

Characterization of Ricci curvature and Ricci flow by Brownian motion

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I. Background

- (Ricci tensor)

$$\text{Ric} \in \Gamma(T^*M \otimes T^*M),$$

$$\text{Ric}_x: T_xM \times T_xM \rightarrow \mathbb{R}, \quad x \in M, \quad \text{bilinear form}$$

- (Scalar curvature)

$$\text{Scal} := \text{trace Ric} \in C^\infty(M)$$

- The metric volume element has the following expansion in geodesic normal coordinates at $p \in M$:

$$d\text{vol}_g = \left(1 - \frac{1}{6} \text{Ric}_{ij}(p) x^i x^j + O(|x|^3)\right) d\text{vol}_{\text{Eucl}}$$

- For the volume of geodesic balls of radius r about p :

$$\text{Volume}(B(p, r)) = \left(1 - \text{Scal}(p) C r^2 + O(r^4)\right) \text{Vol}_{\mathbb{R}^n}(r)$$

where C is a positive constant depending only on $n = \dim M$.

- Why should probabilists care about such things?

Heat flow on a Riemannian manifold

- Let (M, g) be a complete Riemannian manifold and

$$L = \Delta + Z \quad \text{with } Z \in \Gamma(TM)$$

- Let u be a positive solution to

$$\frac{\partial}{\partial t} u = Lu \quad \text{on } M \times \mathbb{R}_+$$

- (Gradient estimate) Find bounds on

$$|\nabla u| \quad \text{or} \quad \frac{|\nabla u|}{u}.$$

- (Harnack inequalities) Compare

$$u(x, s) \quad \text{and} \quad u(y, t).$$

- Heat flow is controlled by lower Ricci curvature bounds.

- There is an exact formula of the differential

$$(\nabla u)(\cdot, t)_x$$

in terms of an L -diffusion starting from x :

$$X_t = X_t^x, \quad t < \zeta(x).$$

- Recall that L -diffusions X_t on M are defined by the property that for each $f \in C_c^\infty(M)$,

$$d(f(X_t)) - (Lf)(X_t) dt = 0$$

(mod differentials of local martingales)

- Denote by

$$\text{Ric}^Z = \text{Ric} - \nabla Z$$

the Bakry-Émery Ricci tensor, i.e.

$$\text{Ric}^Z(X, Y) := \text{Ric}(X, Y) - \langle \nabla_X Z, Y \rangle.$$

- Let

$$\text{Ric}_{//t}^Z := //t^{-1} \circ \text{Ric}_{X_t}^Z \circ //t \in \text{End}(T_x M)$$

where $//t: T_x M \rightarrow T_{X_t} M$ is parallel transport along $X_t = X_t^X$:

$$\begin{array}{ccc} T_x M & \overset{\text{Ric}_{//t}^Z}{\dashrightarrow} & T_x M \\ //t \downarrow & & \uparrow //t^{-1} \\ T_{X_t} M & \xrightarrow{\text{Ric}_{X_t}^Z} & T_{X_t} M \end{array}$$

By convention $\text{Ric}_X^Z(v) = \text{Ric}_X^Z(\cdot, v)^\sharp$ for $v \in T_x M$.

Damped parallel transport

- For $x \in M$ define a linear transformation

$$Q_t: T_x M \rightarrow T_x M$$

as solution to the pathwise ODE

$$\begin{cases} dQ_t = -Q_t \operatorname{Ric}_{//t}^Z dt \\ Q_0 = \operatorname{id}_{T_x M} \end{cases}$$

- In the sequel we need

$$Q_t \circ //t^{-1}: T_{X_t} M \rightarrow T_x M$$

(“damped parallel transport” along X_t)

Theorem (classical probabilistic formulas)

Let $f \in \mathcal{B}_b(M)$ and $u(x, t) = P_t f(x)$ be the (minimal) solution to

$$\frac{\partial}{\partial t} u = Lu, \quad u|_{t=0} = f.$$

- (Semigroup formula) Then $P_t f(x) = \mathbb{E}[f(X_t^x) \mathbf{1}_{\{t < \zeta(x)\}}]$.
- (Derivative formula) If $f \in C_b^1(M)$ and Ric^Z bounded below,

$$(\nabla P_t f)(x) = \mathbb{E}\left[Q_t //_{t}^{-1} \nabla f(X_t^x)\right]$$

- (Bismut formula) If $f \in \mathcal{B}_b(M)$ (no assumption on Ric^Z), then

$$\langle (\nabla P_t f)_x, v \rangle = -\mathbb{E}\left[f(X_t^x) \mathbf{1}_{\{t < \zeta(x)\}} \int_0^t \langle Q_s^* \ell_s, dB_s \rangle\right]$$

for each $v \in T_x M$, where

- $\tau = \tau_D(x) \wedge t$ with $\tau_D(x)$ the first exit time of X_t^x from some relatively compact neighbourhood D of x
- B is a Brownian motion in $T_x M$
- ℓ_t is any adapted process in $T_x M$ with absolutely continuous paths of finite energy such that $\ell_0 = v$ and $\ell_\tau = 0$.

