# MONOTONIC COCYCLES

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ABSTRACT. We develop a "local theory" of multidimensional quasiperiodic $\mathrm{SL}(2,\mathbb{R})$ cocycles which are not homotopic to a constant. It describes a $C^1$-open neighborhood of cocycles of rotations and applies irrespective of arithmetic conditions on the frequency, being much more robust than the local theory of $\mathrm{SL}(2,\mathbb{R})$ cocycles homotopic to a constant. Our analysis is centered around the notion of monotonicity with respect to some dynamical variable. For such *monotonic cocycles*, we obtain a sharp rigidity result, minimality of the projective action, typical nonuniform hyperbolicity, and a surprising result of smoothness of the Lyapunov exponent (while no better than Hölder can be obtained in the case of cocycles homotopic to a constant, and only under arithmetic restrictions). Our work is based on complexification ideas, extended "à la Lyubich" to the smooth setting (through the use of asymptotically holomorphic extensions). We also develop a counterpart of this theory centered around the notion of monotonicity with respect to a parameter variable, which applies to the analysis of $\mathrm{SL}(2,\mathbb{R})$ cocycles over more general dynamical systems and generalizes key aspects of Kotani Theory. We conclude with a more detailed discussion of one-dimensional monotonic cocycles, for which results about rigidity and typical nonuniform hyperbolicity can be globalized using a new result about convergence of renormalization.

CONTENTS



## 1. INTRODUCTION

Let $f : X \to X$ be a homeomorphism of a compact metric space, preserving a probability measure $\mu$. Given a map $A \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$ the skew-product map on $X \times \mathbb{R}^2$ given by $(x, w) \mapsto (f(x), A(x) \cdot w)$, denoted by $(f, A)$, is called an $\mathrm{SL}(2,\mathbb{R})$ cocycle over $f$.

We will be particularly interested in *quasiperiodic cocycles* where $X = \mathbb{R}^d/\mathbb{Z}^d$ and $f$ is a translation, $f(x) = x + \alpha$ for some $\alpha \in \mathbb{R}^d$ and $\mu$ is Lebesgue measure. Since in this case $f$ is a diffeomorphism of a manifold, it makes sense to consider quasiperiodic cocycles with various degrees of smoothness.

Quasiperiodic cocycles $(f, A)$ have been primarily studied in the case where $A$ is homotopic to a constant (in large part because this is the situation arising in the consideration of quasiperiodic Schrödinger operators). One important aspect was the development of a *local theory*, starting with the KAM based work of Dinaburg-Sinai [DS]. This local theory concerns perturbations of the simplest cocycles homotopic to a constant, which are just the constant ones. The development of the KAM approach involves, as usual, non-resonance assumptions which here are coded in arithmetic conditions involving the frequency vector but also the *fibered rotation number*, and a key achievement, due to Eliasson [E], was the development of a local theory covering all cocycles with Diophantine frequency vector. Except for the one-dimensional case, where considerably more can be achieved by a range of techniques (both non-KAM as in [BJ], [AJ], and non-standard KAM, [AFK]), the work of Eliasson remains basically the best description of the local theory in the case of cocycles homotopic to a constant.

One of our goals here is to develop a local theory of cocycles non-homotopic to a constant, covering (in the ergodic case) perturbations of the simplest cocycles arising in this case, which are the $\mathrm{SO}(2, \mathbb{R})$-valued ones. As it will turn out, the theory we develop is considerably more robust then the usual one, in several respects. For instance, the frequency vector plays no role at all in our considerations, and we are able to treat quite low regularity ($C^1$) perturbations. Moreover, several of our conclusions are in a sense also much stronger, and even surprising from the point of view of the intuition developed in the case of cocycles homotopic to a constant. Specific questions addressed in this paper concern the regularity of the Lyapunov exponents, rigidity arising from zero Lyapunov exponents, and minimality of the associated projective action.

Our local theory centers around the crucial property of monotonicity with respect to some phase (dynamical) variable, a kind of twist condition that arises naturally (in the ergodic case) for $\mathrm{SO}(2, \mathbb{R})$-valued cocycles not homotopic to a constant. There is a counterpart, which actually precedes logically the analysis of monotonic cocycles, which describes the consequences of monotonicity with respect to parameter variables, and works for cocycles over general dynamical systems. Our results, abstract and generalize key aspects of the theory of Schrödinger cocycles (particularly Kotani Theory), in particular to the case of non-analytic dependence on parameters. Besides being fundamental to our analysis of monotonic cocycles, we would also like to point out that the larger flexibility afforded by this theory has been recently applied back to address problems about the Schrödinger case [A2].

Our third focus in this paper concerns the analysis of one-dimensional quasiperiodic cocycles non-homotopic to a constant from the *global* point of view. As in [AK], the basic plan is to reduce global questions to local ones by renormalization. To this end, we prove a "convergence of renormalization" result that guarantees that, under a natural (from the point of view of parameter analysis) hypothesis that renormalizations eventually become monotonic. As a consequence, we obtain a global $L^2$-rigidity result, and conclude that typical cocycles non-homotopic to a constant are nonuniformly hyperbolic.

We will next present more formally some key results of each of the three topics mentioned above.

1.1. **Monotonic cocycles.** Below we will use the notation $(f^n, A_n)$ for the $n$-iterate of the cocycle $(f, A)$: thus if $n \geq 1$, $A_n(x) = A(f^{n-1}(x)) \cdots A(x)$. Let us also recall basic definitions of the Lyapunov exponent

$$L(f,A)=\lim_{n\to\infty}\frac{1}{n}\int \ln\|A_n(x)\|dx, \tag{1.1}$$

and of a conjugacy between cocycles $(f,A)$ and $(f,A')$, which is given by a map $B:X\to \mathrm{SL}(2,\mathbb{R})$ satisfying

$$A'(x)=B(f(x))A(x)B(x)^{-1}. \tag{1.2}$$

Now, and for the remaining of this section, fix $d\geq 1$ and let $X=\mathbb{R}^d/\mathbb{Z}^d$. For $\alpha\in\mathbb{R}^d$ denote by $f_\alpha : X\to X$ the map $f_\alpha(x)=x+\alpha$. Since such dynamics are regular, it makes sense to speak of regular cocycles and regular conjugacies.

We say that $A\in C^1(X,\mathrm{SL}(2,\mathbb{R}))$ is *monotonic* if there exists some $w\in\mathbb{R}^d$ such that for every $x\in X$ and $y\in\mathbb{R}^2\setminus\{0\}$, (any determination of) the argument of $A(x+tw)\cdot y$, $t\in\mathbb{R}$, has positive derivative with respect to $t$.

This condition clearly determines a $C^1$-open subset $\mathcal{M}^1$ of $C^1(X,\mathrm{SL}(2,\mathbb{R}))$. Notice that the monotonicity condition only makes reference to $A$ and is thus independent of a frequency vector. For this reason, it is in particular not invariant by conjugacy. Given a frequency vector $\alpha$, it is thus natural to define a set $\mathcal{M}^1_\alpha\subset C^1(X,\mathrm{SL}(2,\mathbb{R}))$ consisting of all $A$ for which there exists $n\geq 1$ such that $(f_\alpha,A)^n$ is $C^1$-conjugated to some monotonic cocycle. We call cocycles $(f_\alpha,A)$ with $A\in\mathcal{M}^1_\alpha$ *premonotonic.* It is also clear that cocycles homotopic to a constant can never be premonotonic.[1] On the other hand, if $(f_\alpha,A)$ is $C^0$ conjugate to a *cocycle of rotations* (that is, to an $\mathrm{SO}(2,\mathbb{R})$-valued one), and $f_\alpha$ is ergodic, then $(f_\alpha,A)$ is premonotonic as long as $A$ is $C^1$ and not homotopic to a constant.

Obviously any cocycle of rotations, or merely measurably conjugate to such, must have a zero Lyapunov exponent. Our first result shows that for regular (pre-) monotonic cocycles, one can go the other way around:

**Theorem 1.1.** *Let $(f_\alpha,A)$ be $C^r$, $r=\infty,\omega$ and premonotonic. If $L(f_\alpha,A)=0$ then $(f_\alpha,A)$ is $C^r$ conjugate to an $\mathrm{SO}(2,\mathbb{R})$-valued cocycle.*

By the previous discussion, Theorem 1.1 contains in it a rigidity result.

**Corollary 1.2.** *Let $f_\alpha$ be ergodic. If $(f_\alpha,A)$ be $C^r$, $r=\infty,\omega$ is non-homotopic to a constant and $C^0$ conjugate to rotations then it is $C^r$-conjugate to rotations.*

Next we look at the dependence of the Lyapunov exponent. We recall first that in the case of cocycles homotopic to a constant, simple examples show that one should never expect more (as far as the modulus of continuity is concerned) than 1/2-Hölder regularity, and Eliasson's local theory does provide precisely this regularity [Am], but only under arithmetic conditions on the frequency vector. In fact, it is easy to see that in general no statement about the modulus of continuity can be made. Even continuity is only known in the analytic case (a deep global result of Bourgain [B]). Thus the following results were very surprising to the authors:

[1] It is somewhat delicate (in the ergodic case) to construct examples of cocycles not homotopic to a constant which are not premonotonic. A non-negligible (positive measure on parameters) set of such examples can be obtained by forcing a certain behavior of the "critical points" that arise in the approach of Lai-Sang Young [Y] (of Benedicks-Carleson [BC] flavor). We will come back to this issue elsewhere.

**Theorem 1.3.** *Let $(f_\alpha, A_s)$ be a one-parameter analytic family of analytic premonotonic cocycles. Then $s \mapsto L(f_\alpha, A_s)$ is analytic.*

**Theorem 1.4.** *Let $(f_{\alpha(s)}, A_s)$ be a one-parameter $C^\infty$ family of $C^\infty$ premonotonic cocycles. Then $s \mapsto L(f_{\alpha(s)}, A_s)$ is $C^\infty$.*

Since the Lyapunov exponent is a regular function, its zero set can be expected to be some kind of variety, hence Lyapunov exponents should be rare unless the equation $L = 0$ is very degenerate. In fact, since the Lyapunov exponent can not become negative, $DL$ must be zero whenever $L = 0$. We are however able to show non-degeneracy of $D^2L$, which implies:

**Theorem 1.5.** *For fixed $\alpha$, a typical $C^r$, $r = \omega, \infty$, premonotonic cocycle over $f_\alpha$ has positive Lyapunov exponent.*[2]

Though we do obtain several other results, particularly addressing less regular situations, we would like to conclude our discussion, at this introduction, with a result of different flavor. Given a cocycle $(f, A)$, we may define its projective action, which is just the projectivized skew-product on $X \times \mathbb{PR}^2$. The topological dynamics of the projective action is a very interesting subject in itself (see for instance [Bj1], [Bj2], [BjJ], [J2]). Here we prove:

**Theorem 1.6.** *If $f_\alpha$ is ergodic and $(f_\alpha, A)$ is premonotonic with $A \in C^{1+\epsilon}$, then the projective action is minimal.*

Let us point out that dynamical notions of monotonicity also make sense for dynamical systems which are not strict translations, such as the skew-shifts, and some of our results can be carried to a larger generality, see Remark 3.1.

1.2. **Monotonicity in the parameter space.** We return now to the consideration of more general dynamics $f : X \to X$. This time we will be interested in parametrized families of cocycles $(f, A_\theta)$, and we will often assume some base regularity of this dependence (just with respect to $\theta$, since nothing beyond continuity can be made sense with respect to the dynamical variable under our hypothesis). Moreover, we will require the dependence of $A_\theta$ on $\theta$ displays monotonicity: assuming that $\theta \mapsto A_\theta$ is $C^1$, this means that for every $x \in X$, $y \in \mathbb{R}^2 \setminus \{0\}$, (any determination of) the argument of $\theta \mapsto A_\theta(x) \cdot y$ has positive derivative.

Let us consider two key examples where monotonicity arises.

Recall that in the dynamical approach to ergodic Schrödinger operators, the basic object considered is a one-parameter family of cocycles, depending on a parameter $E$, of the form $A^{(E)}(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}$. Though the family $(f, A^{(E)})$ is not monotonic in $E$, its second iterate is (which in fact is just as good for our purposes). This family has of course been considered intensively and much is known about it: Kotani Theory ([Ko], [S], and more dynamically [CJ]), in particular, gives much dynamical information about the set of parameters where the Lyapunov exponent vanishes.

Another family type that has been considered is of the form $\theta \mapsto R_\theta A$, where $A$ is arbitrary and $R_\theta \in \mathrm{SO}(2, \mathbb{R})$ is the rotation of angle $2\pi\theta$. This family displays

[2]The set of premonotonic cocycles with zero Lyapunov exponent has infinite codimension when the number of frequencies $d$ is at least 2, and finite codimension when $d = 1$. We will come back to this issue elsewhere.

obvious monotonicity. Some global aspects of this family were first exploited in [H] to yield examples of cocycles with positive Lyapunov exponents: the average Lyapunov exponent, with respect to $\theta$, is zero if and only if $A$ is a cocycle of rotations. In fact, a later refinement [AB] shows that

$$\int_{\mathbb{R}/\mathbb{Z}} L(f, R_\theta A) d\theta = \int_X \ln \frac{\|A\| + \|A\|^{-1}}{2} d\mu(x), \tag{1.3}$$

so the average Lyapunov exponent depends on $A$ through a very simple formula. The dynamical aspects of Kotani Theory have also been extended to such families.

Both examples we mentioned have in common, besides (some) monotonicity, a very nice global behavior of the *holomorphic* dependence on $\theta$ when $\theta$ is complex. Here we will show that a theory can be constructed without taking into account global aspects: in fact even analyticity can be bypassed. But complexification is still fundamental, and what allows us to consider the smooth case is the use of *asymptotically holomorphic* extensions, a tool first used in dynamics by Lyubich [Ly], in the context of unimodal maps.

In doing this, our key motivation has been the understanding of monotonic cocycles. In fact, if $(f_\alpha, A)$ is monotonic in the dynamical sense, then one can construct a monotonic family $(f_\alpha, A_\theta)$ by setting $A_\theta = A(x+\theta w)$, for some $w \in \mathbb{R}^d$. When we change the parameter, we are not really changing the dynamics (merely the coordinates), but we do get something non-trivial out of it, by applying the parameter results we will obtain. For this purpose, we will obtain analogous of Theorems 1.1 and 1.3. Those parametrized versions correspond respectively to a non-global version of (1.3) and to a well known result of Kotani Theory (see [CJ]). Instead of presenting formal versions of those here, we prefer to mention a different Kotani-type application. Let us say that $(f, A)$ is $L^2$-conjugate to rotations if it is measurably conjugate and the conjugacy $B$ satisfies $\int_X \|B(x)\|^2 d\mu(x) < \infty$.

**Theorem 1.7.** *Let $A_\theta \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$, be a one-parameter family which is monotonic and $C^{2+\epsilon}$ in $\theta$. Then for almost every $\theta$, if $L(f, A_\theta) = 0$ then $(f, A_\theta)$ is $L^2$-conjugate to rotations.*

As we will see in the next section, $L^2$-conjugacy to rotations is a fundamental hypothesis in renormalization theory of one-dimensional quasiperiodic cocycles, so in a sense this result enlarges the set of families along which parameter exclusion arguments can be made before applying renormalization. The ability to analyze in this way arbitrary monotonic deformations turns out to be relevant even if one is ultimately interested in the Schrödinger case, see [A2].

1.3. **One-dimensional quasiperiodic cocycles non-homotopic to a constant.** We consider again the quasiperiodic case, but now restrict attention to the one-dimensional case. In this section $X = \mathbb{R}/\mathbb{Z}$ and $f_\alpha$ will always denote an ergodic translation (thus $\alpha \in \mathbb{R} \setminus \mathbb{Q}$).

Renormalization is a classical tool in the analysis of diffeomorphisms of the circle, where it can be used to reduce global questions to local ones [KS]. Application of renormalization ideas to the case of quasiperiodic cocycles has also proved to be fruitful though in this case the renormalization operator does not always lead to the local situation due to the possible presence of positive Lyapunov exponents. It should thus be basically considered as a tool to explore cocycles with zero Lyapunov exponent; see [K1] for the case of $SU(2)$-valued cocycle. However, zero Lyapunov exponents are not a sufficient condition to achieve the global-local reduction.[3]

In [AK], it is shown that the existence of an $L^2$-conjugacy to rotations is enough to guarantee *precompactness* of the renormalization operator, which is used to extract limits which are cocycles of rotations (up to constant conjugacy). Using Kotani Theory and the local (KAM) description of cocycles homotopic to a constant, this yields a dicothomy (under suitable regularity requirements and arithmetic conditions) for typical energies: the associated Schrödinger cocycle has either a positive Lyapunov exponent, or it is conjugate to a cocycle of rotations.[4]

Though cocycles of rotations which are not homotopic to a constant are premonotonic *if the basis is ergodic*, renormalization affects the base dynamics and in particular may lead to non-ergodic limits.

In order to obtain more precise results, we prove here that the limits of renormalization are in fact of a very special kind, namely of the form $x \mapsto R_{\theta+|\deg|x}$, where deg is the topological degree. In particular, if $|\deg| \neq 0$, one does reach monotonicity. We conclude the following global rigidity result:

**Theorem 1.8.** *Let $(f_\alpha, A)$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, be $C^r$, $r = \omega, \infty$, and non-homotopic to a constant. If $(f_\alpha, A)$ is $L^2$-conjugate to rotations, then it is $C^r$-conjugate to rotations.*

Combined with Theorems 1.7 and 1.5 we conclude:

**Theorem 1.9.** *For fixed $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, a typical $C^r$, $r = \omega, \infty$, cocycle over $f_\alpha$ which is not homotopic to a constant has positive Lyapunov exponent.*

*Remark* 1.1. The first result about the *existence* of positive Lyapunov exponents for cocycles non-homotopic to a constant was obtained in [H], for cocycles of a very specific form (with respect to their global holomorphic extensions, say

$$x \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} R_x, \tag{1.4}$$

$\lambda > 1$ (irrespective of the frequency). Later on, Lai-Sang Young [Y] showed that for more general cocycles, but under a largeness condition, for instance perturbations of (1.4) with $\lambda >> 1$, positive Lyapunov exponents have large probability with respect to the choice of the frequency vector. The method used by Young, quite different from ours, of Benedicks-Carleson type [BC], is based on an inductive scheme which loses control of a positive measure set of parameters, and needs some initial condition to start (creating the need for a largeness assumption).

We would like to point out that our more precise results about convergence of renormalization have been recently applied [AFK] to the case of cocycles homotopic to a constant: together with developments in the local theory, it yields the basic [AK] dichotomy without arithmetic conditions.

[3]Particularly, the analysis of the spectrum of the critical almost Mathieu operator, where the Lyapunov exponent is still zero, does not reduce to the local situation.

[4]Since this result indeed assumes at least a Diophantine condition on $\alpha$, and the cocycle is homotopic to a constant, the conclusion is in fact equivalent to the existence of a conjugacy to a *constant* cocycle of rotations.

1.4. **Structure of the paper.** In section 2, we analyze the consequences of monotonicity with respect to parameters. We start with some aspects of the dynamics of certain $\mathrm{SL}(2, \mathbb{C})$ cocycles, whose action by Möbius transformations preserve the upper half plane when going forward, but not necessarily backwards. We also discuss the crucial notion of *variation of the fibered rotation number*, which is necessary even to formulate several key results. After a few simple applications of the complexification idea in the analytic case, we describe an asymptotically holomorphic framework that allows us to exploit the basic monotonicity phenomenon, and we carry out the basic computations of Kotani Theory in this setting.

Basically, monotonicity is used to guarantee that when the parameter turns complex, the dynamics becomes *uniformly hyperbolic*, and everything depends nicely on parameters. The focus is thus to recover some information when the imaginary part approaches zero. In the analytic case, this is done by appealing to theorems of complex analysis (such as Fatou's Theorem on existence of non-tangential limits). In the asymptotically holomorphic setting, we would like to show that the discrepancy from holomorphicity corresponds to a regular correction, say, of the non-tangential limit. There are competing factors though: while the cocycle behaves "more holomorphically" near real parameters, it also behaves "less uniformly hyperbolic". A key point is thus to give an estimate of the resulting regularity (in practice giving up some derivatives in the process). After this, we are in good shape to collect results such as Theorem 1.7.

In section 3 we then move on to the analysis of monotonic cocycles. We prove easily Theorems 1.1 and 1.3 using the results of section 2, and proceed to look at Theorem 1.4, whose proof does not really fall in the same context: since the dynamics changes we have to revisit the estimates regarding the competition between asymptotic holomorphicity and uniform hyperbolicity, incorporating this additional parameter. Using a somewhat different approach, we next discuss results in low regularity, such as continuity of the Lyapunov exponent in the Lipschitz category. We come back to a more regular situation in the proof of Theorem 1.6, where we apply the Schwarz Reflection Principle, or rather, use that it can not be applied. We conclude with an estimate on the second derivative of the Lyapunov exponent, which implies Theorem 1.5.

In section 4 we specify further to the one-dimensional case, but now with a global focus. We introduce formally the renormalization operator and explain how "convergence of renormalization" combined with the local theory indeed implies Theorem 1.8. We conclude with a proof of convergence of renormalization.

We include two appendices. The first discusses an estimate about the conformal barycenter which is used when taking limits of $L^2$-conjugacies to rotations, which is used to avoid unnecessary parameter exclusions. The second gives a proof of transitivity of the projective action for quasiperiodic cocycles non-homotopic to a constant.

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## 2. Monotonicity in parameter space

In this section, $f : X \to X$ is a fixed homeomorphism of a compact metric space $X$, preserving a fixed probability measure $\mu$, assumed to have full support. Given $A \in C^0(X, \mathrm{SL}(2,\mathbb{C}))$, we use the dynamics $f : X \to X$ to define the iterated matrix products $A_n(x)$, $n \in \mathbb{Z}$, by

(2.1)
$$A_n(x) = A(f^{n-1}(x)) \cdot A_{n-1}(x), \quad A_{-n}(x) = A_n(f^{-n}(x))^{-1}, \quad n \geq -1, \quad A_0(x) = \mathrm{id}\,.$$

The Lyapunov exponent is defined by

$$L(A) = \lim_{n\to\infty} \frac{1}{n} \int_X \ln \|A_n(x)\| d\mu(x). \tag{2.2}$$

As discussed in the introduction, our aim here is to develop a "smooth" version of Kotani Theory, centered around the concept of monotonic dependence with respect to the parameter. Let $I \subset \mathbb{R}$ be an interval. We say that a continuous function $f : I \to \mathbb{R}$ is $\epsilon$-monotonic if for every $x \neq x'$ we have

$$\frac{|f(x') - f(x)|}{|x' - x|} \geq \epsilon. \tag{2.3}$$

This definition naturally extends to functions defined on (or taking values on) $\mathbb{R}/\mathbb{Z}$ (by considering lifts) and on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2 \equiv \mathbb{C}$ (by considering the identification with $\mathbb{R}/\mathbb{Z}$ given by $x \mapsto e^{2\pi i x}$). Naturally, we may distinguish between two types of $\epsilon$-monotonicity, increasing or decreasing.

