s)\ \Big[F(Z_s) + \frac{1}{2}\ G(Z_s)\Big]\ ds\ . \end{aligned} \tag{2.13}$$

Fix $\alpha \in \Big[0, \frac{1}{2}\Big[$; then conditions (2.8) and (2.9) are particular cases of the following convergences for every $\varepsilon > 0$:

$$\lim_n\ P\Big(\|\,Y^n - Z\,\|_\alpha > \varepsilon\Big) = 0\ . \tag{2.14}$$

Indeed, setting $F = 0$, $G = \sigma$, $H = 0$ and $B = b - \frac{1}{2}\,(\nabla\sigma)\,\sigma$ we obtain (2.8), while $F = \sigma$, $G = -\sigma$, $H = \sigma$ and $B = b$ yields (2.9).

It is well–known that for $s, t \in [0,1],\ p \in [1,\infty)$,

$$E\left(|\,Z_t - Z_s\,|^{2p}\right) \le C\,|\,t - s\,|^p\ .$$

Thus, by Proposition 2.1, it suffices to check that for any $s,\ t \in [0,1]$, $p \in [1,\infty)$,

$$\sup_n\ E\Big(|\,Y_t^n - Y_s^n\,|^{2p}\Big) \le C\ |\,t - s\,|^p \tag{2.15}$$

and

$$\lim_n\ E\Big(\sup_{0\le i\le 2^n} |\,Y_{i\,2^{-n}}^n - Z_{i\,2^{-n}}\,|^2\Big) = 0\ . \tag{2.16}$$

3. Characterization of the support

In this section we at first check that the moment estimates (2.15) are true; the boundedness of F, G, H and B simplifies the argument. Once (2.15) is checked the proof of (2.16) does not use this boundedness any more.

Proposition 3.1. Let F, G, H and B be bounded coefficients satisfying condition (C), and let (Y_s^n) be solution of (2.12). Then given $p \in [1, \infty)$, there exists a constant C such that for every $s, t \in [0,1]$,

$$\sup_n E\Big(| Y_t^n - Y_s^n |^{2p} \Big) \le C | t - s |^p .$$

Proof: Fix $p \in [1 + \infty)$, $s, t \in \mathbf{R}$. Then for every $n \ge 1$

$$E\left(|Y_t^n - Y_s^n |^{2p} \right) \le C \left(T_1 + T_2 + T_3 + T_4 \right) ,$$

with

$$T_1 = E\Big(| \int_s^t F(Y_u^n)\, dW_u \, |^{2p} \Big) ,$$
$$T_2 = E\Big(| \int_s^t G(Y_u^n)\, \dot\omega_u^n \, du \, |^{2p} \Big) ,$$
$$T_3 = E\Big(| \int_s^t H(Y_u^n)\, \dot h_u^n \, du \, |^{2p} \Big) ,$$
$$T_4 = E\Big(| \int_s^t B(Y_u^n)\, du \, |^{2p} \Big) .$$

Burkholder's inequality together with Schwarz's and Hölder's inequalities imply that

$$T_1 + T_3 + T_4 \le C \, | t - s |^p .$$

Finally, $T_2 \le T_{2,1}(n) + T_{2,2}(n)$, where

$$T_{2,1}(n) = E\Big(\Big| \int_s^t G(Y_{\underline{u}_n}^n)\, \dot\omega_u^n \, du \Big|^{2p} \Big) ,$$
$$T_{2,2}(n) = E\Big(\Big| \int_s^t | G(Y_u^n) - G(Y_{\underline{u}_n}^n) | \, | \dot\omega_u^n | \, du \Big|^{2p} \Big) .$$

Clearly,

$$\sup_n T_{2,1}(n) \le C \, | t - s |^p .$$

Let $a>1,\ b>1,$ be conjugate exponents; Hölder's inequality yields

$$T_{2,2}(n) \le |\,t-s\,|^{2p-1} \int_s^t \Big\{ E\Big(|\,G(Y^n_u) - G(Y^n_{\underline{u}_n})\,|^{2pa} \Big\}^{\frac{1}{a}} \ \Big\{ E\big(|\,\dot{\omega}^n_u\,|^{2pb} \big) \Big\}^{\frac{1}{b}}\, du$$
$$\le |\,t-s|^{2p-1}\ 2^{np} \int_s^t \Big\{ E\left(|Y^n_u - Y^n_{\underline{u}_n}\,|^{2pa} \right) \Big\}^{\frac{1}{a}}\ du\ .$$