A first observation

- Suppose that $\text{Ric}^Z \geq k$ for some $k \in C(M)$, i.e.

$$\text{Ric}^Z(X, X) \geq k(x)|X|^2, \quad X \in T_x M.$$

Then $|Q_t| \leq \exp\left(-\int_0^t k(X_s) ds\right)$ and

$$|\nabla P_t f| \leq \mathbb{E}\left[\exp\left(-\int_0^t k(X_s) ds\right) |\nabla f|(X_t)\right], \quad f \in C_b^1(M).$$

- In particular, if

$$\text{CD}(K, \infty) \quad \text{Ric}^Z(X, X) \geq K|X|^2, \quad X \in TM,$$

for some constant K , then

$$|Q_t| \leq e^{-Kt}$$

and

$$\text{(gradient estimate)} \quad |\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|, \quad f \in C_b^1(M).$$

- Actually the **gradient estimate** is equivalent to $\text{CD}(K, \infty)$.

II. Characterization of bounded Ricci curvature

Our setting

- (Process) X_t is an L -diffusion where

$$L = \Delta + Z \quad \text{with } Z \in \Gamma(TM)$$

- (Curvature tensor) Let

$$\text{Ric}^Z = \text{Ric} - \nabla Z$$

where

$$\text{Ric}^Z(X, Y) = \text{Ric}(X, Y) - \langle \nabla_X Z, Y \rangle.$$

Well-known and classical: Let K be a real constant.

The following conditions are equivalent:

- **(Bakry-Émery lower curvature bound)**

$$\text{CD}(K, \infty) \quad \text{Ric}^Z(X, X) \geq K|X|^2, \quad X \in TM;$$

- **(gradient estimate)** for $p \in [1, \infty[$ and all $f \in C_c^\infty(M)$,

$$|\nabla P_t f|^p \leq e^{-\rho K t} P_t |\nabla f|^p;$$

- **(Poincaré inequality)** for $p \in (1, 2]$ and all $f \in C_c^\infty(M)$,

$$\frac{p}{4(p-1)} \left(P_t f^2 - (P_t f^{2/p})^p \right) \leq \frac{1 - e^{-2Kt}}{2K} P_t |\nabla f|^2;$$

- **(log-Sobolev inequality)** for all $f \in C_c^\infty(M)$,

$$P_t (f^2 \log f^2) - (P_t f^2) \log (P_t f^2) \leq \frac{2(1 - e^{-2Kt})}{K} P_t |\nabla f|^2.$$

Many other equivalent statements, e.g., transportation-cost inequalities; convexity properties of the entropy; Wang's dimension-free Harnack inequalities; Wang's log-Harnack inequalities, ...

Natural questions:

- How to characterize **upper bounds** for Ric^Z ?
- How to characterize **pinched bounds** for Ric^Z ?

Our focus

- For real constants $k_1 \leq k_2$, we want to characterize

$$k_1 \leq \text{Ric}^Z \leq k_2.$$

in terms of functional inequalities for the semigroup P_t .

- Natural extensions:
 - **Pointwise** pinched curvature conditions

$$k_1(x) \leq \text{Ric}_x^Z \leq k_2(x), \quad x \in M$$

- Riemannian manifolds with a **boundary**
- Manifolds evolving under a **geometric flow**

Well-known:

Boundedness of $|\text{Ric}^Z|$, i.e.

$$|\text{Ric}^Z| \leq K,$$

implies certain functional inequalities on path space,
e.g. Capitaine-Hsu-Ledoux (1997), Chen-Wu (2014),
Driver (1992), Hsu (1994)

Boundedness of $|\text{Ric}^Z|$

The problem of characterizing boundedness of Ric^Z has been solved by A. Naber and R. Haslhofer via **analysis on path space**:

Boundedness of $|\text{Ric}^Z| \iff$ functional inequalities on path space

- Aaron Naber, *Characterizations of bounded Ricci curvature on smooth and nonsmooth spaces*, arXiv:1306.6512v4 (2015)
- Robert Haslhofer and Aaron Naber, *Characterizations of the Ricci flow*, J. Eur. Math. Soc. (2018)

Our work:

- Li-Juan Cheng and A.Th.: *Characterization of pinched Ricci curvature by functional inequalities*, J. Geom. Anal. (2017)
- Li-Juan Cheng and A.Th.: *Spectral gap on Riemannian path space over static and evolving manifolds*, J. Funct. Anal. **274** (2018), 959-984

Related work by

- Bo Wu and Shizan Fang (2017)
- Bo Wu and Feng-Yu Wang (2018)

III. Analysis on path space

- For fixed $T > 0$, let $W^T = C([0, T]; M)$ and

$$\mathcal{F}C_{0,T}^\infty = \left\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \dots, \gamma_{t_n}) : \right. \\ \left. 0 < t_1 < \dots < t_n \leq T, f \in C_c^\infty(M^n) \right\}.$$

be the **class of smooth cylindrical functions** on W^T .