We say that a continuous one-parameter family of matrices $A_\theta(\cdot) \in \mathrm{SL}(2,\mathbb{R})$ is $\epsilon$-monotonic if, for every $w \in \mathbb{R}^2 \equiv \mathbb{C}$, the function $\theta \mapsto \frac{A_\theta \cdot w}{\|A_\theta \cdot w\|}$ is $\epsilon$-monotonic.

We will be interested in one-parameter families of $\mathrm{SL}(2,\mathbb{R})$ cocycles displaying monotonicity with respect to the parameter variable. Thus, a continuous one-parameter family $A_\theta \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$ is said to be $\epsilon$-monotonic increasing (respectively, decreasing) if for every $x \in X$, the family $\theta \mapsto A_\theta(x)$ is $\epsilon$-monotonic increasing (respectively, decreasing).

For several results, we will need to assume further regularity with respect to the parameter. Let us say that the family $A_\theta$ is $C^r$ in $\theta \in J$ if $\theta \mapsto A_\theta(x)$ belongs to some fixed compact subset of $C^r(J, \mathrm{SL}(2,\mathbb{R}))$ (compact open topology) for each $x$.

Among the results we will obtain in this section, we highlight the following ones:

**Theorem 2.1.** *Let $A_\theta \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$ be monotonic and $C^{r+1+\epsilon}$, $1 \leq r < \infty$, $C^\infty$, or $C^\omega$ in $\theta$. If $L(A_\theta) = 0$ for every $\theta$ in some open interval $J$ then there exists $B_\theta \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in J$ depending $C^r$, $C^\infty$ or $C^\omega$ on $\theta$ and conjugating $A_\theta$ to a cocycle of rotations.*

**Theorem 2.2.** *Let $A_{\theta,s} \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in \mathbb{R}/\mathbb{Z}$, $s$ a one-dimensional real parameter, be monotonic in $\theta$ and $C^{2r+1+\epsilon}$, $1 \leq r < \infty$, $C^\infty$ or $C^\omega$, in $(\theta, s)$. Then*

$$s \mapsto \int_{\mathbb{R}/\mathbb{Z}} L(A_{\theta,s}) d\theta \tag{2.4}$$

*is $C^r$, $C^\infty$, or $C^\omega$.*

This theorem implies for instance that $A \mapsto \int_{\mathbb{R}/\mathbb{Z}} L(R_\theta A) d\theta$ is an analytic function of $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. Indeed, it can be shown (see [AB]) that

$$\int_{\mathbb{R}/\mathbb{Z}} L(R_\theta A) d\theta = \int_X \ln \frac{\|A(x)\| + \|A(x)\|^{-1}}{2} d\mu(x). \tag{2.5}$$

This generalization beyond families with specific form such as $R_\theta A$ will be crucial when we start to mix phase and parameter in the analysis of quasiperiodic cocycles displaying monotonicity with respect to some *phase* variable.

We will also obtain several other results whose statements depend on the concept of "variation of the fibered rotation number", which we will first need to introduce.

2.1. **Complexification.** Much of the information we will get from matrices in $\mathrm{SL}(2,\mathbb{C})$ will come from their action on the Riemann Sphere $\overline{\mathbb{C}}$ through Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}, \tag{2.6}$$

For $A \in \mathrm{SL}(2,\mathbb{C})$, let $\mathring{A} = QAQ^{-1}$ where

$$Q = \frac{-1}{1+i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \tag{2.7}$$

The map $A \mapsto \mathring{A}$ maps bijectively $SL(2,\mathbb{R})$ to $SU(1,1)$, the real Lie group of matrices $\begin{pmatrix} u & \bar{v} \\ v & \bar{u} \end{pmatrix}$, $u, v \in \mathbb{C}$ such that $|u|^2 - |v|^2 = 1$.[5]

2.2. **Variation of the fibered rotation number.** In the analysis of Schrödinger cocycles, the notion of fibered rotation number plays a fundamental role (often through the analysis of its close cousin, the integrated density of states). In our setting, it turns out that it is not always possible to define "the" fibered rotation number of a cocycle. However, we will be able to define the notion of *variation of the fibered rotation number* along a path.

Define $\Upsilon$ as the space of $\mathrm{SL}(2,\mathbb{C})$ matrices $A$ such that $\mathring{A} \cdot \mathbb{D} \subset \mathbb{D}$ (or equivalently $A \cdot \mathbb{H} \subset \mathbb{H}$).

Let $A \in \Upsilon$. Define $\tau_A : \overline{\mathbb{D}} \to \mathbb{C} \setminus \{0\}$ by

$$\mathring{A} \begin{pmatrix} z \\ 1 \end{pmatrix} = \tau_A(z) \cdot \begin{pmatrix} \mathring{A} \cdot z \\ 1 \end{pmatrix}. \tag{2.8}$$

Since $\overline{\mathbb{D}}$ is simply connected there exists a map $\hat{\tau}_A : \overline{\mathbb{D}} \to \mathbb{C}$ such that $e^{2\pi i \hat{\tau}_A(z)} = \tau_A(z)$; any other lift is obtained by the addition of an integer. If we denote by $\hat{\Upsilon}$ the universal cover of $\Upsilon$ considered as a topological semi-group with unity $\hat{\mathrm{id}}$, there exists a unique continuous map $\hat{\tau} : \hat{\Upsilon} \times \overline{\mathbb{D}} \to \mathbb{C}$ such that $\hat{\tau}(\hat{\mathrm{id}}, z) = 0$ and $e^{2\pi i \hat{\tau}(\hat{A}, z)} = \tau_A(z)$. This map satisfies

$$\hat{\tau}(\hat{A}_2 \hat{A}_1, z) = \hat{\tau}(\hat{A}_2, \mathring{A}_1 \cdot z) + \hat{\tau}(\hat{A}_1, z). \tag{2.9}$$

We note that for any $\hat{A} \in \hat{\Upsilon}$ and any $z, z' \in \overline{\mathbb{D}}$

$$\Im \hat{\tau}(\hat{A}, z) = -\frac{1}{2\pi} |\ln \tau_A(z)|$$

$$|\Re \hat{\tau}(\hat{A}, z) - \Re \hat{\tau}(\hat{A}, z')| < 1/2.$$

The equality is trivial, while the inequality follows from the fact that for any $A$, $\tau_A(\overline{\mathbb{D}})$ is contained in an open half plane (it is enough to observe that if $\mathring{A} =$

[5] If one identifies the complex one-dimensional projective space $\mathbb{CP}^1$ with $\overline{\mathbb{C}}$, by associating to the line through $(0,0) \neq (x,y) \in \mathbb{C}^2$ the complex number $\frac{x-iy}{x+iy}$, the action of $A \in \mathrm{SL}(2,\mathbb{C})$ on $\mathbb{CP}^1$ is given precisely by $z \mapsto \mathring{A} \cdot z$. In this identification, the real one-dimensional projective space $\mathbb{RP}^1$ corresponds to the unit circle $\partial\mathbb{D}$, and $\mathrm{SL}(2,\mathbb{R})$ matrices preserve the unit disk $\mathbb{D}$.

$\begin{pmatrix} u & \overline{v} \\ v & \overline{u} \end{pmatrix}$ then $\tau_A(z) = vz + \overline{u}$, so $\tau_A(\overline{\mathbb{D}})$ does not intersect the line through $i\overline{u}$, since $|u|^2 - |v|^2 = 1$).

Now if $\gamma : [0,1] \to \Upsilon$ is continuous, and $\hat{\gamma} : [0,1] \to \hat{\Upsilon}$ is a continuous lift, we define $\delta_\gamma \hat{\tau}(z_0, z_1) = \hat{\tau}(\hat{\gamma}(1), z_1) - \hat{\tau}(\hat{\gamma}(0), z_0)$; notice that this is independent of the choice of the lift $\hat{\gamma}$.

Let us note a nice composition rule: given $\gamma$ and $\gamma'$, let $\gamma'\gamma(t) = \gamma'(t)\gamma(t)$. Then

$$\delta_{\gamma'\gamma}\hat{\tau}(z_0, z_1) = \delta_{\gamma'}\hat{\tau}(\mathring{\gamma}(0)z_0, \mathring{\gamma}(1)z_1) + \delta_\gamma \hat{\tau}(z_0, z_1). \tag{2.10}$$

Consider now a continuous path $\gamma \in C^0([0,1], C^0(X, \Upsilon)) = C^0([0,1] \times X, \Upsilon)$. Define $\delta_\gamma \xi : X \times \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}$ by $\delta_\gamma \xi(x, z_0, z_1) = \delta_{\gamma_x} \hat{\tau}(z_0, z_1)$, where $\gamma_x(t) = \gamma(t, x)$.

Using the dynamics $f : X \to X$, we define paths $\gamma_n \in C^0([0,1], C^0(X, \Upsilon))$ by putting $\gamma_n(t, x) = \gamma(t, f^{n-1}(x)) \cdots \gamma(t, x)$. Define $\delta_\gamma \xi_n \in C^0(X \times \overline{\mathbb{D}} \times \overline{\mathbb{D}}, \mathbb{C})$ by $\delta_\gamma \xi_n = \frac{1}{n} \delta_{\gamma_n} \xi$. We have an expression for $\delta_\gamma \xi_n$ as a Birkhoff average (for the dynamical system $(x, z_0, z_1) \mapsto (f(x), \mathring{\gamma}(0, x) \cdot z_0, \mathring{\gamma}(1, x) \cdot z_1)$ acting on $X \times \overline{\mathbb{D}} \times \overline{\mathbb{D}}$):

$$\delta_\gamma \xi_n(x, z_0, z_1) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_\gamma \xi(f^k(x), \mathring{\gamma}_k(0, x) \cdot z_0, \mathring{\gamma}_k(1, x) \cdot z_1). \tag{2.11}$$

This is obtained by the composition formula (2.10).

We claim that $\lim_{n\to\infty} \delta_\gamma \xi_n(x, z_0, z_1)$ exists for $\mu$-almost every $x$, and is independent of $z_0, z_1 \in \mathbb{D}$. We will show this by proving convergence for the real and imaginary parts.

Since for every $z_0, z_1 \in \overline{\mathbb{D}}$ we have $|\Re(\delta_\gamma \xi_n(x, z_0, z_1) - \delta_\gamma \xi_n(x, z_0', z_1'))| < \frac{1}{n}$, it follows from Birkhoff Ergodic Theorem that

$$\lim_{n\to\infty} \Re \delta_\gamma \xi_n(x, z_0, z_1) \tag{2.12}$$

exists and is independent of $z_0, z_1 \in \overline{\mathbb{D}}$ for $\mu$-almost every $x$. We call the $\mu$-average of (2.12) the *variation of the fibered rotation number* along $\gamma$ and we denote it by $\delta_\gamma \rho$. It is obviously invariant by homotopy, and it is a continuous function of $\gamma \in C^0([0,1], C^0(X, \Upsilon))$.

*Remark* 2.1. If $\mu$ is ergodic, (2.12) is $\mu$-almost everywhere constant. If $f$ is uniquely ergodic, (2.12) exists for every $x$ and is constant. If the iterates of $f$ are uniformly equicontinuous (for instance, if $f$ is a translation of the torus, but perhaps non-ergodic), (2.12) exists for every $x$, and is a continuous function of $x \in X$.

The convergence of the imaginary part of $\delta_\gamma \xi_n(x, z_0, z_1)$ is somewhat more delicate, and it is certainly less robust. For $A \in C^0(X, \mathrm{SL}(2, \mathbb{C}))$ the Lyapunov exponent is defined by

$$L(A, x) = \lim_{n\to\infty} \frac{1}{n} \ln \|A_n(x)\| \tag{2.13}$$

(which exists $\mu$-almost everywhere by subadditivity). When $A \in C^0(X, \Upsilon)$, a simple application of the Oseledets Theorem shows that for any $z \in \mathbb{D}$

$$L(A, x) = \lim_{n\to\infty} \frac{1}{n} \ln \left\| \mathring{A}_n(x) \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} \right\| \tag{2.14}$$

since $\begin{pmatrix} z \\ 1 \end{pmatrix}$ can not belong to the *stable direction* (this is due to the fact that in that case $A_k(x) \in \Upsilon$ for every $k \geq 0$).

**Lemma 2.3.** *For $\mu$-almost every $x \in X$ and for every $z_0, z_1 \in \mathbb{D}$,*

$$\lim_{n\to\infty} \Im \delta_\gamma \xi_n(x, z_0, z_1) = \frac{1}{2\pi}(L(\gamma(0), x) - L(\gamma(1), x)). \tag{2.15}$$

*Proof.* Notice that if $A \in C^0(X, \mathrm{SL}(2,\mathbb{C}))$

$$\ln \left\| \prod_{k=n-1}^{0} \mathring{A}(f^k(x)) \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} \right\| = \ln |\tau_{A_n(x)}(z)| + \ln \left\| \begin{pmatrix} \prod_{k=n-1}^{0} \mathring{A}(f^k(x)) \cdot z \\ 1 \end{pmatrix} \right\|, \tag{2.16}$$

and the second term in the right hand side is bounded. □

We let $\delta_\gamma \zeta$ be the $\mu$-average of $\lim \delta_\gamma \xi_n(x, 0, 0)$. As we have seen, its real part is $\delta_\gamma \rho$ and its imaginary part is $\frac{1}{2\pi}(L(\gamma(0)) - L(\gamma(1))$, where $L(A)$ is defined as in the introdution:

$$L(A) = \lim_{n\to\infty} \frac{1}{n} \int \ln \|A_n\| d\mu. \tag{2.17}$$

*Remark* 2.2. Notice that $\delta_\gamma \zeta$ behaves well under concatenation: if $\gamma$, $\gamma'$ and $\tilde{\gamma}$ are such that $\gamma(1) = \gamma'(0)$ and $\tilde{\gamma}$ is homotopic to the concatenation of $\gamma$ and $\gamma'$ then $\delta_{\tilde{\gamma}} \zeta = \delta_{\gamma'} \zeta + \delta_\gamma \zeta$.

2.2.1. *Invariant section.* Assume that for every $x \in X$, we have $A(x) \cdot \overline{\mathbb{D}} \subset \mathbb{D}$. In this case, by the Schwarz Lemma, $A(\cdot)$ uniformly contracts the Poincaré metric of the disk and there exists $m \in C^0(X, \mathbb{D})$ satisfying

$$m(f(x)) = \mathring{A}(x) \cdot m(x), \tag{2.18}$$

and we have for every $(x, z) \in X \times \mathbb{D}$,

$$\lim_{n\to\infty} \mathring{A}_n(f^{-n}(x)) \cdot z = m(x). \tag{2.19}$$

Thus

$$\delta_\gamma \zeta = \int \delta_\gamma \xi(x, m_{\gamma(0)}(x), m_{\gamma(1)}(x)) d\mu(x), \tag{2.20}$$

where $m_{\gamma(t)}$ stands for the invariant section corresponding to $A = \gamma(t)$.

Notice that there is another formula for the Lyapunov exponent in terms of $m$. Let $q(x)$ be the norm of the derivative of the holomorphic function $z \mapsto \mathring{A}(x) \cdot z$ at $z = m(x)$, with respect to some conformal Riemannian metric $\|\cdot\|_z$ on $\mathbb{D}$. In this case

$$L = \frac{1}{2} \int -\ln q(x) d\mu(x). \tag{2.21}$$

The most convenient metric to consider is the Poincaré metric $(1 - |z|^2)^{-1}|dz|$, since it enables us to apply the Schwarz Lemma. Notice that whenever

$$\mathring{A} \begin{pmatrix} z \\ 1 \end{pmatrix} = \tau \begin{pmatrix} \tilde{z} \\ 1 \end{pmatrix} \tag{2.22}$$

one has

$$\left| \frac{d\tilde{z}}{dz} \right| = \frac{1}{|\tau|^2}, \qquad q = \left| \frac{d\tilde{z}}{dz} \right| \frac{1 - |z|^2}{1 - |\tilde{z}|^2}, \tag{2.23}$$

and (2.21) can be also obtained as a consequence of (2.18) (which implies that $L = \int \ln |\tau_{A(x)}(m(x))| d\mu(x)$).

The Schwarz Lemma yields, for instance, the following estimate of the Lyapunov exponent. If $\mathring{A}(x)\cdot\mathbb{D}\subset\mathbb{D}_{e^{-\epsilon}}$ for every $x\in X$ then the composition $m\mapsto\tilde m=\mathring{A}(x)\cdot m\mapsto e^{\epsilon}\tilde m$ sends $\mathbb{D}$ to itself and hence weakly contracts Poincaré metric:

$$\frac{|d(e^{\epsilon}\tilde m)|}{1-|e^{\epsilon}\tilde m|^2}\le\frac{|dm|}{1-|m|^2}\tag{2.24}$$

from which one gets

$$\frac{|d\tilde m|}{1-|\tilde m|^2}\le e^{-\epsilon}\frac{1-e^{2\epsilon}|\tilde m|^2}{1-|\tilde m|^2}\frac{|dm|}{1-|m|^2}.\tag{2.25}$$

Whence

$$q(x)^{-1}\ge e^{\epsilon}\frac{1-|m(f(x))|^2}{1-e^{2\epsilon}|m(f(x))|^2}\ge e^{\epsilon}\tag{2.26}$$

so that $L\ge\frac{\epsilon}{2}$.

2.2.2. *Fibered rotation function.* Let us now consider a continuous family $A_\theta\in C^0(X,\Upsilon)$, where $\theta$ belongs to some connected Banach manifold $M$. Since $\delta_\gamma\zeta$ depends only on the homotopy class of $\gamma:[0,1]\to C^0(X,\Upsilon)$, and it behaves well under concatenation, see Remark 2.2, we can define a map $\zeta:\tilde M\to\mathbb{C}$ ($\tilde M$ being the universal cover of $M$) such that for every path $\tilde\gamma:[0,1]\to\tilde M$, we have $\delta_\gamma\zeta=\zeta(\tilde\gamma(1))-\zeta(\tilde\gamma(0))$, where $\gamma(t)=A_{\pi(\tilde\gamma(t))}$ and $\pi:\tilde M\to M$ is the canonical projection. Moreover, we can take $\zeta$ so that for $\pi(\tilde\theta)=\theta$ we have $-2\pi\Im\zeta(\tilde\theta)=L(A_\theta)$. We shall then denote $\rho(\tilde\theta)=\Re\zeta(\tilde\theta)$ or with an abuse of notation $\rho(\theta)$ or $\rho_{A_\theta}$. Though $\rho$ is only defined up to a real constant, it makes sense to speak about the derivative of $\rho$, and if $M=\mathbb{R}$ or $M=\mathbb{R}/\mathbb{Z}$, we may ask whether $\rho$ is monotonic or not.

2.3. **Simple applications.** We now turn to one-parameter continuous families $\theta\mapsto A_\theta(\cdot)\in C^0(X,\mathrm{SL}(2,\mathbb{R}))$. To keep definite and to avoid superfluous notations, we will consider only the case when the parameter space is $\mathbb{R}/\mathbb{Z}$. To fix ideas, we will always assume in the proofs below that $A_\theta(\cdot)$ is monotonic decreasing. Note that due to our identification of $\mathbb{PR}^2$ with $\partial\mathbb{D}$, the projectivization map $\mathbb{S}^1\to\mathbb{PR}^2$ is orientation reversing. Thus if $\theta\mapsto A_\theta(\cdot)$ is monotonic decreasing then for each $x\in X$, $z\in\partial\mathbb{D}$, $\theta\mapsto\mathring{A}_\theta(x)\cdot z$ is monotonic increasing.

Let us show how to deduce the analytic case of Theorem 2.2. Let $\Omega_\delta=\{z\in\mathbb{C}/\mathbb{Z},\,|\Im(z)|<\delta\}$, $\Omega_\delta^\pm=\{z\in\mathbb{C}/\mathbb{Z},\,0<\pm\Im(z)<\delta\}$, and let $\tilde\Omega_\delta$, $\tilde\Omega_\delta^\pm$, be their universal covers.

If $\theta\mapsto A_\theta$ is assumed to be monotonic and analytic, the Cauchy Riemann equations imply that there exists $\delta>0$ such that the analytic extension of $A_\theta$ to $\theta\in\Omega_\delta^+$ satisfies

$$\mathring{A}_\theta\cdot\mathbb{D}\subset\mathbb{D}_{e^{-2\epsilon\Im(\theta)+\kappa(\Im(\theta))}},\quad 0<\Im\theta\le\delta,\tag{2.27}$$

where $\kappa(t)<2\epsilon t$ for $0<t\le\delta$ and $\lim_{t\to0}\kappa(t)/t=0$. As we saw in section 2.2.1, this implies that $L(A_\theta)\ge\epsilon\Im\theta-\frac{1}{2}\kappa(\Im\theta)$.

Indeed, for fixed $z\in\partial\mathbb{D}$, $x\in X$, the Cauchy-Riemann equations applied to the analytic function $F_{x,z}:\Omega_\delta^+\to\mathbb{D}$ defined by $F_{x,z}(\sigma+it)=\mathring{A}_{\sigma+it}(x)\cdot z$ shows that $(\partial_\sigma F_{x,z}(\theta),\partial_t F_{x,z}(\theta))$ is a directed orthogonal basis with vectors of equal length when $\theta\in\mathbb{R}/\mathbb{Z}$. Since $\partial_\sigma F_{x,z}(\theta)$ is tangent to the circle $\partial\mathbb{D}$, and of length at least $2\epsilon$ (by monotonicity), we see that $\partial_t F_{x,z}(\theta)$ is radial (pointing to the origin because of the sign assumption) and of length at least $2\epsilon$ for any $\theta \in \mathbb{R}/\mathbb{Z}$. This proves the above estimate (2.27).

By the previous discussion, we can choose $\delta > 0$ so that for $0 < t \leq \delta$ we have $\mathring{A}_{\sigma+it,s}(x) \cdot \overline{\mathbb{D}} \subset \mathbb{D}$ and we are thus in a situation where the discussion of section 2.2 applies since $A_{\sigma+it,s}(\cdot) \in C^0(X, \Upsilon)$: there exists a function $\zeta : \tilde{\Omega}_\delta^+ \to \mathbb{C}$ such that on the closure of $\tilde{\Omega}_\delta^+$ the map $\theta \mapsto \rho(\theta) = \Re\zeta(\theta)$ is continuous and $\Im\zeta(\theta) = -\frac{1}{2\pi}L(A_\theta)$; also, since $\theta \mapsto A_\theta$ is holomorphic on $\Omega_\delta^+$, the same is true for $\zeta$. We will use the notation $m^+(z,x)$ for the $\mathbb{D}$-valued invariant section of the cocycle $A_z$.

