APPROXIMATION AND SUPPORT THEOREM IN HÖLDER NORM FOR PARABOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

by

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Abstract: The solution $u(t,x)$ of a parabolic stochastic partial differential equation is a random element of the space $\mathcal{C}_{\alpha,\beta}$ of Hölder continuous functions on $[0,T]\times[0,1]$ of order $\alpha=\frac{1}{4}-\varepsilon$ in the time variable and $\beta=\frac{1}{2}-\varepsilon$ in the space variable, for any $\varepsilon>0$. We prove a support theorem in $\mathcal{C}_{\alpha,\beta}$ for the law of u . The proof is based on an approximation procedure in Hölder norm (which should have its own interest) using a space-time polygonal interpolation for the Brownian sheet driving the SPDE, and a sequence of absolutely continuous transformations of the Wiener space.

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0. INTRODUCTION

Consider the stochastic partial differential equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + g\big(u(t,x)\big)\,\dot{W}_{t,x} + f\big(u(t,x)\big)\,, \tag{0.1}$$

$t \in (0\,,\infty)\,,\; x \in (0,1)\,,$ with boundary conditions

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0\,, \tag{0.2}$$

and initial condition $u\,(0,x) = u_0\,(x)\,.$

Here $\big\{\dot{W}_{t,x}\,,\;(t,x) \in [0\,,\infty) \times [0,1]\big\}$ is the space-time white noise, $f, g : \mathbf{R} \longrightarrow \mathbf{R}$ are bounded and Lipschitz and u_0 is some real–valued function defined on $[0,1]$.

The equation (0.1) is formal and a rigorous meaning of this equation is given by means of the evolution equation

$$\begin{aligned} u(t,x) = G_t(x,u_0) &+ \int_0^t \int_0^1 G_{t-s}\,(x,y)\; g\big(u(s,y)\big)\; W(dy\,,\,ds) \\ &+ \int_0^t \int_0^1 G_{t-s}\,(x,y)\; f\big(u(s,y)\big)\, dy\, ds\,, \end{aligned} \tag{0.3}$$

where $G_t(x,y)$ is the fundamental solution of the heat equation with Neumann boundary conditions (0.2) and

$$G_t(x,u_0) = \int_0^1 G_t(x,y)\; u_0(y)\, dy\,.$$

Basic results concerning the existence and uniqueness of solutions for this kind of equations are given in [11]. In particular if the function u_0 is α–Hölder continuous for some $\alpha \in (0,\,\frac{1}{2})$ then the solution u is also Hölder continuous in both variables: α–Hölder continuous in x and $\frac{\alpha}{2}$ –Hölder continuous in t.

The aim of this paper is to give a characterization of the support of the law of u as a probability on the space of Hölder–continuous functions.

Fix $T > 0$ and let $\mathcal{H}$ be the Cameron–Martin space associated with the Brownian sheet $W = \big\{W_{t,x}\,,\;(t,x) \in [0,\,T] \times [0,1]\big\}$, that means, the space of functions $h : [0,\,T] \times [0,1] \longrightarrow \mathbf{R}$ which are absolutely continuous and whose derivative $\dot{h}$ belongs to $L^2\big([0,\,T] \times [0,1]\big)$. For any $h \in \mathcal{H}$ let $S\,(h)$ be the solution of the deterministic evolution equation

$$\begin{aligned} S\,(h)\;(t,x) = G_t(x,\,u_0) &+ \int_0^t \int_0^1 G_{t-s}\,(x,y)\;\big[g\big(S\,(h)\;(s,y)\big)\;\dot{h}\,(s,y) \\ &+ f\big(S\,(h)\;(s,y)\big)\big]\, dy\, ds\,. \end{aligned} \tag{0.4}$$

We prove in Theorem 2.1 that the support of $P \circ u^{-1}$ is the closure in the Hölder topology of the set $\mathcal{S}_{\mathcal{H}} := \left\{ S(h),\ h \in \mathcal{H} \right\}$.

Notice that the stochastic integral in equation (0.3) is an Itô stochastic integral, and not a Stratonovich one (as in the classical theorem of Stroock and Varadhan for diffusion processes). Actually, the Stratonovich integral does not make sense in (0.3) because of an "infinite trace" phenomenon. This is not surprising. Indeed, it is well-known that in (0.3) the space and time variable do not play the same role; actually, we are dealing with an infinite-dimensional process, since $u(t,.)$ is $L^2([0,1])$-valued. This is one of the specific difficulties of this framework.

In the proof of such a characterization we have combined some ideas of [10], [6] and [7] (see also [1], [2] and [3] for related approaches of the support theorem). More precisely, the inclusion support $(P \circ u^{-1}) \subset \overline{\mathcal{S}_{\mathcal{H}}}$ is stated using some adapted approximations of the Brownian sheet W; on the other hand the converse inclusion uses some sequence of absolutely continuous transformations of the canonical probability space $(\Omega, \mathcal{F}, P)$, associated with W. Notice that the sequence of densities of these transformations need not be controlled. In Proposition 2.2 we give an abstract formulation of these ideas. Both inclusions can be deduced from a result on approximation of evolution equations more general than (0.3); this constitutes the core of the work.

The paper is divided in two parts. Section 1 is devoted to establish the main result on approximation. We introduce a sequence of grids, with mesh n^{-1} in the space variable and a^{-n}, $a \in (1,\infty)$, in the time variable; then we associate a sequence of adapted approximations of W by elements of $\mathcal{H}$, say W_n. The general convergence result proved in Theorem 1.13 precises in particular what is the explosive drift perturbation introduced when we replace W by W_n in (0.3). The property $a^{-n} \ll n^{-1}$ is widely used to handle the lack of orthogonality of the stochastic integrals in the space direction.

In Section 2 we prove the support theorem. A suitable choice of the coefficients in the approximation theorem yields both inclusions. Finally, technical results concerning applications of the Garsia-Rodemich-Rumsey lemma in the setting of Hölder norms and the Green function are proved in the Appendices A and B respectively.

All the results of this paper also hold if, instead of the Neumann boundary conditions (0.2), we consider Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0.$$

All positive constants appearing in this article are called C. They may change from one line to the next one.

1. APPROXIMATION IN HÖLDER NORM

Fix $T \in (0,\ \infty)$ and restrict the parameter set to $[0,\ T] \times [0,1]$. For the sake of simplicity we will take $T = 1$. Let $\mathcal{C}^{\alpha}\left([0,1]^2\right)$, $\alpha \in (0, \frac{1}{4})$, denote the set of continuous functions

$\varphi : [0,1]^2 \longrightarrow R$ such that

$$\| \varphi \|_\alpha := \sup_{\substack{0\le t\le 1\\ 0\le x\le 1}} \big| \varphi(t,x) \big| + \sup_{\substack{0\le t,\,\bar t\le 1\\ 0\le x,\,\bar x\le 1\\ (t,x)\neq(\bar t,\bar x)}} \frac{\big| \varphi(t,x) - \varphi(\bar t\,,\,\bar x\,) \big|}{\big(|t-\bar t\,| + |x-\bar x|^2 \big)^\alpha} \tag{1.1}$$

is finite.

Let us consider a sequence of partitions of the parameter set $[0,1]^2$ defined by $\big\{ (k\,a^{-n},\ j\,n^{-1})\,,\ 0 \le k \le a^n - 1\,,\ 0 \le j \le n-1 \big\}\,,\ n \ge 1\,,$ with $a \in (1,\infty)$ and denote by $\Delta_{k,j}$ the rectangle $\big(k\,a^{-n},\ (k+1)\,a^{-n}\big] \times \big(j\,n^{-1}\,,\ (j+1)\,n^{-1}\big]$. Given a point $(t,x) \in [0,1]^2$, set

$$\underline{t}_n = \sup_{0\le k\le a^n-1} \big\{ k\,a^{-n}\,,\ k\,a^{-n} \le t \big\}\ ,$$
$$t_n = \big(\underline{t}_n - a^{-n} \big) \vee 0\ ,$$

and

$$I_n(x) = \big(j\,n^{-1}\,,\ (j+1)\,n^{-1} \big]\ , \tag{1.2}$$

if $x \in \big(j\,n^{-1},\ (j+1)\,n^{-1} \big]$.

We can now define the approximations of the process W that will be used in this paper. For each $n \ge 1$ let W_n be the element of the Cameron-Martin space $\mathcal{H}$ given by

$$\dot W_n\,(t,x) = \begin{cases} n\ a^n\ W\,(\Delta_{k-1,j}) & \text{if} \quad (t,x) \in \Delta_{k,j}\ , \quad 1 \le k \le a^n - 1\,, \\ & \qquad\qquad\qquad\qquad 0 \le j \le n-1\,, \\ 0 & \text{if} \quad (t,x) \in \Delta_{0,j}\ , \quad 0 \le j \le n-1\,. \end{cases} \tag{1.3}$$

Notice that the process $\big\{ \dot W_n(t,x)\,,\ (t,x) \in [0,1]^2 \big\}$ is adapted to the filtration $\big\{ \mathcal{F}_t\,,\ t \in [0,1] \big\}$ generated by the random variables $W(A)$, for any Borel set $A \subset [0,t] \times [0,1]$. Moreover, these approximations satisfy the following moment estimates:

For any $p \in [2,\infty)$,

$$\|\dot W_n(s,y)\|_p \le C n^{\frac12} a^{\frac n2}. \tag{1.4}$$

In all this section, statements concerning the validity of inequalities involving the integer n and a real number $p \ge 1$ are to be understood for $n \ge n_p$, where the integer n_p depends on p, may change from one statement to the next one, and is never specified.

Let $F, H, K, f : \mathbf{R} \longrightarrow \mathbf{R}$ be bounded, Lipschitz functions, and suppose also that H is a $\mathcal{C}^3$ class function with bounded derivatives. We consider the processes $\{ X_n(t,x), (t,x) \in [0,1]^2\}$ and $\{X(t,x), (t,x) \in [0,1]^2\}$ $n \ge 1$, given by

$$X_n(t,x) = G_t(x,u_0) + \int_0^t \int_0^1 G_{t-s}(x,y)\ F\big(X_n(s,y)\big)\ W(dy\,,\,ds)$$

$$+\int_0^t\int_0^1 G_{t-s}(x,y)\,H\big(X_n(s,y)\big)\,W_n(dy,ds)+\int_0^t\int_0^1 G_{t-s}(x,y)\Big\{K\left(X_n(s,y)\right)\,\dot h(s,y)$$
$$+\,f\big(X_n(s,y)\big)-(F\dot H)\left(X_n(s,y)\right)\,b_n(s,y)-(H\,\dot H)\,\left(X_n(s,y)\right)\,c_n(s,y)\Big\}\,dy\,ds\,, \qquad (1.5)$$

where $h\in\mathcal{H}$ and

$$b_n(s,y)=n\;a^n\int_{s_n}^{\underline{s}\,n}\int_{I_n(y)}G_{s-r}(y,z)\;dz\;dr\;, \qquad (1.6)$$

$$c_n(s,y)=n\;a^n\int_{\underline{s}\,n}^{s}\int_{I_n(y)}G_{s-r}(y,z)\;dz\;dr\;, \qquad (1.7)$$

and

$$X(t,x)=G_t(x,u_0)+\int_0^t\int_0^1 G_{t-s}(x,y)\;[F+H]\;\big(X(s,y)\big)\;W(dy\,,\,ds)$$
$$+\int_0^t\int_0^1 G_{t-s}(x,y)\,\big\{K\big(X(s,y)\big)\;\dot h(s,y)+f\,\big(X(s,y)\big)\big\}\,dy\,ds\;. \qquad (1.8)$$

We denote by $\mathcal{H}_b$ the subset of $\mathcal{H}$ consisting of those functions with bounded derivatives. Our aim is to prove the convergence of $\{X_n\,,\ n\ge 1\}$ to X in the norm $\|\cdot\|_\alpha$ defined in (1.1), with $\alpha\in(0\,,\frac14)$ and $h\in\mathcal{H}_b$; this is done in Theorem 1.13 which is the main result of this section. The motivation for this convergence has been to give a unified proof for both inclusions of the support of $P\circ u^{-1}$. This will be made explicit in the next section. Notice also that if $F=K=0$ and $H=g$, the result provides an approximation of the solution of (0.3) by means of a sequence $\{u_n,n\ge 1\}$ defined by

$$u_n(t,x)=G_t(x,u_0)+\int_0^t\int_0^1 G_{t-s}(x,y)g\big(u_n(s,y)\big)W_n(dy,ds)$$
$$+\int_0^t\int_0^1 G_{t-s}(x,y)\big[f\big(u_n(s,y)\big)-\big(g\dot g\big)\big(u_n(s,y)\big)c_n(s,y)\big]dyds.$$

So, the term involving the coefficient c_n corresponds to the *explosive* correction between the Itô and the Stratonovich formulation of the stochastic integral with respect to W.

We start with some preliminary lemmas.

Let $X_n^-(t,x)=G_{t-t_n}\big(x\,,\ X_n(t_n,\,\cdot\,)\big)$. The semi-group property of G implies that

$$X_n(t,x)-X_n^-(t,x)=\int_{t_n}^t\int_0^1 G_{t-s}(x,y)\;F\big(X_n(s,y)\big)\;W(dy\,,\,ds)$$
$$+\int_{t_n}^t\int_0^1 G_{t-s}(x,y)\;H\big(X_n(s,y)\big)\;W_n(dy\,,\,ds)$$
$$+\int_{t_n}^t\int_0^1 G_{t-s}(x,y)\;K_n(s,y)\;dy\,ds\;, \qquad (1.9)$$

with

$$K_n(s,y) = K\left(X_n(s,y)\right)\,\dot h(s,y) + f\left(X_n(s,y)\right) - (F\dot H)(\left(X_n(s,y)\right)\,b_n(s,y)$$
$$-\,(H\,\dot H)\,\left(X_n(s,y)\right)\,c_n(s,y)\ .$$

Notice that

$$\sup_{s,y}\ |\,K_n(s,y)\,| \le C\,n\ . \tag{1.10}$$

Lemma 1.1. For all $p \in (2,\infty)$,

$$\sup_{t,x}\ \|\,X_n(t,x) - X_n^-(t,x)\,\|_p \le C\ a^{-\frac{n}{4}}\ . \tag{1.11}$$

Proof. We have, for all t, x ,

$$E\left(\,|X_n(t,x) - X_n^-(t,x)\,|^p\,\right) \le C\,(T_1 + T_2 + T_3)\ ,$$

with

$$T_1 = E\left(\Big|\int_{t_n}^t\int_0^1 G_{t-s}(x,y)\ F\left(X_n(s,y)\right)\ W(dy\,,\,ds)\Big|^p\right)\ ,$$

$$T_2 = E\left(\Big|\int_{t_n}^t\int_0^1 G_{t-s}(x,y)\ H\left(X_n(s,y)\right)\ W_n(dy\,,\,ds)\Big|^p\right)\ ,$$

and

$$T_3 = E\left(\Big|\int_{t_n}^t\int_0^1 G_{t-s}(x,y)\ K_n(s,y)dy\,ds\Big|^p\right)\ .$$

Since F is bounded, Burkholder's inequality yields (see (B.6))

$$T_1 \le C\Big(\int_{t_n}^t\int_0^1\ G^2_{t-s}(x,y)\ dy\,ds\Big)^{p/2} \le C\ a^{-\frac{np}{4}}\ . \tag{1.12}$$

Lemma B.3 ensures that

$$T_2 \le C\ a^{-n(\frac12-\frac{1}{2p})\,p}\ n^{p/2}\ . \tag{1.13}$$

Finally, by (1.10) and (B.3) we obtain

$$T_3 \le C\ n^p\ a^{-np}\ . \tag{1.14}$$

The estimates given in (1.12), (1.13) and (1.14) imply (1.11). ◇

For $k \ge 1$ set

$$\lambda_n^{(k)}(t,x) = \int_{ka^{-n}\wedge t}^{(k+1)a^{-n}\wedge t}\int_0^1 G_{t-s}(x,y)\ \left[H\big(X_n(s,y)\big) - H\big(X_n^-(s,y)\big)\right]\ W_n(dy\,,\,ds)\ .$$

In order to simplify the notation we will write in the sequel ka^{-n} and $(k+1)\,a^{-n}$ instead of $ka^{-n} \wedge t$ and $(k+1)\,a^{-n} \wedge t$, respectively.

