Characterization of Ricci curvature and Ricci flow by Brownian motion

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I. Motivation

Let (M, g) be a complete Riemannian manifold

- (inner product on tangent spaces)

$$g_x = \langle \cdot, \cdot \rangle_x : \; T_x M \times T_x M \to \mathbb{R}, \quad x \in M$$

- (length of curves) For $\gamma : [a, b] \to M$,

$$\text{Length}(\gamma) = \int_a^b |\dot{\gamma}(t)|\, dt$$

where $|\dot{\gamma}(t)| = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}}$

- (Ricci tensor) Bilinear forms:

$$\mathrm{Ric}_x : \; T_x M \times T_x M \to \mathbb{R}, \quad x \in M$$

- (scalar curvature)

$$\text{Scal} := \mathrm{trace}\, Ric \in C^\infty(M)$$

- The metric volume element has the following expansion in geodesic normal coordinates at $p \in M$:

$$d\mathrm{vol}_g = \Big(1 - \tfrac{1}{6}\,\mathrm{Ric}_{ij}(p)\, x^i x^j + O(|x|^3)\Big)\, d\mathrm{vol}_{\mathrm{Eucl}}$$

- For the volume of geodesic balls of radius r about p:

$$\mathrm{Volume}(B(p,r)) = \Big(1 - \mathrm{Scal}(p)\, C\, r^2 + O(r^4)\Big)\, \mathrm{Vol}_{\mathbb{R}^n}(r)$$

where C is a positive constant depending only on $n = dim\, M$.

- Why should probabilists care about such things?

Heat flow on a Riemannian manifold

- Let (M,g) be a complete Riemannian manifold and

$$L = \Delta + Z \quad \text{with } Z \in \Gamma(TM)$$

- Let u be a positive solution to

$$\frac{\partial}{\partial t} u = Lu \quad \text{on} \quad M \times \mathbb{R}_+$$

- (Gradient estimate) Find bounds on

$$|\nabla u| \ \text{ or } \ \frac{|\nabla u|}{u}.$$

- (Harnack inequalities) Compare

$$u(x,s) \ \text{ and } \ u(y,t).$$

- Why is Ricci curvature important for such questions?

- **Cheng-Yau** (1975)
  Let M be complete and D be some open, relatively compact domain D in M. Assume that u is a positive harmonic function on D:
  $$\Delta u = 0$$
  Then
  $$\frac{|\nabla u|}{u}(x) \leq c(n)\left[\sqrt{K} + \frac{1}{r(x)}\right]$$
  if $\mathrm{Ric}|D \geq -K$, $K \geq 0$ (where $r(x) = \mathrm{dist}(x, \partial D)$ and $n = \dim M$).

The estimate is easy to prove by probabilistic methods, e.g. Arnaudon, Driver, A.Th. (2007).

- For $L = \Delta + Z$ let u be a solution to $\dfrac{\partial}{\partial t} u = Lu$.
- Want to express the differential

$$(\nabla u)(\cdot, t)_x$$

in terms of an L-diffusion starting from x:

$$X_t = X_t^x, \quad t < \zeta(x).$$

- Recall that L-diffusions X_t on M are defined by the property that for each $f \in C_c^\infty(M)$,

$$d(f(X_t)) - (Lf)(X_t)\, dt = 0$$

(mod differentials of local martingales)

- Denote by

$$\mathrm{Ric}^Z = \mathrm{Ric} - \nabla Z$$

the Bakry-Émery Ricci tensor, i.e.

$$\mathrm{Ric}^Z(X,Y) := \mathrm{Ric}(X,Y) - \langle \nabla_X Z, Y\rangle\,.$$

- Let

$$\mathrm{Ric}^Z_{//_t} := //_t^{-1} \circ \mathrm{Ric}^Z_{X_t} \circ //_t \in \mathrm{End}(T_xM)$$

where $//_t\colon T_xM \to T_{X_t}M$ is parallel transport along $X_t = X_t^x$:

$$\begin{tikzcd}
T_xM \arrow[r, dashed, "\mathrm{Ric}^Z_{//_t}"'] \arrow[d, "//_t"'] & T_xM \\
T_{X_t}M \arrow[r, "\mathrm{Ric}^Z_{X_t}"'] & T_{X_t}M \arrow[u, "//_t^{-1}"']
\end{tikzcd}$$

By convention $\mathrm{Ric}^Z_x(v) = \mathrm{Ric}^Z_x(\cdot, v)^\sharp$ for $v \in T_xM$.