Thus the proof of (2.15) is reduced to checking that this estimate holds in the particular case $s=\underline{u}_n$ and $t=u$. The arguments above imply that

$$\sup_s\ E\left(\Big| \int_{\underline{s}_n}^s \Big\{ F(Y^n_u)\ dW_u + H(Y^n_u)\ \dot{h}_u\, du + B(Y^n_u)\ du \Big\} \Big|^{2p} \right) \le C 2^{-np}\ .$$

Therefore, we should check that for every $p\in[1,\infty)$,

$$\sup_s\ E\Big(\Big| \int_{\underline{s}_n}^s G(Y^n_u)\,\dot{\omega}^n_u\, du\, \Big|^{2p} \Big) \le C\, 2^{-np}\ . \tag{3.1}$$

Clearly,

$$E\Big(\Big| \int_{\underline{s}_n}^s G(Y^n_u)\,\dot{\omega}^n_u\ du\,\Big|^{2p} \Big) \le C\, E\left(\left(2^n \int_{\underline{s}_n}^{\underset{\sim}{s}_n} |G(Y^n_u)\,|\, du \right)^{2p} |\,W_{\underline{s}_n} - W_{\underline{s}_n - 2^{-n}\vee 0}\,|^{2p} \right)$$
$$+\ C\, E\left(\left(2^n \int_{\underset{\sim}{s}_n}^s |\,G(Y^n_u)\,|\, du \right)^{2p} |\,W_{\underset{\sim}{s}_n} - W_{\underline{s}_n}\,|^{2p} \right)$$
$$\le\ C \left[E\Big(|\,W_{\underline{s}_n} - W_{\underline{s}_n - 2^{-n}\vee 0}|^{2p} \Big) + E\Big(|\,W_{\underset{\sim}{s}_n} - W_{\underline{s}_n}\,|^{2p} \Big) \right]$$
$$\le\ C\, 2^{-np}\ .$$

Hence (3.1) holds, and this implies $\sup_n T_{2,2}(n) \le C\,|\,t-s\,|^p$. The proof of (2.15) is complete.

◇

Before proving (2.16), we check the following technical results.

Lemma 3.2. Suppose that $(Y^n_\cdot)$ is a sequence of processes such that (2.15) holds. Let f be a globally Lipschitz function; then

$$\lim_n\ E\left(\sup_{1\le k\le 2^n} \Big| \int_0^{k2^{-n}} f(Y^n_{\underline{s}_n})\,\{\dot{\omega}^n_s\, ds - dW_s\} \Big|^2 \right) = 0\ . \tag{3.2}$$

Proof: For fixed n,

$$E\left(\sup_{1\le k\le 2^n}\Big|\int_0^{k2^{-n}} f(Y^n_{\underline{s}_n})\,\{\dot\omega^n_s\,ds-dW_s\}\Big|^2\right)$$

$$=E\left(\sup_{1\le k\le 2^n}\Big|\sum_{i=1}^{k-1} f\big(Y^n_{(i-1)2^{-n}}\big)\,\big[W_{i2^{-n}}-W_{(i-1)2^{-n}}\big]\right.$$

$$\left.-\sum_{i=1}^{k-1} f\big(Y^n_{(i-1)2^{-n}}\big)\,\big[W_{(i+1)2^{-n}}-W_{i2^{-n}}\big]\Big|^2\right)$$

$$=E\left(\sup_{1\le k\le 2^n}\Big|\sum_{i=1}^{k-1}\big[W_{(i+1)2^{-n}}-W_{i2^{-n}}\big]\,\big[f\big(Y^n_{i2^{-n}}\big)-f(Y^n_{(i-1)2^{-n}})\big]\right.$$

$$\left.+f(x)\,W_{2^{-n}}-f\big(Y^n_{(k-2)2^{-n}}\big)\,\big[W_{k2^{-n}}-W_{(k-1)2^{-n}}\big]\Big|^2\right)$$