- Denote

$$X_{[0,T]} = \{X_t : 0 \leq t \leq T\}.$$

- For $F \in \mathcal{F}C_{0,T}^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \dots, \gamma_{t_n})$, the **intrinsic gradient** is defined as

$$D_t^{\prime\prime} F(X_{[0,T]}) = \sum_{i=1}^n \mathbf{1}_{\{t < t_i\}} //_{t,t_i}^{-1} \nabla^i f(X_{t_1}, \dots, X_{t_n}), \quad t \in [0, T],$$

where ∇^i denotes the gradient with respect to the i -th component.

Theorem [A. Naber (2015) and R. Haslhofer and A. Naber (2018)]

The following conditions are equivalent ($K \geq 0$):

- $|\text{Ric}^Z| \leq K$;
- (Gradient inequality on path space) for $F \in \mathcal{F}C_0^\infty$,

$$\left| \nabla_x \mathbb{E}[F(X_{[0,T]}^x)] \right| \leq \mathbb{E}^x \left[|D_0^{\prime\prime} F| + K \int_0^T e^{Kr} |D_r^{\prime\prime} F| dr \right].$$

- (L^2 gradient inequality on path space) for $F \in \mathcal{F}C_0^\infty$,

$$\left| \nabla_x \mathbb{E}[F(X_{[0,T]}^x)] \right|^2 \leq e^{KT} \mathbb{E}^x \left[|D_0^{\prime\prime} F|^2 + K \int_0^T e^{K(r-T)} |D_r^{\prime\prime} F|^2 dr \right].$$

Important observation It is sufficient to check the estimates for very special $F \in \mathcal{F}C_0^\infty$. Namely:

- for $F(X_{[0,T]}^x) = f(X_t^x)$, and
- for 2-point cylindrical functions of the form

$$F(X_{[0,T]}^x) = f(x) - \frac{1}{2} f(X_t^x)$$

From this observation, equivalence of the following two items follows:

- (i) $|\text{Ric}^Z| \leq K$ for $K \geq 0$;
- (ii) for $f \in C_c^\infty(M)$ and $t > 0$,

$$\begin{aligned}
 |\nabla P_t f|^2 &\leq e^{2Kt} P_t |\nabla f|^2 \quad \text{and} \\
 \left| \nabla f - \frac{1}{2} \nabla P_t f \right|^2 &\leq e^{Kt} \mathbb{E} \left[\left| \nabla f - \frac{1}{2} //_{0,t}^{-1} \nabla f(X_t) \right|^2 \right. \\
 &\quad \left. + \frac{1}{4} (e^{Kt} - 1) |\nabla f(X_t)|^2 \right].
 \end{aligned}$$

Remark The inequalities in (ii) can be combined to the single condition:

$$\begin{aligned}
 &|\nabla P_t f|^2 - e^{2Kt} P_t |\nabla f|^2 \\
 &\leq 4 \left((e^{Kt} - 1) |\nabla f|^2 + \langle \nabla f, \nabla P_t f \rangle - \langle \nabla f, e^{Kt} \mathbb{E} [//_{0,t}^{-1} \nabla f(X_t)] \rangle \right) \wedge 0.
 \end{aligned}$$

Theorem (Characterization of pinched Ricci curvature;
Cheng-A.Th. 2017)

Let k_1, k_2 be two real constants such that $k_1 \leq k_2$. The following conditions are equivalent:

(i) $k_1 \leq \text{Ric}^Z \leq k_2$

(ii) (Gradient inequalities) for $f \in C_c^\infty(M)$ and $t > 0$,

$$|\nabla P_t f|^2 - e^{-2k_1 t} P_t |\nabla f|^2 \leq 4 \left[\left(e^{\frac{k_2 - k_1}{2} t} - 1 \right) |\nabla f|^2 + \langle \nabla f, \nabla P_t f \rangle - e^{-k_1 t} \mathbb{E} \langle \nabla f, //_{0,t}^{-1} \nabla f(X_t) \rangle \right] \wedge 0$$

(ii') for $f \in C_c^\infty(M)$ and $t > 0$,

$$|\nabla P_t f|^2 - e^{-2k_1 t} P_t |\nabla f|^2 \leq 4 \left(e^{\frac{k_2 - k_1}{2} t} |\nabla P_t f|^2 - e^{-k_1 t} \mathbb{E} \langle \nabla P_t f, //_{0,t}^{-1} \nabla f(X_t) \rangle \right) \wedge 0$$

Theorem (continuation)