Now if $(\theta, s) \mapsto A_{\theta,s}$ is analytic and if $s$ is in some fixed neighborhood of $s_0$, we can choose $\delta > 0$ so that for $0 < t \leq \delta$ we have $\mathring{A}_{\sigma+it,s}(x) \cdot \overline{\mathbb{D}} \subset \mathbb{D}$. Let then

$$U(t,s) = \int_{\mathbb{R}/\mathbb{Z}} L(A_{\sigma+it,s})d\sigma, \tag{2.28}$$

so that for $t \neq 0$ we have

$$U(t,s) = \int_{X\times\mathbb{R}/\mathbb{Z}} \ln|\tau_{A_{\sigma+it,s}(x)}(m_s^+(\sigma+it,x))|d\mu(x)d\sigma. \tag{2.29}$$

Then for $0 < t < \delta$ the map $s \mapsto U(t,s)$ is analytic. Moreover, the map $(\sigma+it, z) \mapsto \tau_{A_{\sigma+it,s}(x)}(z)$ (defined on $\Omega_\delta^+ \times \mathbb{D}$ is holomorphic and non zero and so $\int_X \ln|\tau_{A_{\sigma+it,s}(x)}(m_s^+(\sigma+it,x))|d\mu(x)$ is harmonic w.r.t $\sigma+it \in \Omega_\delta^+$; its integral w.r.t $\sigma$, given by $U(t,s)$, is thus an affine function of $0 < t < \delta$ since it is harmonic in $\sigma+it$ and does not depend on $\sigma$.

On the other hand, since $\sigma+it \mapsto A_{\sigma+it,s}(x)$ is holomorphic, the functions $\sigma+it \mapsto \ln\|(A_{\sigma+it,s})_n(x)\|$ are subharmonic and the same is true for $\sigma+it \mapsto L(A_{\sigma+it,s})$. Notice that $U(t,s) = U(-t,s)$ since $\sigma+it \mapsto L(A_{\sigma+it,s})$ is real symmetric, so by subharmonicity, $U(t,s)$ is an affine function of $|t|$ for $0 \leq |t| < \delta$ ($t \mapsto U(t,s)$ is convex and thus continuous in $t$). Thus, for $0 < t < \delta$ we have

$$\int_{\mathbb{R}/\mathbb{Z}} L(A_{\theta,s})d\theta = 2U(\frac{t}{2},s) - U(t,s), \tag{2.30}$$

is analytic on $s$, as desired.

*Remark* 2.3. With a little bit more work, one can get the formula

$$\int_{\mathbb{R}/\mathbb{Z}} L(A_{\theta,s})d\theta = U(t,s) - 2\pi t \deg, \tag{2.31}$$

for $0 < t < \delta$, where deg is the variation of the fibered rotation number as $\theta$ runs once around $\mathbb{R}/\mathbb{Z}$.[6] Indeed, for fixed $s$, let $\rho(\sigma+it,s) : \tilde{\Omega}_\delta^+ \to \mathbb{R}$ be a continuous determination of $\rho_{A_{\sigma+it,s}}$, so that $\deg = \rho(\sigma+it+1,s) - \rho(\sigma+it,s)$. Then the function

$$\int_\sigma^{\sigma+1} -\frac{i}{2\pi}L(A_{y+it,s}) + \rho(y+it,s)dy \tag{2.32}$$

is holomorphic in $\sigma+it \in \tilde{\Omega}_\delta^+$ and its real part is an affine function of $\sigma$ of slope deg. Thus the function $U(t,s)$ defined above is an affine function of $0 < t < \delta$ (Cauchy-Riemann) with slope $2\pi \deg$, and (2.31) follows.

[6]It is easy to see that deg is just the $\mu$-average of $\deg(x)$, where $\deg(x)$ is the topological degree of $\theta \mapsto A_{\theta,s}(x)$ as a map $\mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ (in particular, if $X$ is connected, deg is an integer).

Let us now describe another application, the analogous of a basic derivative bound in Kotani Theory. To set it up, let us state an obvious consequence of monotonicity (following directly from the definitions).

**Lemma 2.4.** *Let $A_\theta(\cdot) \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$, be a one-parameter family monotonic decreasing in $\theta$. Then the fibered rotation number is non-increasing as a function of $\theta$.*

**Theorem 2.5.** *Let $A_\theta(\cdot) \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in \mathbb{R}/\mathbb{Z}$, be analytic and monotonic decreasing in $\theta$. For Lebesgue a.e $\theta \in \mathbb{R}/\mathbb{Z}$, if $L(A_\theta) = 0$ then*

$$-\frac{d}{d\theta}\rho(\theta) \geq \frac{\epsilon}{2\pi} > 0, \tag{2.33}$$

*where $\epsilon$ is the monotonicity constant of $\theta \mapsto A_\theta(\cdot)$.*

*Proof.* Using the analyticity in $\theta$ of $A_\theta$, we can conclude that $\theta \mapsto \zeta(\theta)$ can be defined as a holomorphic function $\tilde{\Omega}_\delta^+ \to \mathbb{H}$. We know that the real part of $\zeta(\theta)$ (the "fibered rotation number") is continuous up to the closure. For $\sigma \in \mathbb{R}$,

$$\Im(\zeta(\sigma + it)) = \Im(\zeta(\sigma + i0^+)) + \int_{0^+}^t \partial_s \Im(\zeta(\sigma + is))ds, \tag{2.34}$$

and using the Cauchy-Riemann equations

$$\Im(\zeta(\sigma + it) = \Im(\zeta(\sigma + i0^+)) + \int_{0^+}^t \partial_\sigma \Re\zeta(\sigma + is)ds, \tag{2.35}$$

Since the map $\Re\zeta(\cdot)$ is harmonic on $\Omega_\delta^+$, continuous on the closure of $\tilde{\Omega}_\delta^+$ and its restriction to $\Im\theta = 0$ is non-increasing, one can say[7] that for Lebesgue a.e $\sigma \in \mathbb{R}/\mathbb{Z}$

$$\lim_{s \to 0} \partial_\sigma \Re\zeta(\sigma + is) = \frac{d}{d\sigma}\rho(\sigma). \tag{2.36}$$

Since the Lyapunov exponent is upper semicontinuous (it is by subadditivity the infimum of the continuous functions $\theta \mapsto (1/n) \int_X \ln \|(A_\theta)_n(x)\| d\mu(x))$, if we know additionally that $L(A_\sigma) = 0$ it becomes continuous at the point $\sigma$ and we have

$$-\lim \frac{1}{2\pi t} L(A_{\sigma+it}) = \frac{d}{d\sigma}\rho(\sigma), \tag{2.37}$$

almost surely, and the result follows (since $L(A_{\sigma+it}) \geq \epsilon t - \frac{1}{2}\kappa(t)$). ☐

2.4. **General framework.** After the motivation above, we are ready to introduce a more general framework for the complexification argument.

It may look like analyticity is crucial in order to exploit the complexification approach. This is not the case: in the non-analytic case, we can still complexify the problem using asymptotically holomorphic extensions (this idea is inspired from the work of Lyubich on smooth unimodal maps [Ly]).

Let $\Delta_\delta$ be the space of all continuous families $A_{\sigma+it}(\cdot) \in C^0(X, \mathrm{SL}(2,\mathbb{C}))$, $\sigma+it \in \Omega_\delta$, which are $C^1$ and real-symmetric in $\sigma+it$, satisfying $A_{\sigma+it} \in \operatorname{int} \Upsilon$, $\sigma+it \in \Omega_\delta^+$,

$$\overline{\partial}_z A_z = 0, \quad \text{if } \Im(z) = 0, \tag{2.38}$$

[7]A continuous harmonic function $f$ on the disk is the Poisson integral of its restriction $\rho$ to the boundary of the disk. If $\rho$ is of bounded variation, the tangential derivative $\partial_\sigma f$ is the Poisson integral of the measure $d\rho$. Fatou theorem asserts that for a.e point on the boundary, the radial limit of $\partial_\sigma f$ is $d\rho/d\sigma$. The situation on the strip is easily reduced to the one on the disk.

and such that $\sigma \mapsto A_\sigma(\cdot)$ is monotonic in $\sigma$. Condition (2.38) is an asymptotic holomorphicity assumption, some stronger forms of which we will later introduce.

Let us fix $A \in \Delta_\delta$. Then we have functions $m^+(\sigma+it,x) \in \mathbb{D}$, $\tau^+(\sigma+it,x) \in \mathbb{C} \setminus \{0\}$, $\sigma+it \in \Omega_\delta^+$, $x \in X$, characterized by

$$\mathring{A}_{\sigma+it}(x) \cdot \begin{pmatrix} m^+(\sigma+it,x) \\ 1 \end{pmatrix} = \tau^+(\sigma+it,x) \begin{pmatrix} m^+(\sigma+it,f(x)) \\ 1 \end{pmatrix}. \tag{2.39}$$

Notice that $A_{\sigma+it}(x)^{-1} \in \operatorname{int} \Upsilon$ for $\sigma+it \in \Omega_\delta^-$. Thus we have also functions $m^-(\sigma+it,x)$, $\tau^-(\sigma+it,x)$, $\sigma+it \in \Omega_\delta^-$, $x \in X$, characterized by

$$\mathring{A}_{\sigma+it}(x) \cdot \begin{pmatrix} m^-(\sigma+it,x) \\ 1 \end{pmatrix} = \tau^-(\sigma+it,x) \begin{pmatrix} m^-(\sigma+it,f(x)) \\ 1 \end{pmatrix}. \tag{2.40}$$

Since $A_{\sigma+it}$ is real-symmetric in $\sigma+it$, letting

$$m^+(\sigma+it,x) = \frac{1}{\overline{m^+(\sigma-it,x)}}, \quad \tau^+(\sigma+it,x) = \frac{1}{\overline{\tau^+(\sigma-it,x)}}, \quad \sigma+it \in \Omega_\delta^- \tag{2.41}$$

and

$$m^-(\sigma+it,x) = \frac{1}{\overline{m^-(\sigma-it,x)}}, \quad \tau^-(\sigma+it,x) = \frac{1}{\overline{\tau^-(\sigma-it,x)}}, \quad \sigma+it \in \Omega_\delta^+, \tag{2.42}$$

we have that (2.39) and (2.40) are valid for $\sigma+it \in \Omega_\delta \setminus \mathbb{R}/\mathbb{Z}$.

The following key computation generalizes estimates of Kotani [K2] (see also [S]) and Deift-Simon [DeS].

**Lemma 2.6.** *Let $A \in \Delta_\delta$ and let $\sigma_0 \in \mathbb{R}/\mathbb{Z}$. Then*

(1) *If*

$$\liminf_{t \to 0+} \frac{L(\sigma_0+it)}{t} < \infty \tag{2.43}$$

*then*

$$\liminf_{t \to 0+} \int_X \frac{1}{1-|m^+(\sigma_0+it,x)|^2} d\mu(x) + \int_X \frac{1}{1-|m^-(\sigma_0-it,x)|^2} d\mu(x) < \infty. \tag{2.44}$$

(2) *If*

$$\limsup_{t \to 0+} \frac{L(\sigma_0+it)}{t} < \infty \tag{2.45}$$

*then*

$$\limsup_{t \to 0+} \int_X \frac{1}{1-|m^+(\sigma_0+it,x)|^2} d\mu(x) + \int_X \frac{1}{1-|m^-(\sigma_0-it,x)|^2} d\mu(x) < \infty \tag{2.46}$$

*and*

$$\liminf_{t \to 0+} \int_X |m^+(\sigma_0+it,x) - m^-(\sigma_0-it,x)|^2 d\mu(x) = 0. \tag{2.47}$$

*Proof.* Let us assume that

$$(\partial_t \mathring{A}_{\sigma_0+it}(x)) \mathring{A}_{\sigma_0+it}(x)^{-1} = u(x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + C(\sigma_0+it,x), \tag{2.48}$$

where $u(x) < 0$ and $\lim_{t\to 0} \sup_{x\in X} \|C(\sigma_0 + it, x)\| = 0$. The general case can be reduced to this one by conjugacy. Indeed, if $B : X \to \mathrm{SL}(2,\mathbb{R})$ is continuous, then it is indifferent to prove the results for the original $A$ or for its conjugate $B(f(x))A_{\sigma_0+it}(x)B(x)^{-1}$, so it is enough to select $B$ such that

$$\mathring{B}(f(x))(\partial_t \mathring{A}_{\sigma_0}(x))\mathring{A}_{\sigma_0}(x)^{-1}\mathring{B}(f(x))^{-1} \tag{2.49}$$

is a matrix of the form $\begin{pmatrix} u(x) & 0 \\ 0 & -u(x)\end{pmatrix}$, $u(x) < 0$, for each $x$. Such a $B$ can be found due to the monotonicity hypothesis as we will now show.

Since $\bar\partial_z A_z = 0$ at $\Im z = 0$, we have $(\partial_t \mathring{A})\mathring{A}^{-1} = i(\partial_\sigma \mathring{A})\mathring{A}^{-1}$ at $\sigma + it = \sigma_0$, and since $\mathring{A}_\sigma$ is in $\mathrm{SU}(1,1)$ for any $\sigma \in \mathbb{R}/\mathbb{Z}$, we can write

$$(\partial_t \mathring{A})\mathring{A}^{-1} = \begin{pmatrix} a & -i\bar\nu \\ -i\nu & -a\end{pmatrix}, \qquad a \in \mathbb{R}, \quad \nu \in \mathbb{C} \tag{2.50}$$

at $\sigma + it = \sigma_0$. Moreover, since $\mathring{A}_{\sigma+it} \cdot \mathbb{D} \subset \mathbb{D}_{e^{-\epsilon t + o(t)}}$ and $\mathring{A}_\sigma \cdot \mathbb{D} = \mathbb{D}$, we have

$$\left| \frac{(1 + ta)m - i\bar\nu t + o(t)}{-i\nu m t + (1 - at) + o(t)} \right| < 1 - \epsilon t + o(t) \tag{2.51}$$

for any $m \in \partial\mathbb{D}$ and any small $t > 0$; this implies that $a - |\nu| \sin\phi \le -\epsilon/2$ for any $\phi$ and consequently $a < 0$ and $\det((\partial_t \mathring{A})\mathring{A}^{-1}) = |\nu|^2 - a^2 \le -\epsilon^2/4$. It is then clear that $(\partial_t \mathring{A})\mathring{A}^{-1}$ is conjugated by a matrix in $\mathrm{SU}(1,1)$ to a matrix of the form $\begin{pmatrix} u & 0 \\ 0 & -u\end{pmatrix}$ with $u < 0$. (the sign of $u$ cannot be changed to be positive because the conjugacy is in $\mathrm{SU}(1,1)$).

Let us denote for simplicity $m^+$ for $m^+(\sigma_0 + it, x)$, $m^-$ for $m^-(\sigma_0 + it, x)$, $\tilde m^+$ for $m^+(\sigma_0 + it, f(x))$, $\tilde m^-$ for $m^-(\sigma_0 + it, f(x))$, $\tau^+$ for $\tau^+(\sigma_0 + it, x)$, $\tau^-$ for $\tau^-(\sigma_0 + it, x)$, $A$ for $A_{\sigma_0+it}(x)$, $L$ for $L(\sigma_0 + it)$ and $u$ for $u(x)$.

We now estimate the Lyapunov exponent using (2.21), (2.23) and the fact that $A_{\sigma+it} = \begin{pmatrix} e^{tu} & 0 \\ 0 & e^{-tu}\end{pmatrix} A_\sigma + o(t)$ by evaluating the contraction coefficient $q$ in the Poincaré metric of $\mathbb{D}$. A straightforward computation yields

$$q^{-1} = e^{-2tu+o(t)} \frac{1 - |\tilde m^+|^2}{1 - e^{-4tu}|\tilde m^+|^2} = e^{2tu+o(t)} \frac{e^{-4tu}(1 - |\tilde m^+|^2)}{1 - e^{-4tu}|\tilde m^+|^2}. \tag{2.52}$$

Using that for $r > 0$ and $0 \le s < e^{-r}$ we have

$$\ln\left(\frac{e^r(1 - s)}{1 - e^r s}\right) \ge \frac{r}{1 - s}, \tag{2.53}$$

we get

$$\ln q^{-1} \ge 2tu + o(t) - \frac{4tu}{1 - |\tilde m^+|^2} = -2tu \frac{1 + |\tilde m^+|^2}{1 - |\tilde m^+|^2} + o(t). \tag{2.54}$$

Since $L = \frac{1}{2}\int_X \ln q^{-1} d\mu$, we get

$$L \ge -t \int_X u \frac{1 + |\tilde m^+|^2}{1 - |\tilde m^+|^2} d\mu + o(t). \tag{2.55}$$

An analogous argument yields

$$L \ge -t \int_X u \frac{1 + |\tilde m^-|^{-2}}{1 - |\tilde m^-|^{-2}} d\mu + o(t). \tag{2.56}$$

We conclude that

$$\liminf_{t\to 0+} \frac{L}{t} + \frac{1}{2}\int_X u\left(\frac{1+|\tilde{m}^+|^2}{1-|\tilde{m}^+|^2} + \frac{1+|\tilde{m}^-|^{-2}}{1-|\tilde{m}^-|^{-2}}\right) d\mu \geq 0. \tag{2.57}$$

Since $u < 0$, this gives the first item and the first part of the second item.

Differentiating

$$\mathring{A}^{-1} \cdot \begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix} = \frac{1}{\tau^+}\begin{pmatrix}m^+\\ 1\end{pmatrix} \tag{2.58}$$

with respect to $t$, and applying $-\mathring{A}$ to both sides, we get

$$(\partial_t \mathring{A})\mathring{A}^{-1} \cdot \begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix} - \partial_t \tilde{m}^+ \begin{pmatrix}1\\ 0\end{pmatrix} = \frac{\partial_t \tau^+}{(\tau^+)^2}\mathring{A} \cdot \begin{pmatrix}m^+\\ 1\end{pmatrix} - \frac{1}{\tau^+}\partial_t m^+ \mathring{A} \cdot \begin{pmatrix}1\\ 0\end{pmatrix}. \tag{2.59}$$

Using that

$$\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{m^+ - m^-}\left(\begin{pmatrix}m^+\\ 1\end{pmatrix} - \begin{pmatrix}m^-\\ 1\end{pmatrix}\right) = \frac{1}{\tilde{m}^+ - \tilde{m}^-}\left(\begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix} - \begin{pmatrix}\tilde{m}^-\\ 1\end{pmatrix}\right), \tag{2.60}$$

we get

$$\begin{aligned}(\partial_t \mathring{A})\mathring{A}^{-1} \cdot \begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix} &- \frac{\partial_t \tilde{m}^+}{\tilde{m}^+ - \tilde{m}^-}\left(\begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix} - \begin{pmatrix}\tilde{m}^-\\ 1\end{pmatrix}\right)\\ &= \frac{\partial_t \tau^+}{\tau^+}\begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix} - \frac{\partial_t m^+}{m^+ - m^-}\left(\begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix} - \frac{\tau^-}{\tau^+}\begin{pmatrix}\tilde{m}^-\\ 1\end{pmatrix}\right).\end{aligned} \tag{2.61}$$

On the other hand, we can compute using (2.48)

$$(\partial_t \mathring{A})\mathring{A}^{-1} \cdot \begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix} = u\left(\frac{\tilde{m}^+ + \tilde{m}^-}{\tilde{m}^+ - \tilde{m}^-}\begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix} - \frac{2\tilde{m}^+}{\tilde{m}^+ - \tilde{m}^-}\begin{pmatrix}\tilde{m}^-\\ 1\end{pmatrix}\right) + c^+\begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix} + c^-\begin{pmatrix}\tilde{m}^-\\ 1\end{pmatrix}, \tag{2.62}$$

where $c^+ \equiv c^+(\sigma_0 + it, x)$, $c^- \equiv c^-(\sigma + it, x)$ satisfy

$$|c^\pm(\sigma_0 + it, x)| \leq K\|C(\sigma_0 + it, x)\|\left(\frac{1}{1-|\tilde{m}^+|^2} + \frac{1}{1-|\tilde{m}^-|^{-2}}\right). \tag{2.63}$$

for some constant $K > 0$.

Putting together (2.61) and (2.62), taking the coefficient of $\begin{pmatrix}\tilde{m}^+\\ 1\end{pmatrix}$ and integrating with respect to $\mu$ we get

$$\int_X u\frac{\tilde{m}^+ + \tilde{m}^-}{\tilde{m}^+ - \tilde{m}^-}d\mu + \int_X c^+ d\mu = \int_X \frac{\partial_t \tau^+}{\tau^+}d\mu. \tag{2.64}$$

We can now consider the real part, which gives

$$\int_X u\frac{|\tilde{m}^+|^2|\tilde{m}^-|^{-2} - 1}{\left|\frac{\tilde{m}^+}{\tilde{m}^-} - 1\right|^2}d\mu + \int_X \Re c^+ d\mu = \partial_t L. \tag{2.65}$$

Using (2.63) we conclude

$$\lim_{t\to 0+} -\int_X u\frac{|\tilde{m}^+|^2|\tilde{m}^-|^{-2} - 1}{\left|\frac{\tilde{m}^+}{\tilde{m}^-} - 1\right|^2}d\mu + \partial_t L = 0. \tag{2.66}$$

Write

$$I = \frac{1}{2}\frac{1+|\tilde{m}^+|^2}{1-|\tilde{m}^+|^2} + \frac{1}{2}\frac{1+|\tilde{m}^-|^{-2}}{1-|\tilde{m}^-|^{-2}} + \frac{|\tilde{m}^+|^2|\tilde{m}^-|^{-2}-1}{\left|\frac{\tilde{m}^+}{\tilde{m}^-}-1\right|^2}. \tag{2.67}$$

Using (2.57) and (2.66) we get

$$\liminf_{t\to 0+}\left(\frac{L}{t} - \partial_t L + \int_X uId\mu\right) \geq 0. \tag{2.68}$$

Notice that

$$I \geq \left|\tilde{m}^+ - \frac{1}{\overline{\tilde{m}^-}}\right|^2 \geq 0, \tag{2.69}$$

and we conclude

$$\liminf_{t\to 0+}\left(\frac{L}{t} - \partial_t L\right) \geq \liminf_{t\to 0+} -\int_X uId\mu \geq 0. \tag{2.70}$$

Since $\limsup_{t\to 0+}\frac{L}{t} < \infty$, we must have

$$\liminf_{t\to 0+}\left(\frac{L}{t} - \partial_t L\right) = -\limsup_{t\to 0+}\left(t\partial_t \frac{L}{t}\right) \leq 0, \tag{2.71}$$

so $\liminf_{t\to 0+} -\int_X uId\mu = 0$, and since $-u$ is positive and bounded away from 0 we have $\liminf_{t\to 0+}\int_X Id\mu(x) = 0$, which gives the second part of the second item by (2.69). □

The following estimates will allow us to work in the higher order asymptotically holomorphic setting:

**Lemma 2.7.** *Let $A_{z,s} \in \Delta_\delta$ be a one-parameter family. Assume that $s \mapsto A_{z,s}(x)$ is $C^r$, $1 \leq r < \infty$ and*

$$\|\partial_s^k A_{z,s}(x)\| = O(1), \quad 0 \leq k \leq r. \tag{2.72}$$

*Then*

$$|\partial_s^k m_s^+(z,x)| = O(|\Im(z)|^{-2k+1}), \quad 1 \leq k \leq r. \tag{2.73}$$

*Moreover, if additionally $s \mapsto \overline{\partial}_z A_{z,s}(x)$ is $C^{r-1}$ and we have the estimate*

$$\|\partial_s^k \overline{\partial}_z A_{z,s}(x)\| = o(|\Im(z)|^{\eta-k-1}), \quad 0 \leq k \leq r-1, \tag{2.74}$$

*for some $\eta \in \mathbb{R}$ then*

$$|\partial_s^k \overline{\partial}_z m_s^+(z,x)| = o(|\Im(z)|^{\eta-2k-2}), \quad 0 \leq k \leq r-1. \tag{2.75}$$

*Remark* 2.4. Implicit in the statement of Lemma 2.7 is the existence of the derivatives taken in the left hand sides of (2.73) and (2.75). This is just a consequence of usual normal hyperbolicity theory [HPS], but naturally the estimates depend on the strength of the uniform hyperbolicity, and hence may degenerate as $\Im z \to 0$, so our main concern below is to get the bounds (2.73) and (2.75).

