# Renormalization and disorder : a simple toy model

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Paris 20 June 2018

# My year 2014 - 2015

# at LPMA

# Collaborators

- Hakim and Vannimenus 1992
- Giacomin, Lacoin, Toninelli 2007
- Retaux 2014
- Chen, Hu, Lifshits, Shi 2017
- Dagard 2018

# OUTLINE

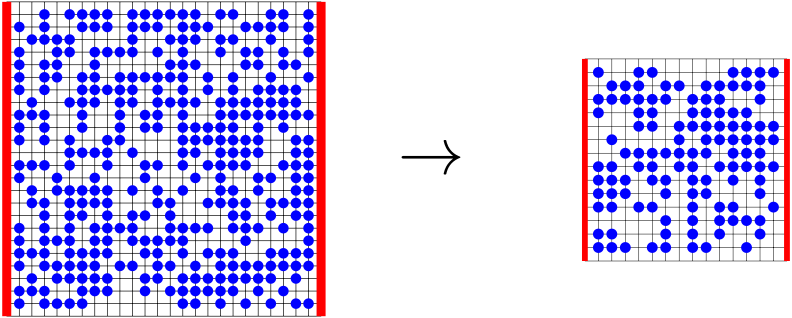
The idea of renormalization

The toy model

The denaturation of DNA problem

Its hierarchical lattice version

# Renormalization: an example, the percolation problem

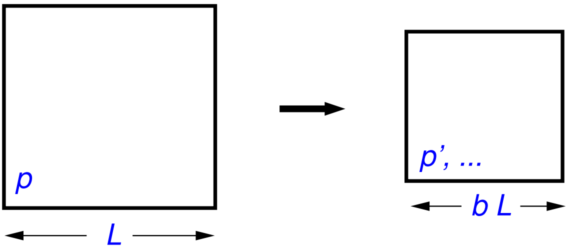


$p$ $\qquad\qquad$ $p'$

$$\boxed{p' = \mathcal{R}(p)}$$

Phase transition $\quad p^* = \mathcal{R}(p^*)$

# The renormalization group




Renormalization transformation

$$\boxed{(p', \cdots) \;=\; \mathcal{R}_b\,(p, \cdots)}$$

Look for the fixed point $\quad (p^*, \cdots) \;=\; \mathcal{R}_b\,(p^* \cdots)$

Linearize $\mathcal{R}_b$ near the fixed point $\quad (p', \cdots) \;=\; \mathcal{L}_b\,(p \cdots)$

$\Rightarrow$ Critical exponents and universality

# A class of models

## Two ingredients:

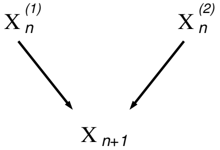
- Start with an infinite sequence i.i.d. random variables

$$X_0^{(1)} \; \dots \; X_0^{(j)} \dots$$

distributed according to a distribution $P_0(X)$.

- A <u>non-linear</u> function $G$ to iterate these variables

$$X_n^{(j)} = G\left(X_n^{(2j-1)} + X_n^{(2j)}\right)$$



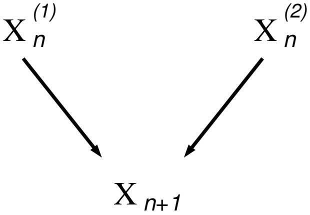

## Questions:

- What is the limiting distribution $P_n(X)$ of the variables $X_n$
- What is the limit of

$$\frac{\langle X_n \rangle}{2^n}$$

# The toy model

Collet, Glaser, Eckmann, Martin 1984




$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \;,\; 0 \,]$$

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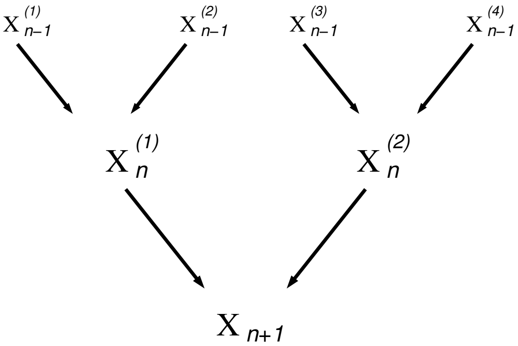