(iii) (Poincaré type inequality) for $f \in C_c^\infty(M)$, $p \in]1, 2]$, $t > 0$,

$$\begin{aligned} & \frac{p(P_t f^2 - (P_t f^{2/p})^p)}{4(p-1)} - \frac{1 - e^{-2k_1 t}}{2k_1} P_t |\nabla f|^2 \\ & \leq 4 \int_0^t \left(e^{\frac{k_2 - k_1}{2}(t-r)} - 1 \right) P_r |\nabla f|^2 \\ & \quad + \mathbb{E} \left\langle \nabla f(X_r), \nabla P_{t-r} f(X_r) - e^{-k_1(t-r)} //_{r,t}^{-1} \nabla f(X_t) \right\rangle dr \wedge 0 \end{aligned}$$

(iv) (Log-Sobolev inequality) for $f \in C_c^\infty(M)$, $t > 0$,

$$\begin{aligned} & \frac{1}{4} \left(P_t (f^2 \log f^2) - P_t f^2 \log P_t f^2 \right) - \frac{1 - e^{-2k_1 t}}{2k_1} P_t |\nabla f|^2 \\ & \leq 4 \int_0^t \left(e^{\frac{k_2 - k_1}{2}(t-r)} - 1 \right) P_r |\nabla f|^2 \\ & \quad + \mathbb{E} \left\langle \nabla f(X_r), \nabla P_{t-r} f(X_r) - e^{-k_1(t-r)} //_{r,t}^{-1} \nabla f(X_t) \right\rangle dr \wedge 0 \end{aligned}$$

The proof uses probabilistic formulas for calculating Ric^Z , e.g. Bakry (1994), von Renesse-Sturm (2005), F-Y Wang (2014).

Lemma

Let $v \in T_x M$ with $|v| = 1$. Let $f \in C_0^\infty(M)$ such that $\nabla f(x) = v$ and $\text{Hess}_f(x) = 0$. Then,

(i) for $p > 0$,

$$\text{Ric}^Z(v, v) = \lim_{t \downarrow 0} \frac{P_t |\nabla f|^p(x) - |\nabla P_t f|^p(x)}{pt}$$

(ii) $\text{Ric}^Z(v, v)$ is also given by the following two limits:

$$\begin{aligned} \text{Ric}^Z(v, v) &= \lim_{t \downarrow 0} \frac{\left\{ \langle \nabla f, \mathbb{E}_{//_{0,t}^{-1}} \nabla f(X_t) \rangle - \langle \nabla f, \nabla P_t f \rangle \right\}(x)}{t} \\ &= \lim_{t \downarrow 0} \frac{\left\{ \langle \nabla P_t f, \mathbb{E}_{//_{0,t}^{-1}} \nabla f(X_t) \rangle - |\nabla P_t f|^2 \right\}(x)}{t} \end{aligned}$$

The theorem can be extended in various ways:

- to characterize variable curvature bounds

$$K_1(x) \leq \text{Ric}^Z(x) \leq K_2(x), \quad x \in M,$$

with functions K_1, K_2 on M

- to manifolds with boundary (reflecting diffusions generated by $L = \Delta + Z$) to characterize

$$K_1(x) \leq \text{Ric}^Z(x) \leq K_2(x), \quad x \in M,$$

$$\sigma_1(x) \leq \text{II}(x) \leq \sigma_2(x), \quad x \in \partial M,$$

in terms of semigroups with Neumann boundary conditions.

The second fundamental form of ∂M is given by

$$\text{II}(X, Y) = -\langle \nabla_X N, Y \rangle, \quad X, Y \in T_x \partial M, \quad x \in \partial M,$$

where N is the inward normal unit vector field on ∂M .

The theorem allows to characterize

- Einstein manifolds (Ric is a multiple of the metric g)
- Ricci solitons ($\text{Ric} + \text{Hess}f = c g$)
- manifolds such that $\text{Ric} = \nabla Z$
- etc

IV. Back to Riemannian path space

Let $F \in \mathcal{F} C_{0,T}^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \dots, \gamma_{t_n})$. Consider the gradients:

- (*intrinsic gradient*)

$$D_t^{\parallel} F(X_{[0,T]}^x) = \sum_{i=1}^n \mathbf{1}_{\{t < t_i\}} \parallel_{t,t_i}^{-1} \nabla_i f(X_{t_1}^x, \dots, X_{t_n}^x);$$

- (*damped gradient*)

$$D_t F(X_{[0,T]}^x) = \sum_{i=1}^n \mathbf{1}_{\{t < t_i\}} Q_{t,t_i} \parallel_{t,t_i}^{-1} \nabla_i f(X_{t_1}^x, \dots, X_{t_n}^x)$$

where $Q_{t,r}$ takes values in the linear automorphisms of $T_{X_t^x} M$ satisfying for fixed $t \geq 0$:

$$\frac{dQ_{t,r}}{dr} = -Q_{t,r} \text{Ric}_{\parallel_{t,r}}^Z, \quad Q_{t,t} = \text{id};$$