*Proof.* Let $F_z^s(x,w) = \mathring{A}_{z,s}(x)\cdot w$, $m_z^s(x) = m_s^+(z,x)$. Our estimates will come from the study of the hyperbolicity of $F$ with respect to the variable $w$, as measured in the Poincaré metric. The way we exploit this hyperbolicity is contained in the following.

**Proposition 2.8.** *There exists $K>0$ such that if $(s,z,x)\mapsto u^s_z(x)$ is any bounded function then*

$$|u^s_z(x)|\leq K|\Im(z)|^{-1}\sup_{y\in X}|u^s_z(f(y))-(\partial_w F^s_z)(y,m^s_z(y))u^s_z(y)|. \tag{2.76}$$

*Proof.* For $s$ and $z$ fixed, let $x$ satisfy

$$\frac{|u^s_z(f(x))|}{1-|m^s_z(f(x))|^2}=M=\sup_{y\in X}\frac{|u^s_z(y)|}{1-|m^s_z(y)|^2} \tag{2.77}$$

(we assume that the suppremum is achieved to keep the argument transparent, the general case is obtained by approximation). Then for every $y$,

$$|u^s_z(y)|\leq\frac{|u^s_z(y)|}{1-|m^s_z(y)|^2}\leq M, \tag{2.78}$$

so it is enough to estimate

$$M\leq K|\Im(z)|^{-1}|u^s_z(f(x))-(\partial_w F^s_z)(x,m^s_z(x))u^s_z(x)|. \tag{2.79}$$

We have

$$\begin{aligned}u^s_z(f(x))-(\partial_w F^s_z)(x,m^s_z(x))u^s_z(x)&=(1-|m^s_z(f(x))|^2)\\ &\cdot\left(\frac{u^s_z(f(x))}{1-|m^s_z(f(x))|^2}-(\partial_w F^s_z)(m^s_z(x))\frac{1-|m^s_z(x)|^2}{1-|m^s_z(f(x))|^2}\frac{u^s_z(x)}{1-|m^s_z(x)|^2}\right).\end{aligned} \tag{2.80}$$

Noticing that by the Schwarz Lemma

$$\left|(\partial_w F^s_z)(x,m^s_z(x))\frac{1-|m^s_z(x)|^2}{1-|m^s_z(f(x))|^2}\right|<1, \tag{2.81}$$

we get

$$\begin{aligned}|u^s_z(f(x))-(\partial_w F^s_z)(x,m^s_z(x))u^s_z(x)|&\geq M(1-|m^s_z(f(x))|^2)\\ &\cdot\left(1-|(\partial_w F^s_z)(x,m^s_z(x))|\frac{1-|m^s_z(x)|^2}{1-|m^s_z(f(x))|^2}\right).\end{aligned} \tag{2.82}$$

The Schwarz Lemma hyperbolicity bound (cf. (2.26))

$$|\partial_w F^s_z(x,m^s_z(x))|\frac{1-|m^s_z(x)|^2}{1-|m^s_z(f(x))|^2}\leq e^{-\epsilon\Im(z)}\frac{1-e^{2\epsilon\Im(z)}|m^s_z(f(x))|^2}{1-|m^s_z(f(x))|^2}, \tag{2.83}$$

for some constant $\epsilon>0$, gives

$$\begin{aligned}(1-|m^s_z(f(x))|^2)&\left(1-|(\partial_w F^s_z)(x,m^s_z(x))|\frac{1-|m^s_z(x)|^2}{1-|m^s_z(f(x))|^2}\right)\\ &\geq 1-e^{-\epsilon\Im z}+(e^{\epsilon\Im z}-1)|m^s_z(f(x))|^2\geq 1-e^{-\epsilon\Im(z)},\end{aligned} \tag{2.84}$$

which together with (2.82) implies (2.79). □

Let us complete the proof of lemma 2.7. Differentiating (taking $\partial^k_s$)

$$m^s_z(f(x))=F^s_z(x,m^s_z(x)) \tag{2.85}$$

(as we are allowed to do, see Remark 2.4), we get

$$\partial_s^k m_z^s(f(x)) = \partial_w F_z^s(x, m_z^s(x)) \cdot \partial_s^k m_z^s(x) + \sum_{\substack{l\geq 0, 1\leq i_1\leq \ldots \leq i_l<k,\\ i_1+\ldots+i_l=j\leq k}} C \cdot \partial_s^{k-j}\partial_w^l F_z^s(x, m_z^s(x)) \cdot \prod_{n=1}^{l} \partial_s^{i_n} m_z^s(x), \tag{2.86}$$

where $C \equiv C(k, i_1, ..., i_l) > 0$ is a constant. Observe that $F_z^s(x, w)$ is a homography in $w$ whose denominator is bounded from below by $(10 \sup_{x\in X} \|A_z(x)\|)^{-1}$ provided $\delta$ is small enough. This comes from the fact that if $z \in \mathbb{R}/\mathbb{Z}$ then $\mathring{A}_z(x) \in SU(1, 1)$, say equal to $\begin{pmatrix} u & \bar{v} \\ v & \bar{u} \end{pmatrix}$ with $|u|^2 - |v|^2 = 1$ and therefore for any $w \in \mathbb{D}$, $|vw + \bar{u}| \geq |u| - |v| \geq 1/(2\max(|u|, |v|)) \geq (1/4\|A_z(x)\|)$. If $\delta$ is small enough, the claim follows. Consequently, all the quantities of the form $\partial_w^l \partial_s^j F_z^s(x, w)$ are uniformly bounded.

Thus, if

$$|\partial_s^j m_z^s(x)| = O(|\Im(z)|^{-(2j-1)}), \quad 1 \leq j \leq k-1, \tag{2.87}$$

we get (either $j = k$ and $l \geq 2$ or $j < k$)

$$|(\partial_s^k m_z^s(f(x)) - \partial_w F_z^s(x, m_z^s(x)) \cdot \partial_s^k m_z^s(x)| = O(|\Im(z)|^{-2(k-1)}), \tag{2.88}$$

which implies by the previous proposition

$$|(\partial_s^k m)(s, z, x)| = O(|\Im(z)|^{-(2k-1)}). \tag{2.89}$$

The first estimate then follows by induction.

Differentiating (taking an extra $\overline{\partial}_z$) (2.86), we get

$$\begin{aligned}\partial_s^k \overline{\partial}_z m_z^s(f(x)) &= \partial_w F_z^s(x, m_z^s(x)) \cdot \partial_s^k \overline{\partial}_z m_z^s(x) \\ &+ \sum_{\substack{l\geq 0, 1\leq i_1\leq \ldots \leq i_l\leq k,\\ i_1+\ldots+i_l=j\leq k}} C \cdot (\partial_s^{k-j} \overline{\partial}_z \partial_w^l F_z^s(x, m_z^s(x)) \cdot \prod_{n=1}^{l} \partial_s^{i_n} m_z^s(x) \\ &+ \sum_{\substack{l\geq 0, 1\leq i_1\leq \ldots \leq i_l<k,\\ i_0\geq 0, i_0+i_1+\ldots+i_l=j\leq k}} D \cdot \partial_s^{k-j} \partial_w^l F_z^s(x, m_z^s(x)) \cdot \partial_s^{i_0} \overline{\partial}_z m_z^s(x) \cdot \prod_{n=1}^{l} \partial_s^{i_n} m_z^s(x),\end{aligned} \tag{2.90}$$

where $D \equiv D(k, i_0, ..., i_l) > 0$ is a constant. Since,

$$|\overline{\partial}_z m_z^s(f(x)) - \partial_w F_z^s(x, m_z^s(x)) \cdot \overline{\partial} m_z^s(x)| = |\overline{\partial}_z F_z^s(x, m_z^s(x))| = o(|\Im(z)|^{\eta-1}), \tag{2.91}$$

we get

$$|\overline{\partial} m_z^s(x)| = o(|\Im(z)|^{\eta-2}), \tag{2.92}$$

Now assuming by induction

$$|\partial_s^j \overline{\partial}_z m_z^s(x)| = o(|\Im(z)|^{\eta-2j-2}), \quad 0 \leq j \leq k-1, \tag{2.93}$$

we get

$$|\partial_s^k \overline{\partial}_z m_z^s(f(x)) - \partial_w F_z^s(x, m_z^s(x)) \cdot \partial_s^k \overline{\partial} m_z^s(x)| = o(|\Im(z)|^{\eta-2k-1}), \tag{2.94}$$

which implies as before

$$|\partial_s^k \overline{\partial}_z m_z^s(x)| = o(|\Im(z)|^{\eta-2k-2}). \tag{2.95}$$

The second estimate then follows by induction. □

*Remark* 2.5. The estimates above are still valid if the parameter space is allowed to be multidimensional, or, more generally, a Banach manifold, but the notation is more cumbersome.

*Remark* 2.6. As a particular case of the previous estimates (zero-dimensional parameter space), if $A \in \Delta_\delta^+$ satisfies

$$\|\overline{\partial}_z A_z(x)\| = o(|\Im(z)|^{\eta-1}) \tag{2.96}$$

then

$$|\overline{\partial}_z m^+(z,x)| = o(|\Im(z)|^{\eta-2}). \tag{2.97}$$

2.5. **Asymptotically holomorphic extensions.** In order to apply the estimates obtained in the previous section to one-parameter families of $\mathrm{SL}(2,\mathbb{R})$ cocycles, we need to consider appropriate asymptotic holomorphic extensions.

For $\eta \in [1,\infty)$, a $C^\eta$ function defined in some neighborhood of $\mathbb{R}/\mathbb{Z}$ satisfying

$$\frac{d^k}{dt^k}\overline{\partial} F(\sigma+it) = 0, \quad \sigma+it \in \mathbb{R},\, k=0,...,[\eta-1] \tag{2.98}$$

is called $\eta$-asymptotically holomorphic.

Let $AH^\eta$ be the set of $\eta$-asymptotically holomorphic functions defined on the whole $\mathbb{C}/\mathbb{Z}$. it is easy to see that one can define (linear) sections $\Phi_\eta$ of the restriction operator $AH^\eta \to C^\eta(\mathbb{R}/\mathbb{Z},\mathbb{C})$. For instance, one can let

$$\Phi_\eta(f)(\sigma+it) = \int K(x) f(\sigma+tx)dx, \tag{2.99}$$

where $K : \mathbb{R} \to \mathbb{C}$ is a $C^\infty$ function with compact support satisfying

$$\int x^k K(x)dx = i^k, \quad k=0,...,[\eta+1]. \tag{2.100}$$

We can also define $\eta$-asymptotically $\mathrm{SL}(2,\mathbb{C})$-valued functions by requiring each coefficient to be asymptotically holomorphic. In order to obtain asymptotically holomorphic extensions of a matrix valued function $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C^\eta(\mathbb{R}/\mathbb{Z},\mathrm{SL}(2,\mathbb{R}))$, it is enough to consider

$$\Phi_\eta(A) = (\Phi_\eta(a)\Phi_\eta(d) - \Phi_\eta(b)\Phi_\eta(c))^{-1/2} \begin{pmatrix} \Phi_\eta(a) & \Phi_\eta(b) \\ \Phi_\eta(c) & \Phi_\eta(d) \end{pmatrix}, \tag{2.101}$$

which is a well defined function $\Omega_\delta \to \mathrm{SL}(2,\mathbb{R})$, where $\delta$ only depends on the $C^1$-norm of $A$.

**Lemma 2.9.** *Let $A_\theta(\cdot) \in C^0(X,\mathrm{SL}(2,R))$, $\theta \in \mathbb{R}/\mathbb{Z}$, be monotonic and $C^\eta$ in $\theta$. Then there exists $\delta > 0$ and an extension $A \in \Delta_\delta$ which is $\eta$-asymotitucally holomorphic in $\theta$.*

*Proof.* Consider the $\eta$-asymptoticaly holomorphic extension given above defined on some $\Omega_\delta$. We just need to check that $A(\sigma+it,x) \in \operatorname{int}\Upsilon$ for $t > 0$ sufficiently small. In the holomorphic case, this was concluded in section 2.3 using the Cauchy-Riemann equation, but the argument just uses it for real values, and thus is valid in the asymptotically holomorphic setting. □

The following result will illustrate the use of higher order asymptotically holomorphic extensions:

**Lemma 2.10.** *If $A_z \in \Delta_\delta$ is $1+\epsilon$-asymptotically holomorphic, then $L(A_{\sigma+it})$ is a continuous function of $0 \le t < \delta$ for almost every $\sigma \in \mathbb{R}/\mathbb{Z}$.*

The proof will involve the following decomposition technique which will play a role in several other arguments. Given a function $u : \Omega_\delta^+ \to \mathbb{C}/\mathbb{Z}$ which is continuous with derivatives in $L^1_{\mathrm{loc}}$ and satisfies

$$|\overline{\partial} u(z)| = O(|\Im(z)|^{\epsilon-1}), \tag{2.102}$$

for some $\epsilon > 0$, let us write a canonical decomposition $u = u_h + u_c$ where $u_h : \Omega_\delta^+ \to \mathbb{C}/\mathbb{Z}$ is holomorphic and $u_c : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ is a real-symmetric continuous function given by the Cauchy transform

$$u_c(z) = \lim_{t\to\infty} \frac{-1}{\pi} \int_{[-t,t]\times[-\delta,\delta]} \frac{\phi(w)}{z-w} dxdy, \tag{2.103}$$

where $\phi(z) = \overline{\partial} u(z)$ if $0 < \Im(z) < \delta$ and $\phi(z) = \overline{\overline{\partial} u(\overline{z})}$ for $0 < -\Im(z) < \delta$.

Notice that if

$$|\overline{\partial} u(z)| = O(|\Im(z)|^{k+\epsilon}), \tag{2.104}$$

then $u_c(z)$ is complex differentiable at each $z \in \mathbb{R}/\mathbb{Z}$, and $u_c : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is $C^{k+1}$.

*Proof of Lemma 2.10.* By Lemma 2.7, $\|\overline{\partial}_z m^+(z,x)\| = O(|\Im z|^{\epsilon-1})$ in $\Omega_+$. Thus we can write a decomposition $m^+ = m_h^+ + m_c^+$ where $z \mapsto m_c^+(z,x)$ is continuous up to $\mathbb{R}/\mathbb{Z}$ (uniformly in $x$), and $m_h^+$ is of course uniformly bounded for $z$ near $\mathbb{R}/\mathbb{Z}$. Thus $m_h^+$ admits non-tangential limits as $\Im z \to 0$ and hence $m^+$ also does. By Fubini's Theorem, for almost every $\sigma$, the right hand side of the expression

$$L(A_{\sigma+it}) = \int_X \ln |\tau_{A_{\sigma+it}(x)}(m^+(\sigma+it,x))| d\mu(x), \tag{2.105}$$

originally defined for $0 < t < \delta$, makes sense up to $t = 0$ and defines a continuous function of $0 \le t < \delta$. Thus we just need to show that we can identify the right hand side (when it makes sense) with $L(A_\theta)$ for $t = 0$. Indeed, if $\theta$ is such that the non-tangential limits $m^+(\theta,x)$ exist, then they provide an invariant section for the cocycle $A_\theta$. Assuming ergodicity, by the Oseledets Theorem the left hand side must be $\pm L(A_\theta)$, and (since it is non-negative (by continuity), it must be $L(A_\theta)$. The general case reduces to this one by ergodic decomposition. □

## 2.6. **Derivative bound.**

Another simple application of the asymptotically holomorphic technique is a generalization of Theorem 2.5.

**Theorem 2.11.** *Let $A_\theta \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in \mathbb{R}/\mathbb{Z}$, be $C^{2+\epsilon}$ and monotonic decreasing in $\theta$. For almost every $\theta \in \mathbb{R}/\mathbb{Z}$, if $L(A_\theta) = 0$ then*

$$-\frac{d}{d\theta}\rho(\theta) \ge \frac{\epsilon_0}{2\pi} > 0, \tag{2.106}$$

*where $\epsilon_0$ is the monotonicity constant of $\theta \mapsto A_\theta$.*

*Proof.* For $\delta > 0$ small, let us denote by $A_z \in \Delta_\delta$, some fixed $2+\epsilon$-asymptotically holomorphic extension of $A_\theta$, thus in particular

$$|\overline{\partial}_z A_z(x)| = O(|\Im(z)|^{1+\epsilon}). \tag{2.107}$$

Notice that estimate (2.27) obtained in section 2.3 in the analytic case is still true since the asymptotically holomorphic extension is complex differentiable on $\mathbb{R}/\mathbb{Z}$. Let us show that our hypothesis imply that for almost every $\sigma \in \mathbb{R}$,

$$\partial_\sigma \rho(\sigma) = \lim_{t\to 0} \frac{L(\sigma+it) - \lim_{t\to 0} L(\sigma+it)}{t}, \tag{2.108}$$

since the result then follows as in Theorem 2.5.

We have

$$|\overline{\partial}_z m^+(z,x)| = O(|\Im(z)|^\epsilon), \tag{2.109}$$

which implies from equation (2.20)

$$|\overline{\partial}_z \zeta(z)| = O(|\Im(z)|^\epsilon) \tag{2.110}$$

as well, when $0 < \Im z < \delta$. Like in Theorem 2.5 we get using the fact that $\zeta$ asymptotically satisfies Cauchy-Riemann equations

$$\Im(\zeta(\sigma+it)) = \Im(\zeta(\sigma+i0^+)) + \int_{0^+}^t \partial_\sigma \Re\zeta(\sigma+is)ds + o(|t|^\eta). \tag{2.111}$$

Notice that by Lemma 2.10, $\Im\zeta(\sigma) = \Im\zeta(\sigma+i0^+)$ for almost every $\sigma \in \mathbb{R}/\mathbb{Z}$. From equation (2.110), decomposing $\zeta = \zeta_h + \zeta_c$, $\zeta_c(z)$ is complex differentiable at $z \in \mathbb{R}/\mathbb{Z}$ and $\zeta_c : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is $C^1$. Since $\Im\zeta > 0$ and $\sigma \mapsto \rho(\sigma) = \lim_{t\to 0+} \Re\zeta(\sigma+it)$ is monotonic, this is enough to conclude that (2.108) holds for almost every $\sigma$. □

For further use, let us remark that an argument analogous to the proof of Theorem 2.11 also gives:

**Proposition 2.12.** *Let $A \in \Delta_\delta$ satisfy*

$$\|\overline{\partial}_z A_z\| = O(|\Im(z)|^{1+\epsilon}). \tag{2.112}$$

*Then, for every $\sigma_0 \in \mathbb{R}/\mathbb{Z}$, if*

$$\limsup_{\sigma\to\sigma_0} \frac{|\rho_{A_\sigma} - \rho_{A_{\sigma_0}}|}{|\sigma-\sigma_0|} < \infty \tag{2.113}$$

*then*

$$\limsup_{t\to 0} \frac{|L(A_{\sigma_0+it}) - L(A_{\sigma_0})|}{|t|} < \infty. \tag{2.114}$$

2.7. **Proof of Theorem 2.2.** It is enough to consider the case of finite differentiability, since we have already proved the analytic case in section 2.3.

Let $A_{z,s} \in \Delta_\delta$ be an asymptotically holomorphic extension of $A_{\theta,s}$ satisfying

$$\|\partial_s^k A_{z,s}(x)\| = O(1), \quad 0 \le k \le 2r, \tag{2.115}$$

$$\|\partial_s^k \overline{\partial}_z A_{z,s}(x)\| = O(|\Im(z)|^{2r-k+\epsilon}), \quad 0 \le k \le 2r. \tag{2.116}$$

Define $U(t,s)$ by formula (2.28). It still satifies (2.29) for $t \neq 0$.

We must show that $s \mapsto U(0,s)$ is $C^k$. By Lemma 2.10, $t \mapsto U(t,s)$ is continuous up to $t=0$, for each $s$. For $t \neq 0$, the functions $s \mapsto U(t,s)$ are in $C^k$, so we just need to show that as $t \to 0$, those functions converge uniformly in $C^k$ if $0 \le k \le r$.

We have the estimate

$$|\partial_s^k \overline{\partial}_z \zeta_s(z)| = O(|\Im(z)|^{2r-2k-1+\epsilon}), \quad 0 \le k \le r-1. \tag{2.117}$$

Fix $0 < \delta' < \delta'' < \delta$. Use Stoke's Theorem to integrate $\zeta_s(z)dz$ on $[0,1] \times [\delta', \delta'']$, and take the imaginary part to get

$$\frac{1}{2\pi}U(\delta', s) = \frac{1}{2\pi}U(\delta'', s) + \int_{X\times\{\delta'<\Im z<\delta''\}} \overline{\partial}_z \zeta_s(z) d\overline{z} \wedge dz + \int_X \int_{\delta'}^{\delta''} \Re\zeta_s(1+it) - \Re\zeta_s(it) dt d\mu(x) \tag{2.118}$$

The first term on the right hand side is a fixed $C^k$ function of $s$. The second is also in $C^k$ and converges in $C^k$ as $\delta' \to 0$ by (2.117). Notice that $\Re\zeta_s(1+it) - \Re\zeta_s(it)$ is independent of $s$ and $t$: it is the integral over $x$ of the topological degree of $A_{z,s}$ as $z$ runs once around $\Omega_\delta^+$. Thus the third term is a linear function $\delta'' - \delta'$ (and independent of $s$). This shows that $U(\delta', s)$ converges uniformly in $C^k$, as desired. □

2.8. **$L^2$-estimates.** We continue with some crucial $L^2$-estimates, which play an important role in renormalization theory, obtaining a somewhat more precise form of Theorem 1.7, Theorem 2.14.

First, let us recall the basic connection between estimates for $\mathbb{D}$-valued invariant sections and the existence of conjugacy to rotations.