Damped parallel transport

- For $x \in M$ define a linear transformation

$$Q_t\colon\ T_xM \to T_xM$$

as solution to the pathwise ODE

$$\begin{cases} dQ_t = -Q_t\,\mathrm{Ric}^Z_{//_t}\,dt \\ Q_0 = \mathrm{id}_{T_xM} \end{cases}$$

- In the sequel we need

$$Q_t \circ //_t^{-1}\colon\ T_{X_t}M \to T_xM$$

("damped parallel transport" along X_t)

Theorem (probabilistic formulas)

Let $f \in \mathscr{B}_b(M)$ and $u(x,t) = P_t f(x)$ be the (minimal) solution to

$$\frac{\partial}{\partial t} u = Lu,\ u|_{t=0} = f.$$

- *(Semigroup formula) Then* $P_t f(x) = \mathbb{E}[f(X_t^x) \mathbb{1}_{\{t<\zeta(x)\}}]$.
- *(Derivative formula) If $f \in C_b^1(M)$ and Ric^Z bounded below,*

$$(\nabla P_t f)(x) = \mathbb{E}\left[Q_t //_t^{-1} \nabla f(X_t^x)\right]$$

- *(Bismut formula) If $f \in \mathscr{B}_b(M)$ (no assumption on Ric), then*

$$\langle (\nabla P_t f)_x, v\rangle = -\mathbb{E}\left[f(X_t^x) \mathbb{1}_{\{t<\zeta(x)\}} \int_0^\tau \left\langle Q_s^* \dot{\ell}_s, dB_s \right\rangle\right]$$

  for each $v \in T_x M$, where
  - $\tau = \tau_D(x) \wedge t$ *with $\tau_D(x)$ the first exit time of X_t^x from some relatively compact neighbourhood D of x*
  - B *is a Brownian motion in* $T_x M$
  - ℓ_t *is any adapted process in $T_x M$ with absolutely continuous paths of finite energy such that $\ell_0 = v$ and $\ell_\tau = 0$.*

An obvious observation

- Suppose that $\mathrm{Ric}^Z \geq k$ for some $k \in C(M)$, i.e.

$$\mathrm{Ric}^Z(X,X) \geq k(x)\,|X|^2, \quad X \in T_x M.$$

Then $$|Q_t| \leq \exp\left(-\int_0^t k(X_s)\,ds\right)$$ and

$$|\nabla P_t f| \leq \exp\left(-\int_0^t k(X_s)\,ds\right) P_t|\nabla f|, \quad f \in C_b^1(M).$$

- In particular, if

$$\mathrm{CD}(K,\infty) \qquad \mathrm{Ric}^Z(X,X) \geq K|X|^2, \quad X \in TM,$$

for some constant K, then

$$|Q_t| \leq e^{-Kt}$$

and

$$\text{(gradient estimate)} \qquad |\nabla P_t f| \leq e^{-Kt} P_t|\nabla f|, \quad f \in C_b^1(M).$$

- Actually the gradient estimate is equivalent to $\mathrm{CD}(K,\infty)$.

II. Characterization of bounded Ricci curvature

Our setting

- (Process) X_t is an L-diffusion where

$$L = \Delta + Z \quad \text{with } Z \in \Gamma(TM)$$

- Assume that $\mathrm{Ric}^Z = \mathrm{Ric} - \nabla Z$

$$\mathrm{Ric}^Z(X,Y) = \mathrm{Ric}(X,Y) - \langle \nabla_X Z, Y \rangle,$$

is bounded below, i.e., for some constant K,

$$\mathrm{Ric}^Z(X,X) \geq K|X|^2, \quad X \in TM.$$

Our focus

- For real constants $k_1 \le k_2$, how to characterize

$$k_1 \le \mathrm{Ric}^Z \le k_2.$$

  in terms of functional inequalities for the semigroup P_t.