Lemma 1.2. For any $p \in [1\,,\,\infty)$ it holds that

$$\sup_{t,x} \Big\| \sum_{k=0}^{a^n-1} \big(\lambda_n^{(k)}(t,x)\big)^2 \Big\|_p \le C\; n^3\; a^{-\frac{n}{2}}\ . \tag{1.15}$$

Proof: By Hölder's inequality we obtain

$$E\left[\Big(\sum_{k=0}^{a^n-1} \big(\lambda_n^{(k)}(t,x)\big)^2 \Big)^p \right] \le a^{n(p-1)} \sum_{k=0}^{a^n-1} E\left[\big(\lambda_n^{(k)}(t,x)\big)^{2p} \right]\ .$$

Moreover, for any $k \ge 0$

$$\begin{aligned} E\left[\big(\lambda_n^{(k)}(t,x)\big)^{2p} \right] &\le C\, n^{2p-1} \sum_{j=0}^{n-1} n^p\; a^{np} \Big\{ E \Big| \int_{k\,a^{-n}}^{(k+1)\,a^{-n}} \int_{j\,n^{-1}}^{(j+1)\,n^{-1}} \\ &\qquad G_{t-s}(x,y)\; |X_n(s,y) - X_n^-(s,y)\,|\; dy\, ds \Big|^{4p} \Big\}^{1/2} \\ &\le C\; n^{3p-1}\; a^{np} \sum_{j=0}^{n-1} a^{-n\;\frac{4p-1}{2}} \\ &\quad \Big\{ \int_{k\,a^{-n}}^{(k+1)\,a^{-n}} \int_{j\,n^{-1}}^{(j+1)\,n^{-1}} G_{t-s}(x,y)\; E\big|X_n(s,y) - X_n^-(s,y)\big|^{4p}\; dy\, ds \Big\}^{1/2} \\ &\le C\, n^{3p}\; a^{-\frac{3np}{2}}\ , \end{aligned}$$

where, in the last two inequalities we have used, first a Hölder inequality with respect to the measure $\mu(dy\,ds) = G_{t-s}(x,y)\,dy\,ds$, and then Lemma 1.1. Consequently (1.15) holds. •

In order to deal with Hölder norms we need a new result in the spirit of Lemma 1.2, but involving increments of the Green function. The following lemma is crucial in the proof of Proposition 1.5, which is used to get rid of the explosive drift correction coefficients appearing in (1.5). We first introduce some notation. For $s, t, \bar{t}, x, \bar{x} \in [0,1]$, set

$$\Gamma\,(t,\,\bar{t},\,x,\,\bar{x};\,s,y) = G_{t-s}(x,y)\; 1_{[0,t]}(s) - G_{\bar{t}-s}(\bar{x},y)\; 1_{[0,\bar{t}]}(s)\ , \tag{1.16}$$

and

$$\begin{aligned} \lambda_n^{(k)}(t,\,\bar{t},\,x,\,\bar{x}) &= \lambda_n^{(k)}(t,x) - \lambda_n^{(k)}(\bar{t},\,\bar{x}) \\ &= \int_{k\,a^{-n}}^{(k+1)a^{-n}} \int_0^1 \Gamma(t,\,\bar{t},\,x,\,\bar{x};\,s,y) \\ &\qquad \times \big[\,H\,\big(X_n(s,y)\big) - H\,\big(X_n^-(s,y)\big)\big]\; W_n(dy\,,\,ds)\ . \end{aligned} \tag{1.17}$$

Lemma 1.3. For any $p \in (1,\infty)$ there exists C such that for every t, $\bar t$, x, $\overline{x}$ and $n \in \mathbf{N}$

$$\Big\| \sum_{k=0}^{a^n-1} \big(\lambda_n^{(k)}(t,\,\bar t,\,x,\,\overline{x})\big)^2 \Big\|_p \leq C\{|x-\overline{x}| + |t-\bar t|^{\frac12}\}\epsilon_n \;, \tag{1.18}$$

with $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Consequently

$$\sup_n \Big\| \sum_{k=0}^{a^n-1} \big(\lambda_n^{(k)}(t,\,\bar t,\,x,\,\overline{x})\big)^2 \Big\|_p \leq C\{\,|x-\overline{x}| + |t-\bar t|^{\frac12}\,\}. \tag{1.19}$$

Proof. Set

$$\lambda_n^{(k)}(t,\,x,\,\overline{x}) = \lambda_n^{(k)}(t,x) - \lambda_n^{(k)}(t,\overline{x}) = \int_{ka^{-n}\wedge t}^{(k+1)a^{-n}\wedge t} \int_0^1 \big[G_{t-s}(x,y) - G_{t-s}(\overline{x},y)\big]$$
$$\big[H\big(X_n(s,y)\big) - H\big(X_n^-(s,y)\big)\big]\; W_n(dy\,,\,ds)\;,$$

and let ℓ be the positive integer such that

$$\ell a^{-n} = \underline{t}_n \leq t < (\ell+1)a^{-n}.$$

We will first prove that for $p \in (1,\infty)$,

$$\sup_t \Big\| \sum_{k=0}^{a^n-1} \big(\lambda_n^{(k)}(t,\,x,\,\overline{x})\big)^2 \Big\|_p = \sup_t \Big\| \sum_{k=0}^{\ell} \big(\lambda_n^{(k)}(t,\,x,\,\overline{x})\big)^2 \Big\|_p \leq C\,|x-\overline{x}|\,\varepsilon_n^{(1)}, \tag{1.20}$$

with $\lim\limits_{n\to\infty} \varepsilon_n^{(1)} = 0$.

Fix $p \in (1,\infty)$ and let γ be such that $\dfrac{1}{2p} + \dfrac{1}{\gamma} = 1$. Then $\gamma \in (1,2)$ and

$$E\Big[\big(\lambda_n^{(k)}\,(t,\,x,\,\overline{x})\big)^{2p}\Big] \leq C\Big(\int_{ka^{-n}}^{(k+1)a^{-n}} \int_0^1 \big[G_{t-s}(x,y) - G_{t-s}(\overline{x}\,,\,y)\big]^\gamma\,dy\,ds\Big)^{\frac{2p}{\gamma}}$$
$$\times E\Big(\int_{ka^{-n}}^{(k+1)a^{-n}} \int_0^1 \big|X_n(s,y) - X_n^-(s,y)\big|^{2p}\,|\dot W_n(s,y)\,|^{2p}\,dy\,ds\Big)$$
$$\leq C\,n^p\;a^{-n+n\,\frac{p}{2}}\Big(\int_{ka^{-n}}^{(k+1)a^{-n}} \int_0^1 \big[G_{t-s}(x,y) - G_{t-s}(\overline{x}\,,\,y)\big]^\gamma\,dy\,ds\Big)^{\frac{2p}{\gamma}}\;. \tag{1.21}$$

Suppose that $\eta = \overline{x} - x > 0$; using statement (i) in Lemma B.2 , we majorize the last integral in the right hand side of (1.21) by $C\,[\,a^{-n}\,\eta^\gamma + \eta^{3-\gamma}\,I(k,\gamma)\,]$, where $I(k,\gamma)$ is defined in (B.15). Because of the explicit estimation of $I(k,\gamma)$ obtained in (B.12) we introduce the positive integer ℓ_0 defined by

$$\ell_0 = \inf\{k \geq 0 : t - (k+1)a^{-n} \leq \eta^2\} \wedge \ell.$$

The estimation (1.21) yields that

$$
\begin{aligned}
S &= \Big\| \sum_{k=0}^{\ell} \big(\lambda_n^{(k)}(t,\,x,\,\overline{x})\,\big)^2 \Big\|_p \\
&\le C\,n\,a^{-\frac{n}{p}+\frac{n}{2}-\frac{2n}{\gamma}}\,\eta^2\,a^n + C\,n\,a^{-\frac{n}{p}+\frac{n}{2}}\,\eta^{2(\frac{3}{\gamma}-1)}\,\sum_{k=0}^{\ell} I(k,\gamma)^{\frac{2}{\gamma}} \\
&\le C\,n\,a^{-\frac{n}{2}}\,\eta^2 + C\,n\,a^{-\frac{n}{p}+\frac{n}{2}}\,\eta^{2(\frac{3}{\gamma}-1)}\,\sum_{i=1}^{3} S_i,
\end{aligned}
\tag{1.22}
$$

where

$$
S_1 = \sum_{k=0}^{(\ell_0-1)\wedge(\ell-2)} I(k,\gamma)^{\frac{2}{\gamma}},
$$

$$
S_2 = \sum_{k=\ell_0+1}^{\ell} I(k,\gamma)^{\frac{2}{\gamma}},
$$

and

$$
S_3 = I(\ell_0,\gamma)^{\frac{2}{\gamma}} + 1_{\{\ell_0=\ell\}}\,I(\ell-1,\gamma)^{\frac{2}{\gamma}}\,,
$$

with the convention that $S_2 = 0$ if $\ell_0 + 1 > \ell$. Using (B.12) and Hölder's inequality, we obtain that

$$
\begin{aligned}
S_1 &= \sum_{k=0}^{(\ell_0-1)\wedge(\ell-2)} \eta^{-2(\frac{3}{2}-\gamma)\frac{2}{\gamma}}\left[(t-ka^{-n})^{\frac{3}{2}-\gamma} - (t-(k+1)a^{-n})^{\frac{3}{2}-\gamma}\right]^{\frac{2}{\gamma}} \\
&= C\,\eta^{4-\frac{6}{\gamma}} \sum_{k=0}^{(\ell_0-1)\wedge(\ell-2)} \left[\int_{t-(k+1)a^{-n}}^{t-ka^{-n}} u^{\frac{1}{2}-\gamma} du\right]^{\frac{2}{\gamma}} \\
&\le C\,\eta^{4-\frac{6}{\gamma}} \left(\int_{t-[\ell_0\wedge(\ell-1)]a^{-n}}^{t} u^{(\frac{1}{2}-\gamma)\frac{2}{\gamma}} du\right) a^{-n(\frac{2}{\gamma}-1)} \\
&\le C\,\eta^{4-\frac{6}{\gamma}}\,a^{-n(\frac{2}{\gamma}-1)}\left[\Big(t-(\ell_0\wedge(\ell-1))\,a^{-n}\Big)^{\frac{1}{\gamma}-1} - t^{\frac{1}{\gamma}-1}\right] \\
&\le C\,\eta^{2-\frac{4}{\gamma}}\,a^{-n(\frac{2}{\gamma}-1)},
\end{aligned}
\tag{1.23}
$$

where the last inequality uses the fact that $t-\ell_0 a^{-n} \ge \eta^2$.

Similarly,

$$
\begin{aligned}
S_2 &= \sum_{k=\ell_0+1}^{\ell} \eta^{-2(\frac{3-\gamma}{2})\frac{2}{\gamma}}\left[(t-ka^{-n})^{\frac{3-\gamma}{2}} - (t-(k+1)a^{-n})^{\frac{3-\gamma}{2}}\right]^{\frac{2}{\gamma}} \\
&= C\,\eta^{-2(\frac{3}{\gamma}-1)} \sum_{k=\ell_0+1}^{\ell} \left(\int_{t-(k+1)a^{-n}}^{t-ka^{-n}} u^{\frac{1-\gamma}{2}} du\right)^{\frac{2}{\gamma}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\,\eta^{-2(\frac{3}{\gamma}-1)}\Big(\int_0^{t-(\ell_0+1)a^{-n}} u^{\frac{1-\gamma}{2}\frac{2}{\gamma}}du\Big)\, a^{-n(\frac{2}{\gamma}-1)} \\
&\leq C\,\eta^{-2(\frac{3}{\gamma}-1)}\,\big(t-(\ell_0+1)a^{-n}\big)^{\frac{1}{\gamma}}\, a^{-n(\frac{2}{\gamma}-1)} \leq C\,\eta^{-2(\frac{3}{\gamma}-1)+\frac{2}{\gamma}}\, a^{-n(\frac{2}{\gamma}-1)}. \qquad (1.24)
\end{aligned}
$$

We finally estimate S_3. Suppose at first that $\ell_0 \leq \ell - 2$; then,

$$
I\,(\ell_0,\gamma) \leq C\ \Big[\Big(\frac{t-\ell_0\,a^{-n}}{\eta^2}\Big)^{\frac{3}{2}-\gamma} - \Big(\frac{t-(\ell_0+1)\,a^{-n}}{\eta^2}\Big)^{\frac{3-\gamma}{2}}\Big].
$$

Since $\frac{3}{2}-\gamma < \frac{3-\gamma}{2}$ and $\frac{t-\ell_0\ a^{-n}}{\eta^2} \geq 1$,

$$
\begin{aligned}
I\,(\ell_0,\gamma)^{\frac{2}{\gamma}} &\leq C\Big[\Big(\frac{t-\ell_0\ a^{-n}}{\eta^2}\Big)^{\frac{3-\gamma}{2}} - \Big(\frac{t-(\ell_0+1)\,a^{-n}}{\eta^2}\Big)^{\frac{3-\gamma}{2}}\Big]^{\frac{2}{\gamma}} \\
&\leq C\,\eta^{-2(\frac{3}{\gamma}-1)}\big(t-(\ell_0+1)a^{-n}\big)^{\frac{1}{\gamma}-1}a^{-n\frac{2}{\gamma}}.
\end{aligned}
$$

Since $\ell_0 \leq \ell-2$, we have that $\eta^2 \geq a^{-n}$ and $t-(\ell_0+1)\,a^{-n} \geq a^{-n}$; therefore,

$$
\begin{aligned}
I(\ell_0,\gamma)^{\frac{2}{\gamma}} &\leq C\,a^{-n(\frac{2}{\gamma}-\frac{1}{\gamma})}\,\eta^{\frac{2}{\gamma}}\,\eta^{-2(\frac{3}{\gamma}-1)}\,a^{-n(\frac{1}{\gamma}-1)} \\
&\leq C\,\eta^{2-\frac{4}{\gamma}}\,a^{-n(\frac{2}{\gamma}-1)}. \qquad (1.25)
\end{aligned}
$$

In order to deal with the cases $\ell_0 = \ell-1$ or $\ell_0 = \ell$, let

$$
\begin{aligned}
I &= \int_0^{\frac{t-t_n}{\eta^2}} \int_{\mathbf{R}} \Big|\,\frac{1}{\sqrt{2\pi r}}\ \exp\Big(-\frac{\xi^2}{4r}\Big) - \frac{1}{\sqrt{2\pi r}}\ \exp\Big(-\frac{(\xi+1)^2}{4r}\Big)\,\Big|\ d\xi\ dr \\
&\leq C\Big[\Big(\frac{t-t_n}{\eta^2}\ \wedge\ 1\Big)^{\frac{3-\gamma}{2}} + 1_{\{t-t_n>\eta^2\}}\ \Big(\big(\frac{t-t_n}{\eta^2}\big)^{\frac{3}{2}-\gamma}-1\Big)\Big].
\end{aligned}
$$

Then, since $\ell_0 = \ell-1$ or $\ell_0 = \ell$, $t-t_n > \eta^2$, so that $\eta^2 \leq 2a^{-n}$. Hence

$$
\begin{aligned}
S_3 &\leq C\Big[1+\big(\frac{t-t_n}{\eta^2}\big)^{\frac{3}{2}-\gamma}-1\Big]^{\frac{2}{\gamma}} \leq C\,\eta^{-\frac{6}{\gamma}+4}a^{-n(\frac{3}{\gamma}-2)} \\
&\leq C\,\eta^{-\frac{6}{\gamma}+2+\frac{2}{\gamma}}a^{-n(\frac{2}{\gamma}-1)}. \qquad (1.26)
\end{aligned}
$$

The inequalities (1.23) to (1.26) yield

$$
n\,a^{-\frac{n}{p}+\frac{n}{2}}\eta^{2(\frac{3}{\gamma}-1)}\sum_{i=1}^{3} S_i \leq Cna^{-\frac{n}{2}}\eta^{\frac{2}{\gamma}}. \qquad (1.27)
$$

Therefore, the inequalities (1.22) and (1.27) yield (1.20) for $p>1$ with

$$
\varepsilon_n^{(1)} = C\,n\,a^{-\frac{n}{2}}.
$$

For any $t \le \bar{t}$ and x, set

$$\lambda_n^{(k)}(t,\bar{t},x) = \lambda_n^{(k)}(\bar{t},x) - \lambda_n^{(k)}(t,x) = \mu_n^{(k)}(t,\bar{t},x) + \nu_n^{(k)}(t,\bar{t},x) \tag{1.28}$$

with

$$\mu_n^{(k)}(t,\bar{t},x) = \int_{ka^{-n}\wedge t}^{(k+1)a^{-n}\wedge t} \int_0^1 \left[G_{\bar{t}-s}(x,y) - G_{t-s}(x,y)\right]$$

$$\left[H\left(X_n(s,y)\right) - H(X_n^-(s,y)\right] W_n(dy\,ds)$$

and

$$\nu_n^{(k)}(t,\bar{t},x) = \int_{ka^{-n}\vee t}^{(k+1)a^{-n}\wedge \bar{t}} \int_0^1 G_{\bar{t}-s}(x,y)\left[H\left(X_n(s,y)\right) - H\left(X_n^-(s,y)\right)\right] W_n(dy\,ds)$$

with the convention $\nu_n^{(k)}(t,\bar{t},x) = 0$ if $k\,a^{-n}\vee t \ge (k+1)a^{-n}\wedge \bar{t}$.
We prove that for $p > 1$

$$\sup_x \Big\| \sum_{k=0}^{a^n-1} \left(\lambda_n^{(k)}\,(t,\bar{t},x)\right)^2 \Big\|_p \le C\,|\bar{t}-t|^{\frac{1}{2}}\,\varepsilon_n^{(2)} \tag{1.29}$$

with $\lim_n \varepsilon_n^{(2)} = 0$. We at first estimate $\nu_n^{(k)}\,(t,\bar{t},x)$. Let $p > 1$ and let $\gamma \in (1,2)$ be again such that $\dfrac{1}{2p} + \dfrac{1}{\gamma} = 1$. Hölder's inequality implies that for every k such that $k\,a^{-n}\vee t < (k+1)\,a^{-n}\wedge\bar{t}$,

$$\begin{aligned}
&\big\| \left(\nu_n^{(k)}\,(t,\bar{t},x)\right)^2 \big\|_p \le C\,\left[\left\{(k+1)\,a^{-n}\wedge\bar{t}\right\} - \left\{k\,a^{-n}\vee t\right\}\right]^{\frac{2p-1}{p}} \\
&\quad\Big[\int_{k\,a^{-n}\vee t}^{(k+1)a^{-n}\wedge\bar{t}} \int_0^1 G_{\bar{t}-s}(x,y)\left(E\left(|X_n(s,y) - X_n^-(s,y)|^{4p}\right)\right)^{\frac{1}{2}}\left(E\,|\dot{W}^n(s,y)\,|^{4p}\right)^{\frac{1}{2}} dy\,ds\Big]^{\frac{1}{p}} \\
&\le C\left[\left\{(k+1)\,a^{-n}\wedge\bar{t}\right\} - \left\{k\,a^{-n}\vee t\right\}\right]^2\,n\,a^{\frac{n}{2}} \\
&\le C\,n\,a^{-\frac{n}{2}}\,\left[\left\{(k+1)\,a^{-n}\wedge\bar{t}\right\} - \left\{k\,a^{-n}\vee t\right\}\right].
\end{aligned}$$

Therefore,

$$\Big\| \sum_{k=0}^{a^n-1} \left(\nu_n^{(k)}\,(t,\bar{t},x)\right)^2 \Big\|_p \le c\,n\,a^{-\frac{n}{2}}\,|\bar{t}-t|\,. \tag{1.30}$$

Thus, the proof of (1.29) reduces to checking

$$\sup_x \Big\| \sum_{k=0}^{a^n-1} \left(\mu_n^{(k)}\,(t,\bar{t},x)\right)^2 \Big\|_p \le C\,|\bar{t}-t|^{\frac{1}{2}}\varepsilon_n^{(2)}\;; \tag{1.31}$$

the arguments are similar to that of (1.20). Thus

$$E\Big(\big(\mu_n^{(k)}(t,\bar t,x)\big)^{2p}\Big)\le C\Big(\int_{k\,a^{-n}}^{(k+1)a^{-n}}\int_0^1\big|G_{\bar t-s}(x,y)-G_{t-s}(x,y)\big|^\gamma\,dy\,ds\Big)^{\frac{2p}{\gamma}}\,n^p\,a^{-n+n\frac{p}{2}}\,. \tag{1.32}$$

Let $h=\bar t-t>0$; then statement (ii) in Lemma B.2 implies that the integral in the right hand side of (1.32) is dominated by $C\,\big[\,a^{-n}\,|\bar t-t\,|+h^{\frac{3-\gamma}{2}}\,J(k,\gamma)\,\big]$, where $J(k,\gamma)$ is defined in (B.16). As previously we introduce the positive integer ℓ_1 defined by

$$\ell_1=\inf\{k\ge 0: t-(k+1)a^{-n}\le h\}\wedge\ell.$$

Hence, we have that

$$\begin{aligned}T=&\sum_{k=0}^{\ell}\;\big\|\,\big(\mu_n^{(k)}(t,\bar t,x)\big)^2\big\|_p\\ &\le C\,n\,a^{-\frac{n}{p}+\frac{n}{2}}\sum_{k=0}^{\ell}\big[a^{-n\frac{2}{\gamma}}\,h^{\frac{2}{\gamma}}+h^{\frac{3-\gamma}{\gamma}}\,J(k,\gamma)^{\frac{2}{\gamma}}\big]\\ &\le C\,n\,a^{-\frac{n}{2}}\,h^{\frac{2}{\gamma}}+C\,n\,a^{-\frac{n}{p}+\frac{n}{2}}\,h^{\frac{3}{\gamma}-1}\sum_{i=1}^{3}T_i,\end{aligned} \tag{1.33}$$

where

$$T_1=\sum_{k=0}^{(\ell_1-1)\wedge(\ell-2)}J(k,\gamma)^{\frac{2}{\gamma}},$$

$$T_2=\sum_{k=\ell_1+1}^{\ell}J(k,\gamma)^{\frac{2}{\gamma}}$$

and

$$T_3=J(\ell_1,\gamma)^{\frac{2}{\gamma}}+1_{\{\ell_1=\ell\}}\,J(\ell-1)^{\frac{2}{\gamma}}\,.$$

We assume that $T_2=0$ whenever $\ell_1+1>\ell$.