**Lemma 2.13.** *Let $A : X \to \mathrm{SL}(2,\mathbb{R})$ be measurable. The following are equivalent:*

1. *There exists a measurable $B : X \to \mathrm{SL}(2,\mathbb{R})$ such that $\int_X \|B(x)\|^2 d\mu(x) < \infty$ and $B(f(x))A(x)B(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$ for almost every $x$,*
2. *There exists a measurable $m : X \to \mathbb{D}$ such that $\int_X \frac{1}{1-|m(x)|^2} d\mu(x) < \infty$ and $\mathring{A}(x) \cdot m(x) = m(f(x))$ for almost every $x$.*

*Proof.* Let $B(x)$ be such that $\mathring{B}(x) = \frac{1}{(1-|m(x)|^2)^{1/2}} \begin{pmatrix} 1 & -m(x) \\ -\overline{m(x)} & 1 \end{pmatrix}$. □

If $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ satisfies the equivalent conditions of the previous lemma, we will say that the corresponding cocycle is $L^2$-conjugate to a cocycle of rotations.

Our aim in this section is to prove the following.

**Theorem 2.14.** *Let $A_\theta \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in \mathbb{R}/\mathbb{Z}$, be $C^{2+\epsilon}$ and monotonic in $\theta$. For every $\theta \in \mathbb{R}/\mathbb{Z}$, if*

$$\limsup_{\theta' \to \theta} \frac{|\rho_{A_{\theta'}} - \rho_{A_\theta}|}{|\theta' - \theta|} < \infty \tag{2.119}$$

*(in particular if $\theta \mapsto \rho_{A_\theta}$ is Lipschitz) and $L(A_\theta) = 0$ then $A_\theta$ is $L^2$-conjugate to a cocycle of rotations.*

We will need a simple compactness result:

**Proposition 2.15.** *Let $A_k \in C^0(X, \Upsilon)$ be a sequence converging to $A$. Assume there exists measurable functions $m_k : X \to \mathbb{D}$ satisfying $\mathring{A}_k(x) \cdot m_k(x) = m_k(f(x))$, such that* $\liminf \int_X \frac{1}{1-|m_k(x)|^2} d\mu(x) < \infty$. *Then there exists a measurable $m : X \to \mathbb{D}$ such that $\mathring{A}(x) \cdot m(x) = m(f(x))$ and $\int_X \frac{1}{1-|m(x)|^2} d\mu(x) < \infty$.*[8]

[8] More generally, we can also let the dynamics vary, thus considering a sequence of $\mu$-preserving homeomorphisms $f_k : X \to X$ converging to $f$.

The proof uses the notion of conformal barycenter [DE], and we leave it for the Appendix A.

**Corollary 2.16.** *Let $A \in \Delta_\delta$. If $\sigma_0 \in \mathbb{R}/\mathbb{Z}$ satisfies*

$$\liminf_{t \to 0} \frac{L(\sigma_0 + it)}{t} < \infty \tag{2.120}$$

*then $A_{\sigma_0}$ is $L^2$-conjugate to a cocycle of rotations.*

*Proof.* The corollary follows from the previous Proposition 2.15 and Lemma 2.6. □

*Proof of Theorem 2.14.* It is enough to apply Proposition 2.12 and the corollary above. □

*Proof of Theorem 1.7.* Since $\theta \mapsto \rho_{A_\theta}$ is monotonic (2.119) holds for almost every $\theta$. We can thus apply Theorem 2.14. □

*Remark* 2.7. If one is only concerned with a result valid for almost every $\theta$ (as in the case of Theorem 1.7), one can bypass the use of the conformal barycenter argument. Indeed, the most usual argument in such situations is to apply the Lemma of Fatou to guarantee convergence of $m^+(\sigma + it, x)$ as $t \to 0+$ for almost every $x$, and then apply Fubini's Lemma to obtain a set of $\sigma$ of full Lebesgue measure for which $\lim_{t \to 0+} m^+(\sigma + it, x)$ exists for almost every $x$.

2.9. **Proof of Theorem 2.1.** We will consider only the finitely differentiable case, the other cases following basically the same argument. We shall assume that $J = \mathbb{R}/\mathbb{Z}$ for simplicity. Consider an asymptotically holomorphic extension of $A_\theta$ satisfying

$$\|\overline{\partial}_z A_z\| = O(|\Im(z)|^{r+\epsilon}). \tag{2.121}$$

Then we have

$$\|\overline{\partial}_z m^+(z,x)\| = O(|\Im(z)|^{r-1+\epsilon}), \quad \Im(z) > 0 \tag{2.122}$$

and analogously

$$\|\overline{\partial}_z m^-(z,x)\| = O(|\Im(z)|^{r-1+\epsilon}), \quad \Im(z) < 0. \tag{2.123}$$

Let

$$\phi(z,x) = \overline{\partial}_z m^\pm(z,x), \quad \pm\Im(z) > 0, \tag{2.124}$$

and let $u : \mathbb{C}/\mathbb{Z} \times X \to \mathbb{C}$ be given by

$$u(z,x) = \lim_{t \to \infty} \frac{-1}{\pi} \int_{[-t,t] \times [-\delta,\delta]} \frac{\phi(w,x)}{z - w} dxdy. \tag{2.125}$$

A compactness argument shows that $u(z,x)$ is continuous on both variables. Moreover, $\mathbb{R}/\mathbb{Z} \ni y \mapsto u(y,x)$ is $C^r$ (uniformly in $x$). Let

$$m(z,x) = m^\pm(z,x), z \in \Omega_\delta^\pm. \tag{2.126}$$

Then $\lim_{t \to 0} m(\sigma + it, x)$ exists for almost every $\sigma$ and almost every $x$ by Lemma 2.6. Thus for almost every $x \in X$, $z \mapsto m(z,x) - u(z,x)$ extends to a holomorphic function defined on $\Omega_\delta$. A compactness argument shows that this holds indeed for all $x \in X$, and that the function $\Omega_\delta \times \mathbb{R}/\mathbb{Z} \ni (z,x) \mapsto m(z,x) - u(z,x)$ is continuous (as in the classical De Concini-Johnson argument [CJ]). It also follows that $\mathbb{R}/\mathbb{Z} \ni y \mapsto m(y,x)$ is $C^r$ (uniformly on $x$). To conclude, it is enough to show that $m(y,x)$ takes values on $\mathbb{D}$.

Let us assume first that $f : X \to X$ is minimal. If $y$ is such that $m(y, x_0) \in \partial\mathbb{D}$ for some $x_0 \in X$, then for every $x \in X$ we also have $m(y, x) \in \partial\mathbb{D}$ (by invariance). However, since $L(A_y) = 0$ for every $y$, $\rho_{A_y}$ is $C^1$ (by Schwarz Reflection and $r \geq 1$), so by Proposition 2.12 and Lemma 2.6, for every $y$ we have

$$\limsup_{t\to 0+} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{1 - |m^+(y + it, x)|^2} d\mu(x) < \infty, \tag{2.127}$$

so by continuity $m(y, x) \in \mathbb{D}$ for almost every $x$.

Let us now consider the general case. Notice that if $\mu'$ is any ergodic invariant probability measure, the Lyapunov exponent $L'(A_y)$ with respect to $\mu'$ is still 0 for every $y \in \mathbb{R}/\mathbb{Z}$. Indeed,

$$L'(A_z) = \pm \int \ln |\tau_{A_z}(m(z, x))| d\mu'(x) \geq 0, \quad z \in \Omega^{\pm}_{\delta}, \tag{2.128}$$

and since $m$ is continuous, we have $\int \ln |\tau_{A_y}(m(y, x))| d\mu'(x) = 0$. On the other hand, since $m$ is an invariant section, if $L'(A_y) \neq 0$ then $\int \ln |\tau_{A_y}(m(y, x))| d\mu'(x)$ coincides with $L'(A_y)$ up to sign, as desired.

Since any minimal set supports an ergodic invariant measure, we can apply the previous argument to conclude that $m(y, x) \in \mathbb{D}$ whenever $x$ belongs to a minimal set. To conclude, notice that for each $y$, the set of all $x$ such that $m(y, x) \in \partial\mathbb{D}$ is compact and invariant, so if it is non-empty it must contain a minimal set. □

## 3. Monotonic cocycles

We now turn to the study of quasiperiodic cocycles presenting monotonicity in phase space. In this section, $d \geq 1$ is a fixed integer and the dynamics $f : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d/\mathbb{Z}^d$ is a translation $f(x) = x + \alpha$, where $\alpha$ is assumed to be fixed except when otherwise noted. The underlying probability measure will be Lebesgue measure.

Given $w \in \mathbb{R}^d$, we shall say that $A \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2, \mathbb{R}))$ is *$w$-monotonic* if $A^w_\theta(x) = A^w(x + \theta w)$ is monotonic decreasing.

We say that $A$ is *monotonic* if it is $w$-monotonic for some $w$ [9]. If $A$ is $C^1$ then the set of $w$ such that $A$ is $w$-monotonic is an open convex cone, hence if non-empty it contains primitive vectors of $\mathbb{Z}^d$, and up to linear automorphism of $\mathbb{R}^d/\mathbb{Z}^d$ it contains $(1, 0, ..., 0)$. Notice that the monotonicity condition implies that for fixed $(x_2, ..., x_d)$, $x_1 \mapsto A(x_1, x_2, ..., x_d)$ is a map $\mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$ with negative topological degree. Thus monotonic cocycles are never homotopic to a constant. Indeed the phenomena we will uncover for monotonic cocycles collide often with the intuition developed for Schrödinger cocycles, which are homotopic to a constant.

A straightforward application of our previous results yields:

**Theorem 3.1.** *Let $A \in C^r(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2, \mathbb{R}))$, $r = \omega, \infty$ be monotonic. If $L(A) = 0$ then $A$ is $C^r$-conjugate to a cocycle of rotations.*

*Proof.* If $A$ is $w$-monotonic, then the family $A^w_\theta(x)$ is obviously $C^r$ in $\theta$ and satisfies $L(A^w_\theta) = L(A)$ for every $\theta$. If $L(A) = 0$, the proof of Theorem 2.1 gives $m^w_\theta \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathbb{D})$, $C^r$ in $\theta$, such that $\mathring{A}^w_\theta(x) \cdot m^w_\theta(x) = m^w_\theta(x + \alpha)$.

We claim that $m^w_\theta(x) = m^w(x + \theta w)$, where $m^w = m^w_0$. Recall that $m^w_\theta(x)$ is obtained, almost everywhere, as a non-tangential limit $\lim_{\epsilon\to 0} m^w_{\theta+\epsilon i}(x)$ where $m^w_{\theta+\epsilon i}$ is the unique $\mathbb{D}$-valued invariant section of an asymptotically holomorphic

[9]We apologize for this collision of definition with the notion of $\epsilon$-monotonic cocycle.

extension of $A^w_\theta$. This asymptotically holomorphic extension can be chosen here to satisfy $A^w_{\theta+\epsilon i}(x) = A^w_{\epsilon i}(x+\theta w)$: this is in fact automatic if we follow the procedure described in section 2.5 for the construction of the asymptotically holomorphic extension. In this case, we get $m^w_{\theta+\epsilon i}(x) = m^w_{\epsilon i}(x+\theta w)$, and hence this equality is satisfied almost surely by the non-tangential limit. Since it is continuous, the claim follows.

Let us now consider another monotonicity vector $w'$. We claim that $m^{w'} = m^w$. Let us first assume ergodicity. In this case, if $m^w \neq m^{w'}$ at some $x \in \mathbb{R}^d/\mathbb{Z}^d$ then this must happen at every $x$, and in fact the hyperbolic distance between $m^{w'}(x)$ and $m^w(x)$ must be some constant $c$, by ergodicity of $f$ and the fact that the projective action preserves the Poincaré distance of the disk. Then we can define $B : \mathbb{R}^d/\mathbb{Z}^d \to \mathrm{PSL}(2,\mathbb{R})$ by $\mathring{B}(x) \cdot m^w(x) = 0$ and $\mathring{B}(x) \cdot m^{w'}(x) \in ti$, where $t > 0$ is such that the hyperbolic distance between 0 and $ti$ is $c$. It follows that $\mathring{B}(f(x))\mathring{A}(x)\mathring{B}(x)^{-1}$ takes 0 to 0 and $ti$ to $ti$, so $B(f(x))A(x)B(x)^{-1}$ is the identity in $\mathrm{PSL}(2,\mathbb{R})$. This is clearly impossible since $A$ is non-homotopic to a constant. In the non-ergodic case, the open set $U$ where $m^{w'}(x) \neq m^w(x)$ is not necessarily everything, but it is foliated by periodic subtori where the dynamics is ergodic. The previous argument shows that there exists a continuous $B : U \to \mathrm{PSL}(2,\mathbb{R})$ such that $B(f(x))A(x)B(x)^{-1} = \mathrm{id}$. By monotonicity, for each $x \in U$ there exists $C = C(x) > 0$ such that for every $\theta > 0$, for any line $l \in \mathbb{PR}^2$, and for every $k \in \mathbb{Z}$, there exists $\gamma = \gamma(x,k,\theta,l)$ with $C^{-1}\theta < \gamma < C\theta$ and $B(f^{k+1}(x))A(f^k(x+\theta w))B(f^k(x))^{-1} \cdot l = R_\gamma \cdot l$. This shows that for every $\epsilon > 0$, $x \in U$, if $\theta > 0$ is sufficiently small then for any line $l \in \mathbb{PR}^2$ the sequence $(B(f^n(x))A_n(x+\theta w)B(x)^{-1})_{n\in\mathbb{Z}}$ is $\epsilon$-dense in $\mathbb{PR}^2$. But this is impossible since for small $\theta > 0$ we have that $B(f^n(x))A_n(x+\theta w)B(x)^{-1} = B(f^n(x))B(f^n(x+\theta w))^{-1}B(x+\theta w)B(x)^{-1}$ is close to the identity for every $n \in \mathbb{Z}$.

We conclude that there exists a single $m \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathbb{D})$ which coincides with all $m^w$'s. Thus for each $w$ in an open cone, $\theta \mapsto m(x+\theta w)$ is $C^r$. By Journé's Theorem [Jo], it is $C^r$ as a function of $x$. □

**Theorem 3.2.** *Let $A_s \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ be a one-parameter family which is $C^r$ in $x$ and $s$, $r = \omega, \infty$. If $A_{s_0}$ is monotonic then $s \mapsto L(A_s)$ is $C^r$ in a neighborhood of $s_0$.*

*Proof.* Let $s_0$ and $w$ be such that $A_{s_0}$ is $w$-monotonic. We may assume that $w$ is a primitive vector of $\mathbb{Z}^d$. Consider the two-parameter family $A_{\theta,s}$, $\theta \in \mathbb{R}/\mathbb{Z}$ given by $A_{\theta,s}(x) = A_s(x+w\theta)$. By Theorem 2.2, the $\theta$-average of $L(A_{\theta,s})$ is $C^r$ in $s$ near $s_0$. But $L(A_{\theta,s}) = L(A_s)$ for every $s$, which gives the result. □

*Remark* 3.1. Much of our analysis generalizes to some other dynamical systems, including the usual skew-shift $(x,y) \mapsto (x+\alpha, y+x)$. Consider a skew-product $f : X \times \mathbb{R}^d/\mathbb{Z}^d \to X \times \mathbb{R}^d/\mathbb{Z}^d$, $f(x,y) = (\phi(x), y+\psi(x))$, and define a cocycle over $f$ to be monotonic if there exists $w \in \mathbb{R}^d$ such that $y \mapsto A(x,y)$ is $w$-monotonic for every $x \in X$. Our arguments imply that if $A$ is monotonic and $C^r$, $r = \omega, \infty$, with respect to the second coordinate then $L(A) = 0$ implies that $A(x,y)$ is $C^0$ conjugated to rotations (the conjugacy being $C^r$ in $y$). We can also show that if $A_s(x,y)$ is a family of monotonic cocycles which is $C^r$ with respect to $(s,y)$ then $s \mapsto L(A_s)$ is $C^r$.

3.1. **Varying the frequency.** Let us briefly allow the dynamics to vary, in order to obtain a result about the regularity of the Lyapunov exponent with respect to such more general perturbations.

We first need a replacement for Lemma 2.7. Given a one-parameter family $A_{z,s} \in \Delta_\delta$ which is monotonic, and a $C^1$ one-parameter family of frequencies $\alpha(s) \in \mathbb{R}^d$, define invariant sections $m_s^+(z,x) \in \mathbb{D}$ so that $\mathring{A}_{z,s}(x) \cdot m_s^+(z,x) = m_s^+(z, x+\alpha(s))$.

**Lemma 3.3.** *Assume that $\alpha(s)$ and $(s,x) \mapsto A_{z,s}(x)$ are $C^r$, $1 \le r < \infty$ and*

$$\|\partial_s^i \partial_x^j A_{z,s}(x)\| = O(1), \quad 0 \le k = i+j \le r. \tag{3.1}$$

*Then*

$$|\partial_s^i \partial_x^j m_s^+(z,x)| = O(|\Im(z)|^{-2k-i+1)}), \quad 1 \le k = i+j \le r. \tag{3.2}$$

*Moreover, if additionally $s \mapsto \overline{\partial} A_{z,s}(x)$ is $C^{r-1}$ and we have the estimate*

$$\|\partial_s^i \partial_x^j \overline{\partial}_z A_{z,s}(x)\| = O(|\Im(z)|^{\eta-k-1}), \quad 0 \le k = i+j \le r-1, \tag{3.3}$$

*for some $\eta \in \mathbb{R}$ then*

$$|\partial_s^i \partial_x^j \overline{\partial}_z m_s^+(z,x)| = O(|\Im(z)|^{\eta-2k-i-2}), \quad 0 \le k = i+j \le r-1. \tag{3.4}$$

*Proof.* One can basically repeat the proof of Lemma 2.7, since the added complications are not very serious. Write $m_z^s(x)$ for $m_s^+(z,x)$ and $F_z^s(x,w)$ for $\mathring{A}_{z,s}(x) \cdot w$.

Consider first the first estimate. Notice that when $i=0$ things reduce to Lemma 2.7. Let us assume by induction (first on $k$ and then on $i$) that we have the desired bounds when $i'+j' < k$ and also for $i'+j'=k$, $0 \le i' < i$. Consider the derivative $\partial_s^i \partial_x^j$ of $m_z^s(x+\alpha(s)) = F_z^s(x, m_z^s(x))$. The left hand side has a main term of the form $(\partial_s^i \partial_x^j m_z^s)(x+\alpha(s))$ and a lower order term of the form

$$\sum_{l<i, l+n \le k}^{i} C \cdot (\partial_s^l (\partial_x^n m_z^s)(x+\alpha(s)), \tag{3.5}$$

with $C$ polynomials (depending on the indices) on the derivatives of $\alpha$ of order at most $i$. The lower order term can be estimated by induction to be $O(|\Im z|^{-(2k+i-2)})$. The right hand side has a main term of the form $\partial_w F_z^s(x, m_z^s(x)) \cdot \partial_s^i \partial_x^j m_z^s(x)$ and a lower order term of the form

$$\sum C \cdot \prod_{n=1}^{l} \partial_s^{i_n} \partial_x^{j_n} m_z^s(x), \tag{3.6}$$

where the sum runs over all $0 \le l \le k$, and $(i_1, j_1) \le \ldots \le (i_l, j_l)$ (lexicographic order) with $i_n, j_n \ge 0$, $1 \le n \le l$, $\sum_{n=1}^l i_n = i' \le i$, $\sum_{n=1}^l j_n = j' \le j$ and $i'+j' < k$, and $C$ are now constant multiples of $\partial_s^{i-i'} \partial_x^{j-j'} \partial_w^l F$. The lower order term can be estimated by induction to be $O(|\Im z|^{-(2k+i-2)})$. Rearranging we get

$$|(\partial_s^i \partial_x^j) m_z^s(f+\alpha(s)) - \partial_w F_z^s(x, m_z^s(x)) \cdot \partial_s^i \partial_x^j m_z^s(x)| = O(|\Im z|^{-(2k+i-2)}), \tag{3.7}$$

and the first estimate follows from Proposition 2.8.

Consider now the second estimate. Let us assume by induction that we have the desired bounds when $i'+j' < k$ and also for $i'+j'=k$, $0 \le i' < i$. We consider the derivative $\partial_s^i \partial_x^j \overline{\partial}_z$ of $m_z^s(x+\alpha(s)) = F_z^s(x, m_z^s(x))$. The left hand side has a main term of the form $(\partial_s^i \partial_x^j \overline{\partial}_z m_z^s)(x+\alpha(s))$ and a lower order term

$$\sum_{l<i, l+m\leq k}^{i} C \cdot (\partial_s^l \partial_x^m \overline{\partial}_z m_z^s)(x+\alpha(s)), \tag{3.8}$$

with $C$ polynomials (depending on the indices) on the derivatives of $\alpha$ of order at most $i$. The lower order term can be estimated by induction to be $O(|\Im z|^{\eta-2k-i-1)})$. The right hand side has one main term $\partial_w F_z^s(x, m_z^s(x)) \cdot \partial_s^i \partial_x^j \overline{\partial}_z m_z^s(x)$ and two lower order terms. The first has the form

$$\sum C \cdot \prod_{n=1}^{l} \partial_s^{i_n} \partial_x^{j_n} m_z^s(x), \tag{3.9}$$

where the sum runs over all $0 \leq l \leq k$, and $(i_1, j_1) \leq \ldots \leq (i_l, j_l)$ (lexicographic order) with $i_n, j_n \geq 0$, $1 \leq n \leq l$, $\sum_{n=1}^{l} i_n = i' \leq i$, $\sum_{n=1}^{l} j_n = j' \leq j$, and $C$ are now constant multiples of $\partial_s^{i-i'} \partial_x^{j-j'} \overline{\partial}_z \partial_w^l F$. Using the first estimate, we see that it is $O(|\Im z|^{\eta-2k-i})$ except when $k = 0$, where it is $O(|\Im z|^{\eta-1})$. The second has the form

$$\sum D \cdot \partial_s^{i_0} \partial_x^{j_0} \overline{\partial}_z m_z^s(x) \prod_{n=1}^{l} \partial_s^{i_n} \partial_x^{j_n} m_z^s(x), \tag{3.10}$$

where the sum runs over all $0 \leq l \leq k$, $0 \leq i_0 \leq i$, $0 \leq j_0 \leq j$, $i_0 + j_0 < k$, and $(i_1, j_1) \leq \ldots \leq (i_l, j_l)$ (lexicographic order) with $i_n, j_n \geq 0$, $1 \leq n \leq l$, $\sum_{n=0}^{l} i_n = i' \leq i$, $\sum_{n=0}^{l} j_n = j' \leq j$, and $D$ are constant multiples of $\partial_s^{i-i'} \partial_x^{j-j'} \partial_w^l F$. This term can be estimated using the first estimate and the induction hypothesis to be $O(|\Im z|^{\eta-2k-i-1})$. Rearranging we get

$$|(\partial_s^i \partial_x^j \overline{\partial}_z) m_z^s(f+\alpha(s)) - \partial_w F_z^s(x, m_z^s(x)) \cdot \partial_s^i \partial_x^j \overline{\partial}_z m_z^s(x)| = O(|\Im z|^{\eta-2k-i-1}), \tag{3.11}$$

and the second estimate follows from Proposition 2.8. □

**Theorem 3.4.** *Let $A_{\theta,s} \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ be monotonic on $\theta$ and $C^\infty$ on $\theta, s$, and let $s \mapsto \alpha(s)$ be $C^\infty$. Then the average with respect to $\theta$ of the Lyapunov exponent of $A_{\theta,s}$ over $x \mapsto x + \alpha(s)$ is a $C^\infty$ function of $s$.*

*Proof.* Fixing $k \geq 0$, choose $\eta$ large and let $A_{z,s} \in \Delta_\delta$ be an $\eta$-asymptotically holomorphic extension of $A_{\theta,s}$. As in the proof of Theorem 2.2, we define for every $s$ and $0 < t < \delta$

$$U(t, s) = \int_{\mathbb{R}/\mathbb{Z}} L(A_{\sigma+it,z}) d\sigma. \tag{3.12}$$

Then for $0 < t < \delta$ the functions $s \mapsto U(t, s)$ are $C^k$ and as $t \to 0$ they converge uniformly in $C^k$. For each fixed $s$, the limit is seen to be the $\theta$-average of the Lyapunov exponent of $A_{\theta,s}$: since $s$ is fixed, we can just apply Lemma 2.10. □

*Remark* 3.2. Even if everything is analytic, we do not, in general, get analytic dependence when varying the frequency by this argument. Analytic dependence should not be expected, since when the frequency is complexified the domain of analyticity does not remain invariant by the dynamics. A special case where analytic dependence holds is when $A_{\theta,s} = R_\theta A_s$, since the dynamics does not influences the $\theta$-average of the Lyapunov exponent in this case.