- Natural extensions:
  - Pointwise pinched curvature conditions

$$k_1(x) \le \mathrm{Ric}_x^Z \le k_2(x), \quad x \in M$$

  - Riemannian manifolds with a boundary
  - Manifolds evolving under a geometric flow

Well-known and classical: Let K be a real constant.
The following conditions are equivalent:

- **(Bakry-Émery lower curvature bound)**

$$\mathrm{CD}(K,\infty) \qquad \mathrm{Ric}^Z(X,X) \geq K|X|^2, \quad X \in TM;$$

- **(gradient estimate)** for $p \in [1,\infty[$ and all $f \in C_c^\infty(M)$,

$$|\nabla P_t f|^p \leq e^{-pKt} P_t |\nabla f|^p;$$

- **(Poincaré inequality)** for $p \in (1,2]$ and all $f \in C_c^\infty(M)$,

$$\frac{p}{4(p-1)}\Big(P_t f^2 - (P_t f^{2/p})^p\Big) \leq \frac{1-e^{-2Kt}}{2K} P_t |\nabla f|^2;$$

- **(log-Sobolev inequality)** for all $f \in C_c^\infty(M)$,

$$P_t(f^2 \log f^2) - (P_t f^2)\log(P_t f^2) \leq \frac{2(1-e^{-2Kt})}{K} P_t |\nabla f|^2.$$

Many other equivalent statements, e.g., transportation-cost inequalities; convexity properties of the entropy; Wang's dimension-free Harnack inequalities; Wang's log-Harnack inequalities, ...

Natural questions:

- How to characterize upper bounds for Ric^Z?
- How to characterize pinched bounds for Ric^Z?

Well-known:

Boundedness of $|\mathrm{Ric}^Z|$, i.e.

$$|\mathrm{Ric}^Z| \leq K,$$

implies certain functional inequalities on path space,

e.g. Capitaine-Hsu-Ledoux (1997), Chen-Wu (2014), Driver (1992), Hsu (1994)

Boundedness of $|\mathrm{Ric}^Z|$

The problem of characterizing boundedness of Ric^Z has been solved by A. Naber and R. Haslhofer via analysis on path space:

Boundedness of $|\mathrm{Ric}^Z| \Longleftrightarrow$ functional inequalities on path space

- Aaron Naber, *Characterizations of bounded Ricci curvature on smooth and nonsmooth spaces*, arXiv:1306.6512v4 (2015)
- Robert Haslhofer and Aaron Naber, Characterizations of the Ricci flow, J. Eur. Math. Soc. (2018)

Our work:

- Li-Juan Cheng and A.Th.: *Characterization of pinched Ricci curvature by functional inequalities*, J. Geom. Anal. (2017)
- Li-Juan Cheng and A.Th.: *Spectral gap on Riemannian path space over static and evolving manifolds*, J. Funct. Anal. **274** (2018), 959-984

III. Analysis on path space

- For fixed $T > 0$, let $W^T = C([0,T]; M)$ and

$$\mathscr{F}C^\infty_{0,T} = \Big\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}) : \\ 0 < t_1 < \ldots < t_n \le T,\ f \in C_c^\infty(M^n) \Big\}.$$

  be the class of smooth cylindrical functions on W^T.

- Denote

$$X_{[0,T]} = \{X_t :\ 0 \le t \le T\}.$$

- For $F \in \mathscr{F}C^\infty_{0,T}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, the intrinsic gradient is defined as

$$D^{//}_t F(X_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t<t_i\}} //^{-1}_{t,t_i} \nabla^i f(X_{t_1}, \ldots, X_{t_n}), \quad t \in [0,T],$$

  where ∇^i denotes the gradient with respect to the i-th component.

Theorem [A. Naber (2015) and R. Haslhofer and A. Naber (2018)]

The following conditions are equivalent ($K \geq 0$):

- $|\mathrm{Ric}^Z| \leq K$;
- (Gradient inequality on path space) for $F \in \mathcal{F}C_0^\infty$,

$$\left|\nabla \mathbb{E}[F(X_{[0,T]})]\right| \leq \mathbb{E}\left[|D_0^{//}F| + K\int_0^T e^{Kr}|D_r^{//}F|\,dr\right].$$

- (L^2 gradient inequality on path space) for $F \in \mathcal{F}C_0^\infty$,

$$\left|\nabla \mathbb{E}[F(X_{[0,T]})]\right|^2 \leq e^{KT}\mathbb{E}\left[|D_0^{//}F|^2 + K\int_0^T e^{Kr}|D_r^{//}F|^2\,dr\right].$$

Important observation It is sufficient to check the estimates for very special $F \in \mathcal{F}C_0^\infty$. Namely:

(i) for $F(X^x_{[0,T]}) = f(X^x_t)$, and

(ii) for 2-point cylindrical functions of the form

$$F(X^x_{[0,T]}) = f(x) - \frac{1}{2}f(X^x_t)$$

From this observation, equivalence of the following two items follows:

(i) $|\mathrm{Ric}^Z| \le K$ for $K \ge 0$;

(ii) for $f \in C_c^\infty(M)$ and $t > 0$,

$$|\nabla P_t f|^2 \le e^{2Kt} P_t |\nabla f|^2 \quad \text{and}$$
$$\left|\nabla f - \frac{1}{2}\nabla P_t f\right|^2 \le e^{Kt}\mathbb{E}\left[\left|\nabla f - \frac{1}{2}//_{0,t}^{-1}\nabla f(X_t)\right|^2 + \frac{1}{4}\left(e^{Kt}-1\right)|\nabla f(X_t)|^2\right].$$

Remark The inequalities in (ii) can be combined to the single condition:

$$|\nabla P_t f|^2 - e^{2Kt} P_t|\nabla f|^2$$
$$\le 4\Big((e^{Kt}-1)|\nabla f|^2 + \langle \nabla f, \nabla P_t f\rangle - \Big\langle \nabla f, e^{Kt}\mathbb{E}[//_{0,t}^{-1}\nabla f(X_t)]\Big\rangle\Big) \wedge 0.$$

Theorem (Characterization of pinched Ricci curvature; Cheng-A.Th. 2017)

Let k_1, k_2 be two real constants such that $k_1 \le k_2$. The following conditions are equivalent:

(i) $k_1 \le \mathrm{Ric}^Z \le k_2$

(ii) (Gradient inequalities) *for $f \in C_c^\infty(M)$ and $t > 0$,*

$$\begin{aligned}|\nabla P_t f|^2 - e^{-2k_1 t} P_t |\nabla f|^2 \le 4\Big[\Big(e^{\frac{k_2-k_1}{2}t} - 1\Big)|\nabla f|^2 \\ + \langle \nabla f, \nabla P_t f\rangle - e^{-k_1 t}\mathbb{E}\Big\langle \nabla f, //_{0,t}^{-1}\nabla f(X_t)\Big\rangle\Big] \wedge 0\end{aligned}$$

(ii') *for $f \in C_c^\infty(M)$ and $t > 0$,*

$$\begin{aligned}&|\nabla P_t f|^2 - e^{-2k_1 t} P_t |\nabla f|^2 \\ &\le 4\Big(e^{\frac{k_2-k_1}{2}t}|\nabla P_t f|^2 - e^{-k_1 t}\mathbb{E}\Big\langle \nabla P_t f, //_{0,t}^{-1}\nabla f(X_t)\Big\rangle\Big) \wedge 0\end{aligned}$$

Theorem (continuation)

(iii) (Poincaré type inequality) *for* $f \in C_c^\infty(M)$, $p \in]1,2]$, $t > 0$,

$$\frac{p(P_t f^2 - (P_t f^{2/p})^p)}{4(p-1)} - \frac{1-e^{-2k_1 t}}{2k_1} P_t|\nabla f|^2$$
$$\leq 4\int_0^t \left(e^{\frac{k_2-k_1}{2}(t-r)} - 1\right) P_r|\nabla f|^2$$
$$+\mathbb{E}\left\langle \nabla f(X_r), \nabla P_{t-r} f(X_r) - e^{-k_1(t-r)} //_{r,t}^{-1} \nabla f(X_t)\right\rangle dr \wedge 0$$

(iv) (Log-Sobolev inequality) *for* $f \in C_c^\infty(M)$, $t > 0$,

$$\frac{1}{4}\left(P_t(f^2\log f^2) - P_t f^2 \log P_t f^2\right) - \frac{1-e^{-2k_1 t}}{2k_1} P_t|\nabla f|^2$$
$$\leq 4\int_0^t \left(e^{\frac{k_2-k_1}{2}(t-r)} - 1\right) P_r|\nabla f|^2$$
$$+\mathbb{E}\left\langle \nabla f(X_r), \nabla P_{t-r} f(X_r) - e^{-k_1(t-r)} //_{r,t}^{-1} \nabla f(X_t)\right\rangle dr \wedge 0$$

The proof uses probabilistic formulas for calculating Ric^Z, e.g. Bakry (1994), von Renesse-Sturm (2005), Wang (2014).