$$\le C\,\left(T^n_1+T^n_2+T^n_3\right)\,,$$

where

$$T^n_1=E\left(\sup_{1\le k\le 2^n}\Big|\int_0^{k2^{-n}}\big[f(Y^n_{\tilde{s}_n})-f(Y^n_{\underline{s}_n})\big]\,dW_s\Big|^2\right)\,,$$

$$T^n_2=f(x)\;E\big(W^2_{2^{-n}}\big)\,,$$

$$T^n_3=E\left(\sup_{1\le k\le 2^n} f^2\big(Y^n_{(k-2)2^{-n}}\big)\,\big[W_{k2^{-n}}-W_{(k-1)2^{-n}}\big]^2\right)\,.$$

Proposition 3.1 implies that $T^n_1\le C\,2^{-n}$ clearly $T^n_2\le C\,2^{-n}$. For any $t\in[0,1]$, set

$$M^n_t=\int_0^t f(Y_{\underline{s}_n})\,dW_s\,.$$

Proposition 3.1 implies that $\sup_n\,\sup_t\,E\big(|\,Y^n_t|)^{2p}\big)<\infty$ for any $p\in[1,\infty)$. Hence by Burkholder's inequality, for any $p\in[1,\infty)$, there exists C such that for every $0\le s<t\le 1$,

$$\sup_n\,E\big(|\,M^n_t-M^n_s\,|^{2p}\big)\le C\,|\,t-s\,|^p\,.$$

Hence by Proposition 2.1, letting $p=3$, $\beta=\frac{1}{4}<\frac{1}{3}$, we have that

$$\sup_n\;P\Big(\sup_{0<i\le 2^n}|\,M^n_{i2^{-n}}-M^n_{(i-1)2^{-n}}\,|>\lambda\Big)$$

$$\le\sup_n\,P\left(\sup_{s\ne t}\frac{|\,M^n_t-M^n_s\,|}{|\,t-s\,|^\beta}\ge\lambda\,2^{-n}\right)\le C\,\lambda^{-6}\,2^{-\frac{3n}{2}}\,.$$

Hence

$$E\Big(\sup_{0<i\le 2^n} |\,M^n_{i2^{-n}} - M^n_{(i-1)2^{-n}}\,|^2\Big) \le \frac{1}{n^2} + 2\int_{1/n}^{\infty} \lambda\, P\Big(\sup_i |\,M^n_{i\,2^{-n}} - M^n_{(i-1)2^{-n}}\,|\lambda\Big)\, d\lambda$$

$$\le n^{-2} + \frac{1}{2}\, n^4\, 2^{-\frac{3n}{2}} \;.$$

Therefore $\lim_n T_3^n = 0$, which completes the proof of (3.2). ◇

Lemma 3.3. Let $(J^n_t;\, t\in[0,1])$ be a sequence of measurable processes such that there exists $p\in]1,\infty)$, $C>0$ and a sequence $\alpha(n)$ such that

$$\lim_n \alpha(n) = 0\;, \quad \text{and} \quad \sup_t E\big(|J^n_t|^{2p}\big) \le \alpha(n)\, 2^{-np}\;. \tag{3.3}$$

Then

$$\lim_n E\left(\sup_{1\le k\le 2^n} \Big|\int_0^{k2^{-n}} |\,J^n_s\, \dot\omega^n_s\,|\, ds\Big|^2\right) = 0\;. \tag{3.4}$$

Proof: Let $p>1$ and $q>1$ be conjugate exponents. By Hölder's inequality

$$E\left(\sup_{1\le k\le 2^n} \Big|\int_0^{k2^{-n}} |\,J^n_s\, \dot\omega^n_s\,|\; ds\Big|^2\right) \le \Big\{E\int_0^1 |\,J^n_s\,|^{2p}\; ds\Big\}^{\frac1p}\;\Big\{E\int_0^1 |\,\dot\omega^n_s\,|^{2q}\; ds\Big\}^{\frac1q}$$

$$\le\; C\big(\alpha(n)\,2^{-np}\big)^{\frac1p}\;2^n = C\,\alpha(n)^{\frac1p}\;;$$

this clearly yields (3.4). ◇

The following proposition proves the validity of (2.16).