**Theorem 3.5.** *If $A_s \in C^\infty(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ is monotonic and $C^\infty$ on $x$ and $s$, and $s \mapsto \alpha(s)$ is $C^\infty$, then the Lyapunov exponent of $A_s$ as a cocycle over $x \mapsto x+\alpha(s)$ is a $C^\infty$ function of $s$.*

*Proof.* Assume that $A$ is $w$-monotonic with $w$ a primitive vector of $\mathbb{Z}^d$ and consider the family $A_{\theta,s} = A_s(x+\theta w)$. The Lyapunov exponents of $A_{\theta,s}$ and of $A_s$ (both considered as cocycles over the same $x \mapsto x+\alpha(s)$) are obviously equal for every $\theta$. The result follows by the previous theorem. □

3.2. **Low regularity considerations.** We return to the consideration of a fixed translation dynamics, but now focus on trying to obtain conclusions at low regularity.

Consider some $A \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ which is $w$-monotonic, and let $A^w_\theta(x) = A(x+\theta w)$. It follows directly from the definitions that $\rho_{A^w_\theta}$ is an affine function of $\theta$ with negative slope $\deg^w$. In fact $\deg^w$ can be explicitly given as a linear function of $w$ which only depends on topological data: $\deg^w = \langle l, w\rangle$ where $l \in \mathbb{Z}^d$ is the unique integer such that $A$ is homotopic to $x \mapsto R_{\langle l,x\rangle}$.[10]

In particular, $\theta \mapsto \rho_{A^w_\theta}$ is Lipschitz. More generally, we have the following result.

**Lemma 3.6.** *Let us consider a one-parameter family $A_\theta \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$. If for some $\theta_0$, $A_{\theta_0}$ is a $w$-monotonic cocycle, and*

$$K = \limsup_{\theta\to\theta_0} \frac{1}{|\theta-\theta_0|} \sup_x \|A_\theta(x) - A_{\theta_0}(x)\| < \infty \tag{3.13}$$

*then*

$$\limsup_{\theta\to\theta_0} \frac{1}{|\theta-\theta_0|} |\rho_{A_\theta} - \rho_{A_{\theta_0}}| \le K', \tag{3.14}$$

*where $K'$ depends on $K$, the monotonicity constant of $A^w_{\theta_0}$, $\|A_{\theta_0}\|_{C^0}$ and $\deg^w$.*

*Proof.* The hypothesis imply that for $h$ close to 0 and $z \in \partial\mathbb{D}$, $\mathring{A}_{\theta_0+h}(x)\cdot z$ lies in the shortest segment of $\partial\mathbb{D}$ determined by $\mathring{A}_{\theta_0}(x-Chw)\cdot z$ and $\mathring{A}_{\theta_0}(x+Chw)\cdot z$, for some $C>0$. This implies that $\rho_{A_{\theta_0+h}}$ lies between $\rho_{A_{\theta_0}(\cdot+Chw)}$ and $\rho_{A_{\theta_0}(\cdot-Chw)}$,[11] that is, in the segment $[\rho_{A_{\theta_0}} + Ch\deg^w, \rho_{A_{\theta_0}} - Ch\deg^w]$, and the result follows. □

Thus, if $A$ is monotonic then $\theta \mapsto R_{-\theta}A$ is a monotonic family with Lipschitz rotation number. In low regularity, it may be preferable to work with this family, because it is always analytic in $\theta$. As an application, we have the following result which is a direct consequence of Theorem 2.14 (if we were to use only the family $\theta \mapsto A^w_\theta$, we would need to assume further regularity).

**Theorem 3.7.** *Let $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be monotonic. If $L(A)=0$ then $A$ is $L^2$-conjugate to a cocycle of rotations.*

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[10]To see this, define $\delta_{v,w}\xi_n = \delta_\gamma\xi_n$ and $\delta_{v,w}\zeta = \delta_\gamma\zeta$ where $\gamma$ is any path homotopic to $\gamma(t,x) = A(x+v+tw)$. Notice that $\rho_{A^w_\theta}$ is given, up to additive constant, by $\delta_{0,\theta w}\zeta$. Let us show that $\delta_{v,w}\zeta = \langle l, w\rangle$. It is clear that $\delta_{v,w}\xi_n(x,z_0,z_1) = \delta_{0,w}(x+v,z_0,z_1)$, so that $\delta_{v,w}\zeta$ does not depend on $v$. By Remark 2.2, $\delta_{v,w}\zeta + \delta_{v+w,w'}\zeta = \delta_{v,w+w'}\zeta$. This shows that $\delta_{v,w}\zeta$ is a linear function of $w$. Moreover, for $w \in \mathbb{Z}^d$ we have exactly $\delta_{v,w}\xi_n = \langle l,w\rangle n$, so that $\delta_{v,w}\zeta = \langle l, w\rangle$.

[11]Indeed we can construct a monotonic decreasing family $\tilde{A}_t \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$, $t \in [0,1]$, such that $\tilde{A}_0(\cdot) = A_{\theta_0}(\cdot - Chw))$, $\tilde{A}_1(\cdot) = A_{\theta_0}(\cdot + Chw)$ and $\tilde{A}_{1/2} = A_{\theta_0+h}$, and $\tilde{A}_t$ is close to $A_{\theta_0}$ for every $t$ (so that we remain in a region where there is a continuous determination of $\rho$). Monotonicity of this family gives $\rho_{\tilde{A}_1} \le \rho_{\tilde{A}_{1/2}} \le \rho_{\tilde{A}_0}$.

It also allows us to get continuity results in Lipschitz open sets of cocycles. For the family $R_{-\theta}A$, $e^{2\pi i\theta} \mapsto e^{-2\pi i\rho(\theta)-L(\theta)}$, $\Im\theta > 0$, defines an univalent function from $\mathbb{D} \setminus \{0\}$ to $\mathbb{D} \setminus \{0\}$ (see [CJ]).This gives an harmonic conjugacy relation between the Lyapunov exponent and the fibered rotation number. If the fibered rotation number turns out to be Lipschitz, then the Lyapunov exponent (as a function of the circle) has derivative in $L^1$ and in fact $\partial_\theta L$ is a zero-average BMO function (since it is basically the Hilbert trasform of the derivative of the fibered rotation number, which is in $L^\infty$). This argument also shows that the BMO norm of $\partial_\theta L$ can be bounded in terms of the Lipschitz constant of $\rho$.

**Theorem 3.8.** *Let $\epsilon > 0$ be fixed. The Lyapunov exponent is a continuous function of $\epsilon$-monotonic cocycles (the frequency may be varied as well).*

*Proof.* Let $A^{(n)} \to A$ be a sequence of $\epsilon$-monotonic cocycles converging in $C^0$. We allow $A^{(n)}$ to be regarded as cocycles over $x \mapsto x + \alpha_n$ and $A$ over $x \mapsto x + \alpha$, as long as $\alpha_n \to \alpha$. By the previous lemma, there exists $C > 0$ such that the fibered rotation number of $\theta \mapsto R_{-\theta}A^{(n)}$ is a $C$-Lipschitz function for all $n$, and by the discussion above, $\theta \mapsto L_n(\theta) = L(R_{-\theta}A^{(n)})$ is uniformly equicontinuous. Thus we may assume $L_n \to L_\infty$ in $C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. By upper semicontinuity of the Lyapunov exponent, $\theta \mapsto L(R_{-\theta}A) - L_\infty(\theta)$ is a non-negative continuous function, which we must show to be identically zero. By [AB],

$$\begin{aligned}(3.15)\quad & \int_{\mathbb{R}/\mathbb{Z}} L(R_{-\theta}A) - L(R_{-\theta}A^{(n)})d\theta \\ &= \int_{\mathbb{R}/\mathbb{Z}} \ln \frac{\|A(x)\| + \|A(x)\|^{-1}}{2} - \ln \frac{\|A^{(n)}(x)\| + \|A^{(n)}(x)\|^{-1}}{2} dx,\end{aligned}$$

so $\int_{\mathbb{R}/\mathbb{Z}} L(R_{-\theta}A) - L_\infty(\theta)d\theta = \lim_{n\to\infty} \int_{\mathbb{R}/\mathbb{Z}} L(R_{-\theta}A) - L(R_{-\theta}A^{(n)})d\theta = 0.$ □

3.3. **Minimality.** Here we are interested in considering the dynamics of cocycles from the topological point of view. For this, one considers the cocycle given by some $A$ as a map, still denoted by $(f_\alpha, A)$, from $\mathbb{R}^d/\mathbb{Z}^d \times \partial\mathbb{D} \to \mathbb{R}^d/\mathbb{Z}^d \times \partial\mathbb{D}$ (a $d+1$-dimensional torus) given by $(x, w) \mapsto (x + \alpha, \mathring{A}(x) \cdot w)$, where $\partial\mathbb{D}$ is identified with $\mathbb{PR}^2$ in the usual way. Below we consider only the case where $x \mapsto x + \alpha$ is ergodic.

It can be shown (see [KKHO]) that if $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ is not homotopic to the identity then for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $(f_\alpha, A)$ is transitive, and an adapted argument for the multidimensional case is given in Appendix B. Though proofs of transitivity just involve simple topological arguments, the following question seems much harder:

*Problem* 3.1. Is $(f_\alpha, A)$ minimal whenever $A \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2, \mathbb{R}))$ is non-homotopic to the identity?

Of course the same problem still makes sense under additional smoothness assumptions. In this section we will show that the complexification methods allow one to address the local case, at least if one assumes enough smoothness.

Let us first discuss some known results on the minimal sets of non-uniformly hyperbolic cocycles (we follow the presentation of Herman [H], but the results are due to Johnson [J1]). If $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ and $L(A) > 0$ then it follows from Oseledets Theorem that there exist two measurable functions $u, s : \mathbb{R}^d/\mathbb{Z}^d \to \partial\mathbb{D}$

(the unstable and stable directions) such that $\mathring{A}(\theta)\cdot u(\theta) = u(\theta+\alpha)$ and $\mathring{A}(\theta)\cdot s(\theta) = s(\theta+\alpha)$ and for almost every $\theta$, for every $w \in \overline{\mathbb{C}}$, if $w \neq s(\theta)$ then $|\mathring{A}_n(\theta)\cdot w - u(\theta+n\alpha)| \to 0$ exponentially fast, and if $w \neq u(\theta)$ then $|\mathring{A}_n(\theta-n\alpha)^{-1}\cdot w - s(\theta-n\alpha)| \to 0$ exponentially fast as $n \to \infty$. It follows (from unique ergodicity of $\theta \mapsto \theta+\alpha$) that there are exactly two ergodic invariant measures on $\mathbb{R}^d/\mathbb{Z}^d \times \partial\mathbb{D}$, the push-forwards of Lebesgue measure on $\mathbb{R}^d/\mathbb{Z}^d$ by $\theta \mapsto (\theta, u(\theta))$ and $\theta \mapsto (\theta, s(\theta))$, which we denote by $\mu_u$ and $\mu_s$. Let us denote their (compact) support by $K_u$ and $K_s$. It follows that any minimal set for $(f_\alpha, A)$ coincides with either $K_u$ or $K_s$. Moreover, if $K_u \neq K_s$ then $A$ would necessarily be uniformly hyperbolic. So assuming $A$ to be non-homotopic to the identity with $L(A) > 0$ we get that $K_u = K_s$ is the unique minimal set of $(f_\alpha, A)$.

**Theorem 3.9.** *Let $A \in C^{1+\epsilon}(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ be monotonic. Then $(f_\alpha, A)$ is minimal.*

*Proof.* If $L(A) = 0$ then $(f_\alpha, A)$ is $C^0$ conjugate to a cocycle of rotations, by the argument of the proof of Theorem 2.1.[12]

For a cocycle of rotations, transitivity obviously implies minimality, so the result follows from Proposition B.1.

Let now $L(A) > 0$. We consider the analytic case, the smooth case being analogous. Up to coordinate change, we may assume that $A$ is $w$-monotonic where $w = (1, 0, ..., 0)$. Let $m : \Omega_\delta^+ \times \mathbb{R}^{d-1}/\mathbb{Z}^{d-1} \to \mathbb{D}$ satisfy $\mathring{A}(z)\cdot m(z) = m(z+\alpha)$. Then for every $x_2, ..., x_n$ and for almost every $x_1$, $m(x_1, ..., x_n) = \lim_{t\to 0} m(x_1+ti, x_2, ..., x_n)$ exists. Since $L(A) > 0$, for almost every $x \in \mathbb{R}^d/\mathbb{Z}^d$, $m(x) \in \partial\mathbb{D}$ and $m|\mathbb{R}^d/\mathbb{Z}^d$ coincides with either the unstable or stable directions $u$, $s$ defined above.[13]

We claim that for every open set of the form $J \times U \subset \mathbb{R}^d/\mathbb{Z}^d$ with $J \subset \mathbb{R}/\mathbb{Z}$, and any interval $J' \subset \partial\mathbb{D}$, there exists a positive measure set of $x \in J \times U$ such that $m(x) \in J'$: by the previous discussion, the unique minimal set $K_u = K_s$ of $(f_\alpha, A)$ must intersect $J \times U \times J'$, and since it is arbitrary we must have $K_u = K_s = \mathbb{R}^d/\mathbb{Z}^d \times \partial\mathbb{D}$.

Suppose by contradiction that the claim does not hold. Up to a change of coordinates, we may assume that $-1 \in J'$. Then for almost every $y \in U$, $z \mapsto (1-m(z,y))(1+m(z,y))$ is a holomorphic function on $\Omega_\delta^+$ with positive real part, bounded near $J$ and whose non-tangential limits are purely imaginary on $J$; by the Schwarz Reflection Principle there is a holomorphic extension to $\Omega_\delta \backslash (\mathbb{R}/\mathbb{Z} \backslash J)$. Thus $x_1 \mapsto m(x_1, ..., x_n)$ is analytic on $J$ for almost every $(x_2, ..., x_n) \in U$. By invariance we have $x_1 \mapsto m(x_1, ..., x_n)$ analytic on $\mathbb{R}/\mathbb{Z}$ for almost every $(x_2, ..., x_n)$. The topological degree of $x_1 \mapsto m(x_1, ..., x_n)$ is an integer valued measurable function $\deg(x_1, x_2, ..., x_n)$ which does not depend on $x_1$. But $\deg(x+\alpha) = \deg(x) + \deg$, where deg is the topological degree of $x_1 \mapsto \mathring{A}(x_1, 0, ..., 0)\cdot 1$. Since $A$ is $w$-monotonic with $w = (1, 0, ..., 0)$, $\deg < 0$. But by Poincaré recurrence $\deg(x+n\alpha)$ takes the same value infinitely many times, for almost every $x$, so $\deg = 0$, contradiction. □

3.4. **Premonotonic cocycles.** As remarked in the introduction, the concept of monotonicity is not dynamically natural. The easiest way to extend the concept of

[12] Theorem 2.1 gives a $C^1$ conjugacy under a slightly stronger $C^{2+\epsilon}$ condition. Under $C^{1+\epsilon}$, it still gives a continuous invariant section $m : \mathbb{R}^d/\mathbb{Z}^d \to \overline{\mathbb{D}}$, which implies the $C^0$ conjugacy, unless the invariant section is real, i.e., it lies in $\partial\mathbb{D}$. However, this last possibility is impossible here, since $A$ is non-homotopic to a constant so it can not admit invariant continuous sections in $\partial\mathbb{D}$.

[13] It is easy to see that it actually coincides with $u$, but this will not play a role here.

monotonicity is the following. We say that a cocycle $A \in C^1(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ is *premonotonic* if some iterate is $C^1$ conjugate to a monotonic cocycle: there exist $n \geq 1$ and $B \in C^1(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ such that $B(x+n\alpha)A_n(x)B(x)^{-1}$ is monotonic. This happens if and only if some iterate of $(f_\alpha, A)$ is real-analytic conjugate to a monotonic cocycle (any $C^1$-perturbation of $B$ which is real analytic will do). Notice that premonotonic cocycles are $C^1$-stable, and there is even stability with respect to perturbations of the frequency vector defining the dynamics in the basis.

Cocycles of rotations over ergodic translations which are not homotopic to a constant provide the simplest examples of premonotonic cocycles. Given a cocycle $A \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$, let $[A]$ be the unique cocycle homotopic to $A$ of the form $[A](x) = R_{\langle l,x\rangle}$ with $l = l^A \in \mathbb{Z}^d$. Notice that $A$ is not homotopic to a constant if and only if $l \neq 0$, and in this case $[A]$ is $l$-monotonic.

**Lemma 3.10.** *Let $A \in C^r(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$, $1 \leq r \leq \infty$ or $r = \omega$, and let $x \mapsto x+\alpha$ be ergodic on $\mathbb{R}^d/\mathbb{Z}^d$. If $(f_\alpha, A)$ is $C^r$-conjugate to a cocycle of rotations then there exists a sequence $B^{(n)} \in C^r(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ such that $A^{(n)} \to [A]$ in the $C^r$-topology, where $A^{(n)}(x) = B^{(n)}(x+\alpha)A(x)B^{(n)}(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$.*

*Proof.* By definition, we may assume that $A$ is itself a cocycle or rotations (the homotopy class being clearly conjugacy invariant). Thus let $A(x) = [A](x)R_{\phi(x)}$, where $\phi : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ is $C^r$. Let $c = \int \phi(x)dx$. Since $\alpha$ is irrational, we can consider a sequence $l_k \in \mathbb{Z}^d$ such that $\langle \alpha, l_k\rangle \to c \bmod 1$.

Let us consider a sequence $\phi^{(n)} : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ of trigonometric polynomials converging to $\phi$ in $C^r$ and with $\int \phi^{(n)}(x)dx = c$. Since $x \mapsto x+\alpha$ is an ergodic translation, it is easy to define, using Fourier series, trigonometric polynomials $\psi^{(n)} : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ such that $\phi^{(n)}(x) = -\psi^{(n)}(x+\alpha) + \psi^{(n)}(x) + c$. Letting $C^{(n)}(x) = R_{\psi^{(n)}(x)}$, we see that $C^{(n)}(x+\alpha)R_{\phi(x)}C^{(n)}(x)^{-1}$ is $C^r$ close to $R_c$. Consider now $B^{(n)}(x) = R_{-\langle l_{k_n},x\rangle}C^{(n)}(x)$. If we choose $k_n \to \infty$ very slowly, we will have $B^{(n)}(x+\alpha)A(x)B^{(n)}(x)^{-1} \to [A](x)$ in $C^r$, as desired. $\square$

While premonotonicity is only *a priori* invariant under $C^1$-conjugacies, we have:

**Theorem 3.11.** *Let $A \in C^1(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ be non homotopic to the identity, and let $f_\alpha : x \mapsto x+\alpha$ be ergodic on $\mathbb{R}^d/\mathbb{Z}^d$. If $(f_\alpha, A)$ is $C^0$-conjugate to a cocycle of rotations, then $(f_\alpha, A)$ is premonotonic.*

*Proof.* Up to isometric automorphism of $\mathbb{R}^d/\mathbb{Z}^d$, we may assume that the first coordinate $l_1$ of $l = l^A$ is positive.

Denote by $B : \mathbb{R}^d/\mathbb{Z}^d \to \mathrm{SL}(2,\mathbb{R})$ the $C^0$-map such that $R(\cdot) := B(\cdot+\alpha)A(\cdot)B(\cdot)^{-1}$ takes it values in the group of rotations. We will identify $\mathbb{PR}^2$ with $\mathbb{R}/\mathbb{Z}$ so that the projective action of rotations corresponds to translations. Let $F_n : \mathbb{R}^d/\mathbb{Z}^d \times \mathbb{PR}^2 \to \mathbb{R}^d/\mathbb{Z}^d \times \mathbb{PR}^2$ be the projective action, $F_n(x,y) = (x+n\alpha, A_n(x)\cdot y)$. For $(x,y) \in \mathbb{R}^d/\mathbb{Z}^d \times \mathbb{PR}^2$, let $a_n(x,y) = \partial_{x_1}(A_n(x)\cdot y)$, $b_n(x,y) = \partial_y(A_n(x)\cdot y)$, $p(x,y) = \partial_y(B(x)\cdot y)$, $q_n = a_n p \circ F^n$, $q = q_1$. We claim that $b_n = \frac{p}{p\circ F^n}$ and

$$q_n = \sum_{k=0}^{n-1} q \circ F^k. \tag{3.16}$$

Indeed, from the definition of $B$ and $R$, one can write for any $x$, $B(x+n\alpha)\cdot(A_n(x)\cdot y) = R_n(x)\cdot(B_n(x)\cdot y)$ and thus taking derivatives with respect to $y$, $p(x+n\alpha, A_n(x)\cdot y)b_n(x,y) = p(x,y)$ which is the relation $b_n = \frac{p}{p\circ F^n}$. For (3.16) we just write $A_{n+1}(x) \cdot y = A(x + n\alpha) \cdot (A_n(x) \cdot y)$ and take derivatives with respect to $x_1$ to get $a_{n+1} = a_1 \circ F^n + (b_1 \circ F^n)a_n$; since we have just seen that $b_1 \circ F^n = (p \circ F^n)/(p \circ F^{n+1})$ we have $a_{n+1}(p \circ F^{n+1}) = (a_1 p \circ F) \circ F^n + a_n(p \circ F^n)$ which obviously gives (3.16).