Using (B.14) and Hölder's inequality, we have

$$\begin{aligned}T_1\le C&\sum_{k=0}^{(\ell_1-1)\wedge(\ell-2)}h^{-\frac{3}{2}(1-\gamma)\frac{2}{\gamma}}\,\big[(t-ka^{-n})^{\frac{3}{2}(1-\gamma)}-(t-(k+1)a^{-n})^{\frac{3}{2}(1-\gamma)}\big]^{\frac{2}{\gamma}}\\ &\le C\,h^{3-\frac{3}{\gamma}}\sum_{k=0}^{(\ell_1-1)\wedge(\ell-2)}\Big[\int_{t-(k+1)a^{-n}}^{t-ka^{-n}}u^{\frac{1}{2}-\frac{3\gamma}{2}}du\Big]^{\frac{2}{\gamma}}\\ &\le C\,h^{3-\frac{3}{\gamma}}\Big(\int_{t-[\ell_1\wedge(\ell-1)]\,a^{-n}}^{t}u^{\frac{1}{\gamma}-3}du\Big)\,a^{-n(\frac{2}{\gamma}-1)}\\ &\le C\,h^{3-\frac{3}{\gamma}}\,a^{-n(\frac{2}{\gamma}-1)}\Big[\Big(t-[\ell_1\wedge(\ell-1)]a^{-n}\Big)^{\frac{1}{\gamma}-2}-t^{\frac{1}{\gamma}-2}\Big]\\ &\le C\,h^{1-\frac{2}{\gamma}}\,a^{-n(\frac{2}{\gamma}-1)}.\end{aligned} \tag{1.34}$$

Similarly,

$$
\begin{aligned}
T_2 &\le C \sum_{k=\ell_1+1}^{\ell} h^{-(\frac{3}{\gamma}-1)} \big[(t-ka^{-n})^{\frac{3-\gamma}{2}} - (t-(k+1)a^{-n})^{\frac{3-\gamma}{2}}\big]^{\frac{2}{\gamma}} \\
&\le C\, h^{-(\frac{3}{\gamma}-1)} \sum_{k=\ell_1+1}^{\ell} \Big(\int_{t-(k+1)a^{-n}}^{t-ka^{-n}} u^{\frac{1-\gamma}{2}}\, du\Big)^{\frac{2}{\gamma}} \\
&\le C\, h^{-(\frac{3}{\gamma}-1)} \Big(\int_0^{t-(\ell_1+1)a^{-n}} u^{\frac{1-\gamma}{2}\frac{2}{\gamma}} du\Big)\, a^{-n(\frac{2}{\gamma}-1)} \\
&\le C\, h^{-(\frac{3}{\gamma}-1)} \big(t-(\ell_1+1)a^{-n}\big)^{\frac{1}{\gamma}}\, a^{-n(\frac{2}{\gamma}-1)} \le C\, h^{-(\frac{3}{\gamma}-1)+\frac{1}{\gamma}} a^{-n(\frac{2}{\gamma}-1)} . \qquad (1.35)
\end{aligned}
$$

To estimate T_3, we at first suppose that $\ell_1 \le \ell - 2$; then the mean value theorem yields

$$
\begin{aligned}
J(\ell_1,\gamma) &\le C \Big[-\Big(\frac{t-\ell_1 a^{-n}}{h}\Big)^{\frac{3}{2}(1-\gamma)} + 1 + 1 - \Big(\frac{t-(\ell_1+1)\, a^{-n}}{h}\Big)^{\frac{3-\gamma}{2}} \Big] \\
&\le C \Big[\frac{a^{-n}}{h} + \frac{a^{-n}}{h} \Big(\frac{t-(\ell_1+1)\, a^{-n}}{h}\Big)^{\frac{1}{2}-\frac{\gamma}{2}} \Big].
\end{aligned}
$$

Since $\ell_1 \le \ell - 2$, we have that $h \ge a^{-n}$, and $t-(\ell_1+1)\, a^{-n} \ge a^{-n}$. Therefore, if $\ell_1 \le \ell - 2$,

$$
J(\ell_1,\gamma)^{\frac{2}{\gamma}} \le C\, h^{-\frac{2}{\gamma}}\, a^{\frac{-2n}{\gamma}} \le C\, a^{-n(\frac{2}{\gamma}-1)}\, h^{1-\frac{2}{\gamma}}. \qquad (1.36)
$$

Finally, consider the cases $\ell_1 = \ell - 1$ or $\ell_1 = \ell$, and let

$$
\begin{aligned}
J &= \int_0^{\frac{t-t_n}{h}} \int_{\mathbf{R}} \Big| \frac{1}{\sqrt{2\pi(v+1)}} \exp\Big(-\frac{z^2}{4(v+1)}\Big) - \frac{1}{\sqrt{2\pi v}} \exp\Big(-\frac{z^2}{4v}\Big)\Big|^{\gamma} dz\, dv \\
&\le C \Big[\Big(\frac{t-t_n}{h} \wedge 1\Big)^{\frac{3-\gamma}{2}} + 1_{\{t-t_n>h\}} \Big(1 - \big(\frac{t-t_n}{h}\big)^{\frac{3}{2}(1-\gamma)}\Big)\Big] .
\end{aligned}
$$

Since $t - t_n > h$ it holds that $h < 2a^{-n}$ and

$$
T_3 \le C\, h^{1-\frac{2}{\gamma}}\, a^{-n(\frac{2}{\gamma}-1)}. \qquad (1.37)
$$

The inequalities (1.34) to (1.37) imply that

$$
n\, a^{-\frac{n}{p}+\frac{n}{2}}\, h^{\frac{3}{\gamma}-1} \sum_{i=1}^{3} T_i \le C\, n\, a^{-\frac{n}{2}}\, h^{\frac{1}{\gamma}} \le C\, n\, a^{-\frac{n}{2}}\, h^{\frac{1}{2}} . \qquad (1.38)
$$

The inequalities (1.33) and (1.38) imply that (1.31) and (1.29) hold with $\varepsilon_n^{(2)} = n\, a^{-\frac{n}{2}}$; this concludes the proof of the lemma. ◇

Set

$$\phi_n(s,y) = E\left(H\left(X_n(s,y)\right)\dot{W}_n(s,y)\,/\mathcal{F}_{s_n}\right) - \left[\,(F\dot{H})\left(X_n(s,y)\right)b_n(s,y) \right.$$
$$\left. + (H\,\dot{H})\,\left(X_n(s,y)\right)\,c_n(s,y)\right]\,, \qquad (1.39)$$

where b_n and c_n are given in (1.6) and (1.7) respectively. Our aim is to prove that

$$\lim_{n\to\infty}\ \sup_{s,y}\ \|\,\phi_n(s,y)\,\|_p = 0\ . \qquad (1.40)$$

This will be done in three steps. We at first show an estimate for $\sup_{s,y}\|\,\phi_n(s,y)\,\|_p$ in Lemma 1.4. This enables us to prove L_p estimates of $X_n(t,x) - X_n(\overline{t},\overline{x})$ with constants depending on n, which will be used to complete the proof of (1.40) by improving the estimates of Lemma 1.4.

Lemma 1.4. For any $p\in[1,\,\infty)$ it holds that

$$\sup_{s,y}\ \|\,\phi_n(s,y)\,\|_p \le C\ n\ . \qquad (1.41)$$

Proof. We consider the Taylor expansion

$$H\left(X_n(s,y)\right) = H\left(X_n^-(s,y)\right) + \dot{H}\left(X_n^-(s,y)\right)\,\left(X_n(s,y) - X_n^-(s,y)\right) + R_n(s,y)\ ,$$

with $|\,R_n(s,y)\,| \le C\,|\,X_n(s,y) - X_n^-(s,y)\,|^2$. Then

$$\|\,\phi_n(s,y)\,\|_p \le \sum_{j=1}^{2}\ \|\,\varphi_n^j(s,y)\,\|_p\ ,$$

where

$$\varphi_n^1(s,y) = E\Big(\dot{H}\left(X_n^-(s,y)\right)\,\left(X_n(s,y) - X_n^-(s,y)\right)\dot{W}_n(s,y)\,/\mathcal{F}_{s_n}\Big)$$
$$-\left[\,(F\dot{H})\,\left(X_n(s,y)\right)b_n(s,y) + (H\dot{H})\,\left(X_n(s,y)\right)\,c_n(s,y)\right]\,,$$

$$\varphi_n^2(s,y) = E\left(R_n(s,y)\,\dot{W}_n(s,y)\,/\mathcal{F}_{s_n}\right)\ .$$

Set $I_n^j(s,y) = E\,|\varphi_n^j(s,y)\,|^p$, $j=1,2$. The identity (1.9) yields

$$I_n^1(s,y) \le C\ \sum_{j=1}^{3}\ I_n^{1,j}(s,y)\ ,$$

where

$$I_n^{1,1}(s,y) = E\Bigg[\Big|\, E\big(\dot{H}\big(X_n^-(s,y)\big)\Big(\int_{s_n}^{s}\int_0^1 \; G_{s-r}\,(y,z)\; F\big(X_n(r,z)\big)\; W(dz,dr)\Big)$$

$$\times\, \dot{W}_n(s,y)\,/\mathcal{F}_{s_n}\big) - (F\dot{H})\; \big(X_n(s,y)\big)\; b_n(s,y)\Big|^p\Bigg]\;,$$

$$I_n^{1,2}(s,y) = E\Bigg[\Big|\, E\big(\dot{H}\big(X_n^-(s,y)\big)\Big(\int_{s_n}^{s}\int_0^1 \; G_{s-r}\,(y,z)\; H\big(X_n(r,z)\big)\; W_n(dz,dr)\Big)$$

$$\times\, \dot{W}_n(s,y)\,/\mathcal{F}_{s_n}\big) - (H\dot{H})\; \big(X_n(s,y)\big)\; c_n(s,y)\Big|^p\Bigg]\;,$$

$$I_n^{1,3}(s,y) = E\Bigg[\,|\, E\Big(\dot{H}\big(X_n^-(s,y)\big)\Big(\int_{s_n}^{s}\int_0^1 \; G_{s-r}\,(y,z)\; K_n(y,z)\; dz\,dr\Big)$$

$$\times\, \dot{W}_n(s,y)\,/\mathcal{F}_{s_n}\Big)\Big|^p\Bigg]\;.$$

The estimate (1.10) yields

$$\begin{aligned} I_n^{1,3}(s,y) &\le Cn^p\Big(\int_{s_n}^{s}\int_0^1 G_{s-r}\,(y,z)\; dy\,dz\Big)^p\; n^{\frac{p}{2}}\; a^{\frac{np}{2}} \\ &\le C\; n^{\frac{3}{2}p}\; a^{-\frac{n}{2}p}\;. \end{aligned} \tag{1.42}$$

We will now deal with $I_n^{1,1}(s,y)$. For $I_n(y)$ defined in (1.2), first notice that

$$E\Big[\,\dot{H}\big(X_n^-(s,y)\big)\Big(\int_{\underline{s}_n}^{s}\;\int_0^1 G_{s-r}\,(y,z)\; F\big(X_n(r,z)\big)\; W(dz,dr)\Big)$$

$$\times\, \dot{W}_n(s,y)\,/\mathcal{F}_{s_n}\Big] = 0\;,$$

and

$$E\Big[\dot{H}\big(X_n^-(s,y)\big)\int_{s_n}^{\underline{s}_n}\;\int_{I_n^c(y)} G_{s-r}\,(y,z)\; F\big(X_n(r,z)\big)\; W(dz,dr)$$

$$\times\, \dot{W}_n(s,y)\,/\mathcal{F}_{s_n}\Big] = 0\;.$$

Hence

$$I_n^{1,1}(s,y) = E\Big[\,\big|E\big(\dot{H}\big(X_n^-(s,y)\big)\int_{s_n}^{\underline{s}_n}\;\int_{I_n(y)} G_{s-r}\,(y,z)\; F\big(X_n(r,z)\big)\; W(dz,dr)$$

$$\times \dot{W}_n(s,y)\,/\mathcal{F}_{s_n}\big) - (F\dot{H})\,\big(X_n(s,y)\big)\; b_n(s,y)\,\big|^p\,\Big]$$

$$= E\Big(\Big|\,n\; a^n \int_{s_n}^{\underline{s}_n} \int_{I_n(y)} G_{s-r}\,(y,z)\;\big[\dot{H}\,\big(X_n^-(s,y)\big)\; E\big(F\big(X_n(r,z)\big)\,/\mathcal{F}_{s_n}\,\big)$$

$$- (F\dot{H})\,\big(X_n(s,y)\big)\,\big]\,dz\,dr\,\Big|^p\Big)\;. \qquad (1.43)$$

We have

$$I_n^{1,1}(s,y) \le C\; E\Big(\Big|\,n\; a^n \int_{s_n}^{\underline{s}_n} \int_{I_n(y)} G_{s-r}\,(y,z)\;|F\big(X_n(r,z)\big) - F\big(X_n(s,y)\big)\,|$$

$$dz\,dr\,\Big|^p\Big) + C\; n^p\; a^{-n\frac{p}{4}}\;. \qquad (1.44)$$

Indeed, consider the following decomposition

$$I_n^{1,1}(s,y) \le C\Big(\,I_n^{1,1,1}(s,y) + I_n^{1,1,2}(s,y)\,\Big),$$

with

$$I_n^{1,1,1}(s,y) = E\Big[\,\Big|\,na^n \int_{s_n}^{\underline{s}_n} \int_{I_n(y)} G_{s-r}(y,z)\Big(\dot{H}\big(X_n^-(s,y)\big) - \dot{H}\big(X_n(s,y)\big)\Big)$$

$$F\big(X_n(s,y)\big)\,dzdr\,\Big|^p\,\Big],$$

and

$$I_n^{1,1,2}(s,y) = E\Big[\,\Big|\,na^n \int_{s_n}^{\underline{s}_n} \int_{I_n(y)} G_{s-r}(y,z)\,\dot{H}\big(X_n^-(s,y)\big)\Big\{\,E\big(F\big(X_n(r,z)\big)/\mathcal{F}_{s_n}\big)$$

$$- F\big(X_n(s,y)\big)\,\Big\}\,dzdr\,\Big|^p\,\Big].$$

Since $\dot{H}$ is Lipschitz, Lemma 1.1 shows that

$$I_n^{1,1,1}(s,y) \le C\,E\Big(\Big|n\; a^n\;|\,X_n(s,y)\; - X_n^-(s,y)\,|\int_{s_n}^{\underline{s}_n} \int_{I_n(y)} G_{s-r}\,(y,z)\,dz\,dr\,\Big|^p\Big)$$

$$\le C\; n^p\; a^{-\frac{np}{4}}.$$

Furthermore, by Hölder's inequality

$$I_n^{1,1,2}(s,y) \le E\Big(\Big|n\; a^n \int_{s_n}^{\underline{s}_n} \int_{I_n(y)} G_{s-r}(y,z)\,\Big[\,|\,F\big(X_n^-(r,z)\big) - F\big(X_n(r,z)\big)\,|$$

$$+ \big|E\big(F\big(X_n(r,z)\,\big) - F\big(X_n^-(r,z)\big)/\mathcal{F}_{s_n}\big)\big|$$

$$+ |F\big(X_n(r,z)\big) - F\big(X_n(s,y)\big)\,|\,\Big]\,dzdr\Big|^p\Big) \le$$

$$\le Cn^p\; a^{np}\; a^{-n(p-1)} \int_{s_n}^{\underline{s}_n} \int_{I_n(y)} G_{s-r}\,(y,z)\; E\,\big(|X_n(r,z) - X_n^-(r,z)\,|^p\big)\,dz\,dr$$

$$+ E\Big(\Big|na^n \int_{s_n}^{\underline{s}_n} \int_{I_n(y)} G_{s-r}(y,z)\,|F\big(X_n(r,z)\big) - F\big(X_n(s,y)\big)\Big|\,dzdr\,\Big|^p\Big)\,. \qquad (1.45)$$

Then, Lemma 1.1 and the boundedness of F ensure

$$I_n^{1,1,2}(s,y) \le C\big(n^p a^{-\frac{np}{4}} + n^p\big).$$

Hence

$$I_n^{1,1}(s,y) \le C\ n^p\ . \tag{1.46}$$

This estimation will be improved in the sequel.