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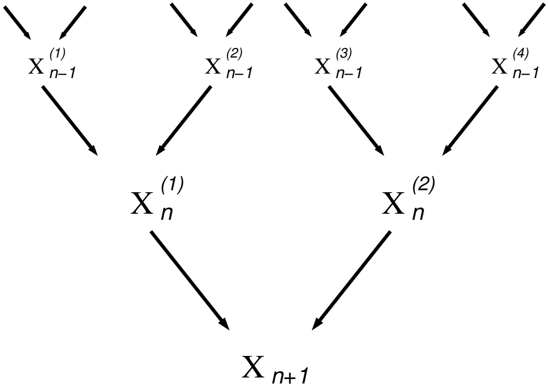


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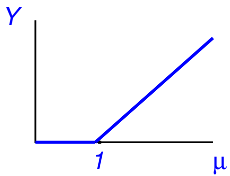
# Main question

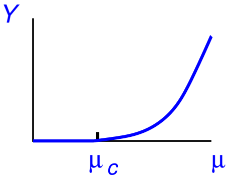
$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \;,\; 0\,]$$

What is the limit $Y$ *of* $\dfrac{X_n}{2^n}$

$$P_0(x) = \delta_{x,\mu}$$




$$P_0(x) = (1-\lambda)\,\delta_{x,0} + \lambda\,\delta_{x,\mu}$$

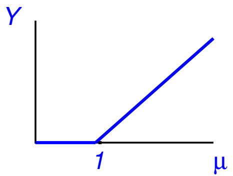


# Main question

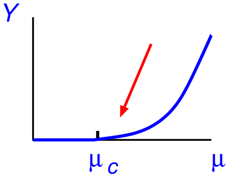
$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \;,\; 0\,]$$

What is the limit $Y$ *of* $\dfrac{X_n}{2^n}$

$$P_0(x) = \delta_{X-\mu}$$




$$P_0(x) = (1-\lambda)\,\delta_X + \lambda\,\delta_{X-\mu}$$



# Renormalization

$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \,,\ 0\,]$$

Exact renormalization (special case $X_n$ are integers)

$P_0(x)$ is given ; $P_n \rightarrow P_{n+1}$

Define the generating function $H_n(z) = \sum_X P_n(X)\, z^X$

$$H_{n+1}(z) = \frac{H_n(z)^2 - H_n(0)^2}{z} + H_n(0)^2$$

# A few facts

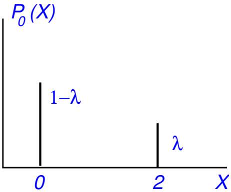
$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \,,\, 0 \,]$$

$$\boxed{H_{n+1}(z) = \frac{H_n(z)^2 - H_n(0)^2}{z} + H_n(0)^2}$$

- A phase transition

Collet, Glaser, Eckmann, Martin 1984

For example




$$H_0(z) = 1 - \lambda + \lambda z^2$$

$$\Rightarrow \quad \boxed{\lambda_c = \frac{1}{5}}$$

- A one parameter family of fixed points

- None of them is accessible

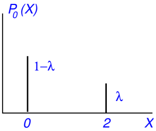
- A phase transition of the Berezinski Kosterlitz Thouless type

# The critical behavior

$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \;,\; 0\,] \qquad H_n(z) = \sum_X P_n(X) z^X$$

A phase transition given by $\boxed{2H'(2) - H(2) = 0}$

Collet, Glaser, Eckmann, Martin 1984




$$H_0(z) = 1 - \lambda + \lambda z^2 \qquad \Rightarrow \qquad \boxed{\lambda_c = \frac{1}{5}}$$

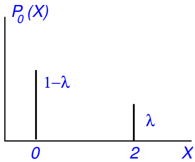
$$2H'(2) - H(2) \equiv \lambda - \lambda_c$$

| $2H'(2) - H(2) \leq 0$ | $2H'(2) - H(2) > 0$ |
| --- | --- |
| $\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \to 0$ | $\boxed{\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \simeq \exp\left[-\frac{A}{\sqrt{\lambda - \lambda_c}}\right]}$ |