Proposition 3.4. Assume that F, G, H and B satisfy (C) and that the solution $(Y^n_\cdot)$ of (2.12) satisfies (2.15). Let $(Z_\cdot)$ be the solution of (2.13). Then

$$\lim_n E\Big(\sup_{0\le i\le 2^n} |\,Y^n_{i2^{-n}} - Z_{i2^{-n}}|^2\Big) = 0\;.$$

Proof: Let $n\ge 1$, $t = k\,2^{-n}$; then

$$Y^n_t - Z_t = \int_0^t \big[(F+G)\,(Y^n_{\underline{s}_n}) - (F+G)\,(Z_{\underline{s}_n})\big]\; dW_s + \int_0^t \big[H(Y^n_{\underline{s}_n}) - H(Z_{\underline{s}_n})\big]\,\dot h_s\, ds$$

$$+ \int_0^t \left\{\Big[B + (\nabla G)\,F + \frac12\,(\nabla G)\,G\Big]\,(Y^n_{\underline{s}_n}) - \Big[B + (\nabla G)\,F + \frac12\,(\nabla G)\,G\Big]\;(Z_{\underline{s}_n})\right\}\, ds$$

$$+ \sum_{\alpha=1}^{5} A_\alpha(t)\;,$$

where

$$A_1^n(t) = \int_0^t \left[F(Y_s^n) - F(Y_{\underline{s}_n}^n) - (F+G)(Z_s) + (F+G)(Z_{\underline{s}_n})\right] dW_s \ ,$$

$$A_2^n(t) = \int_0^t \left[H(Y_s^n) - H(Y_{\underline{s}_n}^n) - H(Z_s) + H(Z_{\underline{s}_n})\right] \dot{h}_s \, ds \ ,$$

$$A_3^n(t) = \int_0^t \Big[B(Y_s^n) - B(Y_{\underline{s}_n}^n) - \big[B + (\nabla G)\,F + \frac{1}{2}(\nabla G)\,G\big] \ (Z_s)$$

$$+ \big[B + (\nabla G)\,F + \frac{1}{2}(\nabla G)\,G\big]\,(Z_{\underline{s}_n})\Big] \ ds \ ,$$

$$A_4^n(t) = \int_0^t G(Y_{\underline{s}_n}^n) \ \{\dot{\omega}_s^n \ ds - dW_s\} \ ,$$

$$A_5^n(t) = \int_0^t \left[G(Y_s^n) - G(Y_{\underline{s}_n}^n)\right] \ \dot{\omega}_s^n \ ds - \int_0^t \Big[(\nabla G)\,F + \frac{1}{2}(\nabla G)\,G\Big] \ (Y_{\underline{s}_n}^n) \ ds \ .$$

Gronwall's lemma applied to the function $\varphi(t) = E\Big(\sup_{i2^{-n}\le t} \big|Y_{i2^{-n}}^n - Z_{i2^{-n}}\big|^2\Big)$ implies that

$$E\Big(\sup_{0\le i\le 2^n} \big|Y_{i2^{-n}}^n - Z_{i2^{-n}}|^2\Big) \le C \sum_{\alpha=1}^{5} \ E\Big(\sup_{0\le i\le 2^n} |A_\alpha^n(i2^{-n})|^2\Big) \ .$$

Burkholder's inequality and Proposition 3.1 imply that $E\Big(\sup_t |A_1^n(t)|^2\Big) \le C\,2^{-n}$, since (2.13) is a particular case of (2.12).

Schwarz's inequality yields $E\Big(\sup_t \ |A_2^n(t)|^2\Big) \le C\|\,h\,\|_{\mathcal{H}}^2 \ 2^{-n}$. Since B, $(\nabla G)\,F$ and $(\nabla G)\,G$ are Lipschitz, Proposition 3.1 implies $E\Big(\sup_t |A_3^n(t)|^2\Big) < \infty$. Lemma 3.2 yields $\lim_n E\Big(\sup_{0\le k\le 2^n} |A_4^n(k2^{-n})|^2\Big) = 0$. Therefore the proof of (2.16) reduces to check that

$$\lim_n \ E\Bigg(\sup_{0\le k\le 2^n} \bigg| \int_0^{k2^{-n}} \left[G(Y_s^n) - G(Y_{\underline{s}_n}^n)\right] \ \dot{\omega}_s^n \ ds$$

$$- \int_0^{k2^{-n}} \Big[(\nabla G)\,F + \frac{1}{2}(\nabla G)\,G\Big] \ (Y_{\underline{s}_n}^n) \ ds\bigg|^2\Bigg) = 0 \ . \qquad (3.5)$$

