Monotonicity for $\lambda \le 1/2$

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The speed ℓ_λ of the λ-biased random walk on a Galton–watson tree is given by

$$\ell_\lambda = \mathrm{E}\left[\frac{(\nu-\lambda)\beta_0}{\lambda - 1 + \sum_{i=0}^{\nu}\beta_i}\right] \Big/ \mathrm{E}\left[\frac{(\nu+\lambda)\beta_0}{\lambda - 1 + \sum_{i=0}^{\nu}\beta_i}\right]. \tag{0.1}$$

The $(\beta_i, i \ge 0)$ are i.i.d copies of the conductance β, and ν is independent of the β's and is the offspring distribution associated to the Galton-Watson tree.

1 The derivative of β when $\lambda < 1$

In this section, we write β_λ to make the dependency in λ explicit. One way to construct β_λ is the following: For $n \ge 1$, define $\beta_{\lambda,n}(x) = 1$ if $|x| = n$. Then, for any vertex $|x| < n$, define

$$\beta_{\lambda,n}(x) := \frac{\sum_{i=1}^{\nu(x)} \beta_{\lambda,n}(xi)}{\lambda + \sum_{i=1}^{\nu(x)} \beta_{\lambda,n}(xi)}. \tag{1.1}$$

Here, xi is the i-th child of x and $\nu(x)$ is the number of children of x. Then, $\beta_\lambda(e)$ (e being the root) is the almost sure limit of $\beta_{\lambda,n}(e)$ as $n \to \infty$. It is easy to check that $\lambda \to \beta_{\lambda,n}(x)$ has a continuous derivative for any n and $|x| \le n$. Write

$$\begin{aligned} A_n(x) &:= \frac{\lambda}{(\lambda + \sum_{i=1}^{\nu(x)} \beta_{\lambda,n}(xi))^2} \\ B_n(x) &:= \frac{\sum_{i=1}^{\nu(x)} \beta_{\lambda,n}(xi)}{(\lambda + \sum_{i=1}^{\nu(x)} \beta_{\lambda,n}(xi))^2}. \end{aligned}$$

Using the fact that $\beta_\lambda \ge 1 - \lambda$, we check that

$$A_n(x) \le \frac{\lambda}{\lambda + \sum_{i=1}^{\nu(x)} \beta_{\lambda,n}(xi)} \tag{1.2}$$

$$B_n(x) \le \beta_{\lambda,n}(x). \tag{1.3}$$

Derivating (1.1) yields that for any $|x| < n$,

$$-\beta'_{\lambda,n}(x) = A_n(x) \sum_{i=1}^{\nu(x)} -\beta'_{\lambda,n}(xi) + B_n(x)$$

where $\beta'_{\lambda,n}(x)$ is the derivative in λ of $\lambda \to \beta_{\lambda,n}(x)$. We get that

$$-\beta'_{\lambda,n}(e) = \sum_{k=0}^{n-1} \sum_{|x|=k} B_n(x) \prod_{i=0}^{k-1} A_n(x_i) \tag{1.4}$$

where x_i is the ancestor at generation i of x. Similarly, in view of (1.1), we have that for any $k \in [0, n-1]$,

$$\beta_{\lambda,n}(e) = \sum_{|x|=k} \beta_{\lambda,n}(x) \prod_{i=0}^{k-1} \frac{1}{\lambda + \sum_{j=1}^{\nu(x_i)} \beta_{\lambda,n}(x_i j)}$$

($x_i j$ is the j-th child of the ancestor x_i). From (1.2) and (1.3), we deduce that for any $k \in [0, n-1]$,

$$\sum_{|x|=k} B_n(x) \prod_{i=0}^{k-1} A_n(x_i) \leq \sum_{|x|=k} \beta_{\lambda,n}(x) \prod_{i=0}^{k-1} \frac{\lambda}{\lambda + \sum_{j=1}^{\nu(x_i)} \beta_{\lambda,n}(x_i j)} = \lambda^k \beta_{\lambda,n}(e). \tag{1.5}$$

Let us go back to (1.4). Observe that, for each vertex x, $A_n(x)$ and $B_n(x)$ converge as $n \to \infty$, this uniformly in $\lambda \in (0, 1-\varepsilon)$ (for this notice that $\beta_{\lambda,n}(x) - \beta_\lambda(x)$ is less than the probability that the random walk starting at level n touches the vertex x, then use coupling with a 1-d biased random walk). We deduce by dominated convergence (see (1.5)) that $\beta'_{\lambda,n}(e)$ converges uniformly in $\lambda \in (0, 1-\varepsilon)$ as $n \to \infty$ to some limit, say F_λ. In particular $\lambda \to F_\lambda$ is continuous on $(0, 1-\varepsilon)$. Moreover, summing (1.5) over $k \in [0, n-1]$, we have that

$$|\beta'_{\lambda,n}(e)| \leq \frac{\beta_{\lambda,n}(e)}{1-\lambda}.$$

By dominated convergence $\int_0^\lambda \beta'_{s,n}(e) ds$ converges to $\int_0^\lambda F_s ds$ which is also $\beta_\lambda(e) - 1$. Hence $\beta_\lambda(e)$ is differentiable for $\lambda \in (0, 1)$ and

$$|\beta'_\lambda(e)| \leq \frac{\beta_\lambda(e)}{1-\lambda}. \tag{1.6}$$

2 The derivative of the speed

By symmetry, we can rewrite (0.1) as

$$\ell_\lambda = \mathrm{E}\left[\frac{\nu-\lambda}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right]\Big/\mathrm{E}\left[\frac{\nu+\lambda}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right] =: \frac{f(\lambda)}{g(\lambda)}.$$

Beware that we are back to the old notation where the dependency in λ is hidden. We write β' for the derivative in λ of β. We compute that $f'(\lambda)$ is

$$-\mathrm{E}\left[\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right] - \mathrm{E}\left[\frac{\nu-\lambda}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{(\lambda-1+\sum_{i=0}^{\nu}\beta_i)^2}\right]$$

and $g'(\lambda)$ is

$$\mathrm{E}\left[\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right] - \mathrm{E}\left[\frac{\nu+\lambda}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{(\lambda-1+\sum_{i=0}^{\nu}\beta_i)^2}\right].$$

We have $\ell'_\lambda = \frac{f'(\lambda)g(\lambda)-f(\lambda)g'(\lambda)}{g(\lambda)^2}$. We find that $\ell'_\lambda \le 0$ is equivalent with

$$\mathrm{E}\left[\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right]\mathrm{E}\left[\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{(\lambda-1+\sum_{i=0}^{\nu}\beta_i)^2}\right] \tag{2.1}$$

$$- \quad \mathrm{E}\left[\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right]\mathrm{E}\left[\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{(\lambda-1+\sum_{i=0}^{\nu}\beta_i)^2}\right] \tag{2.2}$$

$$\le \quad \frac{1}{\lambda}\mathrm{E}\left[\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right]\mathrm{E}\left[\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right] \tag{2.3}$$

We know that $\beta+(1-\lambda)\beta' \ge 0$ by (1.6). Hence the expectations in (2.2) are positive. Since $\beta' \le 0$, (2.1) is less than (taking $\lambda < 1$)

$$\begin{aligned} &\mathrm{E}\left[\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right]\mathrm{E}\left[\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{(\lambda-1+\sum_{i=0}^{\nu}\beta_i)^2}\right] \\ \le \quad &\frac{1}{1-\lambda}\mathrm{E}\left[\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right]\mathrm{E}\left[\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right] \end{aligned}$$

which is less than (2.3) if $\lambda \le 1/2$.