Lemma

Let $v \in T_xM$ with $|v| = 1$. Let $f \in C_0^\infty(M)$ such that $\nabla f(x) = v$ and $\mathrm{Hess}_f(x) = 0$. Then,

(i) *for $p > 0$,*

$$\mathrm{Ric}^Z(v,v) = \lim_{t\to 0} \frac{P_t|\nabla f|^p(x) - |\nabla P_t f|^p(x)}{pt}$$

(ii) $\mathrm{Ric}^Z(v,v)$ *is also given by the following two limits:*

$$\begin{aligned}\mathrm{Ric}^Z(v,v) &= \lim_{t\to 0} \frac{\left\{\left\langle \nabla f, \mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle - \langle \nabla f, \nabla P_t f\rangle\right\}(x)}{t} \\ &= \lim_{t\to 0} \frac{\left\{\left\langle \nabla P_t f, \mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle - |\nabla P_t f|^2\right\}(x)}{t}\end{aligned}$$

The theorem can be extended in various ways:

- to characterize variable curvature bounds

$$K_1(x) \le \mathrm{Ric}^Z(x) \le K_2(x), \quad x \in M,$$

  with functions K_1, K_2 on M
- to manifolds with boundary (reflecting diffusions generated by $L = \Delta + Z$) to characterize

$$K_1(x) \le \mathrm{Ric}^Z(x) \le K_2(x), \quad x \in M,$$
$$\sigma_1(x) \le \mathrm{II}(x) \le \sigma_2(x), \quad x \in \partial M,$$

  in terms of semigroups with Neumann boundary conditions.

  The second fundamental form of ∂M is given by

$$\mathrm{II}(X,Y) = -\langle \nabla_X N, Y \rangle, \quad X, Y \in T_x\partial M,\ x \in \partial M,$$

  where N is the inward normal unit vector field on ∂M.

The theorem allows to characterize

- Einstein manifolds (Ric is a multiple of the metric g)
- Ricci solitons ($\mathrm{Ric} + \mathrm{Hess} f = c\, g$)
- manifolds such that $\mathrm{Ric} = \nabla Z$
- etc

IV. Back to Riemannian path space

For $F \in \mathscr{F}C^\infty_{0,T}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, let

$$D_t^{//} F(X^x_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t<t_i\}} //_{t,t_i}^{-1} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n})$$

be the *intrinsic gradient*, and

$$D_t F(X^x_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t<t_i\}} Q_{t,t_i} //_{t,t_i}^{-1} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n})$$

the *damped gradient*, where $Q_{t,r}$ takes values in the linear automorphisms of $T_{X^x_t} M$ satisfying for fixed $t \geq 0$:

$$\frac{dQ_{t,r}}{dr} = -Q_{t,r} \,\mathrm{Ric}^Z_{//_{t,r}}, \quad Q_{t,t} = \mathrm{id}.$$

- Let $\mathcal{L}$ be the Ornstein-Uhlenbeck operator defined as generator associated to the Dirichlet form

$$\mathcal{E}(F,F) = \mathbb{E}\left[\int_0^T |D_t^{//} F|^2(X_{[0,T]})\, dt\right].$$

- It is well-known that a log-Sobolev inequality

$$\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \le 2H(T,k_1,k_2) \int_0^T |D_t^{//} F|^2(X_{[0,T]})\, dt$$

or a Poincaré inequality

$$\mathbb{E}[F - \mathbb{E}[F]) \le H(T,k_1,k_2) \int_0^T |D_t^{//} F|^2(X_{[0,T]})\, dt$$

for some explicit bound $H(T,k_1,k_2)$, are equivalent to the spectral gap-lower bound $H(T,k_1,k_2)^{-1}$ for the operator $\mathcal{L}$.
- For constants $k_1 \le k_2$ let

$$\hat{D}_t^{//} F(X_{[0,T]}^x) = \sum_{i=1}^n \mathbb{1}_{\{t \le t_i\}}\, e^{-\frac{k_1+k_2}{2}(t_i - t)} //_{t,t_i}^{-1} \nabla_i f(X_{t_1}^x, \ldots, X_{t_n}^x)$$

Theorem (Path space characterization of pinched curvature)

The following conditions are equivalent:

(i) $k_1 \le \mathrm{Ric}^Z \le k_2$;

(ii) *for any* $F \in \mathscr{F}C_{0,T}^\infty$,

$$\left|\nabla_x \mathbb{E}F(X_{[0,T]}^x)\right| \le \mathbb{E}|\hat{D}_0^{//}F| + \frac{k_2-k_1}{2}\int_0^T e^{-k_1 s}\,\mathbb{E}|\hat{D}_s^{//}F|\,ds;$$