Let us now consider $I_n^{1,2}(s,y)$. We have

$$E\Big(\dot H\left(X_n^-(s,y)\right)\,\dot W_n(s,y)\int_{s_n}^{s}\int_0^1 G_{s-r}\left(y,z\right)\,H\big(X_n(r,z)\big)\,W_n(dz\,dr)\,/\mathcal{F}_{s_n}\Big)=$$
$$E\Big(\dot H\left(X_n^-(s,y)\right)\,\dot W_n(s,y)\int_{s_n}^{s}\int_0^1 G_{s-r}\left(y,z\right)\,H\big(X_n^-(r,z)\big)\,W_n(dz\,dr)\,/\mathcal{F}_{s_n}\Big)+\Delta_n(s,y)\ ,$$

where using Lemma B.3 we obtain

$$E\left(|\Delta_n(s,y)|^p\right) \le C\ n^p\ a^{-n\frac{p-1}{4}}\ .$$

On the other hand,

$$E\Big(\dot H\left(X_n^-(s,y)\right)\,\dot W_n(s,y)\int_{s_n}^{\underline{s}_n}\int_0^1 G_{s-r}\left(y,z\right)\,H\big(X_n^-(r,z)\big)\,W_n(dz,dr)\,/\mathcal{F}_{s_n}\Big)=0\ ,$$

and

$$E\Big(\dot H\left(X_n^-(s,y)\right)\,\dot W_n(s,y)\int_{\underline{s}_n}^{s}\int_{I_n(y)^c} G_{s-r}\left(y,z\right)\,H\big(X_n^-(r,z)\big)\,W_n(dz,dr)\,/\mathcal{F}_{s_n}\Big)=0\ .$$

Consequently, arguments similar to those used to study $I_n^{1,1}(s,y)$ yield

$$\begin{aligned}
I_n^{1,2}(s,y) &\le C\ E\Big[\Big|\,E\Big(\dot H\left(X_n^-(s,y)\right)\,\dot W_n(s,y)\int_{\underline{s}_n}^{s}\int_{I_n(y)} G_{s-r}\left(y,z\right)\,H\big(X_n^-(r,z)\big)\\
&\qquad\qquad W_n(dz,dr)\quad/\mathcal{F}_{s_n}\Big)-(\dot H\,H)\,\big(X_n(s,y)\big)\,c_n(s,y)\Big|^p\Big]+C\ n^p\ a^{-n\frac{p-1}{4}}\\
&= C\ E\Big(\Big|n\ a^n\int_{\underline{s}_n}^{s}\int_{I_n(y)} G_{s-r}\left(y,z\right)\,\Big[\dot H\left(X_n^-(s,y)\right)\,H\big(X_n^-(r,z)\big)\\
&\qquad\qquad -(\dot H\,H)\,\big(X_n(s,y)\big)\Big]\,dz\,dr\Big|^p\Big)+C\ n^p\ a^{-n\frac{p-1}{4}}\\
&\le C\ E\Big(\Big|n\ a^n\int_{\underline{s}_n}^{s}\int_{I_n(y)} G_{s-r}\left(y,z\right)\,\big|H\left(X_n(r,z)\right)-H\left(X_n(s,y)\right)\big|\,dz\,dr\,\Big|^p\Big)\\
&\quad + C\ n^p\ a^{-n\frac{p-1}{4}}.
\end{aligned} \tag{1.47}$$

Since H is bounded,

$$I_n^{1,2}(s,y) \le C\, n^p\ , \tag{1.48}$$

which will also be improved later on. The inequalities (1.42), (1.46) and (1.48) show that

$$\sup_{s,y}\ I_n^1(s,y) \le C\, n^p\ . \tag{1.49}$$

Finally, Jensen's and Schwarz's inequalities together with Lemma 1.1 show that

$$I_n^2(s,y) \le C\ E\big(\,|\dot W_n(s,y)\,|^{2p}\big)^{\frac12}\ E\big(\,|X_n(s,y) - X_n^-(s,y)\,|^{4p}\big)^{\frac12} \le C\ n^{\frac{p}{2}}\ . \tag{1.50}$$

Hence, (1.49) and (1.50) imply that

$$\sup_{s,y}\ \|\,\phi_n(s,y)\,\|_p \le C\ n\ . \qquad \diamond$$

We now prove moment estimates of $X_n(t,x) - X_n(\bar t,\bar x)$ with constants depending on n.

Proposition 1.5. For any $p\in(2,\infty)$, we have

$$\|\,X_n(t,x) - X_n(\bar t,\bar x)\,\|_p \le C\ n\,\big(\,|t-\bar t\,|^{\frac14} + |x-\bar x\,|^{\frac12}\,\big)\ . \tag{1.51}$$

Proof. Fix $p\in(2,\infty)$; then

$$\|\,X_n(t,x) - X_n(\bar t\,,\,\bar x\,)\,\|_p \le \sum_{i=1}^4\ \|\,\psi_n^i(t,\bar t,x,\bar x)\,\|_p\ ,$$

where

$$\begin{aligned}
\psi_n^1(t,\bar t,x,\bar x) =& \int_0^1\!\!\int_0^1 \Gamma\,(t,\bar t,x,\bar x\,;\,s,y)\ \Big\{F\,\big(X_n(s,y)\,\big)\ W(dy\,ds)\\
&+ H\,\big(X_n^-(s,y)\,\big)\ W_n(dy\,ds)\Big\}\ ,\\
\psi_n^2(t,\bar t,x,\bar x) =& \int_0^1\!\!\int_0^1 \Gamma\,(t,\bar t,x,\bar x\,;\,s,y)\ \Big[H\,\big(X_n(s,y)\,\big) - H\big(X_n^-(s,y)\,\big)\Big]\ W_n(dy\,ds)\\
&- \sum_{k=0}^{a^n-1} E\Big(\int_{ka^{-n}}^{(k+1)a^{-n}}\int_0^1 \Gamma\,(t,\bar t,x,\bar x\,;\,s,y)\ \Big[H\big(X_n(s,y)\,\big) - H\big(X_n^-(s,y)\,\big)\Big]\\
&\quad W_n(dy\,ds)\,/\mathcal F_{(k-1)a^{-n}}\Big)\ ,\\
\psi_n^3(t,\bar t,x,\bar x) =& \int_0^1\!\!\int_0^1 \Gamma\,(t,\bar t,x,\bar x\,;\,s,y)\ \Big[K\,\big(X_n(s,y)\,\big)\ \dot h(s,y) + f\big(X_n(s,y)\,\big)\Big]\,dy\,ds
\end{aligned}$$

and

$$\begin{aligned}
\psi_n^4(t,\bar t,x,\bar x) =& \int_0^1\!\!\int_0^1 \Gamma\,(t,\bar t,x,\bar x\,;\,s,y)\ \Big[E\big(H\big(X_n(s,y)\,\big)\ \dot W_n(s,y)\,/\mathcal F_{s_n}\big)\\
&- \dot H\,\big(X_n(s,y)\,\big)\Big\{F\,\big(X_n(s,y)\,\big)\ b_n(s,y) + H\,\big(X_n(s,y)\,\big)\ c_n(s,y)\Big\}\Big]\,dy\,ds\ .
\end{aligned}$$

Since $\psi_n^1(t,\,\overline{t},x,\overline{x})$ is a stochastic integral, Burkholder's inequality and Lemma B.1 imply

$$\big\|\,\psi_n^1(t,\,\overline{t},x,\,\overline{x}\,)\,\big\|_p\le C\,\Big(\int_0^1\int_0^1\ \Gamma\,(t,\,\overline{t},x,\,\overline{x}\,;\,s,y)^2\ dy\,ds\Big)^{\frac12}\le C\,\big(\,|x-\overline{x}|^{\frac12}+|t-\overline{t}|^{\frac14}\,\big)\ . \tag{1.52}$$

The term $\psi_n^2(t,\,\overline{t},x,\,\overline{x})$ can be written as follows:

$$\psi_n^2(t,\,\overline{t},x,\,\overline{x})=\sum_k\Big\{\,\lambda_n^{(k)}\,(t,\,\overline{t},x,\,\overline{x})-E\,\big(\,\lambda_n^{(k)}\,(t,\,\overline{t},x,\,\overline{x})\,/\mathcal{F}_{(k-1)a^{-n}}\big)\Big\}\ ,$$

where $\lambda_n^{(k)}(t,\,\overline{t},x,\,\overline{x})$ has been defined in (1.17). The discrete Burkholder inequality and Lemma 1.3 $\big($see (1.19)$\big)$ yield

$$\big\|\,\psi_n^2(t,\,\overline{t},x,\,\overline{x})\,\big\|_p\le C\,\big(\,|x-\overline{x}|^{\frac12}+|t-\overline{t}|^{\frac14}\,\big)\ . \tag{1.53}$$

Clearly Lemma B.1 implies

$$\big\|\,\psi_n^3(t,\,\overline{t},x,\,\overline{x})\,\big\|_p\le C\,\big(\,|x-\overline{x}|^{\frac12}+|t-\overline{t}|^{\frac14}\,\big)\ . \tag{1.54}$$

Finally,

$$\psi_n^4(t,\,\overline{t},x,\,\overline{x})=\int_0^1\int_0^1\ \Gamma\,(t,\,\overline{t},x,\,\overline{x}\,;\,s,y)\ \phi_n(s,y)\ dy\,ds\ ,$$

with $\phi_n(s,y)$ defined in (1.39).
Schwarz's and Hölder's inequalities, then Lemmas 1.4 and B.1 imply

$$\begin{aligned}\big\|\,\psi_n^4(t,\,\overline{t},x,\,\overline{x})\,\big\|_p&\le\Big(\int_0^1\int_0^1\ \Gamma\,(t,\,\overline{t},x,\,\overline{x}\,;\,s,y)^2\,dy\,ds\Big)^{\frac12}\ \sup_{s,y}\ \big\|\,\phi_n(s,y)\,\big\|_p\\&\le C\ n\,\big(\,|t-\overline{t}|^{\frac14}+|x-\overline{x}|^{\frac12}\,\big).\end{aligned} \tag{1.55}$$

The estimates (1.52) to (1.55) imply (1.51). ◇

The preceding proposition enables us to improve the inequalities (1.46) and (1.48) using (1.44) and (1.47) respectively. The additional tool is given in the next lemma.

Lemma 1.6. For any $p\in(2,\,\infty)$,

$$\Big\|\int_{s_n}^s\int_{I_n(y)}G_{s-r}\,(y,z)\ \big|X_n(r,z)-X_n(s,y)\big|\,dz\,dr\,\Big\|_p\le C\ n\ a^{-\frac{5n}{4}}\ . \tag{1.56}$$

Proof. Fix $p\in(2,\,\infty)$; then

$$\begin{aligned}&E\,\Big(\Big|\int_{s_n}^s\int_{I_n(y)}\ G_{s-r}\,(y,z)\ |X_n(r,z)-X_n(s,y)|\,dz\,dr\,\Big|^p\Big)\le\\&\quad\le a^{-n(p-1)}\int_{s_n}^s\int_{I_n(y)}\ G_{s-r}\,(y,z)\ E\,\big(\,|X_n(r,z)-X_n(s,y)\,|^p\,\big)\,dz\,dr\ .\end{aligned} \tag{1.57}$$

By Proposition 1.5, the right hand side of (1.57) is bounded by

$$
\begin{aligned}
C\ a^{-n(p-1)} \Big[\int_{s_n}^{s} \int_{I_n(y)} & G_{s-r}(y,z)\ n^p\ |y-z|^{\frac{p}{2}}\ dz\ dr \\
& + \int_{s_n}^{s} \int_{I_n(y)} G_{s-r}(y,z)\ n^p\ |s-r|^{\frac{p}{4}}\ dz\ dr \Big] \\
\le C\ a^{-n(p-1)}\ n^p \Big[& \int_{s_n}^{s} \int_{\mathbf{R}} \frac{1}{\sqrt{2\pi(s-r)}}\ |z|^{\frac{p}{2}}\ \exp\Big(- \frac{z^2}{2(s-r)} \Big)\, dz\, dr \\
& + \int_{s_n}^{s} \int_{\mathbf{R}} a^{-n\frac{p}{4}}\ G_{s-r}(y,z)\, dz\, dr \Big] \\
\le C\ a^{-n(p-1)}\ n^p \Big[& \int_{s_n}^{s} (s-r)^{\frac{p}{4}}\ dr + a^{-n-\frac{np}{4}} \Big] \le C\ n^p\ a^{-\frac{5np}{4}}\ .
\end{aligned}
$$

◇

Remark 1.7. Consider the inequalities (1.44) and (1.47) in the proof of Lemma 1.4. The Lipschitz property of F and H together with Lemma 1.6 gives

$$
\begin{aligned}
I_n^{1,1}(s,y) + I_n^{1,2}(s,y) \le C\ E\Big(& \Big| n\ a^n \int_{s_n}^{\underline{s}_n} \int_{I_n(y)} G_{s-r}(y,z)\ \big|X_n(r,z) - X_n(s,y)\big|\ dz\, dr \Big|^p \Big) \\
& + C\ n^p\ a^{-n\frac{p-1}{4}} \\
& \le C\ n^p\ a^{np}\ n^p\ a^{-\frac{5np}{4}} + C\ n^p\ a^{-n\frac{p-1}{4}} \le C\ n^{2p}\ a^{-n\frac{p-1}{4}}\ .
\end{aligned} \tag{1.58}
$$

Thus the estimate (1.49) is improved as follows:

$$
\sup_{s,y}\ I_n^1(s,y) \le C\ n^{2p}\ a^{-n\frac{p-1}{4}}\ . \tag{1.59}
$$

The improvement of (1.50) requires a Taylor expansion of order three of H around $X_n^-(s,y)$. The next lemma deals with the corresponding term of order 2.

Lemma 1.8. For any $p \in (2, \infty)$,

$$
\sup_{s,y} \big\| E\big(|X_n(s,y) - X_n^-(s,y)|^2\ \dot{W}_n(s,y) / \mathcal{F}_{s_n} \big) \big\|_p \le C\ n^4\ a^{-\frac{n}{4}}\ . \tag{1.60}
$$

Proof. By the identity (1.9) we have

$$
E\big(|X_n(s,y) - X_n^-(s,y)|^2\ \dot{W}_n(s,y) / \mathcal{F}_{s_n} \big) = \sum_{i=1}^{6} T_n^i(s,y)\ ,
$$

where

$$T_n^1(s,y) = E\Big[\,\dot W_n(s,y)\Big(\int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\;F\big(X_n(r,z)\big)\;W\,(dz\,dr)\Big)^2/\mathcal{F}_{s_n}\Big]\;,$$

$$T_n^2(s,y) = E\Big[\,\dot W_n(s,y)\Big(\int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\;H\big(X_n(r,z)\big)\;W_n\,(dz\,dr)\Big)^2/\mathcal{F}_{s_n}\Big]\;,$$

$$T_n^3(s,y) = E\Big[\,\dot W_n(s,y)\Big(\int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\;K_n(r,z)\;dz\,dr\Big)^2/\mathcal{F}_{s_n}\Big]\;,$$

$$T_n^4(s,y) = 2\,E\Big[\,\dot W_n(s,y)\Big(\int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\;F\big(X_n(r,z)\big)\;W\,(dz\,dr)\Big)$$
$$\Big(\int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\;H\big(X_n(r,z)\big)\;W_n\,(dz\,dr)\Big)/\mathcal{F}_{s_n}\Big]\;,$$

$$T_n^5(s,y) = 2\,E\Big[\,\dot W_n(s,y)\Big(\int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\;F\big(X_n(r,z)\big)\;W\,(dz\,dr)\Big)$$
$$\Big(\int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\;K_n(r,z)\;dz\,dr\Big)/\mathcal{F}_{s_n}\Big]\;,$$

$$T_n^6(s,y) = 2\,E\Big[\dot W_n(s,y)\Big(\int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\;H\big(X_n(r,z)\big)\;W_n\,(dz\,dr)\Big)$$
$$\Big(\int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\;K_n(r,z)\;dz\,dr\Big)/\mathcal{F}_{s_n}\Big]\;.$$

For any $p\in(1,\,\infty)$ and every random variable $Y_n(s,y)\in\bigcap_{1<q<\infty} L_q$ we have, by Jensen's and Schwarz's inequalities,

$$\big\|\,E\big(\,|\dot W_n(s,y)|\;Y_n(s,y)^2\,/\mathcal{F}_{s_n}\big)\big\|_p \le n^{\frac12}\;a^{\frac n2}\;\big\|\,Y_n(s,y)\,\big\|_{4p}^2\;. \tag{1.61}$$

Consider $Y_n^1(s,y) = \int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\,F\big(X_n(r,z)\big)\,W(dz,dr)$; Burkholder's inequality implies that

$$\big\|\,Y_n^1(s,y)\,\big\|_p \le C\;a^{-\frac n4}\;.$$

Hence

$$\big\|\,T_n^1(s,y)\,\big\|_p \le C\;n^{\frac12}\;. \tag{1.62}$$

For $Y_n^2(s,y) = \int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\,H\big(X_n(r,z)\big)\,W_n(dz\,dr)$, Lemma B.3 implies that $\|Y_n^2(s,y)\|_p \le C\;a^{-n(\frac12-\frac1{2p})}\,n^{\frac12}$ and hence

$$\|\,T_n^2(s,y)\,\|_p \le C\;n^{\frac32}\;a^{-n(\frac12-\frac1{4p})}\;. \tag{1.63}$$

Finally, if $Y_n^3(s,y) = \int_{s_n}^s\int_0^1 G_{s-r}\,(y,z)\,K_n(r,z)\,dz\,dr$, we have $\|\,Y_n^3(s,y)\,\|_p \le C\;n\;a^{-n}$, and hence

$$\|\,T_n^3(s,y)\,\|_p \le C\;n^{\frac52}\;a^{-\frac{3n}{2}}\;. \tag{1.64}$$

Schwarz's inequality implies that

$$\| T_n^4(s,y) \|_p + \| T_n^5(s,y) \|_p + \| T_n^6(s,y) \|_p \le C \; n^4 \; a^{-n(\frac{1}{2}-\frac{1}{8p})} \; . \tag{1.65}$$