- (*balanced gradient*) For constants $k_1 \leq k_2$ let

$$\hat{D}_t^{\parallel} F(X_{[0,T]}^x) = \sum_{i=1}^n \mathbf{1}_{\{t \leq t_i\}} e^{-\frac{k_1+k_2}{2}(t_i-t)} \parallel_{t,t_i}^{-1} \nabla_i f(X_{t_1}^x, \dots, X_{t_n}^x).$$

- Let \mathcal{L} be the Ornstein-Uhlenbeck operator defined as generator associated to the Dirichlet form

$$\mathcal{E}(F, F) = \mathbb{E} \left[\int_0^T |D_t'' F|^2(X_{[0,T]}) dt \right].$$

- It is well-known that a log-Sobolev inequality

$$\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \leq 2H(T, k_1, k_2) \int_0^T |D_t'' F|^2(X_{[0,T]}) dt$$

or a Poincaré inequality

$$\mathbb{E}[F - \mathbb{E}[F]]^2 \leq H(T, k_1, k_2) \int_0^T |D_t'' F|^2(X_{[0,T]}) dt$$

for some explicit bound $H(T, k_1, k_2)$, are equivalent to the spectral gap-lower bound $H(T, k_1, k_2)^{-1}$ for the operator \mathcal{L} .

Theorem (Path space characterization of pinched curvature)

The following conditions are equivalent:

(i) $k_1 \leq \text{Ric}^Z \leq k_2$;

(ii) for any $F \in \mathcal{F}C_{0,T}^\infty$,

$$|\nabla_x \mathbb{E} F(X_{[0,T]}^x)| \leq \mathbb{E} |\hat{D}_0 // F| + \frac{k_2 - k_1}{2} \int_0^T e^{-k_1 s} \mathbb{E} |\hat{D}_s // F| ds;$$

(iii) for any $F \in \mathcal{F}C_{0,T}^\infty$ and $t_1 < t_2$ in $[0, T]$,

$$\begin{aligned} & \mathbb{E} \left[\mathbb{E}[F^2(X_{[0,T]}) | \mathcal{F}_{t_2}] \log \mathbb{E}[F^2(X_{[0,T]}) | \mathcal{F}_{t_2}] \right. \\ & \quad \left. - \mathbb{E}[F^2(X_{[0,T]}) | \mathcal{F}_{t_1}] \log \mathbb{E}[F^2(X_{[0,T]}) | \mathcal{F}_{t_1}] \right] \\ & \leq 2 \int_{t_1}^{t_2} \left(1 + \frac{k_2 - k_1}{2} \int_t^T e^{-k_1(s-t)} ds \right) \\ & \quad \times \left(\mathbb{E} |\hat{D}_t // F|^2 + \frac{k_2 - k_1}{2} \int_t^T e^{-k_1(s-t)} \mathbb{E} |\hat{D}_s // F|^2 ds \right) dt. \end{aligned}$$

Theorem

Assume $k_1 \leq \text{Ric}^Z \leq k_2$. Then

$$\text{gap}(\mathcal{L})^{-1} \leq C(T, k_1, |k_1| \vee |k_2|) \\ \wedge \left[C\left(T, k_1, \frac{k_2 - k_1}{2}\right) \times C\left(T, \frac{k_1 + k_2}{2}, \frac{|k_1 + k_2|}{2}\right) \right]$$

where

$$C(T, K_1, K_2) \\ = \begin{cases} 1 + K_2 T + \frac{K_2^2 T^2}{2}, & K_1 = 0; \\ (1 + \beta)^2 - \beta \sqrt{(2 + \beta)(2 + 2\beta - \beta e^{-K_1 T})} e^{-K_1 T/2}, & K_1 > 0; \\ \frac{1}{2} + \frac{1}{2} (1 + \beta(1 - e^{-K_1 T}))^2, & K_1 < 0. \end{cases}$$

with $\beta = K_2/K_1$.

V. Riemannian metrics evolving in time

Let $g(t)$ be a C^1 family of Riemannian metrics on a manifold M , $t \in I$ where $I = [0, T^*[$ or \mathbb{R}_+ .