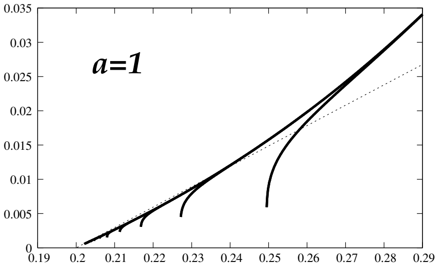
D., Retaux 2014

# An essential singularity ?

$$\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \simeq \exp\left[-\frac{A}{\sqrt{\lambda-\lambda_c}}\right] \quad \Leftrightarrow \quad \left(\log \frac{\langle X_n \rangle}{2^n}\right)^{-2} \propto (\lambda - \lambda_c)$$



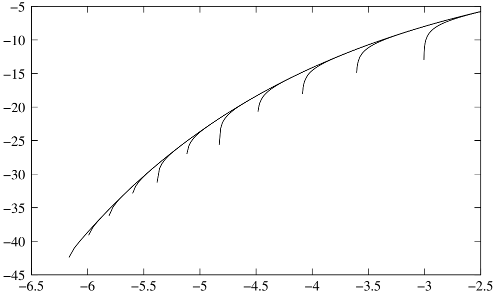

$$\lambda_c = \frac{1}{5}$$




$$\left(\log \frac{\langle X_n \rangle}{2^n}\right)^{-2} \quad \text{versus} \quad \lambda$$

# A power law singularity ?

$$\frac{\langle X_n \rangle}{2^n} \propto (\lambda - \lambda_c)^\gamma \quad \Leftrightarrow \quad \log \frac{\langle X_n \rangle}{2^n} \sim \gamma \log(\lambda - \lambda_c)$$




$$\log \frac{\langle X_n \rangle}{2^n} \quad \text{versus} \quad \log(\lambda - \lambda_c)$$

Chen Dagard D. Hu Lifshits Shi 2018

Initial distribution $P_0(X) = (1-\lambda)\delta_X + \lambda Q(X)$

Particular case: $Q(X) \sim \dfrac{C}{X^\alpha \, 2^X}$

- If $Q(X)$ decays fast enough $(\alpha > 4)$ then $\lambda_c > 0$ and

$$\lim_n \frac{\langle X_n \rangle}{2^n} = \exp\left( -\frac{1}{(\lambda - \lambda_c)^{\frac{1}{2}+o(1)}} \right)$$

- If $2 < \alpha < 4$ then $\lambda_c > 0$

$$\lim_n \frac{\langle X_n \rangle}{2^n} = \exp\left( -\frac{1}{(\lambda - \lambda_c)^{\nu+o(1)}} \right) \qquad \text{with } \nu = \frac{1}{\alpha - 2}$$

- If $\alpha \leq 2$ then $\lambda_c = 0$ (because $H'(2) = \infty$)

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Hu Shi 2017

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$$\text{Particular case:} \quad Q(X) \sim \frac{C}{X^\alpha \; 2^X}$$

$$\boxed{\alpha \leq 2 \quad \Rightarrow \quad \lambda_c = 0}$$

- If $\alpha < 2$

$$\boxed{\lim_n \frac{\langle X_n \rangle}{2^n} = \exp\left(-\frac{1}{\lambda^{\nu+o(1)}}\right)} \quad \text{with} \quad \nu = \frac{1}{2-\alpha}$$

- If $\alpha = 2$ then $\lambda_c > 0$

$$\boxed{\lim_n \frac{\langle X_n \rangle}{2^n} = \exp\left(-\exp\left[\frac{C+o(1)}{\lambda}\right]\right)}$$

# A ?? special ?? family of initial conditions

D., Retaux 2014

$$P_n(X) = 2^{-X} \lambda \, R(\sqrt{\lambda} X, \sqrt{\lambda}\, n) \qquad \textit{for} \qquad X > 0$$

and

$$P_n(0) = 1 - \textstyle\sum_{X \geq 1} P_n(X)$$

then one can show that for $\lambda$ small

$$\boxed{\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2} \int_0^x R(x_1,\tau)\; R(x - x_1,\tau)\; dx_1}$$