(iii) *for any* $F \in \mathscr{F}C_{0,T}^\infty$ *and* $t_1 < t_2$ *in* $[0,T]$,

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\log \mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\Big] \\
&\qquad -\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\log \mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\Big] \\
&\le 2\int_{t_1}^{t_2}\left(1+\frac{k_2-k_1}{2}\int_t^T e^{-k_1(s-t)}ds\right) \\
&\qquad \times\left(\mathbb{E}|\hat{D}_t^{//}F|^2 + \frac{k_2-k_1}{2}\int_t^T e^{-k_1(s-t)}\mathbb{E}|\hat{D}_s^{//}F|^2 ds\right)dt.
\end{aligned}$$

Theorem

Assume $k_1 \le \mathrm{Ric}^Z \le k_2$. *Then*

$$\mathrm{gap}(\mathcal{L})^{-1} \le C(T, k_1, |k_1| \vee |k_2|) \\ \wedge \left[C\left(T, k_1, \tfrac{k_2 - k_1}{2}\right) \times C\left(T, \tfrac{k_1 + k_2}{2}, \tfrac{|k_1 + k_2|}{2}\right)\right]$$

where

$$C(T, K_1, K_2) \\ = \begin{cases} 1 + K_2 T + \frac{K_2^2 T^2}{2}, & K_1 = 0; \\ (1+\beta)^2 - \beta\sqrt{(2+\beta)\left(2 + 2\beta - \beta e^{-K_1 T}\right)} e^{-K_1 T/2}, & K_1 > 0; \\ \frac{1}{2} + \frac{1}{2}\left(1 + \beta(1 - e^{-K_1 T})\right)^2, & K_1 < 0. \end{cases}$$

with $\beta = K_2 / K_1$.

V. Time-dependent Riemannian metrics

Let $g(t)$ be a C^1 family of Riemannian metrics on a manifold M, $t \in I$ where $I = [0, T^*[$ or $\mathbb{R}_+$.

- A continuous adapted process X is called Brownian motion with respect to $g(t)$ if

  $\forall f \in C^\infty(M)$,

  $$d(f(X_t)) - (\Delta_{g(t)} f)(X_t)\, dt = 0 \quad \text{(mod loc mart)}$$

- We call X shortly a $g(t)$-Brownian motion on M.

- We use the notation

  $$X_t = X_t^{(x,s)}, \quad t \geq s, \quad \text{if } X_s = x.$$

Geometries evolving in time: Deformation of Riemannian metrics $g(t)$ under certain evolution equations

Eminent example Ricci flow (R. Hamilton, 1982)

- Start with a given metric g_0 on M and let it evolve under

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)}, \quad g(0) = g_0$$

- Idea behind Ricci flow: Ricci flow works as heat equation on the space of Riemannian metrics.
- For instance, in terms of local coordinates x_i, if $\Delta x_i = 0$, then

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta g_{ij} \ + \text{lower order terms.}$$

- The scalar curvature $\mathrm{Scal} := \mathrm{trace}\,\mathrm{Ric}$ satisfies the reaction-diffusion equation

$$\frac{\partial}{\partial t}\mathrm{Scal} = \Delta \mathrm{Scal} + 2|\mathrm{Ric}|^2.$$

Depending on the sign $\pm$ in

$$\frac{\partial}{\partial t} g(t) = \pm 2\,\mathrm{Ric}_{g(t)}, \quad g(0) = g_0$$

we talk about backward/forward Ricci flow.

Heat equation under moving Riemannian metrics

- Study the heat equation under Ricci flow
- Consider positive solutions u to the heat equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u - \Delta_{g(t)}u = 0 \\ \dfrac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}_{g(t)} \end{cases}$$

or to the conjugate heat equation

$$\begin{cases} \dfrac{\partial}{\partial t}u + \Delta_{g(t)}u - \mathrm{Scal}(t,\cdot)u = 0 \\ \dfrac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}_{g(t)} \end{cases}$$