It remains to improve (1.62).
Set

$$Z_n(s,y) = E\Big[\dot W_n(s,y) \; \Big(\int_{\underline{s}_n}^s \int_0^1 G_{s-r}\,(y,z)\; F\big(X_n^-(r,z)\big)\; W(dz,dr) \Big)^2 / \mathcal{F}_{s_n} \Big] \; .$$

We have $Z_n(s,y) = 0$; indeed,

$$\begin{aligned} Z_n(s,y) &= E\Big[\dot W_n(s,y) \; E\Big(\int_{\underline{s}_n}^s \int_0^1 G_{s-r}^2\,(y,z)\; F\,\big(X_n^-(x,z)\big)^2 \; dz\,dr \,/\mathcal{F}_{\underline{s}_n} \Big) / \mathcal{F}_{s_n} \Big] \\ &= \Big(\int_{\underline{s}_n}^s \int_0^1 \; G_{s-r}^2\,(y,z)\; F\,\big(X_n^-(r,z)\big)^2\, dz\,dr \Big) \; E\,\big(\dot W^n(s,y)\,/\mathcal{F}_{s_n}\big) \; . \end{aligned}$$

Moreover,

$$\begin{aligned} E\Big[\dot W_n(s,y) \; &\Big(\int_{s_n}^{\underline{s}_n} \int_0^1 G_{s-r}\,(y,z)\; F\,\big(X_n^-(r,z)\big)\; W(dz,dr) \Big) \\ &\Big(\int_{\underline{s}_n}^s \int_0^1 G_{s-r}\,(y,z)\; F\,\big(X_n^-(r,z)\big)\; W(dz,dr) \Big) / \mathcal{F}_{s_n} \Big] = 0 \; . \end{aligned}$$

Hence,

$$T_n^1(s,y) = E\Big[\dot W_n(s,y) \Big(\int_{s_n}^{\underline{s}_n} \int_0^1 G_{s-r}(y,z) F\big(X_n(r,z)\big) W(dz,dr) \Big)^2 / \mathcal{F}_{s_n} \Big] .$$

We want to show that for $p > 2$

$$\Big\| T_n^1(s,y) \Big\|_p \le C \; n \; a^{-\frac{1}{4}n} \; . \tag{1.66}$$

Notice that we can replace $T_n^1(s,y)$ by

$$T_n^{1,1}(s,y) = E\Big(\dot W_n(s,y) \Big(\int_{s_n}^{\underline{s}_n} \int_0^1 G_{s-r}(y,z) F\big(X_n^-(r,z)\big) W(dz,dr) \Big)^2 / \mathcal{F}_{s_n} \Big).$$

More precisely,

$$\| T_n^1(s,y) - T_n^{1,1}(s,y) \|_p \le C n^{\frac{1}{2}} a^{-\frac{n}{4}} . \tag{1.67}$$

Indeed, set

$$Y_n^4(s,y) = \int_{s_n}^{\underline{s}_n} \int_0^1 G_{s-r}(y,z) F\big(X_n(r,z)\big) W(dz,dr)$$

and

$$Y_n^5(s,y)=\int_{s_n}^{\underline{s}_n}\int_0^1 G_{s-r}(y,z)F\big(X_n^-(r,z)\big)W(dz,dr).$$

Then for $p>2$,

$$\begin{aligned}&\|Y_n^4(s,y)-Y_n^5(s,y)\|_p\\&=\|\int_{s_n}^{\underline{s}_n}\int_0^1 G_{s-r}(y,z)\big[F\big(X_n(r,z)\big)-F\big(X_n^-(r,z)\big)\big]W(dz,dr)\|_p\\&\le Ca^{-\frac{n}{2}}.\end{aligned}$$

Furthermore, since F is bounded,

$$\|Y_n^4(s,y)\|_p+\|Y_n^5(s,y)\|_p\le Ca^{-\frac{n}{4}}.$$

Then,

$$\begin{aligned}&\|T_n^1(s,y)-T_n^{1,1}(s,y)\|_p=\|E\Big(\dot W_n(s,y)\big(Y_n^4(s,y)^2-Y_n^5(s,y)^2\big)/\mathcal{F}_{s_n}\Big)\|_p\\&\le\|\dot W_n(s,y)\|_{2p}\|Y_n^4(s,y)^2-Y_n^5(s,y)^2\|_{2p}\\&\le Cn^{\frac12}a^{\frac{n}{2}}\|Y_n^4(s,y)+Y_n^5(s,y)\|_{4p}\|Y_n^4(s,y)-Y_n^5(s,y)\|_{4p}\\&\le Cn^{\frac12}a^{\frac{n}{2}}a^{-\frac{n}{4}}a^{-\frac{n}{2}}=Cn^{\frac12}a^{-\frac{n}{4}},\end{aligned}$$

which proves (1.67). Fix $s\in(0,1]$ and consider the stochastic processes $\big\{N_u\,,\ u\in(s_n,\,\underline{s}_n]\big\}$ and $\big\{M_u\,,\ u\in(s_n,\,\underline{s}_n]\big\}$ given by

$$\begin{aligned}N_u&=W\left(\,(s_n,u]\times I_n(y)\,\right)\,,\\M_u&=\int_{s_n}^u\int_0^1 G_{s-r}\,(y,z)\;F\left(X_n^-(r,z)\,\right)\;W(dz,dr)\,,\end{aligned}$$

respectively. Using the Itô formula for $f:\mathbf{R}^2\longrightarrow\mathbf{R}$ defined by $f(x,y)=x\,y^2$, we obtain

$$\begin{aligned}&E\Big[\,\dot W_n(s,y)\,\Big[\,\int_{s_n}^{\underline{s}_n}\int_0^1 G_{s-r}\,(y,z)\;F\left(X_n^-(r,z)\,\right)\;W(dz,dr)\Big)^2/\mathcal{F}_{s_n}\,\Big]\\&=n\;a^nE\left[\,f(N_{\underline{s}_n}\,,\,M_{\underline{s}_n})\,/\mathcal{F}_{s_n}\right]=n\;a^n\left[\,Z_n^1(s,y)+Z_n^2(s,y)\,\right],\end{aligned}$$

where

$$\begin{aligned}Z_n^1(s,y)=2\;E\Big[\,\int_{s_n}^{\underline{s}_n}\;\Big(\int_{s_n}^u\int_0^1 G_{s-r}\,(y,z)\;F\left(X_n^-(r,z)\,\right)\;W(dz,dr)\Big)\\\times\Big(\int_{I_n(y)}G_{s-u}\,(y,z)\;F\left(X_n^-(u,z)\,\right)dz\Big)\,du\,/\mathcal{F}_{s_n}\,\Big]\end{aligned}$$

and

$$Z_n^2(s,y) = E\Big[\int_{s_n}^{\underline{s}_n} W\big((s_n, u] \times I_n(y) \big) \Big(\int_0^1 G_{s-u}^2(y,z)\ F\big(X_n^-(u,z)\big)^2\, dz \Big)\, du \,/\mathcal{F}_{s_n} \Big].$$

Notice that, since $\int_0^1 G_{s-u}^2\,(y,z)\ F\big(X_n^-(u,z)\big)^2\, dz$ is $\mathcal{F}_{s_n}$–measurable, $Z_n^2(s,y) = 0$. Consequently,

$$\big\| T_n^{1,1}(s,y) \big\|_p = n\ a^n\ \big\| Z_n^1(s,y) \big\|_p\ .$$

Jensen's, Hölder's and Burkholder's inequality imply

$$\big\| Z_n^1(s,y) \big\|_p \le C\ a^{-n} \Big(\int_{s_n}^{\underline{s}_n} \int_0^1 G_{s-r}^2\,(y,z)\, dz\, dr \Big)^{\frac{1}{2}} \le C\ a^{-\frac{5}{4}\,n}\ ,$$

and therefore (1.66) holds true. The estimates (1.63) to (1.66) yield

$$\sum_{i=1}^{6}\ \big\| T_n^i(s,y) \big\|_p \le C\ n^4\ a^{-\frac{n}{4}}\ ,$$

and thus, the proof of inequality (1.60) is complete. ◇

It is now possible to obtain a much more precise result than that stated in Lemma 1.4.

Proposition 1.9. Let $p \in (2, \infty)$ and $\{\phi_n(s,y)\,,\ (s,y) \in (0,1] \times [0,1]\,\}$ be the process defined in (1.39). It holds that

$$\sup_{s,y} \big\| \phi_n(s,y) \big\|_p \le C\ n^4\ a^{-\frac{n}{4}\frac{p-1}{p}}. \tag{1.68}$$

Proof. As for the proof of Lemma 1.4 we consider a Taylor expansion of $H\big(X_n(s,y)\big)$ around $X_n^-(s,y)$, but this time up to the third order, that is

$$\begin{aligned} H\big(X_n(s,y)\big) &= H\big(X_n^-(s,y)\big) + \dot H\big(X_n^-(s,y)\big)\ \big(X_n(s,y) - X_n^-(s,y)\big) \\ &\quad + \frac{1}{2}\ \ddot H\big(X_n^-(s,y)\big)\ \big(X_n(s,y) - X_n^-(s,y)\big)^2 + r_n(s,y)\ , \end{aligned}$$

with

$$r_n(s,y) \le C\ \big| X_n(s,y) - X_n^-(s,y) \big|^3\ .$$

Then

$$\big\| \phi_n(s,y) \big\|_p \le C\ \sum_{j=1}^{3}\ \big\| \overline{\varphi}_n^j(s,y) \big\|_p\ ,$$

where

$$\overline{\varphi}_n^1(s,y) = \varphi_n^1(s,y) \ ,$$

as in the proof of Lemma 1.4, while

$$\overline{\varphi}_n^2(s,y) = \ddot{H}\left(X_n^-(s,y)\right)\ E\Big(\dot{W}_n(s,y)\ \left(X_n(s,y) - X_n^-(s,y)\right)^2 / \mathcal{F}_{s_n}\Big),$$

and

$$\overline{\varphi}_n^3(s,y) = E\left(\dot{W}_n(s,y)\ r_n(s,y) / \mathcal{F}_{s_n}\right).$$

Remark 1.7 $\big($see(1.59)$\big)$ yields

$$\left\| \overline{\varphi}_n^1(s,y) \right\|_p = \left(I_n^1(s,y)\right)^{\frac{1}{p}} \le C\ n^2\ a^{-\frac{n}{4}\frac{p-1}{p}} \ . \tag{1.69}$$

Moreover, Lemma 1.8 implies

$$\left\| \overline{\varphi}_n^2(s,y) \right\|_p \le C\ n^4\ a^{-\frac{n}{4}} \ . \tag{1.70}$$

Finally, Jensen's and Schwarz's inequalities together with Lemma 1.1 show that

$$\begin{aligned} \left\| \overline{\varphi}_n^3(s,y) \right\|_p &\le C \left\| \dot{W}_n(s,y) \right\|_{2p} \left\| \left(X_n(s,y) - X_n^-(s,y)\right)^3 \right\|_{2p} \\ &\le C\ n^{\frac{1}{2}}\ a^{\frac{n}{2}}\ a^{-\frac{3}{4}n} = C\ n^{\frac{1}{2}}\ a^{-\frac{n}{4}} \ . \end{aligned} \tag{1.71}$$

Consequently, (1.69) to (1.71) give the assertion (1.68). ◇

In the next proposition we will establish L^p–estimates of $X_n(t,x) - X_n(\overline{t}\,,\overline{x})$ similar to those proved in Proposition 1.5, but with constants which no longer depend on n.

Theorem 1.10. For each $p \in (2,\infty)$, there exists $C > 0$ such that

$$\sup_n \left\| X_n(t,x) - X_n(\overline{t}\,,\overline{x}) \right\|_p \le C\left(|t-\overline{t}|^{\frac{1}{4}} + |x-\overline{x}|^{\frac{1}{2}}\right) \ . \tag{1.72}$$

Proof. It suffices to check that the estimate (1.55) in the proof of Proposition 1.5 can be improved, as follows.

$$\left\| \psi_n^4(t,\overline{t},x,\overline{x}) \right\|_p \le C\left(|t-\overline{t}|^{\frac{1}{2}} + |x-\overline{x}|\right) \ . \tag{1.73}$$

But this is an immediate consequence of the estimate of $\sup_{s,y} \left\| \phi_n(s,y) \right\|$ provided by (1.68).

◇

We now prove the convergence of $\big(X_n(t,x),\ n\ge 1\big)$ to $X(t,x)$ in L^p for fixed (t,x).

Theorem 1.11. For any $p\in[1,\infty)$, any $(t,x)\in[0,1]^2$,

$$\lim_n \big\| X_n(t,x)-X(t,x)\big\|_p = 0\ . \tag{1.74}$$

Proof. We decompose the difference $X_n(t,x)-X(t,x)$ into several terms:

$$\begin{aligned}X_n(t,x)-X(t,x) &= \int_0^t\int_0^1 G_{t-s}(x,y)\Big\{(F+H)\big(X_n(s,y)\big)-(F+H)\big(X(s,y)\big)\Big\}\\ &\quad W(dy,ds)+\int_0^t\int_0^1 G_{t-s}(x,y)\Big\{\big[K\big(X_n(s,y)-K\big(X(s,y)\big)\big]\ \dot h(s,y)\\ &\quad +\big[f\big(X_n(s,y)\big)-f\big(X(s,y)\big)\big]\Big\}\,dy\,ds+\delta_n(t,x)\end{aligned}$$

where

$$\begin{aligned}\delta_n(t,x) &= \int_0^t\int_0^1 G_{t-s}(x,y)\ H\big(X_n(s,y)\big)\ \big[W_n(dy,ds)-W(dy,ds)\big]\\ &\quad -\int_0^t\int_0^1 G_{t-s}(x,y)\Big[(F\dot H)\big(X_n(s,y)\big)\,b_n(s,y)+\\ &\quad +(H\dot H)\big(X_n(s,y)\big)\,c_n(s,y)\Big]\,dy\,ds\,.\end{aligned}$$

Fix $p\in(6,\infty)$, and let q satisfy $\frac{2}{p}+\frac{1}{q}=1$; then Burkholder's and Hölder's inequalities imply that

$$\begin{aligned}E\big(\big|X_n(t,x)-X(t,x)\big|^p\big) &\le C\ E\Big(\Big|\int_0^t\int_0^1 G^2_{t-s}(x,y)\,\big|X_n(s,y)-X(s,y)\big|^2\,dy\,ds\Big|^{\frac{p}{2}}\Big)\\ &\quad +C\ E\big(\big|\delta_n(t,x)\big|^p\big)\\ &\le C\Big(\int_0^t\int_0^1 G^{2q}_{t-s}(x,y)dyds\Big)^{\frac{p}{2q}}\int_0^t\int_0^1 E\big(\big|X_n(s,y)-X(s,y)\big|^p\big)\,dy\,ds\\ &\quad +C\ E\big(\big|\delta_n(t,x)\big|^p\big),\end{aligned}$$

Hence,

$$\begin{aligned}\sup_x E\big(\big|X_n(t,x)-X(t,x)\big|^p\big) &\le C\sup_x\ \big\|\delta_n(t,x)\big\|_p^p\\ &\quad +C\int_0^t\sup_x E\big(\big|X_n(s,x)-X(s,x)\big|^p\big)\,ds\,.\end{aligned}$$

Gronwall's lemma implies that

$$\sup_x\ \big\|X_n(t,x)-X(t,x)\big\|_p\le C\ \sup_x\ \big\|\delta_n(s,x)\big\|_p\ . \tag{1.75}$$

Set $\delta_n(t,x) = \sum_{j=1}^{4} \delta_n^{(j)}(t,x)$, where

$$\delta_n^{(1)}(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) \left[H\big(X_n^-(s,y)\big) W_n(dy,ds) - H\big(X_n^-(s,y)\big) W(dy,ds) \right],$$
$$\delta_n^{(2)}(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) \left[H\big(X_n^-(s,y)\big) - H\big(X_n(s,y)\big) \right] W(dy,ds),$$
$$\delta_n^{(3)}(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) \left[H\big(X_n(s,y)\big) - H\big(X_n^-(s,y)\big) \right] W_n(dy,ds)$$
$$- \int_0^t \int_0^1 G_{t-s}(x,y)\, E\Big(H\big(X_n(s,y)\big)\, \dot{W}_n(s,y) / \mathcal{F}_{s_n} \Big)\, dy\, ds,$$

and

$$\delta_n^{(4)}(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) \Big\{ E\Big(H\big(X_n(s,y)\big)\, \dot{W}_n(s,y) / \mathcal{F}_{s_n} \Big)$$
$$- (\dot{H}\,F)\big(X_n(s,y)\big)\, b_n(s,y) - (\dot{H}\,H)\big(X_n(s,y)\big)\, c_n(s,y) \Big\}\, dy\, ds.$$

We next prove that $\lim_n \sup_{t,x} \| \delta_n^{(1)}(t,x) \|_p = 0$. To this end, we first introduce some notations in order to write $\delta_n^{(1)}(t,x)$ as a stochastic integral. Let τ_n be the transformation defined on real–valued functions by $\tau_n\ \rho(s) = \rho\big((s + a^{-n}) \wedge 1\big)$. We also consider the orthogonal projection from $L^2\big([0,1]^2\big)$ on the subspace generated by the indicator functions of rectangles $\Delta_{k,j} = \big(ka^{-n},\ (k+1)\,a^{-n}\big] \times \big(jn^{-1},\ (j+1)n^{-1}\big]$, $0 \le k \le a^n - 1$, $0 \le j \le n-1$, which will be denoted by π_n. Then

$$\delta_n^{(1)}(t,x) = \int_0^t \int_0^1 \Big\{ \pi_n\Big[\tau_n\Big(1_{[0,t]}(\cdot)\ G_{t-\cdot}(x,\cdot)\ H\big(X_n^-(\cdot\,,\cdot)\big)\Big)\Big](s,y)$$
$$- 1_{[0,t]}(s,y)\ G_{t-s}(x,y)\ H\big(X_n^-(s,y)\big) \Big\}\ W(dy,ds).$$