- A continuous adapted process X is called **Brownian motion with respect to $g(t)$** if

$$\forall f \in C^\infty(M),$$

$$d(f(X_t)) - (\Delta_{g(t)} f)(X_t) dt = 0 \quad (\text{mod loc mart})$$

- We call X shortly a **$g(t)$ -Brownian motion** on M .
- We use the notation

$$X_t = X_t^{(x,s)}, \quad t \geq s, \quad \text{if } X_s = x.$$

Deformation of Riemannian metrics $g(t)$ under certain evolution equations

Eminent example Ricci flow (R. Hamilton, 1982)

- Start with a given metric g_0 on M and let it evolve under

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)}, \quad g(0) = g_0$$

- For instance, in terms of local coordinates x_i , if $\Delta x_i = 0$, then

$$\operatorname{Ric}_{ij} = -\frac{1}{2} \Delta g_{ij} + \text{lower order terms.}$$

- Idea behind Ricci flow: Ricci flow works as heat equation on the space of Riemannian metrics.
- The scalar curvature $\operatorname{Scal} := \operatorname{trace} \operatorname{Ric}$ satisfies the reaction-diffusion equation

$$\frac{\partial}{\partial t} \operatorname{Scal} = \Delta \operatorname{Scal} + 2|\operatorname{Ric}|^2.$$

Depending on the sign \pm in

$$\frac{\partial}{\partial t}g(t) = \pm 2\text{Ric}_{g(t)}, \quad g(0) = g_0$$

we talk about **backward/forward Ricci flow**.

Heat equation under moving Riemannian metrics

- Study the heat equation under Ricci flow
- Consider positive solutions u to the heat equation:

$$\begin{cases} \frac{\partial}{\partial t} u - \Delta_{g(t)} u = 0 \\ \frac{\partial}{\partial t} g(t) = -2\text{Ric}_{g(t)} \end{cases}$$

or to the conjugate heat equation

$$\begin{cases} \frac{\partial}{\partial t} u + \Delta_{g(t)} u - \text{Scal}(t, \cdot) u = 0 \\ \frac{\partial}{\partial t} g(t) = -2\text{Ric}_{g(t)} \end{cases}$$

Brownian motion on $(M, g(t))$

- Let $\mathbb{M} := M \times I$ be space time and consider the tangent bundle TM over \mathbb{M} :

$$TM \xrightarrow{\pi} \mathbb{M}, \quad \pi \text{ projection.}$$

- There is a natural *space-time connection* on TM , considered as bundle over space-time \mathbb{M} , defined by

$$\nabla_X Y = \nabla_X^{g_t} Y \quad \text{and} \quad \nabla_{\partial_t} Y = \partial_t Y + \frac{1}{2}(\partial_t g_t)(Y, \cdot)^{\sharp_{g_t}}, \quad g_t = g(t).$$

- This connection is compatible with the metric, i.e.

$$\frac{d}{dt} |Y|_{g_t}^2 = 2 \langle Y, \nabla_{\partial_t} Y \rangle_{g_t}$$

- The connection allows to define parallel transport along curves, but *curves in space-time* \mathbb{M} , typically of the form

$$\gamma_t = (x_t, t), \quad t \in [0, T].$$

- Let $(M, g_t)_{t \in I}$ be an evolving manifold where $[0, T] \subset I \subset \mathbb{R}_+$.
- Stochastic development of Euclidean Brownian motion gives **space-time Brownian motions**

$$(X_t, t)$$

where X_t is a g_t -Brownian motion, together with a notion parallel transport along X_t (by construction consisting of isometries!):

$$//_{r,s}: (T_{x_r}M, g_r) \rightarrow (T_{x_s}M, g_s), \quad 0 \leq r \leq s \leq T.$$

- We write

$$X_t = X_t^{(x,s)}, \quad t \geq s,$$

where X_t is a g_t -Brownian motion (with generator Δ_t) starting from x at time s .

Main probabilistic ingredients

- **(semigroup)** $P_{s,t}f(x) := \mathbb{E}[f(X_t^{(x,s)})]$ for $s \leq t$ in I .
- **(gradient formula)**

$$\nabla^s P_{s,t}f(x) = \mathbb{E}\left[Q_{s,t} //_{s,t}^{-1} \nabla^t f(X_t^{(x,s)})\right], \quad 0 \leq s \leq t,$$

where $Q_{s,t} \in \text{Aut}(T_{X_s}M)$ is constructed as solution to the (pathwise) equation:

$$\frac{dQ_{s,t}}{dt} = -Q_{s,t} \mathcal{R} //_{s,t}, \quad Q_{s,s} = \text{id}.$$

where

$$\mathcal{R} //_{s,t} = //_{s,t}^{-1} \left(\text{Ric}_{g_t} - \frac{1}{2} \partial_t g_t \right) //_{s,t}.$$

- We see that $Q_{s,t} = \text{identity}$ if and only if the metric evolves by (backward) Ricci flow.
- This explains why Riemannian manifolds evolving under Ricci flow share many properties of Ricci flat static manifolds.