Let $e_1 = (1, ..., 0)$. Notice that $a_n(x, y) > 0$ for all $(x, y)$ is equivalent to $-e_1$-monotonicity of $A_n$. Below we will prove that $q_n \to \infty$, and hence $a_n \to \infty$, uniformly in $(x, y)$, giving the premonotonicity of $(f_\alpha, A)$.

For $x_0, x \in \mathbb{R}^d/\mathbb{Z}^d$, $y \in \mathbb{PR}^2$

$$d_n^\epsilon(x_0, x, y) = \int_0^\epsilon p(x_0 + n\alpha, A_n(x + te_1) \cdot y) a_n(x + te_1, y) dt \tag{3.17}$$

gives the oriented length of the path $\gamma|[0, \epsilon]$, where $\gamma = \gamma_{n,x_0,x,y} : \mathbb{R} \to \mathbb{PR}^2$ is given by $\gamma(t) = B(x_0 + n\alpha)A_n(x + te_1) \cdot y$ (the oriented length can be defined as $\hat{\gamma}(\epsilon) - \hat{\gamma}(0)$ where $\hat{\gamma} : \mathbb{R} \to \mathbb{R}$ is a lift to the universal cover). Especially, for any $y, y' \in \mathbb{PR}^2$ we must have

$$-1 < d_n^\epsilon(x_0, x, y) - d_n^\epsilon(x_0, x, y') < 1, \tag{3.18}$$

since when $y \neq y'$ we must have $\gamma_{n,x_0,x,y}(t) \neq \gamma_{n,x_0,x,y'}(t)$ for every $t \in \mathbb{R}$.

Let $H(x, y) = (x, B(x) \cdot y)$ and let $G = H \circ F \circ H^{-1}$. Since $G$ is topologically conjugate to $F$, and Lebesgue measure on $(x, y)$ is invariant for $G$, $H_*^{-1}\mathrm{Leb}$ is an invariant measure for $F$ equivalent to Leb. Thus for Lebesgue almost every $(x, y)$,

$$\hat{q}(x, y) = \lim \frac{1}{n} \sum_{k=0}^{n-1} q \circ F^k(x, y) \tag{3.19}$$

exists. Since $\hat{q}$ is measurable, Lebesgue almost every $(x, y)$ is a measurable continuity point along the $x_1$ direction. Especially, for almost every $(x, y)$ we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \lim_{n \to \infty} \frac{1}{n} d_n^\epsilon(x, x, y) = \hat{q}(x, y). \tag{3.20}$$

Thus $\hat{q}(x, y)$ is almost surely independent of $y$, and since $x \mapsto x + \alpha$ is ergodic, $\hat{q}(x, y)$ is almost surely independent of $x$ and $y$.

Notice that $G$ commutes with shifts in the second coordinate $T_t(x, y) = (x, y+t)$. Thus any ergodic invariant measure $\mu$ for $G$ gives rise to a one-parameter family of ergodic invariant measures $\mu_t = (T_t)_*\mu$. By unique ergodicity of $x \mapsto x+\alpha$, all those measures project down to Lebesgue measure on $x$. It follows that $\int_{\mathbb{R}/\mathbb{Z}} \mu_t dt = \mathrm{Leb}$. By uniqueness of the ergodic decomposition, it follows that all ergodic invariant measures are of the form $\mu_t$, for some $t \in \mathbb{R}$ (for any fixed $\mu$).

Since $\mu_t$ depends continuously on $t \in \mathbb{R}/\mathbb{Z}$ (with respect to the weak-$*$ topology) and $\int_{\mathbb{R}/\mathbb{Z}} \mu_t dt = \mathrm{Leb}$, the fact that $\hat{q} \circ H^{-1} = \lim \frac{1}{n} \sum_{k=0}^{n-1} q \circ H^{-1} \circ G^k$ is almost everywhere constant implies that $\int q \circ H^{-1} d\mu_t$ is independent of $t$. Since $q \circ H^{-1}$ has constant average with respect to all ergodic invariant measures, the Birkhoff averages of $q \circ H^{-1}$ converge uniformly to a constant limit. Thus $\frac{q_n}{n} \to \int q \circ H^{-1} d\mathrm{Leb}$ uniformly.

To conclude, we must show that $\int q \circ H^{-1} d\mathrm{Leb} > 0$. If this is not the case, then for every $n$ sufficiently large we will have $\frac{a_n}{n} < \frac{1}{2}$. But this is impossible because of the identity $\frac{1}{n} \int_0^1 a_n(t, x_2, ..., x_n, y) dt = l_1$ which is a positive integer. $\square$

The definition of premonotonicity is such that results proved for monotonic cocycles extend easily to this larger setting. Let us comment in more detail on the results stated in the introduction which involve premonotonicity (except for Theorem 1.5, which we discuss in the next section).

Theorem 1.3 follows from Theorem 3.2 and Theorem 1.4 follows from Theorem 3.5 (as the Lyapunov exponent is well behaved when taking conjugacies and iterates).

In order to derive Theorem 1.1 from Theorem 3.1, it is enough to notice that if a $C^r$ cocycle $(f_\alpha, A)$ admits an iterate which is $C^r$ conjugate to a cocycle of rotations, then $(f_\alpha, A)$ is itself $C^r$-conjugate to a cocycle of rotations.[14]

The proof (if not the statement) of Corollary 1.2 also involves premonotonicity: it follows from Theorem 1.1 and Theorem 3.11.

Theorem 1.6 follows from Theorem 3.9 (since minimality of any iterate implies minimality).

3.5. **Non-uniform hyperbolicity for typical premonotonic cocycles.** In this section, we will only consider, for simplicity, the case of $C^r$ cocycles with $r = \infty$ or $\omega$. Then the Lyapunov exponent is indeed a $C^r$ function of premonotonic cocycles (while we have only carried out the formal arguments for the dependence of the Lyapunov exponent along one-parameter families, it is clear the estimates go through to the infinite dimensional parametrization). Since the Lyapunov exponent $L$ takes non-negative values, we must have $DL = 0$ whenever $L = 0$. Here we are going to show that, in the case of premonotonic cocycle, if $L = 0$ then $D^2 L \neq 0$. This implies that $\{L = 0\}$ is a subvariety of positive codimension in the space of premonotonic cocycles and completes the proof of Theorem 1.5.

If $B \in C^r(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$, then the conjugacy operator $A \mapsto A'$, $A'(x) = B(x+\alpha)A(x)B(x)^{-1}$ is a $C^r$ diffeomorphism in $C^r(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$. Since the Lyapunov exponent is clearly invariant by conjugacy, it suffices to check that any premonotonic cocycle $A$ with $L(A) = 0$ is conjugate to some $A'$ such that $D^2 L(A') \neq 0$. But $C^r$ premonotonic cocycles with zero Lyapunov exponent are $C^r$ conjugate to cocycles of rotations by Theorem 1.1, and in fact, by Lemma 3.10, those may be chosen arbitrarily close to a cocycle of the form $x \mapsto [A](x) = R_{\langle l,x\rangle}$ with $l \neq 0$. Since the Lyapunov exponent is $C^2$ near $[A]$, it suffices to show $D^2 L([A]) \neq 0$. We will in fact give a simple estimate implying the existence of cocycles near $[A]$ with a quadratic lower bound on the Lyapunov exponent.

For a matrix $s \in \mathrm{sl}(2,\mathbb{R})$, let $s_1, s_2, s_3$ be such that $s = \begin{pmatrix} s_1 & s_2 + s_3 \\ s_2 - s_3 & -s_1 \end{pmatrix}$.

**Lemma 3.12.** *Let* $l \in \mathbb{Z}^d \backslash \{0\}$, $s \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{sl}(2,\mathbb{R}))$, *and define* $A_{\theta,t} \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in \mathbb{R}/\mathbb{Z}$, $t \in \mathbb{R}$ *by* $A_{\theta,t}(x) = R_{\langle l,x\rangle} e^{ts(\langle l,x\rangle - \theta)}$. *Then*

$$\lim_{t \to 0} \frac{2}{t^2} \int_{\mathbb{R}/\mathbb{Z}} L(A_{\theta,t}) d\theta = \int_{\mathbb{R}/\mathbb{Z}} s_1^2(\theta) + s_2^2(\theta) d\theta. \tag{3.21}$$

*In particular, the limit is zero if and only if $s$ takes values in* $\mathrm{so}(2,\mathbb{R})$.

[14]This is most easily seen by working with $C^r$ invariant sections (which arise from and give rise to a conjugacy to rotations in the usual way). If $m \in C^r(\mathbb{R}^d/\mathbb{Z}^d, \mathbb{D})$ satisfies $\mathring{A}_n(x) \cdot m(x) = m(x + n\alpha)$, let $m_j(x) = \mathring{A}_j(x - j\alpha) \cdot m(x - j\alpha)$. Then $m_{j+n} = m_j$ and $\mathring{A}(x) \cdot m_j(x) = m_{j+1}(x)$. For each $x \in \mathbb{R}^d/\mathbb{Z}^d$, let $m_*(x)$ minimizes the sum of the squares of the hyperbolic distances (in $\mathbb{D}$) to $(m_j(x))_{j=0}^{n-1}$: this is a well defined $C^r$ function of $x$ by strict convexity. Then $\mathring{A}(x) \cdot m_*(x) = m_*(x + \alpha)$.

*Proof.* Let $C_{t,\theta}(x) = R_\theta C_t$ where $C_t(x) = R_{\langle l,x\rangle} e^{ts(\langle l,x\rangle)}$. Notice that $A_{\theta,t}(x + \theta\frac{l}{\|l\|^2}) = C_{t,\theta}(x)$. So $L(A_{\theta,t}) = L(C_{t,\theta})$. By [AB],

$$\int_{\mathbb{R}/\mathbb{Z}} L(C_{t,\theta})d\theta = \int_{\mathbb{R}^d/\mathbb{Z}^d} \ln \frac{\|C_t(x)\| + \|C_t(x)\|^{-1}}{2} dx = \int_{\mathbb{R}/\mathbb{Z}} \ln \frac{\|e^{ts(\theta)}\| + \|e^{ts(\theta)}\|^{-1}}{2} d\theta. \tag{3.22}$$

On the other hand, a direct computation shows that

$$\lim_{t\to 0} \frac{2}{t^2} \int_{\mathbb{R}/\mathbb{Z}} \ln \frac{\|e^{ts(\theta)}\| + \|e^{ts(\theta)}\|^{-1}}{2} d\theta = \int_{\mathbb{R}/\mathbb{Z}} s_1^2(\theta) + s_2^2(\theta) d\theta. \tag{3.23}$$

The result follows. □

Choosing, say, $s_1(\theta) = \cos 2\pi\theta$, $s_2 = s_3 = 0$, we see that the family $A_{\theta,t}$ is an analytic family (on $\theta$ and $t$) of analytic cocycles such that $A_{0,t}$ is constant equal to $x \mapsto R_{\langle l,x\rangle}$. The previous lemma then implies that $D^2L$ (in either setting, analytic or smooth) does not vanish on $x \mapsto R_{\langle l,x\rangle}$, as desired.

## 4. One-frequency cocycles: renormalization and rigidity

We continue our investigations of quasiperiodic cocycles, but now specify to the case of one frequency. Though the number of frequencies is quite irrelevant in the analysis of monotonic cocycles, in the one-frequency case we will be able to obtain global consequences from our local analysis, by means of renormalization, a tool that is not as effective when several frequencies are involved.

Below we will only consider cocycles over irrational rotations. To highlight the dependence on the base dynamics, through this section a cocycle will be specified by a pair $(\alpha, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$.

After defining the renormalization operator, we are going to show that if $(\alpha, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is $L^2$-conjugate to rotations, then it admits a "renormalization representative" $(\alpha', A') \in (\mathbb{R}\setminus\mathbb{Q}) \times C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ (seen as a cocycle over some different irrational rotation) with $A'$ $C^1$-close to $x \mapsto R_{\theta+\deg x}$ for some $\theta$ (here deg is the topological degree of $A$). Moreover, if $A$ is $C^r$, $A'$ can be chosen to be $C^r$. The dynamics of $A$ and $A'$ can be related, in particular if $L(\alpha, A) = 0$ then $L(\alpha', A') = 0$ and if $(\alpha', A')$ is $C^r$ conjugate to rotations then $(\alpha, A)$ is also $C^r$-conjugate to rotations.

Now, if $A$ is not homotopic to a constant, $\deg \neq 0$, so $A'$ is monotonic. This leads to our main global rigidity result in the one-dimensional case, Theorem 1.8.

Let us note that by our analysis of one-parameter families, Theorem 1.8 implies:

**Theorem 4.1.** *Let $(\alpha, A_\theta)$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $A \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $r = \infty, \omega$, be a one-parameter family which is monotonic and $C^{2+\epsilon}$ in $\theta$. If the $A_\theta$ are non-homotopic to a constant then for almost every $\theta$, either $L(A_\theta) > 0$ or $A_\theta$ is $C^r$ conjugate to rotations.*

*Proof.* By Theorem 1.7, for almost every $\theta$ with $L(\alpha, A_\theta) = 0$, $(\alpha, A)$ is $L^2$ conjugate to rotations. By Theorem 1.8, they must be actually $C^r$ conjugate to rotations. □

4.1. **Renormalization.** In this section we recall some basic facts on renormalization. We refer to [AK] for the proofs and further details.

Let $(\alpha, A) \in ((0,1) \setminus \mathbb{Q}) \times C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be a cocycle Let $p_n/q_n$ be the continued fraction approximants of $\alpha$ and let $\beta_n = (-1)^n(q_n\alpha - p_n)$, $\alpha_n = \beta_n/\beta_{n-1}$. Thus $\alpha_n = G^n(\alpha)$ where $G(\alpha) = \{\alpha^{-1}\} = \alpha^{-1} - [\alpha^{-1}]$ is the Gauss map.

Classically, the dynamical systems $x \mapsto x+\alpha_n$ can be interpreted as the sequence renormalization of $x \mapsto x+\alpha$. We would like to produce, starting from $(\alpha, A)$, a sequence $A^{(n)} \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ such that $(\alpha_n, A^{(n)})$ can be interpreted as the sequence of renormalizations of $(\alpha, A)$. However, this can not be done canonically, and to define renormalization one must introduce *commuting pairs*.

Fixing $x_* \in \mathbb{R}/\mathbb{Z}$, we associate to $(\alpha, A)$ a sequence of pairs $(A^{(n,0)}, A^{(n,1)}) \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$, by

$$A^{(n,0)}(x) = A_{(-1)^{n-1}q_{n-1}}(x_* + \beta_{n-1}x), \tag{4.1}$$

$$A^{(n,1)}(x) = A_{(-1)^n q_n}(x_* + \beta_{n-1}x). \tag{4.2}$$

We should regard $A^{(n,0)}$ and $A^{(n,1)}$ as defining cocycles over the dynamics on $\mathbb{R}$ given by $x \mapsto x+1$ and $x \mapsto x+\alpha_n$. It is easy to see that $A^{(n,1)}(x+1)A^{(n,0)}(x) = A^{(n,0)}(x+\alpha_n)A^{(n,1)}(x)$, which expresses the commutation of the cocycles. We call $((1, A^{(n,0)}), (\alpha_n, A^{(n,1)}))$ the $n$-th *renormalization* of $(\alpha, A)$ around $x_*$.

The dynamics of $A^{(n,0)}$ (and of $A^{(n,1)}$ as well) is trivial, since all orbits go to infinity. In fact we can always define ([AK], Lemma 4.1) a (non-canonical) *normalizing map* associated to $(1, A^{(n,0)})$, that is, some $B^{(n)} \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$ such that $B^{(n)}(x+1)A^{(n,0)}(x)B^{(n)}(x)^{-1} = \mathrm{id}$.

Because of the commutation relation, it follows that if $B^{(n)}$ is a normalizing map for $(1, A^{(n,0)})$, then $A^{(n)}(x) = B^{(n)}(x+\alpha_n)A^{(n,1)}(x)B^{(n)}(x)^{-1}$ satisfies $A^{(n)}(x+1) = A^{(n)}(x)$. Thus $A^{(n)}$ can be seen as an element of $C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, and $(\alpha_n, A^{(n)})$ is called a *representative* of the $n$-th renormalization of $(\alpha, A)$.

Of course, choosing a different normalizing map $\tilde{B}^{(n)}$ leads to a possibly different $\tilde{A}^{(n)}$. But it is easy to see that $C = \tilde{B}^{(n)}(B^{(n)})^{-1}$ is 1-periodic, which implies that $\tilde{A}^{(n)}(x) = C(x+\alpha_n)A^{(n)}(x)C(x)^{-1}$, expressing the fact that all renormalization representatives are conjugate, and in fact any element of the conjugacy class of $A^{(n)}$ arises as a renormalization representative.

Now, if $A$ is $C^r$, $1 \leq r \leq \infty$ or $r = \omega$, the normalizing maps may be chosen to be $C^r$ as well ([AK], Lemma 4.1). Hence we may restrict considerations to renormalization representatives obtained by the use of a $C^r$ normalizing map, which we call $C^r$-renormalization representatives. Such $C^r$-renormalization representatives are defined up to $C^r$-conjugacy.

The dynamics of $(\alpha, A)$ and of its renormalization representatives are of course intimately related. For instance:

**Proposition 4.2.** *If a $C^r$-renormalization representative $(\alpha_n, A^{(n)})$ is $C^r$ conjugate to rotations, then $(\alpha, A)$ is $C^r$ conjugate to rotations.*

*Proof.* Let $B^{(n)}$ be a $C^r$-normalizing map for $(1, A^{(n,0)})$ such that we have, for every $x \in \mathbb{R}$, $B^{(n)}(x+\alpha_n)A^{(n,1)}B^{(n)}(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$, and let $B'(x_* + \beta_{n-1}x) = B^{(n)}(x)$. Note that $A^{(n,1)}_{q_{n-1}}(x+q_n)A^{(n,0)}_{q_n}(x) = \mathrm{id}$ so that $B^{(n)}(x+q_n+\alpha_n q_{n-1})B^{(n)}(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$ for every $x \in \mathbb{R}$. Writing $\tilde{A}^{(0)}(x) = B'(x+1)B'(x)^{-1}$ and using that that $\frac{1}{\beta_{n-1}} = q_n + \alpha_n q_{n-1}$, we see that $\tilde{A}^{(0)}(x) \in \mathrm{SO}(2,\mathbb{R})$ for every $x \in \mathbb{R}$. An analogous argument shows that $\tilde{A}^{(1)}(x) = B'(x+\alpha)A(x)B'(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$ for every $x \in \mathbb{R}$. As remarked in the beginning of the proof of Lemma 4.4 of [AK], a simpler version of Lemma 4.1 of [AK] shows the existence of an $\mathrm{SO}(2,\mathbb{R})$-valued $C^r$-normalizing map $\tilde{B}$ for $(1, \tilde{A}^{(0)})$. Then $B(x) = \tilde{B}(x)B'(x)$ is 1-periodic and $B(x+\alpha)A(x)B(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$ for every $x \in \mathbb{R}$. ∎

4.2. **Convergence of renormalization.** A weak version of convergence of renormalization can be stated as follows:

**Theorem 4.3.** *Let $(\alpha, A)$ be a $C^r$ cocycle, $1 \leq r \leq \infty$ or $r = \omega$, over an irrational rotation, and let* deg *be the topological degree of $A$. If $(\alpha, A)$ is $L^2$-conjugate to rotations then there exist a sequence of $C^r$-renormalization representatives $(\alpha_n, A^{(n)})$ and $\theta_n \in \mathbb{R}$, such that $R_{-\theta_n-(-1)^n \deg x} A^{(n)}(x) \to \mathrm{id}$ in $C^r$.*[15]

*Proof of Theorem 1.8.* If $(\alpha, A)$ is non-homotopic to a constant, then $\deg \neq 0$. By Theorem 4.3, it admits a monotonic $C^r$-renormalization representative, which is $C^r$-conjugate to rotations by Theorem 2.1. By Proposition 4.2, $(\alpha, A)$ is $C^r$-conjugate to rotations as well. □

We call the convergence given by Theorem 4.3 weak because it does not say anything about the normalizing map leading to the "nice" renormalization representative. The strong form of convergence is the following:

**Theorem 4.4.** *Let $(\alpha, A)$ be a $C^r$ cocycle, $1 \leq r \leq \infty$ or $r = \omega$, over an irrational rotation. If $(\alpha, A)$ is $L^2$-conjugate to rotations then for almost every $x_* \in \mathbb{R}/\mathbb{Z}$ there exists $B(x_*) \in \mathrm{SL}(2, \mathbb{R})$, and a sequence of affine functions with bounded linear coefficients $\phi^{(n,0)}, \phi^{(n,1)} : \mathbb{R} \to \mathbb{R}$ such that*

$$R_{-\phi^{(n,0)}(x)} B(x_*) A^{(n,0)}(x) B^{-1}(x_*) \to \mathrm{id} \tag{4.3}$$

*and*

$$R_{-\phi^{(n,1)}(x)} B(x_*) A^{(n,1)}(x) B^{-1}(x_*) \to \mathrm{id}, \tag{4.4}$$

*in $C^r$.*

In [AK] is is shown that if $A$ is $C^r$, then there exists $x_*$ and $B(x_*) \in \mathrm{SL}(2, \mathbb{R})$ such that $B(x_*) A^{(n,i)}(x) B^{-1}(x_*)$, $i = 0, 1$, approaches $\mathrm{SO}(2, \mathbb{R})$-valued functions in the $C^r$ topology for $r = \infty, \omega$, or $C^{r-1}$ if $1 \leq r < \infty$. While computations in [AK] are "local", the more precise version obtained, based on the recent work [A1], takes into account global aspects of the (asymptotically) holomorphic extensions of matrix products. This complex variables proof turns out to be simpler and more powerful than our original real variables approach, which shows that if $A$ is $C^1$ then the oscillations of the derivative of $B(x_*) A^{(n,i)}(x) B^{-1}(x_*)$ become less pronounced as $n \to \infty$ (due to cancellations appearing through the Ergodic Theorem).

We will prove Theorem 4.4 in the next section. For the moment, we will just relate it to Theorem 4.3.