Set

$$\delta_n^{(1,1)}(t,x) = \int_0^t \int_0^1 \Big\{ \pi_n\Big[\tau_n\Big(1_{[0,t]}(\cdot)\ G_{t-\cdot}(x,\cdot)\ H\big(X_n^-(\cdot\,,\cdot)\big)\Big)\Big](s,y)$$
$$- \pi_n\left[1_{[0,t]}(\cdot)\ G_{t-\cdot}(x,\cdot)\ H\left(X_n^-(\cdot\,,\cdot)\right)\right](s,y) \Big\} W(dy,ds),$$
$$\delta_n^{(1,2)}(t,x) = \int_0^t \int_0^1 \Big\{ \pi_n\Big[1_{[0,t]}(\cdot)\ G_{t-\cdot}(x,\cdot)\ H\big(X_n^-(\cdot\,,\cdot)\big)\Big](s,y)$$
$$- 1_{[0,t]}(s)\ G_{t-s}(x,y)\ H\big(X_n^-(s,y)\big) \Big\}\ W(dy,ds).$$

Burkholder's inequality yields

$$E\Big(\big|\,\delta_n^{(1,1)}(t,x)\big|^p\Big) \le C\Bigg\{\Big|\int_0^{t-a^{-n}}\int_0^1 \big|\,G_{t-(a^{-n}+s)}\,(x,y) - G_{t-s}\,(x,y)\,\big|^2\,dy\,ds\,\Big|^{\frac{p}{2}}$$

$$+ E\,\Big(\Big|\int_0^{t-a^{-n}}\int_0^1 G^2_{t-s}\,(x,y)\,\big|X_n^-(s+a^{-n},y) - X_n^-(s,y)\big|^2\,dy\,ds\,\Big|^{\frac{p}{2}}\Big)\Bigg\}.$$

Using Lemma B.1 we obtain

$$\Big|\int_0^{t-a^{-n}}\int_0^1 \big|G_{t-(s+a^{-n})}(x,y) - G_{t-s}(x,y)\big|^2 dyds\Big|^{\frac{p}{2}} \le Ca^{-\frac{np}{4}}.$$

Moreover, since $p > 6$ and $\frac{2}{p}+\frac{1}{q}=1$, we have that $2q < 3$. Thus

$$E\Big(\Big|\int_0^{t-a^{-n}}\int_0^1 G^2_{t-s}(x,y)\,|X_n^-(s+a^{-n},y) - X_n^-(s,y)|^2 dyds\Big|^{\frac{p}{2}}\Big)$$

$$\le \Big(\int_0^{t-a^{-n}}\int_0^1 G^{2q}_{t-s}(x,y)\,dy\;ds\Big)^{\frac{p}{2q}} \sup_{s,y}\|X_n^-(s+a^{-n},y) - X_n^-(s,y)\|_p^p.$$

$$\le C\,\sup_{s,y}\|X_n^-(s+a^{-n},y) - X_n^-(s,y)\|_p^p \le Ca^{-\frac{np}{4}}\ ,$$

by Lemma 1.1 and Theorem 1.10. Consequently,

$$\sup_{t,x} E\big(|\delta_n^{(1,1)}(t,x)|^p\big) \le Ca^{-n\frac{p}{4}}. \tag{1.76}$$

Set

$$\delta_n^{(1,2)}(t,x) = \delta_n^{1,2,1}(t,x) + \delta_n^{1,2,2}(t,x),$$

with

$$\delta_n^{1,2,1}(t,x) = \int_0^t\int_0^1 \Big\{\sum_{k=0}^{[a^nt]}\sum_{j=0}^{n-1}\Big(\int_{\Delta_{kj}} n\,a^n\,[G_{t-r}(x,z) - G_{t-s}(x,y)]H\big(X_n^-(s,y)\big)\,dzdr\Big)$$
$$1_{\Delta_{kj}}(s,y)\,\Big\}W(dy,ds)\,,$$

and

$$\delta_n^{1,2,2}(t,x) = \int_0^t\int_0^1 \Big\{\sum_{k=0}^{[a^nt]}\sum_{j=0}^{n-1}\Big(\int_{\Delta_{kj}} n\,a^n\,\big[H\big(X_n^-(r,z)\big) - H\big(X_n^-(s,y)\big)\big]G_{t-r}(x,z)\,dzdr\Big)$$
$$1_{\Delta_{kj}}(s,y)\Big\}\,W(dy,ds),$$

Burkholder's inequality yields

$$E\big(\,|\delta_n^{1,2,1}(t,x)|^p\,\big) \le C\Big(\int_0^t\int_0^1 \big|\,\pi_n\big(G_{t-.}(x,.)\big) - G_{t-s}(x,y)\big|^2\,dyds\,\Big)^{\frac{p}{2}}.$$

For every $(t,x)\in[0,1]^2$ the sequence $\{\|\pi_n\big(G_{t-.}(x,.)\big) - G_{t-.}(x,.)\|_{L^2([0,1]^2)}, n\ge 1\}$ decreases to zero as n goes to infinity. By Dini's theorem this convergence is uniform in (t,x). Hence,

$$\lim_{n\to\infty}\sup_{(t,x)\in[0,1]^2}\|\delta_n^{1,2,1}(t,x)\|_p = 0. \tag{1.77}$$

Applying Burkholder's inequality and then Fubini's theorem we obtain

$$\begin{aligned}
&E\big(\,|\delta_n^{1,2,2}(t,x)|^p\,\big)\\
&\le CE\Big[\,\Big(\sum_{k=0}^{[a^nt]}\sum_{j=0}^{n-1}\int_{\Delta_{kj}}\Big\{\int_{\Delta_{kj}} na^n|X_n^-(r,z)) - X_n^-(s,y)|^2 G_{t-r}^2(x,z)\,dzdr\Big\}\,dyds\Big)^{\frac{p}{2}}\,\Big]\\
&= CE\Big[\Big(\sum_{k=0}^{[a^nt]}\sum_{j=0}^{n-1}\int_{\Delta_{kj}} G_{t-r}^2(x,z)\Big\{\int_{\Delta_{kj}} na^n\,|X_n^-(r,z) - X_n^-(s,y)|^2\,dyds\Big\}\,dzdr\Big)^{\frac{p}{2}}\,\Big]\\
&= CE\Big[\Big(\int_0^t\int_0^1 G_{t-r}^2(x,z)\sum_{k=0}^{[a^nt]}\sum_{j=0}^{n-1}\\
&\qquad\qquad \Big\{\int_{\Delta_{kj}} na^n|X_n^-(r,z) - X_n^-(s,y)|^2\,dyds\Big\}\,1_{\Delta_{kj}}(z,r)\,dzdr\Big)^{\frac{p}{2}}\,\Big]\\
&\le C\Big(\int_0^t\int_0^1 G_{t-r}^{2q}(x,z)dzdr\Big)^{\frac{p}{2q}}\\
&\qquad \int_0^t\int_0^1\sum_{k=0}^{[a^nt]}\sum_{j=0}^{n-1}\Big(\int_{\Delta_{kj}} na^n\|X_n^-(r,z) - X_n^-(s,y)\|_p^p\,dyds\Big)\,1_{\Delta_{kj}}(r,z)\,dzdr\\
&\le C\sup_{(r,z),(s,y)\in\Delta_{kj}}\|X_n^-(r,z) - X_n^-(s,y)\|_p^p.
\end{aligned}$$

Lemma 1.1 and Theorem 1.10 yield

$$\begin{aligned}
\|X_n^-(r,z) - X_n^-(s,y)\|_p^p &\le C\Big(\|X_n^-(r,z) - X_n(r,z))\|_p^p\\
&\qquad + \|X_n(r,z) - X_n(s,y)\|_p^p + \|X_n(s,y) - X_n^-(s,y)\|_p^p\Big)\\
&\le C\big(a^{-n\frac{p}{4}} + a^{-n\frac{p}{4}} + n^{-\frac{p}{2}}\big).
\end{aligned}$$

Therefore

$$\sup_{t,x}\|\delta_n^{1,2,2}(t,x)\|_p^p \le Cn^{-\frac{1}{2}}. \tag{1.78}$$

The estimates (1.76) to (1.78) imply

$$\sup_{t,x} \left\| \delta_n^{(1)} (t,x) \right\|_p \le C\ n^{-\frac{1}{2}}\ . \tag{1.79}$$

Clearly by Burkholder's inequality and Lemma 1.1 we have

$$\sup_{t,x}\ \left\| \delta_n^{(2)} (t,x) \right\|_p \le C\ a^{-\frac{n}{4}}\ . \tag{1.80}$$

The discrete Burkholder inequality and Lemma 1.2 show that

$$\sup_{t,x}\ \left\| \delta_n^{(3)} (t,x) \right\|_p \le C\ n^{\frac{3}{2}}\, a^{-\frac{n}{4}}\ . \tag{1.81}$$

Let p and γ be conjugate exponents and let ϕ_n be defined by (1.39) . Since $p \in (6, \infty)$ we have that $\gamma \in (1, \frac{6}{5})$. Hölder's inequality and Proposition 1.9 yield

$$\begin{aligned}\sup_{t,x}\ \left\| \delta_n^{(4)} (t,x) \right\|_p &\le \Big(\int_0^t \int_0^1 G_{t-s}^{\gamma} (x,y)\, dy\, ds \Big)^{\frac{1}{\gamma}}\ \sup_{s,y} \left\| \phi_n (s,y) \right\|_p \\ &\le C\ n^4\ a^{-\frac{n}{4}\frac{p-1}{p}}\ . \end{aligned} \tag{1.82}$$

Consequently, the inequalities (1.75) and (1.79) to (1.82) give the desired convergence. $\diamond$

We finally prove the main result of this section, that is the convergence of the sequence $(X_n\ ,\ n \ge 1)$ to X in the space $\mathcal{C}^\alpha$ of α–Hölder continuous functions on $[0,1]^2, \alpha \in (0,\ \frac{1}{4})$. It is a straightforward consequence of Theorems 1.10 and 1.11.

Remark 1.12 Given $\alpha \in (0, \frac{1}{2})$ consider the separable subspace H_0^α of $\mathcal{C}^\alpha\big([0,1]^2\big)$ consisting on functions φ vanishing on the axes and such that

$$|\varphi(t,x) - \varphi(s,y)| = o\big((|t-s| + |x-y|^2)^\alpha\big),$$

when $|t-s| + |x-y|$ goes to zero.

We notice that equation (1.8) is a particular case of (1.5) corresponding to $H = 0$. Consequently the estimate given in (1.72) is also valid for the process X.

Then, Theorem 1.10 and an easy extension of Théorème A in [4] to a two-parameter case ensure that, almost surely, the paths of $X_n - X$ belong to the separable Banach space H_0^α, for $\alpha \in (0, \frac{1}{4})$. Consequently, although $\mathcal{C}^\alpha\big([0,1]^2\big)$ is not separable, $X_n - X$ is a $\mathcal{C}^\alpha\big([0,1]^2\big)$-valued *random variable.*

Theorem 1.13. For any $\alpha \in (0,\ \dfrac{1}{4})$ and $p \in [1, \infty)$,

$$\lim_{n\to\infty} \big(X_n - X\big) = 0$$

in $L^p\big(\Omega\,;\ \mathcal{C}^\alpha([0,1]^2)\big)$.

Proof. Fix $\alpha \in (0,\ \dfrac{1}{4})$ and fix $p_0 \in (1,+\infty)$ such that $\frac{2}{p_0} < \frac{1}{4} - \alpha$. We apply Lemma A.1 to the sequence $Y_n = X_n - X$. Theorem 1.11 yields the validity of condition (P1), while Theorem 1.10 (see also the remark 1.12) ensures (P2) with $2+\gamma = \frac{p_0}{4}$. Consequently

$$\lim_{n\to\infty} E\big(\,\|X_n - X\|_\alpha^p\,\big) = 0,$$

for $\alpha \in (0, \frac{1}{4})$ and any $p \in (1,\infty)$. ◇

2. SUPPORT THEOREM

The goal of this section is to prove the following theorem which describes the support of the law of the process u given by (0.3).

Theorem 2.1. Let $f, g : \mathbf{R} \longrightarrow \mathbf{R}$ be bounded Lipschitz functions, f of class $\mathcal{C}^3$ with bounded derivatives up to order 3. Let $u_0 \in \mathcal{C}^{2\alpha}(\,[0,1]\,)$ for some $\alpha \in (0,\ \dfrac{1}{4})$ and let $\big(\,u(t,x)\,;\ (t,x) \in [0,\ \infty) \times [0,1]\,\big)$ be the process solution of (0.3). Then the support of $P\circ u^{-1}$, as a probability on $\mathcal{C}^\alpha([0,1]^2)$, is the closure of the set $\mathcal{S}_{\mathcal{H}} = \{S(h); h \in \mathcal{H}\}$, where $S(h)$ is the solution of (0.4) .

In the proof of this theorem we use a method provided by the next proposition.

Proposition 2.2. Let $(\mathbf{B}, \|\ \|)$ be a separable Banach space, $\mathcal{H}_0 \subset \mathcal{H}$, and $F : \Omega \longrightarrow \mathbf{B}$.
(i) Let $\xi_1 : \mathcal{H}_0 \longrightarrow \mathbf{B}$ be measurable and assume that there exists a sequence of random variables $H_n : \Omega \longrightarrow \mathcal{H}_0$ such that for any $\varepsilon > 0$,

$$\lim_n\ P\left(\,\big\|\,F(\omega) - \xi_1\big(H_n(\omega)\,\big)\,\big\| > \varepsilon\,\right) = 0\ . \tag{2.1}$$

Then

$$\text{support}\ (P \circ\ F^{-1}) \subset \overline{\xi_1(\mathcal{H}_0)}\ . \tag{2.2}$$

(ii) Let $\xi_2 : \mathcal{H}_0 \longrightarrow \mathbf{B}$ be measurable and suppose that for each $h \in \mathcal{H}_0$ there exists a sequence of measurable transformations $T_n^h\ : \Omega \longrightarrow \Omega$ such that $P\circ(T_n^h)^{-1} \ll P$, and for every $\varepsilon > 0$

$$\limsup_n\ P\ \Big(\,\big\|\,F\big(T_n^h\,(\omega)\,\big) - \xi_2(h)\,\big\| < \varepsilon\,\Big) > 0\ . \tag{2.3}$$

Then

$$\text{support}\ (P \circ\ F^{-1}) \supset \overline{\xi_2(\mathcal{H}_0)}\ . \tag{2.4}$$

Proof. Although this proposition has already been proved in [7], we recall the main arguments for the sake of completeness. Part (i) is standard. As for (ii), we have to check that for each $h \in \mathcal{H}_0$ and each $\varepsilon > 0$, $P\Big(\| F(\omega) - \xi_2(h) \| < \varepsilon\Big) > 0$. Since $P \circ (T_n^h)^{-1} \ll P$, this is a consequence of $P\Big(\| F\big(T_n^h(\omega)\big) - \xi_2(h) \| < \varepsilon\Big) > 0$ for some $n > 0$; (2.3) ensures the existence of such an integer n . ◇

Remark. In the previous proposition the separability of $\mathbf{B}$ is required in order to guarantee the measurability of the map $\omega \to \| F(\omega) - \xi_1(H_n(\omega)) \|$. In our setting, $\mathbf{B} = \mathcal{C}^\alpha([0,1]^2)$ is not separable. However, in our applications $F(\omega) - \xi_1\big(H_n(\omega)\big)$ takes, almost surely, its values in some separable subspace $\mathbf{B}_0$ of $\mathcal{C}^\alpha\big([0,1]^2\big)$ (see for instance Remark 1.12).

We can now apply Proposition 2.2 in order to prove Theorem 2.1, using the convergence result stated in Theorem 1.13 .