Taylor's formula implies that

$$\Big| G(Y_s^n) - G(Y_{\underline{s}_n}^n) - (\nabla G)\,(Y_{\underline{s}_n}^n)\,\big[Y_s^n - Y_{\underline{s}_n}^n\big]\Big| \le C \ |Y_s^n - Y_{\underline{s}_n}^n|^2 \ .$$

Set

$$\phi_n(s) = \int_{\underline{s}_n}^{s} \Big\{\big[F(Y_u^n) - F(Y_{\underline{s}_n}^n)\big] \ dW_u + \big[G(Y_u^n) - G(Y_{\underline{s}_n}^n)\big] \ \dot{\omega}_u^n \ du$$

$$+ H(Y_u^n)\,\dot{h}_u \, du + B(Y_u^n) \ du\Big\} \ ;$$

then

$$E\Big(\sup_{1\le k\le 2^n} |\,A_5^n(k2^{-n}\,|^2\Big) \le C\,\sum_{\alpha=1}^{6}\,T_\alpha^n\ ,$$

where

$$\begin{aligned}
T_1^n &= E\Big(\sup_{1\le k\le 2^n}\Big|\int_0^{k2^{-n}} |\,Y_s^n - Y_{\underline{s}_n}^n|^2\ |\dot\omega_s^n|\ ds\Big|^2\Big)\ ,\\
T_2^n &= E\Big(\sup_{1\le k\le 2^n}\Big|\int_0^{k2^{-n}} |\,\nabla G(Y_s^n)\,\phi_n(s)|\ |\dot\omega_s^n|\ ds\Big|^2\Big)\ ,\\
T_3^n &= E\Bigg(\sup_{1\le k\le 2^n}\Big|\int_0^{k2^{-n}} (\nabla G)\,F(Y_{\underline{s}_n}^n)\Big(\int_{\underline{s}_n}^{\tilde{s}_n} dW_u\Big)\,\dot\omega_s^n\ ds\\
&\quad -\int_0^{k2^{-n}} (\nabla G)\,F(Y_{\underline{s}_n}^n)\,ds\Big|^2\Bigg)\ ,\\
T_4^n &= E\Bigg(\sup_{1\le k\le 2^n}\Big|\int_0^{k2^{-n}} (\nabla G)\,F(Y_{\underline{s}_n}^n)\Big(\int_{\tilde{s}_n}^{s} dW_u\Big)\,\dot\omega_s^n\ ds\Big|^2\Bigg)\ ,\\
T_5^n &= E\Bigg(\sup_{1\le k\le 2^n}\Big|\int_0^{k2^{-n}} (\nabla G)\,G(Y_{\underline{s}_n}^n)\Big(\int_{\underline{s}_n}^{\tilde{s}_n} \dot\omega_u^n\ du\Big)\ \dot\omega_s^n\ ds\Big|^2\Bigg)\ ,\\
T_6^n &= E\Bigg(\sup_{1\le k\le 2^n}\Big|\int_0^{k2^{-n}} (\nabla G)\,G(Y_{\underline{s}_n}^n)\Big(\int_{\tilde{s}_n}^{s} \dot\omega_u^n\ du\Big)\ \dot\omega_s^n\ ds\\
&\quad -\frac{1}{2}\int_0^{k2^{-n}} (\nabla G)\ G(Y_{\underline{s}_n}^n)\ ds\Big|^2\Bigg)\ .
\end{aligned}$$

Proposition 3.1 and Lemma 3.3 imply $\lim_n\ T_1^n = 0$. Set $J_s^n = \nabla G(Y_s^n)\ \phi_n(s)$; then Proposition 3.1 and Hölder's inequality implies that if $a > 1,\ b > 1$ are conjugate exponents for any $p \in [1,\infty)$ and $s \in [0,1]$,

$$\begin{aligned}
E(|J_s^n|^{2p}) &\le \Big\{E\big(|\nabla G(Y_s^n)|^{2pa}\big)\Big\}^{\frac{1}{a}}\Big\{E\big(|\phi_n(s)|^{2pb}\big)\Big\}^{\frac{1}{b}}\\
&\le C\Big\{E\big(|\phi_n(s)|^{2pb}\big)\Big\}^{\frac{1}{b}}\ .
\end{aligned}$$