Still a difficult problem

Criticality

$$\int_0 R(X,\tau)\, X dX = 1$$

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\; R(x-x_1,\tau)\; dx_1$$

For

$$R(x,\tau) = A(\tau)\exp[-B(\tau)x]$$

one gets

$$\frac{dA(\tau)}{d\tau} = -B(\tau)A(\tau) \quad ; \quad \frac{dB(\tau)}{d\tau} = -\frac{A(\tau)}{2}$$

Kosterlitz Thouless renormalization

$$R(x,\tau) = 4\frac{k^2}{\sin(k(\tau+\tau_0))^2}\;\exp\left[-\frac{2k\;x}{\tan(k(\tau+\tau_0))}\right]$$

($k \to 0$ is the critical case)

# Other solutions

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\, R(x-x_1,\tau)\, dx_1$$

Physical solutions

$$R = \sum_{i=1}^{n} A_i(\tau)\, e^{-B_i(\tau)x}$$

with

$$\frac{dB_i}{d\tau} = -\frac{A_i}{2} \quad ; \quad \frac{dA_i}{d\tau} = -B_iA_i - \sum_{j\neq i}\frac{A_iA_j}{B_i - B_j}$$

Unphysical solutions

$$R = \frac{4}{\tau^2} e^{-\frac{3x}{\tau}} \left[3\cos\left(\frac{\sqrt{3}x}{\tau}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}x}{\tau}\right)\right]$$

# Other solutions

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\ R(x-x_1,\tau)\ dx_1$$

Scaling solutions along the critical manifold ($\int R(X)\ X\ dX = 1$)

$$R = \frac{1}{t^2}\, G\left(\frac{x}{t}\right)$$

Then $G$ should satisfy

- $G(z) = 4\, e^{-2z}$
- Taking the the Laplace transform $H(p) = \int_0^\infty G(z)\, e^{-pz}\, dz$

$$H(p) - pH'(p) + pH(p) + \frac{1}{2}H(p)^2 - G(0) = 0$$

This is a non-linear equation!

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\; R(x-x_1,\tau)\; dx_1$$

$$R = \frac{1}{t^2}\, G\left(\frac{x}{t}\right) \qquad\qquad H(p) = \int_0^\infty G(z)\, e^{-pz}\, dz$$

Introducing a function $y(p)$ such that

$$H(p) = -1 - p - i\,p\,\frac{y'(ip/2)}{y(ip/2)}$$

then $y$ is solution of

$$p^2 y'' + py' + \left(p^2 - \beta^2\right) y = 0$$

with

$$\beta^2 = \frac{1}{4} + \frac{G(0)}{2}$$

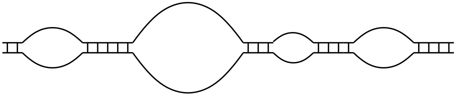
So $y(p)$ is a Bessel function !

$$G(x) \sim x^{-\alpha} \qquad \text{for} \quad x \to \infty$$

where $\alpha = 1 + 2\beta$

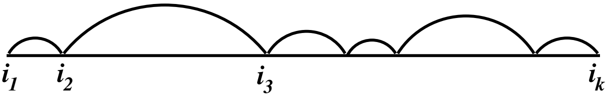
# The Poland Scheraga model

Model of DNA denaturation



Model of depinning

# The Poland Scheraga model

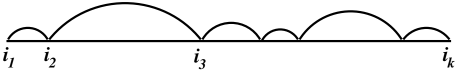



- the contact energy at position $i$ is $\epsilon_i$
- the weight of a loop of length $n$ is

$$\omega(n) \sim \frac{1}{n^c}$$

$$Z_L = \sum_{k \geq 2} \sum_{1 < i_2 \cdots < i_{k-1} < L} \omega(i_2 - i_1) \cdots \omega(i_k - i_{k-1}) \exp\left[ -\frac{\epsilon_{i_1} + \epsilon_{i_2} + \cdots \epsilon_{i_k}}{T} \right]$$