Proof of Theorem 2.1:
Let $\{X_n(t,x), (t,x) \in [0,1]^2\}$ be the solution of the following equation

$$X_n(t,x) = G_t(x,u_0) + \int_0^t \int_0^1 G_{t-s}(x,y)\, g\big(X_n(s,y)\big)\, W_n(dy,ds) + \int_0^t \int_0^1 G_{t-s}(x,y)\, \big\{ f\big(X_n(s,y)\big) - (g\,\dot g)\big(X_n(s,y)\big)\, c_n(s,y) \big\}\, dy\, ds\ , \qquad (2.5)$$

where

$$c_n(s,y) = n\, a^n \int_{\underline{s}_n}^{s} \int_{I_n(y)} G_{s-r}(y,z)\, dz\, dr\ ,$$

Let $\mathcal{H}_0 = \mathcal{H}_b$, the subset of $\mathcal{H}$ of functions with bounded first derivatives. Set $\xi_1(h) = S_{|\mathcal{H}_b}(h)$, where $S(h)$ has been defined in (0.4), and let $H_n : \Omega \longrightarrow \mathcal{H}_b$ be given by

$$\dot H_n(\omega)\ (s,y) = \dot W_n(s,y) - \dot g\big(X_n(s,y)\big)\, c_n(s,y)\ . \qquad (2.6)$$

Then $S(H_n)$ satisfies the evolution equation

$$S(H_n)\,(t,x) = G_t(x,u_0) + \int_0^t \int_0^1 G_{t-s}(x,y)\, g\big(S(H_n)\,(s,y)\big) \big\{ \dot W_n(s,y) - \dot g\big(X_n(s,y)\big)\, c_n(s,y) \big\}\, dy\, ds + \int_0^t \int_0^1 G_{t-s}(x,y)\, f\big(S\big(H_n(s,y)\big)\, dy\, ds\ . \qquad (2.7)$$

By uniqueness of the solution of (2.7), $X_n = S(H_n)$. Hence, by Theorem 1.13 with $F = K = 0$ and $H = g$, the sequence $\{S(H_n), n \geq 1\}$ converges to u in $L^p\big(\Omega; \mathcal{C}^\alpha([0,1]^2)\big)$. Thus, condition (2.1) of Proposition 2.2 holds, and consequently

$$\text{support } P \circ u^{-1} \subset \overline{\mathcal{S}_{\mathcal{H}_b}}$$

We now introduce, for $h \in \mathcal{H}_b$, the process X_n defined by the equation :

$$\begin{aligned} X_n(t,x) = G_t\,(x,u_0) + &\int_0^t \int_0^1 G_{t-s}\,(x,y)\; g\,\big(X_n(s,y)\,\big)\; W(dy,ds) \\ &- \int_0^t \int_0^1 G_{t-s}(x,y)\; g\,\big(X_n(s,y)\,\big)\; W_n(dy,ds) + \\ &+ \int_0^t \int_0^1 G_{t-s}(x,y)\;\big\{\, g\,\big(X_n(s,y)\,\big)\; \dot h(s,y) + f\,\big(X_n(s,y)\,\big) \\ &- (g\,\dot g)\;\big(X_n(s,y)\,\big)\;\big[c_n(s,y) - b_n(s,y)\,\big]\,\big\}\; dy\,ds\;, \end{aligned} \tag{2.8}$$

where

$$b_n(s,y) = n\; a^n \int_{s_n}^{\underline{s}_n} \int_{I_n(y)} G_{s-r}(y,z)\; dz\; dr\;.$$

Let $K_n(\omega)$ be the element of $\mathcal{H}_b$ defined by

$$\dot K_n(\omega)(s,y) = \dot h(s,y) - \dot g\,\big(X_n(s,y)\,\big)\;[\,c_n(s,y) - b_n(s,y)\,], \tag{2.9}$$

and let $T_n^h : \Omega \longrightarrow \Omega$ be given by

$$T_n^h(\omega) = \omega - \omega_n + K_n(\omega)\;;$$

Girsanov's theorem implies that $P \circ (T_n^h)^{-1} \ll P$. Furthermore, if (Z_n) is the sequence of processes defined by $Z_n(\omega) = u \circ T_n^h(\omega)$, then

$$\begin{aligned} Z_n(t,x) = G_t\,(x,u_0) + &\int_0^t \int_0^1 G_{t-s}\,(x,y)\; g\,\big(Z_n(s,y)\,\big)\; W(dy,ds) \\ - &\int_0^t \int_0^1 G_{t-s}\,(x,y)\; g\,\big(Z_n(s,y)\,\big)\; W_n(dy\,ds) + \int_0^t \int_0^1 G_{t-s}\,(x,y)\; g\,\big(Z_n(s,y)\,\big) \\ &\big\{\,\dot h(s,y) - \dot g\,\big(X_n(s,y)\,\big)\;\big[\,c_n(s,y) - b_n(s,y)\,\big]\,\big\}\; dy\,ds \\ + &\int_0^t \int_0^1 G_{t-s}\,(x,y)\; f\,\big(Z_n(s,y)\big)\; dy\,ds\;. \end{aligned} \tag{2.10}$$

Then, by uniqueness of the solution of (2.10) we have that $X_n = u \circ T_n^h$. Furthermore, Theorem 1.13 applied with $F = K = g$ and $H = -g$ implies that $(X_n\,,\; n \geq 1)$ converges in $L^p\big(\,\Omega\,;\;\mathcal{C}^\alpha[0,1]^2)\big)$ to the process X defined by

$$\begin{aligned} X(t,x) = G_t(x,\,u_0) + &\int_0^t \int_0^1\; G_{t-s}(x,y)\, g\,\big(X(s,y)\,\big)\; \dot h(s,y)\; dy\,ds \\ + &\int_0^t \int_0^1\; G_{t-s}(x,y)\, f\,\big(X(s,y)\,\big)\; dy\,ds\;, \end{aligned}$$

that is $X = S(h)$. Consequently, the assumption (ii) of Proposition 2.2 is satisfied with $\mathcal{H}_0 = \mathcal{H}_b$, $\xi_2 = S_{|\mathcal{H}_b}$. Hence,

$$\text{support}\, P \circ u^{-1} \supset \overline{\mathcal{S}_{\mathcal{H}_b}}\,.$$

To conclude the proof of the proposition, it remains to check that the closures of $\mathcal{S}_{\mathcal{H}_b}$ and $\mathcal{S}_{\mathcal{H}}$ in $\mathcal{C}^\alpha([0,1]^2)$ coincide. Since $\mathcal{H}_b$ is dense in $\mathcal{H}$, it suffices to check that, given any $M > 0$ and $\alpha \in (0, \frac{1}{4})$, there exists $C > 0$ such that for any $h_1, h_2 \in \mathcal{H}$ with $\|h_1\|_{\mathcal{H}} \vee \|h_2\|_{\mathcal{H}} \le M$,

$$\|\, S(h_2) - S(h_1) \,\|_\alpha \le C\, \|h_2 - h_1\|_{\mathcal{H}} \tag{2.11}$$

Given $(t,x) \in [0,1]^2$ it holds that

$$\begin{aligned}
\big|\, S(h_2)(t,x) - S(h_1)\,(t,x)\,\big|^2 &\le C\, \|\, h_2 - h_1\,\|_{\mathcal{H}}^2 \int_0^t\!\int_0^1 G_{t-s}^2\,(x,y)\; dy\, ds \\
&+ C\,\big(\,1 + \|h_1\|_{\mathcal{H}}^2\,\big) \int_0^t\!\int_0^1 G_{t-s}^2\,(x,y)\; \big|\, S(h_2)\,(s,y) - S(h_1)\,(s,y)\,\big|^2\, dy\, ds \\
&\le C\, \|\, h_2 - h_1\,\|_{\mathcal{H}}^2 + C \int_0^t \frac{1}{\sqrt{t-s}} \sup_y \big|\, S(h_2)\,(s,y) - S(h_1)\,(s,y)\,\big|^2\, ds\,,
\end{aligned}$$

with some constant C depending on M. A generalized version of Gronwall's lemma (see e.g. [11]) applied to the function

$$\psi(t) = \sup_x \,\big|\, S(h_2)\,(t,x) - S(h_1)\,(t,x)\,\big|^2$$

implies that

$$\sup_{(t,x)\,\in\,[0,1]^2} \big|\, S(h_2)\;(t,x) - S(h_1)\,(t,x)\,\big| \le C\; \|h_2 - h_1\|_{\mathcal{H}}\;, \tag{2.12}$$

Let (t,x) and $(\,\overline{t}\,,\,\overline{x}\,)$ belong to $[0,1]^2$, and set

$$\hat{G}_{t-s}(x,y) = G_{t-s}(x,y) 1_{[0,t]}(s).$$

Then using (2.12) and Lemma B.1 we obtain that

$$\begin{aligned}
&\big|\,\big[\,S(h_2)\,(t,x) - S(h_1)\,(t,x)\,\big] - \big[\,S(h_2)\,(\,\overline{t}\,,\,\overline{x}\,) - S(h_1)\,(\,\overline{t}\,,\,\overline{x}\,)\,\big]\,\big| \le \\
&\le C \int_0^1\!\int_0^1 \Big\{\,\big|\,\hat{G}_{t-s}\,(x,y) - \hat{G}_{t-s}\,(\overline{x},y)\,\big| + \big|\hat{G}_{t-s}\,(\,\overline{x},y) - \hat{G}_{\overline{t}-s}\,(\,\overline{x},y)\,\big|\,\Big\} \\
&\qquad\qquad \big|\,S(h_2)\,(s,y) - S(h_1)\,(s,y)\,\big|\,\big(1 + |\dot{h}_1\,(s,y)|\,\big)\; dy\, ds \\
&\quad + C \int_0^1\!\int_0^1 \Big\{\,\big|\,\hat{G}_{t-s}\,(x,y) - \hat{G}_{t-s}\,(\overline{x},y)\,\big| + \big|\hat{G}_{t-s}\,(\,\overline{x},y) - \hat{G}_{\overline{t}-s}\,(\,\overline{x},y)\,\big|\,\Big\} \\
&\qquad\qquad \big|\dot{h}_2\,(s,y) - \dot{h}_1(s,y)\big|\; dy\, ds \\
&\le C\,\Big(\int_0^1\!\int_0^1 \big|\,\hat{G}_{t-s}\,(x,y) - \hat{G}_{t-s}\,(\overline{x},y)\,\big|^2 dy\, ds\Big)^{\frac{1}{2}}\; \big\|h_2 - h_1\big\|_{\mathcal{H}} \\
&\quad + C\,\Big(\int_0^1\!\int_0^1 \big|\,\hat{G}_{t-s}\,(\,\overline{x}\,,y) - \hat{G}_{\overline{t}-s}\,(\overline{x},y)\,\big|^2 dy\, ds\Big)^{\frac{1}{2}}\; \big\|h_2 - h_1\big\|_{\mathcal{H}} \\
&\le C\,\big(\,|x - \overline{x}|^{\frac{1}{2}} + |t - \overline{t}|^{\frac{1}{4}}\,\big)\; \|h_2 - h_1\|_{\mathcal{H}}\;.
\end{aligned} \tag{2.13}$$

The inequalities (2.12) and (2.13) yield (2.11), which completes the proof of the theorem. $\diamond$

APPENDIX A : HÖLDER NORMS

The following lemma is a straightforward consequence of the Garsia–Rodemich–Rumsey theorem. See also [7] for a similar result.

Lemma A1 Let $\big(Y_n(t,x)\,;\ (t,x)\in[0,1]^2\big)$ be a sequence of $\mathbf{R}^m$–valued stochastic processes and let $p\in(1,\infty)$ satisfy the following assumptions

(P 1) For any $(t,x)\in[0,1]^2$,

$$\lim_n E\left(\left|Y_n(t,x)\right|^p\right)=0\ .$$

(P 2) There exists $\gamma>0$ such that for any (t,x) and $(\,\bar t\,,\,\overline{x}\,)$,

$$\sup_n E\left(\left|Y_n(t,x)-Y_n(\,\bar t\,,\,\overline{x}\,)\right|^p\right)\le C\,\left(|t-\bar t|+|x-\overline{x}|^2\right)^{2+\gamma}\ .$$

Then for any $\alpha\in(0,\ \dfrac{\gamma}{p})$ and any $r\in[1,p)$,

$$\lim_n E\left(\|Y_n\|_\alpha^r\right)=0\ .$$

Proof. Let $z=(t,x)$, $\overline{z}=(\bar t,\overline{x})$ and set $\|z-\overline{z}\|=|t-\bar t|+|x-\overline{x}|^2$. By the Garsia–Rodemich–Rumsey lemma, for any $\beta<\dfrac{\gamma}{p}$, there exists C such that, for every $\lambda>0$,

$$\sup_n P\Big(\sup_{z\neq\overline{z}}\frac{|\,Y_n(z)-Y_n(\,\overline{z}\,)\,|}{\|z-\overline{z}\|^\beta}>\lambda\Big)\le C\,\lambda^{-p}\ . \tag{A.1}$$

Fix a strictly positive integer n_0 (to be specified later on) and set $z_{ij}=\big(\frac{i}{n_0},\frac{j}{n_0}\big)$, $\Delta_{ij}=(\frac{i}{n_0},\frac{i+1}{n_0}]\times(\frac{j}{n_0},\frac{j+1}{n_0}]$, $0\le i,j\le n_0$, and $T=[0,1]^2$.
For every $\lambda>0$ condition (P1) implies that

$$P\Big(\sup_{0\le i,j\le n_0}|\,Y_n(z_{ij})\,|\ge\frac{\lambda}{2}\Big)\le\sum_{0\le i,j\le n_0}P\Big(\big|\,Y_n(z_{ij})\,\big|\ge\frac{\lambda}{2}\Big)\le\frac{C\,(n_0)\;\varepsilon\,(n)}{\lambda^p}\ , \tag{A.2}$$

where $C\,(n_0)$ is a constant depending on n_0 and $\lim_n\ \varepsilon\,(n)=0$. Hence for any $\lambda>0$ and $\alpha<\dfrac{\gamma}{p}$, (A.1) and (A.2) imply

$$P\Big(\sup_{z\in T}|\,Y_n(z)\,|\ge\lambda\Big)\le P\Big(\sup_{0\le i,j\le n_0}|\,Y_n(z_{ij})\,|\ge\frac{\lambda}{2}\Big)$$

$$
\begin{aligned}
&+ P\Big(\sup_{0\le i,j<n_0} \ \sup_{z\in\Delta_{ij}} \ \big| Y_n(z) - Y_n(z_{ij}) \big| \ge \frac{\lambda}{2} \Big) \\
&\le \frac{C\,(n_0)\;\varepsilon(n)}{\lambda^p} + P\Big(\sup_{z\neq\overline{z}} \frac{|\,Y_n(z) - Y_n(\,\overline{z}\,)\,|}{\|z-\overline{z}\|^\alpha} \ge C\;\lambda\;n_0^\alpha\Big) \\
&\le \frac{C\,(n_0)\;\varepsilon\,(n)}{\lambda^p} + \frac{C}{n_0^{\alpha p}\;\lambda^p}\ . \qquad (A.3)
\end{aligned}
$$

Fix a strictly positive integer n_1 and let $\alpha < \dfrac{\gamma}{p}$. Then for every $\lambda > 0$,

$$
P\Big(\sup_{z\neq\overline{z}} \frac{|\,Y_n(z) - Y_n(\,\overline{z}\,)\,|}{\|\,z-\overline{z}\|^\alpha} \ge \lambda\Big) \le A_n + B_n\ ,
$$

with

$$
\begin{aligned}
A_n &= P\Big(\sup_{0<\,\|z-\overline{z}\|\le n_1^{-1}} \frac{|\,Y_n(z) - Y_n(\,\overline{z}\,)\,|}{\|\,z-\overline{z}\|^\alpha} \ge \lambda\Big)\ , \\
B_n &= P\Big(\sup_{\|z-\overline{z}\|> n_1^{-1}} \frac{|\,Y_n(z) - Y_n(\,\overline{z}\,)\,|}{\|\,z-\overline{z}\|^\alpha} \ge \lambda\Big)\ .
\end{aligned}
$$

Let $\delta > 0$ be such that $\alpha+\delta < \dfrac{\gamma}{p}$. Then (A.1) yields

$$
A_n \le P\Big(\sup_{z\neq\overline{z}} \frac{|\,Y_n(z) - Y_n(\,\overline{z}\,)\,|}{\|z-\overline{z}\|^{\alpha+\delta}} \ge \lambda\;n_1^\delta\Big) \le C\;\lambda^{-p}\;n_1^{-\delta p}\ . \qquad (A.4)
$$

Furthermore, (A.3) implies that

$$
B_n \le P\big(\sup_{z\neq\overline{z}} \ \big|\,Y_n(z) - Y_n(\,\overline{z}\,)\big| \ge \lambda\;n_1^{-\alpha}\big) \le \frac{C\,(n_0)\;\varepsilon\,(n)\;n_1^{\alpha p}}{\lambda^p} + C\;\Big(\frac{n_1}{n_0}\Big)^{\alpha p}\;\frac{1}{\lambda^p}\ . \qquad (A.5)
$$

Therefore, the inequalities (A.3) to (A.5) yield that for any $\lambda > 0$, $\alpha < \dfrac{\gamma}{p}$ and $0 < \delta < \dfrac{\gamma}{p} - \alpha$,

$$
P\,\big(\,\|Y_n\|_\alpha \ge \lambda\,\big) \le C\,\Big[\frac{C\,(n_0)\;n_1^{\alpha p}\;\varepsilon\,(n)}{\lambda^p} + \Big(\frac{n_1}{n_0}\Big)^{\alpha p}\;\frac{1}{\lambda^p}\,\Big]\ .
$$

Thus Fubini's theorem implies that for any $a > 0$, $r \in [1,p)$,

$$
\begin{aligned}
E\,\big(\,\|Y_n\|_\alpha^r\,\big) &\le 2a^r + \int_a^{+\infty} r\lambda^{r-1}\;P\big(\,\|Y_n\|_\alpha \ge \lambda\,\big)\;d\,\lambda \\
&\le 2a^r + C\Big[C(n_0)\,n_1^{\alpha p}\;\varepsilon\,(n) + \Big(\frac{n_1}{n_0}\Big)^{\alpha p}\,\Big]\;a^{r-p}\,.
\end{aligned}
$$

Fix $\varepsilon > 0$, and choose $a = \varepsilon$, $n_0 = n_1^2$ such that $\dfrac{1}{{n_1}^{\alpha p}} \le \varepsilon^{1+p-r}$ and finally let N be such that for $n \ge N$, $\varepsilon\,(n)\;C\,(n_0)\;n_1^{\alpha p} \le \varepsilon^{1+p-r}$. Then for $n \ge N$,

$$
E\,\big(\,\|Y_n\|_\alpha^r\,\big) \le \varepsilon^r + C\,\varepsilon\,,
$$

which completes the proof of the lemma. ◇

The following lemma shows that under proper regularity conditions on u_0, the trajectories of the solution X_n of (1.5) almost surely belong to $\mathcal{C}^\alpha\left([0,1]^2\right)$ for any $0 \le \alpha < \dfrac{1}{4}$; see [9] for a similar result.

Lemma A 2. Let u_0 be a 2α-Hölder continuous real function for $0 < \alpha < \frac{1}{4}$; then the solution X_n of (1.5) belongs to $\mathcal{C}^\alpha\left([0,1]^2\right)$ almost surely.