- Write Ric_t , ∇^t for the Ricci tensor, Levi-Civita connection with respect to g_t , respectively.
- We allow the g_t -Brownian motions to have a drift Z_t .
- For a smooth family $(Z_t)_{t \in I}$ of vector fields on M let

$$L_t = \Delta_t + Z_t.$$

- The “Ricci tensor”

$$\mathcal{R}_t(X, Y) := \text{Ric}_t(X, Y) - \frac{1}{2}(\partial_t g_t)(X, Y)$$

then generalizes to

$$\mathcal{R}_t^Z(X, Y) := \text{Ric}_t(X, Y) - \langle \nabla_X^t Z_t, Y \rangle_t - \frac{1}{2}(\partial_t g_t)(X, Y)$$

- For $f \in C_b(M)$, we write again

$$P_{s,t} f(x) := \mathbb{E}[f(X_t^{(x,s)})] = \mathbb{E}^{(x,s)}[f(X_t)], \quad 0 \leq s \leq t \text{ in } I,$$

where $X_t^{(x,s)}$ is a L_t -diffusion starting from x at time s .

Theorem (Cheng-A.Th. 2017)

Let $(t, x) \mapsto K_1(t, x)$ and $(t, x) \mapsto K_2(t, x)$ be two continuous functions on $I \times M$ such that $K_1 \leq K_2$ (satisfying some weak integrability conditions).

The following statements are equivalent:

(i) the curvature \mathcal{R}_t^Z satisfies

$$K_1(t, x) \leq \mathcal{R}_t^Z(x) \leq K_2(t, x), \quad (t, x) \in I \times M;$$

(ii) for $f \in C_0^\infty(M)$ and $0 \leq s \leq t$ in I ,

$$\begin{aligned} & |\nabla^s P_{s,t} f|_s^2 - \mathbb{E}^{(x,s)} \left[e^{-2 \int_s^t K_1(r, X_r) dr} |\nabla^t f|_t^2(X_t) \right] \\ & \leq 4 \left[\left(\mathbb{E}^{(x,s)} e^{\frac{1}{2} \int_s^t (K_2(r, X_r) - K_1(r, X_r)) dr} - 1 \right) |\nabla^s f|_s^2 + \langle \nabla^s f, \nabla^s P_{s,t} f \rangle_s \right. \\ & \quad \left. - \langle \nabla^s f, \mathbb{E}^{(x,s)} \left[e^{-\int_s^t K_1(r, X_r) dr} //_{s,t}^{-1} \nabla^t f(X_t) \right] \rangle_s \right] \wedge 0; \end{aligned}$$

Theorem—cont.

(iii) for $f \in C_0^\infty(M)$, $p \in (1, 2]$ and $0 \leq s \leq t$ in I ,

$$\begin{aligned} & \frac{p(P_{s,t}f^2 - (P_{s,t}f^{2/p})^p)}{4(p-1)} - \mathbb{E}^{(x,s)} \left[\int_s^t e^{-2 \int_r^t K_1(\tau, X_\tau) d\tau} dr \times |\nabla^t f|_t^2(X_t) \right] \\ & \leq 4 \int_s^t \left[\mathbb{E}^{(x,s)} e^{\frac{1}{2} \int_r^t (K_2(\tau, X_\tau) - K_1(\tau, X_\tau)) d\tau} - 1 \right] P_{s,r} |\nabla^r f|_r^2 \\ & \quad + \mathbb{E}^{(x,s)} \left\langle \nabla^r f(X_r), \nabla^r P_{r,t} f(X_r) - e^{-\int_r^t K_1(\tau, X_\tau) d\tau} //_{r,t}^{-1} \nabla^t f(X_t) \right\rangle_r dr \wedge 0; \end{aligned}$$

(iv) for $f \in C_0^\infty(M)$ and $0 \leq s \leq t$ in I ,

$$\begin{aligned} & \frac{1}{4} (P_{s,t}(f^2 \log f^2) - P_{s,t}f^2 \log P_{s,t}f^2) \\ & - \mathbb{E}^{(x,s)} \left[\int_s^t e^{-2 \int_r^t K_1(\tau, X_\tau) d\tau} dr \times |\nabla^t f|_t^2(X_t) \right] \\ & \leq 4 \int_s^t \left[\mathbb{E}^{(x,s)} e^{\frac{1}{2} \int_r^t (K_2(\tau, X_\tau) - K_1(\tau, X_\tau)) d\tau} - 1 \right] P_{s,r} |\nabla^r f|_r^2 \\ & \quad + \mathbb{E}^{(x,s)} \left\langle \nabla^r f(X_r), \nabla^r P_{r,t} f(X_r) - e^{-\int_r^t K_1(\tau, X_\tau) d\tau} //_{r,t}^{-1} \nabla^t f(X_t) \right\rangle_r dr \wedge 0. \end{aligned}$$

Corollary [Cheng-A.Th. 2017]

Let $(t, x) \mapsto K(t, x)$ be some continuous function on $I \times M$. The following statements are equivalent to each other:

(i) the family $(M, g_t)_{t \in I}$ evolves by

$$\frac{1}{2} \partial_t g_t = \text{Ric}_t - \nabla^t Z_t - K(t, \cdot) g_t, \quad t \in I;$$