Therefore, in order to apply Lemma 3.3, it suffices to check that $\sup_s\ E\big(|\phi_n(s)|^{2p}\big) \le \alpha(n)2^{-np}$, with $\lim_n\ \alpha(n) = 0$. Burkholder's and Hölder's inequalities and Proposition 3.1 yield

$$E|\phi_n(s)|^{2p} \le C\,E\Bigg[\Big|\int_{\underline{s}_n}^{s} |Y_u^n - Y_{\underline{s}_n}^n|^2\ du\Big|^p +$$

$$
\begin{aligned}
&+ 2^n \Big(\int_{\underline{s}_n}^{\underset{\sim}{s}_n} |Y_u^n - Y^n_{\underline{s}_n}|^{2p} \; du \Big) \, |W_{\underline{s}_n} - W_{(\underline{s}_n - 2^{-n}) \vee 0}|^{2p} \\
&+ 2^n \Big(\int_{\underset{\sim}{s}_n}^{s} |Y_u^n - Y^n_{\underline{s}_n}|^{2p} \; du \Big) \, |W_{\underset{\sim}{s}_n} - W_{\underline{s}_n}|^{2p} \\
&+ \Big(\sup \big\{ \big(\int_I |\dot{h}_u|^2 du) \big\} \big)^p ; \lambda(I) \leq 2^{1-n} \big\} \, 2^{-n(p-1)} \\
&+ 2^{-n(2p-1)} \Big) \int_{\underline{s}_n}^{s} \big(1 + |Y_u^n|^{2p}\big) du \Big] \\
&\leq C \; 2^{-np} \alpha(n) \; ,
\end{aligned}
$$

where $\alpha(n) = 2^{-np} + \sup\{\big(\int_I |\dot{h}_u|^2 du\big)^p ; \lambda(I) \leq 2^{1-n}\}$, which tends to zero as n tend to ∞. Thus Lemma 3.3 implies that $\lim_n T_2^n = 0$.

Since $Y^n_{(i-1)2^{-n}}$ and $\big[W_{(i+1)2^{-n}} - W_{i2^{-n}}\big]$ are independent,

$$
\begin{aligned}
T_3^n &= E\Bigg(\sup_{1 \leq k \leq 2^n} \Big| \sum_{i=0}^{(k-2)\vee 0} (\nabla G) \, F(Y^n_{(i-1)2^{-n} \vee 0}) \; \big[\big(W_{(i+1)2^{-n}} - W_{i2^{-n}} \big)^2 - 2^{-n} \big] \Big|^2 \Bigg) \\
&\leq \sum_{i=0}^{2^n - 2} E\Big((\nabla G) \, F(Y^n_{(i-1)2^{-n} \vee 0})^2 \Big) \; E\Big[|\big(W_{(i+1)2^{-n}} - W_{i2^{-n}} \big)^2 - 2^{-n}|^2 \Big] \\
&\leq C \; 2^n \; 2^{-2n} \; ,
\end{aligned}
$$

so that $\lim_n \; T_3^n = 0$. A similar computation yields

$$
\begin{aligned}
T_6^n = E\Bigg(&\sup_{2 \leq k \leq 2^n} \Big| \sum_{i=0}^{k-2} (\nabla G) \, G(Y^n_{i2^{-n}}) \\
&\Big\{ \Big(2^{2n} \int_{(i+1)2^{-n}}^{(i+2)2^{-n}} \int_{(i+1)2^{-n}}^{s} du \; ds \Big) \; \big[W_{(i+1)2^{-n}} - W_{i2^{-n}} \big]^2 - \frac{1}{2} \; 2^{-n} \Big\} \Big|^2 \Bigg) \\
\leq C \; 2^n \; 2^{-2n} &\underset{n \to \infty}{\longrightarrow} \; 0 \; .
\end{aligned}
$$