Proof. Theorem 1.10 together with the Garsia–Rodemich–Rumsey lemma clearly implies that $X_n(t,x) - G_t(x,\, u_0)$ a.s. belongs to $\mathcal{C}^\alpha\left([0,1]^2\right)$ for $0 \le \alpha < \dfrac{1}{4}$. Thus, it suffices to check the regularity of $G_t(x,\, u_0)$. Fix $0 \le s \le t$, $x \in [0,1]$. The semi-group property of G implies that

$$\begin{aligned} G_t(x,u_0) - G_s(x,u_0) &= \int_0^1 \int_0^1 G_s(x,y)\, G_{t-s}(y,z)\, u_0(z)\, dy\, dz - \int_0^1 G_s(x,y)\, u_0(y)\, dy \\ &= \int_0^1 G_s(x,y) \Big(\int_0^1 G_{t-s}(y,z)\, \big[u_0(z) - u_0(y) \big]\, dz \Big)\, dy\ . \end{aligned}$$

Hence

$$\begin{aligned} \big| G_t(x,u_0) - G_s(x,u_0) \big| &\le C \int_0^1 G_s(x,y) \int_0^1 G_{t-s}(y,z)\, |z-y|^{2\alpha}\, dz\, dy \\ &\le C \int_0^1 G_s(x,y)\, |t-s|^\alpha\, dy = C\, |t-s|^\alpha\ . \qquad (A.6) \end{aligned}$$

Finally, the definition of $G_t(x,z)$ shows that

$$G_t(x,z) = \varphi_t(z-x) + \varphi_t(z+x)$$

where

$$\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} \sum_{n\in\mathbf{Z}} \exp\Big(-\frac{(x-2n)^2}{4\,t} \Big)\ .$$

Notice that $\varphi_t(\cdot)$ is an even function of period 2 ; hence $\varphi_t(x) = \varphi_t(2-x)$. Let $\eta = y - x$, and assume that $\eta > 0$ without loss of generality. Then

$$\begin{aligned} \big| G_t(x,u_0) - G_t(y,u_0) \big| &\le \Big| \int_0^1 \big[\varphi_t(z-x) - \varphi_t(z-y) \big]\, u_0(z)\, dz \\ &\qquad + \int_0^1 \big[\varphi_t(z+x) - \varphi_t(z+y) \big]\, u_0(z)\, dz \Big| \\ &\le \int_0^{1-\eta} \varphi_t(z-x)\, |u_0(z) - u_0(z+\eta)|\, dz + \int_\eta^1 \varphi_t(z+x)\, |\, u_0(z) - u_0(z-\eta)\, |\, dz \end{aligned}$$

$$
\begin{aligned}
&+ \quad \Big| \int_0^{\eta} \varphi_t(z+x)\, u_0(z)\, dz - \int_{-\eta}^{0} \varphi_t(z-x)\, u_0(z+\eta)\, dz \Big| \\
&+ \quad \Big| \int_{1-\eta}^{1} \varphi_t(z-x)\, u_0(z)\, dz - \int_{1}^{1+\eta} \varphi_t(z+x)\, u_0(z-\eta)\, dz \Big| \\
&\leq C\, \eta^{2\alpha} \int_0^1 G_t(x,z)\, dz + \int_0^{\eta} \varphi_t(z+x)\ |\, u_0(z) - u_0(\eta - z)\,|\, dz \\
&\qquad + \Big| \int_0^{\eta} \big[\varphi_t(1-z-x)\, u_0(1-z) - \varphi_t(z+1+x)\, u_0(1+z-\eta) \big]\, dz \Big| \\
&\leq C\, \eta^{2\alpha} + C \int_0^{\eta} \Big[\ \varphi_t(z+x) |\, u_0(z) - u_0(\eta - z)\,| \\
&\qquad\qquad + \varphi_t(1-z-x)\,|\, u_0(1-z) - u_0(1+z-\eta)\,|\, \Big]\, dz \\
&\leq C\, \eta^{2\alpha} + C \int_0^{\eta} \big[\, G_t(x,z) + G_t(x, 1-z)\,\big]\, (\eta - 2z)^{2\alpha}\, dz \leq C\, |x-y|^{2\alpha}\ . \qquad (A.7)
\end{aligned}
$$

Hence using (A.6) and (A.7), we obtain that for any (t,x) and $(\,\overline{t}\,,\,\overline{x}\,)$ in $[0,1]^2$,

$$
|\, G_t(x, u_0) - G_{\overline{t}}\,(\,\overline{x}, u_0)\,| \leq C\, \big(\, |t - \overline{t}\,| + |x - \overline{x}\,|\,\big)^{\alpha}.
$$

This completes the proof of the lemma. ◇

APPENDIX B : THE GREEN FUNCTION

In this section several properties concerning the fundamental solution of the heat equation either with Neumann or with Dirichlet conditions will be proved, that is, for the functions defined as follows,

$$
G_t(x,y) = \frac{1}{\sqrt{2\,\pi\, t}} \sum_{n \in \mathbf{Z}} \Big\{ \exp\Big(- \frac{(y-x-2n)^2}{4\,t} \Big) + \exp\Big(- \frac{(y+x-2n)^2}{4\,t} \Big) \Big\}\ , \quad (B.1)
$$

and

$$
G_t(x,y) = \frac{1}{\sqrt{2\,\pi\, t}} \sum_{n \in \mathbf{Z}} \Big\{ \exp\Big(- \frac{(y-x-2n)^2}{4\,t} \Big) - \exp\Big(- \frac{(y+x-2n)^2}{4\,t} \Big) \Big\}\ , \quad (B.2)
$$

respectively.

First we recall some well–known facts that will be used repeatedly in the sequel. For instance:

(a) For any $(t,x) \in (0,\, \infty) \times [0,1]$

$$
\int_0^1 G_t(x,y)\ dy = 1. \qquad (B.3)
$$

(b) Semi-group property

$$\int_0^1 G_t(x,y)\, G_s(y,z)\, dy = G_{s+t}(x,z)\ , \tag{B.4}$$

for any $s,t\in(0,\infty)$, $x,z\in[0,1]$.
(c) There exists a constant C such that for every $(t,x,y)\in(0,\infty)\times[0,1]^2$,

$$G_t(x,y) \le \frac{C}{\sqrt{t}}\ \exp\Big(-\frac{(y-x)^2}{2\,t}\Big)\ . \tag{B.5}$$

A consequence of property (c) is the following
(d) For any $q\in(1,3)$,

$$\int_{t_n}^t \int_0^1 G_{t-s}(x,y)^q\ ds\, dy \le C\ a^{-n\frac{3-q}{2}}\ . \tag{B.6}$$

In the sequel $G_t(x,y)$ will be the Green function defined by (B.1); however all the results also hold for (B.2). In order to deal with the singularities of $G_t(x,y)$ the following decomposition will be useful.

$$\begin{aligned} G_t(x,z) = \Big\{ &\frac{1}{\sqrt{2\,\pi\,t}}\ \exp\Big(-\frac{(z-x)^2}{4\,t}\Big) + \frac{1}{\sqrt{2\,\pi\,t}}\ \exp\Big(-\frac{(z+x)^2}{4\,t}\Big) \\ &+ \frac{1}{\sqrt{2\,\pi\,t}}\ \exp\Big(-\frac{(z+x-2)^2}{4\,t}\Big) + H_t(x,z)\Big\}\ , \end{aligned} \tag{B.7}$$

where $H_t(x,z)$ is a smooth function in (t,x,z). The following lemma provides standard regularity properties of u ; the proof, similar to that in Walsh [11], is omitted.

Lemma B.1.

a) Let $\alpha\in(\frac{3}{2}, 3)$. For any $x,y,t\in[0,1]$.

$$I(\alpha) := \int_0^t \int_0^1 \big|\, G_{t-s}(x,z) - G_{t-s}(y,z)\,\big|^\alpha\ dz\, ds \le C\,\big(\,|x-y|^{3-\alpha}\big)\ . \tag{B.8}$$

(b) For any $\alpha\in(1,3)$, $s,t,x\in[0,1]$ with $s\le t$,

$$J(\alpha) := \int_0^s \int_0^1 \Big|G_{t-r}(x,y) - G_{s-r}(x,y)\Big|^\alpha\ dy\, dr \le C\,\big(\,|t-s|^{\frac{3-\alpha}{2}}\big)\ , \tag{B.9}$$

$$K(\alpha) := \int_s^t \int_0^1 \big|\, G_{t-r}(x,y)\,\big|^\alpha\ dy\, dr \le C\,\big(\,|t-s|^{\frac{3-\alpha}{2}}\big)\ . \tag{B.10}$$

The following lemma provides more precise information on the increments of G. For any $t\in[0,1]$ and $k\in\{0,1,\cdots,a^n-1\}$, we write $k\,a^{-n}$ instead of $k\,a^{-n}\wedge t$.

Lemma B.2. Let $\gamma \in (1,3)$, and let $k \in \{0, \cdots, a^{-n}\}$ be an integer.
(i) Let $\eta = |\,\overline{x} - x| > 0$; then it holds that

$$\int_{ka^{-n}}^{(k+1)a^{-n}} \int_0^1 \left| G_{t-s}(x,y) - G_{t-s}(\overline{x},y) \right|^{\gamma} dy\, ds \le \; C \left[a^{-n} \; \eta^{\gamma} + \eta^{3-\gamma} I\,(k,\gamma) \right] , \quad (B.11)$$

with

$$I(k,\gamma) \le \Big\{ \Big[\Big(\frac{t-ka^{-n}}{\eta^2} \wedge 1 \Big)^{\frac{3-\gamma}{2}} - \Big(\frac{t-(k+1)a^{-n}}{\eta^2} \wedge 1 \Big)^{\frac{3-\gamma}{2}} \Big] + \Big[\Big(\frac{t-ka^{-n}}{\eta^2} \vee 1 \Big)^{\frac{3}{2}-\gamma} - \Big(\frac{t-(k+1)a^{-n}}{\eta^2} \vee 1 \Big)^{\frac{3}{2}-\gamma} \Big] \Big\} . \quad (B.12)$$

(ii) Let $h = \overline{t} - t > 0$; then

$$\int_{ka^{-n}}^{(k+1)a^{-n}} \int_0^1 |G_{\overline{t}-s}\,(x,y) - G_{t-s}\,(x,y)|^{\gamma} \, dy\, ds \le C \, \left[a^{-n}\,|\,\overline{t} - t|^{\gamma} + h^{\frac{3-\gamma}{2}} \; J(k,\gamma) \right] , \; (B.13)$$

with

$$J(k,\gamma) \le C \; \Bigg[\Big\{ \Big(\frac{t-ka^{-n}}{h} \; \wedge 1 \Big)^{\frac{3-\gamma}{2}} - \Big(\frac{t-(k+1)\,a^{-n}}{h} \wedge 1 \Big)^{\frac{3-\gamma}{2}} \Big\} + \Big\{ - \Big(\frac{t-k\,a^{-n}}{h} \vee 1 \Big)^{\frac{3}{2}(1-\gamma)} + \Big(\frac{t-(k+1)\,a^{-n}}{h} \vee 1 \Big)^{\frac{3}{2}(1-\gamma)} \Big\} \Bigg] . \; (B.14)$$

Proof. (i) The identity (B.7) yields that

$$\begin{aligned} |\, G_t(\overline{x},y) - G_t(x,y)\,| &\le C\,\eta + \frac{1}{\sqrt{2\,\pi\,t}} \; \Big| \exp\Big(- \frac{(\overline{x}-y)^2}{4\,t} \Big) - \exp\Big(- \frac{(x-y)^2}{4\,t} \Big) \Big| \\ &+ \frac{1}{\sqrt{2\,\pi\,t}} \; \Big| \exp\Big(- \frac{(\overline{x}+y)^2}{4\,t} \Big) - \exp\Big(- \frac{(x+y)^2}{4\,t} \Big) \Big| \\ &+ \frac{1}{\sqrt{2\,\pi\,t}} \; \Big| \exp\Big(- \frac{(\overline{x}+y-2)^2}{4\,t} \Big) - \exp\Big(- \frac{(x+y-2)^2}{4\,t} \Big) \Big| , \end{aligned}$$

Consider the change of variables defined by $t - s = \eta^2\, r$ and $x - y = \eta\; \xi$ (respectively $x + y = \eta\; \xi$ and $x + y - 2 = \eta\; \xi$). Then (B.11) holds with

$$I\,(k,\gamma) = \int_{\frac{t-(k+1)a^{-n}}{\eta^2}}^{\frac{t-k\,a^{-n}}{\eta^2}} \int_{\mathbf{R}} \Big| \frac{1}{\sqrt{2\pi r}} \; \exp\Big(- \frac{\xi^2}{4r} \Big) - \frac{1}{\sqrt{2\pi r}} \; \exp\Big(- \frac{(\xi+1)^2}{4r} \Big) \Big|^{\gamma} d\xi\, dr \,. \; (B.15)$$

The mean value theorem applied for $r>1$ yields that

$$I(k,\gamma)\le C\Bigg[\int_{\frac{t-(k+1)a^{-n}}{\eta^2}\wedge 1}^{\frac{t-ka^{-n}}{\eta^2}\wedge 1} r^{-\frac{\gamma}{2}+\frac12}\Big\{\int_{\mathbf R}\frac{1}{\sqrt{2\pi r}}\;\Big|\exp\Big(-\frac{\xi^2\gamma}{4r}\Big)+\exp\Big(-\frac{(\xi+1)^2\gamma}{4r}\Big)\Big|\,d\xi\Big\}dr$$
$$+\int_{\frac{t-(k+1)a^{-n}}{\eta^2}\vee 1}^{\frac{t-ka^{-n}}{\eta^2}\vee 1} r^{-\frac32\gamma+\frac12}\Big\{\int_{\mathbf R}\frac{1}{\sqrt{2\pi r}}\big(|\xi|^\gamma+|\xi+1|^\gamma\big)\times\Big[\exp\Big(-\frac{\gamma\xi^2}{4r}\Big)+\exp\Big(-\frac{\gamma(\xi+1)^2}{4r}\Big)\Big]\,d\xi\Big\}\,dr\Bigg],$$

which clearly implies (B.12).

(ii) The identity (B.7) yields that

$$\big|\,G_{\overline{t}-s}(x,y)-G_{t-s}(x,y)\,\big|\le C\,h$$
$$+\Big|\frac{1}{\sqrt{2\,\pi\,(\overline{t}-s)}}\;\exp\Big(-\frac{(y-x)^2}{4\,(\overline{t}-s)}\Big)-\frac{1}{\sqrt{2\,\pi\,(t-s)}}\;\exp\Big(-\frac{(y-x)^2}{4\,(t-s)}\Big)\Big|$$
$$+\Big|\frac{1}{\sqrt{2\,\pi\,(\overline{t}-s)}}\;\exp\Big(-\frac{(y+x)^2}{4\,(\overline{t}-s)}\Big)-\frac{1}{\sqrt{2\,\pi\,(t-s)}}\;\exp\Big(-\frac{(y+x)^2}{4\,(t-s)}\Big)\Big|$$
$$+\Big|\frac{1}{\sqrt{2\,\pi\,(\overline{t}-s)}}\;\exp\Big(-\frac{(x+y-2)^2}{4\,(\overline{t}-s)}\Big)-\frac{1}{\sqrt{2\,\pi\,(t-s)}}\;\exp\Big(-\frac{(x+y-2)^2}{4\,(t-s)}\Big)\Big|\,.$$

Consider the change of variables defined by $t-s=h\,v$, and $y-x=\sqrt{h}\;z$ (respectively $y+x=\sqrt{h}\;z$ and $y+x-2=\sqrt{h}\;z$). Then (B.13) holds with

$$J(k,\gamma)=\int_{\frac{t-(k+1)a^{-n}}{h}}^{\frac{t-ka^{-n}}{h}}\int_{\mathbf R}\;\Big|\frac{1}{\sqrt{2\pi(v+1)}}\;\exp\Big(-\frac{z^2}{4(v+1)}\Big)-\frac{1}{\sqrt{2\pi v}}\;\exp\Big(-\frac{z^2}{4v}\Big)\Big|^\gamma dz\,dv\,. \qquad (B.16)$$

Furthermore, the mean value theorem applied for $v>1$ implies that

$$J(k,\gamma)\le C\;h^{\frac{3-\gamma}{2}}\Bigg[\int_{\frac{(t-(k+1)a^{-n}}{h}\wedge 1}^{\frac{t-ka^{-n}}{h}\wedge 1} v^{-\frac{\gamma}{2}+\frac12}\int_{\mathbf R}\Big\{\frac{1}{\sqrt{2\pi v}}\;\exp\Big(-\frac{z^2\gamma}{4v}\Big)+\frac{1}{\sqrt{2\pi(v+1)}}\;\exp\Big(-\frac{z^2\gamma}{4(v+1)}\Big)\Big\}\,dz\,dv$$
$$+\int_{\frac{(t-(k+1)a^{-n}}{h}\vee 1}^{\frac{t-ka^{-n}}{h}\vee 1} v^{-\frac{5\gamma}{2}+\frac12}\int_{\mathbf R}\frac{1}{\sqrt{2\pi(v+1)}}\big(z^{2\gamma}+(v+1)^\gamma\big)\;\exp\Big(-\frac{z^2\gamma}{4(v+1)}\Big)dz\,dv\Bigg]$$

which clearly implies (B.14) . ◇

Lemma B.3. For any measurable process $\Phi = \{\, \Phi(t,x)\,,\ (t,x) \in [0,1]^2 \,\}$, and any $p \in (\frac{3}{2}, +\infty)$,

$$\Big\| \int_{t_n}^t \int_0^1 G_{t-s}\,(x,y)\ \Phi\,(s,y)\ |\dot W_n(s,y)|\ dy\,ds\,\Big\|_p \\ \le\ C\ a^{-n(\frac{1}{2}-\frac{1}{2p})}\ n^{\frac{1}{2}}\ \sup_{s,y} \big\|\,\Phi\,(s,y)\,\big\|_{2p}\ . \qquad (B.17)$$

Proof. Let p and q be conjugate exponents ; then $q \in (1,3)$ and Hölder's inequality implies that the left hand side of (B.17) is bounded by $(\alpha_n\ \beta_n)^{\frac{1}{p}}$, where

$$\alpha_n = \Big(\int_{t_n}^t \int_0^1 G_{t-s}\,(x,y)^q\ dy\,ds \Big)^{\frac{p}{q}}\ ,$$

$$\beta_n = E \int_{t_n}^t \int_0^1 \Big| \Phi\,(s,y)\ |\dot W_n(s,y)|\,\Big|^p\ dy\,ds\ .$$

Property (B.6) yields

$$\alpha_n \le C\ a^{-n(p-\frac{3}{2})}\ .$$

Moreover, by Schwarz's inequality

$$\beta_n \le \int_{t_n}^t \int_0^1 \Big\{ E\big(\,\big|\,\Phi\,(s,y)\,\big|^{2p}\,\big)\, E\big(\,\big|\,\dot W_n(s,y)\,\big|^{2p}\,\big) \Big\}^{\frac{1}{2}}\ dy\,ds \\ \le C\ \sup_{s,y}\ \big\|\,\Phi\,(s,y)\,\big\|_{2p}^p\ n^{\frac{p}{2}}\ a^{n\frac{p}{2}}\ a^{-n}\ .$$

Hence (B.17) is established. ◇

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