# Phase transition in the Poland Scheraga model



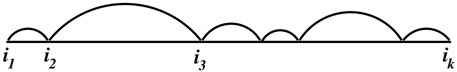

$$Z_L = \sum_{k \geq 2} \sum_{1 < i_2 \cdots < i_{k-1} < L} \omega(i_2 - i_1) \cdots \omega(i_k - i_{k-1}) \exp\left[-\frac{\epsilon_{i_1} + \epsilon_{i_2} + \cdots \epsilon_{i_k}}{T}\right]$$

The free energy $F_L = \log Z_L$ In the thermodynamic limit

$$f_\infty = \lim_{L \to \infty} \frac{F_L}{L}$$

- $T > T_c$ $f_\infty = 0$ the unpinned phase
- $T < T_c$ $f_\infty > 0$ the pinned phase

# Phase transition in the pure case $\epsilon_i = \epsilon$




$T_c$ is known

$$\exp\left[-\frac{\epsilon}{T_c}\right] = \sum_{n\geq 1} \omega(n)$$

- For $c > 2$ the transition is first order

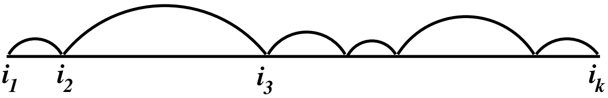
$$f_\infty \sim (T_c - T)$$

- For $1 < c < 2$ the transition is second order

$$f_\infty \sim (T_c - T)^{\frac{1}{c-1}}$$

- For $c < 1$ no transition

# The Poland Scheraga model with disorder




the weight of a loop of length $n$ is

$$\omega(n) \sim \frac{1}{n^c}$$

$$Z_L = \sum_{k \geq 2} \sum_{1 < i_2 \cdots < i_{k-1} < L} \omega(i_2 - i_1) \cdots \omega(i_k - i_{k-1}) \exp\left[ -\frac{\epsilon_{i_1} + \epsilon_{i_2} + \cdots \epsilon_{i_k}}{T} \right]$$

The $\epsilon_i$'s are i.i.d.

# Some important results in the disordered case

Alexander, Berger, Giacomin, Lacoin, Toninelli , $\cdots$

Giacomin, Toninelli 2006

The transition is always smooth

Tang, Chaté 2001

Strong disorder $\Rightarrow$ infinite order transition

# The hierarchical lattice

- $L = 2^n$
- all loops have lengths $2^k$ with $k = 0, 2, 3, \cdots$
- $Z_0^{(j)} = \exp\left[-\frac{\epsilon_j}{T}\right]$

$$Z_{2L} = \frac{Z_L^{(1)} Z_L^{(2)} + b - 1}{b}$$

$$X_n = \log Z_{2^n} \qquad ; \qquad f_\infty = \lim_{n\to\infty} \frac{X_n}{2^n}$$

$f_\infty > 0$ is the pinned phase ; $f_\infty = 0$ is the unpinned phase

# The hierarchical lattice ($X_n = \log Z_{2^n}$)

$$X_{n+1} = G\left(X_n^{(1)} + X_n^{(2)}\right) \quad \Leftarrow \quad Z_{2L} = \frac{Z_L^{(1)} Z_L^{(2)} + b - 1}{b}$$

with

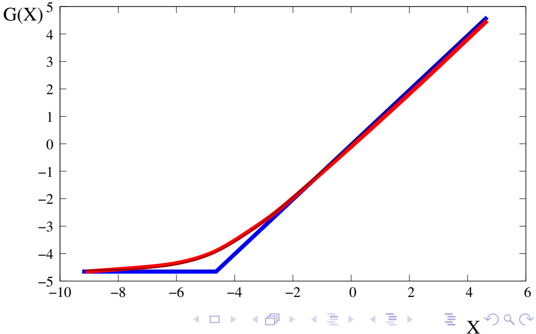
$$G(X) = X + \log\left(\frac{1 + (b-1)e^{-X}}{b}\right)$$

## The toy model

$$X_{n+1} = G\left(X_n^{(1)} + X_n^{(2)}\right)$$

with

$$G(X) = \max(X, -a)$$



# Conclusion

- Mathematical rigor
- Analysis of

$$\frac{\partial r(x,\tau)}{\partial \tau} = \frac{\partial r(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x r(x_1,\tau)\, r(x-x_1,\tau)\, dx_1$$

- Going back to the hierarchical model
- Going back to the Poland Scheraga model