Finally, by Doob's inequality and Proposition 3.1,

$$
\begin{aligned}
T_4^n &\leq E\Bigg(\Big| \int_0^1 2^n \, (\underset{\sim}{s}_n + 2^{-n} - s) \, (\nabla G) \, F(Y^n_{\underset{\sim}{s}_n}) \, \big[W_{\underset{\sim}{s}_n} - W_{\underline{s}_n} \big] \; dW_s \Big|^2 \Bigg) \\
&\leq C \int_0^1 E\Big(|(\nabla G) \, F(Y^n_{\underset{\sim}{s}_n})|^2 \Big) \; E\Big(|W_{\underset{\sim}{s}_n} - W_{\underline{s}_n}|^2 \Big) \; ds \\
&\leq C \, 2^{-n} \; ;
\end{aligned}
$$

and for conjugate exponents $a>1$ and $b>1$,

$$\begin{aligned} T_5^n &\le E\Big(\Big|\int_0^1 (\nabla G)\,G(Y^n_{\underline{s}_n})\Big(\int_{\underline{s}_n}^{\underset{\sim}{s}_n} \dot\omega_u^n\,du\Big)\,dW_s\Big|^2\Big)\\ &\le C\int_0^1 E\Big((\nabla G)\,G(Y^n_{\underline{s}_n})^{2a}\Big)^{\frac1a}\,E\Big(\big|W_{\underline{s}_n}-W_{(\underline{s}_n-2^{-n})\vee 0}\big|^{2b}\Big)^{\frac1b}\,ds\\ &\le C\,2^{-n}. \end{aligned}$$

This completes the proof of $\lim_n E\Big(\sup_{1\le k\le 2^n}|\,A_5^n(k2^{-n})\,|^2\Big)=0$, and hence that of the proposition. ◇

Proposition 3.1 and 3.4 prove that (2.14) holds. Therefore Proposition 2.2 gives the following characterization of the support of the law of the diffusion X.

Theorem 3.5. Let σ and b be functions such that condition (H) is satisfied, and let X be the diffusion solution of (1.1). Then, for any $\alpha\in[0,\frac12[$ the support of the probability $P\circ X^{-1}$ in $\mathcal{C}^\alpha([0,1],\mathbf{R}^m)$ is the closure $\mathcal{S}$ of the set $\{S(h),\,h\in\mathcal{H}\}$, where $S(h)$ is given by (1.3).

REFERENCES

[1] S. Aida, S. Kusuoka and D. Stroock, On the Support of Wiener Functionals, Preprint, 1991.

[2] G. Ben Arous and M. Gradinaru, Normes Hölderiennes et support des diffusions, C.R. Acad. Sc. Paris, t. 316, Ser 1 n. 3, 283-286, 1993.

[3] G. Ben Arous, M. Gradinaru and M. Ledoux, Hölder norms and the support theorem for diffusions, preprint.

[4] I. Gyöngy and T. Pröhle, On the approximation of stochastic differential equations and on Stroock–Varadhan's support theorem, Computers Math. Applic, 19, 65–70 (1990).

[5] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, Amsterdam, Oxford, New York: North Holland; Tokyo: Kodansha 1981.

[6] V. Mackevičius, On the Support of the Solution of Stochastic Differential Equations, Lietuvos Matematikow Rinkings XXXVI (1), 91–98 (1986).

[7] A. Millet and M. Sanz–Solé, The Support of an Hyperbolic Stochastic Partial Differential Equation, Prépublication du Laboratoire de Probabilités de l'Université Paris VI n.° 150, 1993.

[8] D. W. Stroock and S. R. S. Varadhan, On the Support of Diffusion Processes with Applications to the Strong Maximum Principle, Proc. Sixth Berkeley Symp. Math. Statist. Prob. III, 333–359, Univ. California Press, Berkeley, 1972.

[9] D. W. Stroock and S. R. S. Varadhan, Multidimensional processes, Springer–Verlag, Berlin Heildelberg, New York, 1979.

A. Millet
Université Paris X
and
Laboratoire de Probabilités
URA 224, Université Paris VI
4, place Jussieu
75252 PARIS Cedex 05
FRANCE

M. Sanz–Solé
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via 585
08007 BARCELONA
SPAIN