*Proof of Theorem 4.3.*

Let $B(x_*)$, $\phi^{(n,0)} = a_{n,0}x + b_{n,0}$ and $\phi^{(n,1)} = a_{n,1}x + b_{n,1}$ be as in Theorem 4.4. Let $n$ be large and let $\tilde{B}(x) = R_{-(a_{n,0}\frac{x^2-x}{2} + b_{n,0}x)} B(x_*)$. Then $\tilde{A}(x) = \tilde{B}(x+1) A^{(n,0)}(x) \tilde{B}(x)^{-1}$ is $C^r$-close to the identity and $\tilde{B}(x+\alpha_n) A^{(n,1)}(x) \tilde{B}(x)^{-1}$ is $C^r$-close to $R_{\psi^{(n)}(x)}$, where $\psi^{(n)}(x) = (-a_{n,0}\alpha_n + a_{n,1})x + (b_{n,1} - \alpha_n b_{n,0} - a_{n,0}\frac{\alpha_n^2 - \alpha_n}{2})$. By Lemma 4.1 of [AK], there exists $C \in C^r(\mathbb{R}, \mathrm{SL}(2, \mathbb{R}))$ which is $C^r$-close to the identity such that $C(x+1)\tilde{A}(x)C(x)^{-1} = \mathrm{id}$. Set $B^{(n)} = C\tilde{B}$. Then $B^{(n)}$ is a $C^r$ normalizing map for $A^{(n,0)}$ and $A^{(n)}(x) = B^{(n)}(x+\alpha_n) A^{(n,1)} B^{(n)}(x)^{-1}$ is a $C^r$ renormalization representative close to $R_{\psi_n(x)}$. Since $(\alpha_n, A^{(n)})$ is a renormalization representative of $(\alpha, A)$, the topological degree of $A^{(n)}$ is $(-1)^n \deg$ (compute

[15] In fact, as the proof will show the convergence holds uniformly on any compact subsets of larger and larger complex strips.

directly the degree of an $n$-th renormalization representative of $(\alpha, R_{\deg x})$, which will be automatically homotopic to $(\alpha, A^{(n)})$, or see [AK], Appendix A). Thus the linear coefficient of $\psi_n$ must be close to $(-1)^n \deg$ and $A^{(n)}(x)$ must be $C^r$-close $R_{\theta_n+(-1)^n \deg x}$ for some $\theta_n \in \mathbb{R}$. □

4.3. **Proof of Theorem 4.4.** The complex variables proof given below follows basically [A1], which uses "renormalization in parameter space" as an approach to the local distribution of zeros of orthogonal polynomials, originally treated in [ALS] with a different technique. We translate the argument of [A1] to the usual renormalization operator in the analytic case, and then use asymptotically holomorphic extensions to address the non-analytic case.

We consider first the analytic case. Let $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$ be a measurable map with $\|B\|^2 \in L^2$ and for any $x \in \mathbb{R}/\mathbb{Z}$, $B(x+\alpha)A(x)B(x)^{-1} \in \mathrm{SO}(2, \mathbb{R})$. Let $S(x) = \sup_{n\geq 1} \frac{1}{n} \sum_{k=0}^{n-1} \|B(x+k\alpha)\|^2$, which is finite almost everywhere by the Maximal Ergodic Theorem. Assume that $A$ has a holomorphic extension which is Lipschitz in $\Omega_\delta$.

**Lemma 4.5.** *There exists $C > 0$ such that if $x_0 \in \mathbb{R}/\mathbb{Z}$ then for every $k \geq 1$ and $z \in \Omega_\delta$ we have*

$$\|A_k(x_0)^{-1}(A_k(x) - A_k(x_0))\| \leq e^{C\|B(x_0\|^2 S(x_0)k|x-x_0|} - 1. \tag{4.5}$$

*Proof.* The proof is the same as the proof of Lemma 3.1 of [AK] (there only the case $x \in \mathbb{R}/\mathbb{Z}$ is considered, but the proof works equally well for the complex extension). □

Suppose now that $x_*$ is a measurable continuity point of $S$ and $B$ (this means that $x_*$ is a Lebesgue density point of $\{|S(x) - S(x_*)| < \epsilon\}$ and of $\{\|B(x) - B(x_*)\| < \epsilon\}$ for every $\epsilon > 0$). Then we get the estimate

$$\|A_{(-1)^n q_n}(x)\| \leq \inf_{x_0' - x_* \in [-\frac{d}{q_n}, +\frac{d}{q_n}]} C(x_*) e^{C(x_*)q_n|x-x_0'|}, \tag{4.6}$$

for every $d > 0$, as long as $n > n_0(d)$. The argument is as in Lemma 3.3 of [AK]: if $n$ is large, the measurable continuity hypothesis implies that for every $x_0' \in [x_* - \frac{d}{q_n}, x_* + \frac{d}{q_n}]$ we can locate $x_0$ with $|x_0' - x_0| \leq \frac{1}{q_n}$ and such that $B(x_0), B(x_0 + \beta_n)$ are close to $B(x_*)$ and $S(x_0), S(x_0 + \beta_n)$ are close to $S(x_*)$, and then apply Lemma 4.5 to estimate either $\|A_{q_n}(x_0)^{-1}(A_{q_n}(x) - A_{q_n(x_0)}\|$ (if $n$ is even), or $\|A_{-q_n}(x_0)(A_{-q_n}(x)^{-1} - A_{-q_n}(x_0)^{-1})\|$ (if $n$ is odd), using also the bound $\|A_{(-1)^n q_n}(x_0)\| \leq \|B(x_0)\| \|B(x_0 + \beta_{n-1})\|$.

This estimate implies, since $q_{n-1} < q_n < \beta_{n-1}^{-1}$,

$$\|A^{(n,i)}(x)\| \leq \inf_{x_0 \in [-d,d]} Ce^{C|x-x_0|}, \quad i = 0, 1, \quad x \in \Omega_{\delta/\beta_{n-1}}, \quad x_0 \in [-d, d]. \tag{4.7}$$

It follows that the sequences $A^{(n,i)}$ are precompact in $C^\omega$, and the limits are entire functions $\tilde{A}$ with $\|\tilde{A}(z)\| \leq Ce^{C|\Im z|}$. We now show that $B(x_*)\tilde{A}(x)B(x_*)^{-1}$ must be of the form $R_{\tilde{\phi}(x)}$ with $\tilde{\phi}$ affine with bounded linear coefficient.

Indeed, Lemma 3.4 of [AK] shows that limits $\tilde{A}$ of the $A^{(n,i)}$ satisfy $B(x_*)\tilde{A}(x)B(x_*)^{-1} \in \mathrm{SO}(2, \mathbb{R})$, $x \in \mathbb{R}$. It follows that we can write $\tilde{A}(z) = B(x_*)^{-1} R_{\tilde{\phi}(x)} B(x_*)$ for some entire function $\tilde{\phi} : \mathbb{C} \to \mathbb{C}$, satisfying the estimate $|\Im \tilde{\phi}(z)| \leq C + C|\Im z|$ (since $\Im \tilde{\phi} = 0$ on the real axis). This implies that $\tilde{\phi}$ is affine with bounded linear coefficient.

We consider now the $C^r$ case, $1 \leq r < \infty$, since it implies the $C^\infty$ case. Consider an $r$-asymptotically holomorphic extension of $A$ to some $\Omega_\delta$, and let $x_*$ be selected as in the analytic case. The asymptotically holomorphic extension is in particular Lipschitz in $\Omega_\delta$, thus estimate (4.7) still holds. For $0 < \epsilon \leq \delta$, let us denote by $\|\cdot\|_{C^{r-1}_\epsilon}$ the $C^{r-1}$ norm of the restriction to $\Omega_\epsilon$ of a function defined on $\Omega_\delta$.

**Lemma 4.6.** *Suppose that $x_0 \in \mathbb{R}/\mathbb{Z}$ and $k \geq 1$ satisfy $S(x_0), \|B(x_0)\|, \|B(x_0 + k\alpha)\| \leq C_0$. Then there exists $C > 0$ (depending on $C_0$ and $\|A\|_{C^r_\delta}$, but not on $x_0$), such that if $z \in \Omega_\epsilon$ with $0 < \epsilon \leq \delta$ then*

$$\max_{0 \leq s \leq r-1} \|D^s \overline{\partial}_z A_k(z)\| \leq C k^r e^{Ck|z-x_0|} \|\overline{\partial}_z A\|_{C^{r-1}_\epsilon} \tag{4.8}$$

*(D stands for the full derivative).*

*Proof.* The proof is the same as that of Lemma 3.2 of [AK] which estimates the real derivatives of matrix products: the consideration of the complex extension is again harmless, and the incorporation of a $\overline{\partial}_z$ in the estimates is straightforward. □

By the same measurable continuity argument given above, we obtain

$$\max_{0 \leq s \leq r-1} \|D^s \overline{\partial}_z A_{(-1)^n q_n}(z)\| \leq \inf_{x_0 - x_* \in [-\frac{d}{q_n}, \frac{d}{q_n}]} C q_n^r e^{C q_n |z-x_0|} \|\overline{\partial}_z A\|_{C^{r-1}_\epsilon}, \quad z \in \Omega_\epsilon, \tag{4.9}$$

which yields

$$\max_{0 \leq s \leq r-1} \|D^s \overline{\partial}_z A^{(n,i)}(z)\| \leq \inf_{x_0 \in [-d,d]} C e^{C|z-x_0|} \|\overline{\partial}_z A\|_{C^{r-1}_\epsilon}, \quad z \in \Omega_{\epsilon/\beta_{n-1}}, \tag{4.10}$$

This implies that we can write $A^{(n,i)} = A_c^{(n,i)} + A_h^{(n,i)}$ where each term is defined in an increasing sequence of disks $D_n$ with $\cup D_n = \mathbb{C}$, $A_h^{(n,i)}$ are matrix valued (not necessarily $\mathrm{SL}(2,\mathbb{C})$) holomorphic functions and form precompact sequences with limits satisfying $\|\tilde{A}(z)\| \leq C e^{C|z|}$, and $A_c^{(n,i)}$ are $C^r$ matrix valued functions with $C^r$ norm going to 0. It follows that $A^{(n,i)}$ are precompact in $C^r$ and the limits are entire functions (necessarily $\mathrm{SL}(2,\mathbb{C})$ valued now) satisfying $\|\tilde{A}(z)\| \leq C e^{C|\Im z|}$. By the same argument of the analytic case, the limits have the form $B(x_*)^{-1} R_{\tilde{\phi}(z)} B(x_*)$, where $\tilde{\phi}$ is affine with bounded linear coefficient.

## Appendix A. Conformal barycenter

Let $\mathcal{M}$ be the set of probability measures on $\mathbb{D}$, and for $\mu \in \mathcal{M}$, let $\Phi(\mu) = \int_{\mathbb{D}} \frac{1}{1-|z|^2} d\mu(z)$. For $w \in \mathbb{D}$, let $\Phi_w(\mu) = \Phi(\mu')$ where $\mu'$ is the pushforward of $\mu$ by some Moebius transformation of $\mathbb{D}$ taking $w$ to 0. Notice that if $\Phi(\mu) < \infty$ then $\Phi_w(\mu) < \infty$ for every $w$. For every $1 \leq K < \infty$, let $\mathcal{M}_K = \{\mu \in \mathcal{M}, \Phi(\mu) \leq K\}$, and let $\mathcal{M}_\infty = \cup \mathcal{M}_K$. Notice that $\mathcal{M}_K$ is compact in the weak-* topology for every $K < \infty$.

The next proposition can be proved using the conformal barycenter of Douady-Earle [DE]. The construction is sufficiently simple for us to give the details here.

**Proposition A.1.** *There exists a Borelian function $\mathcal{B} : \mathcal{M} \to \mathbb{D}$, equivariant with respect to Möebius transformations of $\mathbb{D}$ and such that $\Phi(\delta_{\mathcal{B}(\mu)}) \leq \Phi(\mu)$.*

*Proof.* Following an idea of Yoccoz, let us define a pairing $\mathbb{D} \times \mathbb{D} \to \mathbb{D}$ by setting $z * w$ as the midpoint of the hyperbolic geodesic passing through $z$ and $w$ if $z \neq w$, and $z * z = z$. This pairing is continuous and equivariant, and we have

$$u_s(z,w) \equiv \Phi_s\left(\frac{\delta_z + \delta_w}{2}\right) - \Phi_s(\delta_{z*w}) \geq 0, \tag{A.1}$$

with equality if and only if $z = w$. Notice that

$$u_s(z,s) = (2\Phi_s(\delta_{z*s}) - 1)(\Phi_s(\delta_{z*s}) - 1) \geq \Phi_s(\delta_{z*s}) - 1. \tag{A.2}$$

Extend the pairing $*$ to $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ linearly. Thus

$$\mu * \nu = \int_{\mathbb{D}\times\mathbb{D}} \delta_{z*w} d\mu(z) d\nu(w). \tag{A.3}$$

If $\mu, \nu \in \mathcal{M}_\infty$ then

$$u_s(\mu,\nu) \equiv \Phi_s(\frac{1}{2}(\mu+\nu)) - \Phi_s(\mu * \nu) = \int_{\mathbb{D}\times\mathbb{D}} u_s(z,w) d\mu(z) d\nu(w) \geq 0, \tag{A.4}$$

with equality if and only if $\mu = \nu$ is a Dirac mass. Notice that $u_s : \mathcal{M}\times\mathcal{M} \to [0,\infty]$ is lower semicontinuous, so if $\mu_k \to \mu$ and $u_s(\mu_k,\mu_k) \to 0$ then $\mu$ is a Dirac mass. If $\mu_k \to \delta_s$ we have

$$\limsup_{k\to\infty} u_s(\mu_k,\mu_k) \geq \limsup_{k\to\infty} u_s(\mu_k,\delta_s) \geq \limsup_{k\to\infty} \int_{\mathbb{D}} \Phi_s(\delta_{z*s}) - 1 d\mu_k(z), \tag{A.5}$$

and in particular if additionally $\lim u_s(\mu_k,\mu_k) = 0$ then $\lim \Phi_s(\mu_k * \delta_s) = 1$.

Given $\mu \in \mathcal{M}$, define $\mu^{(k)}$ inductively by $\mu^{(0)} = \mu$ and $\mu^k = \mu^{(k-1)} * \mu^{(k-1)}$. If $\mu \in \mathcal{M}_\infty$ then $\mu^{(k)} \in \mathcal{M}_\infty$ and we have $\Phi(\mu^{(k+1)}) = \Phi(\mu^{(k)}) - u(\mu^{(k)},\mu^{(k)})$. Thus $u_s(\mu^{(k)},\mu^{(k)}) \to 0$, and any limit of $\mu^{(k)}$ (which exists by compactness) must be a Dirac mass. Moreover, if $\mu^{(n_k)} \to \delta_s$ then $\Phi_s(\mu^{(n_k)} * \delta_s) \to 1$, so $\Phi_s(\mu^{(n)} * \delta_s) \to 1$ as well and $\delta_s$ must be the unique limit of $\mu^{(n)}$. Now we can set $\mathcal{B}(\mu) = s$, which is clearly Borelian.[16] □

The estimates above allow us to obtain compactness result for invariant sections of cocycles. For instance, we have the following.

**Proposition A.2.** *Let $f_k : X \to X$ be a sequence of homeomorphisms of $X$ preserving a probability measure $\mu$ and converging uniformly to a homeomorphism $f : X \to X$. Let $A_k \in C^0(X,\Upsilon)$ be a sequence converging to $A \in C^0(X,\Upsilon)$. Assume there exists measurable $m_k : X \to \mathbb{D}$ satisfying $A_k(x) \cdot m_k(x) = m_k(f_k(x))$, such that*

$$H \equiv \liminf_{K,k\to\infty} \int_X \min\{K, \frac{1}{1-|m_k(x)|^2}\} d\mu(x) < \infty. \tag{A.6}$$

*Then there exists a measurable $m : X \to \mathbb{D}$ such that $A(x) \cdot m(x) = m(f(x))$ and $\int_X \frac{1}{1-|m(x)|^2} d\mu(x) \leq H$.*

*Proof.* Let $X_{K,k} = \{x \in X, \frac{1}{1-|m_k(x)|^2} < K\}$, and let $\nu_{K,k} = \int_{X_{K,k}} \delta_{m_k(x)} d\mu(x)$. Let $\nu$ be any limit of $\nu_{K,k}$ along a sequence $K_i \to \infty$, $k_i \to \infty$ attaining the $\liminf$ in (A.6). Then $\nu$ is a probability measure which projects onto $\mu$ and satisfies $\int_{X\times\mathbb{D}} \frac{1}{1-|z|^2} d\nu(x,z) \leq H$. Let $\nu_x$, $x \in \mathbb{R}/\mathbb{Z}$ be a desintegration of $\nu$:

[16] Although we do not need this fact, it is easy to see that $\mathcal{B}$ is continuous in each $\mathcal{M}_K$, $1 \leq K < \infty$.

$\int_{X\times\mathbb{D}}\phi(x,z)d\nu(x,z)=\int_X(\int_{\mathbb{D}}\phi(x,z)d\nu_x(z))d\mu(x)$. Then $\nu_{f(x)}$ is the pushforward of $\nu_x$ by $w\mapsto\mathring{A}(x)\cdot w$, and $\nu_x\in\mathcal{M}_\infty$ for $\mu$-almost every $x$. Let $m(x)=\mathcal{B}(\nu_x)$. Then $m(f(x))=\mathring{A}(x)\cdot m(x)$ and we have $\int\frac{1}{1-|m(x)|^2}d\mu(x)\leq\int\int\frac{1}{1-|z|^2}d\nu_x(z)d\mu(x)\leq H$. □

## Appendix B. Transitivity of the projective action

We follow the notation of section 3.3. Our goal is to show the transitivity of the projective action of multidimensional quasiperiodic cocycles which are not homotopic to the identity. The one-dimensional case was considered in [KKHO], and in fact the topological ideas that make the one-dimensional argument work are easily implemented in the multidimensional case as well. Let $f_\alpha:x\mapsto x+\alpha$ be an ergodic translation in $\mathbb{R}^d/\mathbb{Z}^d$.

**Proposition B.1.** *Let $A\in C^0(\mathbb{R}^d/\mathbb{Z}^d,\mathrm{SL}(2,\mathbb{R}))$ be non-homotopic to the identity. If $f_\alpha:x\mapsto x+\alpha$ be an ergodic translation in $\mathbb{R}^d/\mathbb{Z}^d$ then $(f_\alpha,A)$ is transitive on $\mathbb{R}^d/\mathbb{Z}^d\times\partial\mathbb{D}$.*

*Proof.* Up to change of coordinate, we may assume that $x_1\mapsto A(x_1,...,x_d)$ has positive degree $\deg\geq 1$. To prove transitivity, it is enough to show that for any open set $U\subset\mathbb{R}^d/\mathbb{Z}^d\times\partial\mathbb{D}$, the set $\cup_{k\geq 0}(f_\alpha,A)^k(U)$ is dense in $\mathbb{R}^d/\mathbb{Z}^d\times\partial\mathbb{D}$.

We will actually show a stronger statement. Let $\Pi_1:\mathbb{R}^d/\mathbb{Z}^d\times\partial\mathbb{D}\to\mathbb{R}/\mathbb{Z}$, $\Pi_2:\mathbb{R}^d/\mathbb{Z}^d\times\partial\mathbb{D}\to\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$ and $\Pi_3:\mathbb{R}^d/\mathbb{Z}^d\times\partial\mathbb{D}\to\partial\mathbb{D}$ be given by $\Pi_1(x_1,...,x_d,z)=x_1$, $\Pi_2(x_1,...,x_d,z)=(x_2,...,x_d)$ and $\Pi_3(x_1,...,x_d,z)=z$.

Let $0<\epsilon<1/10$. Let us say that a point $x\in\mathbb{R}^d/\mathbb{Z}^d$ is $\epsilon$-short if there exists a sequence of paths $\gamma_n:[0,1]\to\mathbb{R}^d/\mathbb{Z}^d\times\partial\mathbb{D}$ such that $\Pi_1\circ\gamma_n(t)$ converges uniformly to $\Pi_1(x)+\epsilon t$, $\Pi_2\circ\gamma_n(t)$ converges uniformly to $\Pi_2(x)$, and $\Pi_3((f_\alpha,A)^k(\gamma_n(t)))$ has (algebraic) length at most $2\pi-1/10$ for every $k\geq 0$. It is clear that the set of $\epsilon$-short $x$ is forward invariant and closed, so for each $\epsilon$, either every point is $\epsilon$-short or no point is $\epsilon$-short.

If for every $\epsilon>0$ there is no point which is $\epsilon$-short, then for any $(x,z)=(x_1,...,x_d,z)$ and for every $\delta>0$, there exists $k\geq 0$ such that, letting $J_\delta(x,z)=[x_1,x_1+\delta]\times\{(x_2,...,x_d,z)\}$, we have $|\Pi_3(f_\alpha,A)^k(J_\delta(x,z))|>2\pi-\delta$. It follows that for every $\delta_0$, the closure of $\cup_{k\geq 0}(f_\alpha,A)^k(J_{\delta_0}(x,z))$ contains some circle $\{y\}\times\partial\mathbb{D}$. Since this set is also forward invariant, it must contain also the circles $\{y+l\alpha\}\times\partial\mathbb{D}$ for every $l\geq 0$, and hence, by minimality of $x\mapsto x+\alpha$, the whole $\mathbb{R}^d/\mathbb{Z}^d\times\partial\mathbb{D}$. Since $(x,z)$ and $\delta_0>0$ are arbitrary, transitivity follows.

Assume now that there exists $\epsilon>0$ such that every point is $\epsilon$-short. We may assume that $\epsilon=\frac{1}{k}$ for some $k\geq 2$. Let $\gamma_{n,i}$, $1\leq i\leq k$, $n\geq 1$, be the sequences of paths associated to $(\frac{i-1}{k},0,...,0)$. We define a sequence of paths $\tilde\gamma_n:\mathbb{R}/\mathbb{Z}\to\mathbb{R}^d/\mathbb{Z}^d\times\partial\mathbb{D}$ so that

1. $\tilde\gamma_n|[(i-1)/k,(3i-2)/3k]$ is given by $\tilde\gamma_n(t)=\gamma_{n,i}(3kt-3i+3)$,
2. $\tilde\gamma_n|[(3i-2)/3k,(3i-1)/3k]$ is such that $\Pi_1\circ\tilde\gamma_n$ and $\Pi_2\circ\tilde\gamma_n$ are constant and $\Pi_3\circ\tilde\gamma_n$ is a homeomorphism,
3. the diameter of the image of $\tilde\gamma_n|[(3i-1)/3k,i/k]$ converges to 0.

One readily checks that these properties imply that for every $l\geq 0$, if $n$ is sufficiently large, then $\Pi_1\circ(f_\alpha,A)^l\circ\tilde\gamma_n$ has topological degree 1, $\Pi_2\circ(f_\alpha,A)^l\circ\tilde\gamma_n$ is homotopic to a constant and $\Pi_3\circ(f_\alpha,A)^l\circ\tilde\gamma_n$ has topological degree $\deg_{l,n}$ satisfying $|\deg_{l,n}|\leq 2k-1$. But $\deg_{l+1,n}=\deg_{l,n}+\deg\geq\deg_{l,n}+1$ for every $l$

and $n$, since $A$ is not homotopic to the identity. Thus for large $n$ we have both $\deg_{4k,n} - \deg_{0,n} \geq 4k$ and $|\deg_{4k,n}|, |\deg_{0,n}| \leq 2k-1$, a contradiction. □

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