(ii) for $f \in C_0^\infty(M)$ and $0 \leq s \leq t$ in I ,

$$\begin{aligned} & |\nabla^s P_{s,t} f|_s^2 - \mathbb{E}^{(x,s)} \left[e^{-2 \int_s^t K(r, X_r) dr} |\nabla^t f|_t^2(X_t) \right] \\ & \leq 4 \left[|\nabla^s P_{s,t} f|_s - \left\langle \nabla^s P_{s,t} f, \mathbb{E}^{(x,s)} \left[e^{-\int_s^t K(r, X_r) dr} //_{s,t}^{-1} \nabla^t f(X_t) \right] \right\rangle_s \right] \wedge 0; \end{aligned}$$

(iii) version of a **Poincaré inequality**

(iv) version of a **log-Sobolev inequality**

- If $Z_t \equiv 0$ and $K \equiv 0$, the results characterize solutions to the **Ricci flow**; see Haslhofer and Naber (2018) for characterizations on **path space**.
- We have

$$\frac{1}{2} \partial_t g_t = \text{Ric}_t, \quad t \in I$$

if and only if for $f \in C_0^\infty(M)$ and $0 \leq s \leq t$ in I ,

$$\begin{aligned} & |\nabla^s P_{s,t} f|_s^2 - P_{s,t} |\nabla^t f|_t^2 \\ & \leq 4 \left[|\nabla^s P_{s,t} f|_s^2 - \left\langle \nabla^s P_{s,t} f, \mathbb{E}^{(x,s)} \left[\int_{s,t}^{-1} \nabla^t f(X_t) \right] \right\rangle_s \right] \wedge 0; \end{aligned}$$

- Consider the **heat equation under Ricci flow**:

$$\begin{cases} \frac{\partial}{\partial t} u - \Delta_{g_t} u = 0 \\ \frac{\partial}{\partial t} g_t - 2 \operatorname{Ric}_{g_t} = 0 \end{cases}$$

To deal with the forward Ricci flow, we reparametrize the metric:

$$\hat{g}_t := g_{T-t}$$

- As before, if $u(\cdot, s) = f$, we write

$$u(x, t) = (P_{s,t} f)(x), \quad 0 \leq s \leq t \text{ in } I.$$

Supersolutions to the Ricci flow

For a smooth family $(M, g(t))_{t \in I}$ of Riemannian metrics are equivalent:

- $(M, g(t))_{t \in I}$ is a supersolution to the Ricci flow, i.e.

$$2 \operatorname{Ric}_{g(t)} - \frac{\partial}{\partial t} g(t) \geq 0.$$

- For each $f \in C_c^\infty(M)$ the heat flow on $(M, g(t))_{t \in I}$ satisfies

$$|\nabla^s P_{s,t} f|_{g(s)} \leq P_{s,t} |\nabla^t f|_{g(t)}, \quad 0 \leq s < t \text{ in } I.$$

- For each $f \in C_c^\infty(M)$ the heat flow on $(M, g(t))_{t \in I}$ satisfies

$$|\nabla^s P_{s,t} f|_{g(s)}^2 \leq P_{s,t} |\nabla^t f|_{g(t)}^2, \quad 0 \leq s < t \text{ in } I.$$

- Let $\mathcal{P}^{(x,s)}M$ be the space of continuous paths on M , starting in x at time s and $\mathbb{P}^{(x,s)}$ the probability measure on it, induced by the (inhomogeneous) BM

$$X_t^{(x,s)}, \quad t \geq s.$$

- For a cylindrical function F on $\mathcal{P}^{(x,s)}M$ with

$$F(\gamma) = f(\gamma_{t_1}, \dots, \gamma_{t_r}), \quad s \leq t_1 < \dots < t_r \leq t,$$

consider the **intrinsic gradient** defined as

$$D_s^{//} F(X_{[s,t]}) = \sum_{i=1}^r //_{s,t_i}^{-1} (\nabla_{g(t_i)}^i f)(X_{t_1}, \dots, X_{t_r}),$$

where ∇^i denotes the gradient with respect to the i -th component.

Characterization of solutions to the Ricci flow

For a smooth family $(M, g(t))_{t \in I}$ of Riemannian metrics are equivalent:

- $(M, g(t))_{t \in I}$ is a solution to the Ricci flow, i.e.

$$\frac{\partial}{\partial t} g(t) - 2 \operatorname{Ric}_{g(t)} = 0.$$

- For each cylindrical function $F: \mathcal{P}^{(x,s)} M \rightarrow \mathbb{R}$,

$$|\nabla_x^s \mathbb{E}^{(x,s)} F| \leq \mathbb{E}^{(x,s)} |D_s^{//} F|.$$

- For each cylindrical function $F: \mathcal{P}^{(x,s)} M \rightarrow \mathbb{R}$,

$$|\nabla_x^s \mathbb{E}^{(x,s)} F|^2 \leq \mathbb{E}^{(x,s)} |D_s^{//} F|^2.$$

Here $|\cdot| = |\cdot|_{g(s)}$.