Brownian motion on $(M, g(t))$

- Let $\mathbb{M} := M \times I$ be space time and consider the tangent bundle TM over $\mathbb{M}$:
$$TM \xrightarrow{\pi} \mathbb{M}, \quad \pi \text{ projection.}$$
- There is a natural *space-time connection* on TM, considered as bundle over space-time $\mathbb{M}$, defined by
$$\nabla_X Y = \nabla_X^{g_t} Y \quad \text{and} \quad \nabla_{\partial_t} Y = \partial_t Y + \frac{1}{2}(\partial_t g_t)(Y, \cdot)^{\sharp g_t}, \quad g_t = g(t).$$
- This connection is compatible with the metric, i.e.
$$\frac{d}{dt}|Y|^2_{g_t} = 2\langle Y, \nabla_{\partial_t} Y\rangle_{g_t}$$
- The connection allows to define parallel transport along curves, but curves in space-time $\mathbb{M}$, typically of the form
$$\gamma_t = (x_t, t), \quad t \in [0, T].$$

- Let $(M, g_t)_{t \in I}$ be an evolving manifold where $[0, T] \subset I \subset \mathbb{R}_+$.
- Stochastic development of Euclidean Brownian motion gives space-time Brownian motions

$$(X_t, t)$$

  where X_t is a g_t-Brownian motion, together with a notion parallel transport along X_t (by construction consisting of isometries!):

$$//_{r,s} \colon (T_{X_r}M, g_r) \to (T_{X_s}M, g_s), \quad 0 \le r \le s \le T.$$

- We write

$$X_t = X_t^{(x,s)}, \quad t \ge s.$$

Main probabilistic ingredients

- **(semigroup)** $P_{s,t}f(x) := \mathbb{E}[f(X_t^{(x,s)})]$ for $s \le t$ in I.
- **(gradient formula)**

$$\nabla^s P_{s,t}f(x) = \mathbb{E}\Big[Q_{s,t} //_{s,t}^{-1} \nabla^t f(X_t^{(x,s)})\Big], \quad 0 \le s \le t,$$

  where $Q_{s,t} \in \mathrm{Aut}(T_{X_s}M)$ is constructed as solution to the (pathwise) equation:

$$\frac{dQ_{s,t}}{dt} = -Q_{s,t}\mathcal{R}_{//_{s,t}}, \quad Q_{s,s} = \mathrm{id}.$$

  where

$$\mathcal{R}_{//_{s,t}} = //_{s,t}^{-1}\left(\mathrm{Ric}_{g_t} - \frac{1}{2}\partial_t g_t\right)//_{s,t}.$$

- We see that $Q_{s,t} =$ identity if and only if the metric evolves by (backward) Ricci flow.
- This explains why Riemannian manifolds evolving under Ricci flow share many properties of Ricci flat static manifolds.

- Write Ric_t, ∇^t for the Ricci tensor, Levi-Civita connection with respect to g_t, respectively.
- Let $(Z_t)_{t\in I}$ be a smooth family of vector fields on M and

$$L_t = \Delta_t + Z_t.$$

- We may allow the g_t-Brownian motions to have a drift Z_t.
- The "Ricci tensor"

$$\mathcal{R}_t(X,Y) := \mathrm{Ric}_t(X,Y) - \frac{1}{2}(\partial_t g_t)(X,Y)$$

then generalizes to

$$\mathcal{R}_t^Z(X,Y) := \mathrm{Ric}_t(X,Y) - \left\langle \nabla_X^t Z_t, Y \right\rangle_t - \frac{1}{2}(\partial_t g_t)(X,Y)$$

- For $f \in C_b(M)$, write again

$$P_{s,t}f(x) := \mathbb{E}\left[f(X_t^{(x,s)})\right] = \mathbb{E}^{(x,s)}\left[f(X_t)\right], \quad 0 \le s \le t \text{ in } I,$$

where $X_t^{(x,s)}$ is a L_t-diffusion starting from x at time s.

Theorem (Cheng-A.Th. 2017)

Let $(t,x)\mapsto K_1(t,x)$ and $(t,x)\mapsto K_2(t,x)$ be two continuous functions on $I\times M$ such that $K_1\le K_2$ (satisfying some weak integrability conditions).

The following statements are equivalent:

(i) *the curvature $\mathcal{R}_t^Z$ satisfies*

$$K_1(t,x)\le \mathcal{R}_t^Z(x)\le K_2(t,x),\quad (t,x)\in I\times M;$$

(ii) *for $f\in C_0^\infty(M)$ and $0\le s\le t$ in I,*

$$\begin{aligned}
&|\nabla^s P_{s,t}f|_s^2-\mathbb{E}^{(x,s)}\Big[e^{-2\int_s^t K_1(r,X_r)\,dr}|\nabla^t f|_t^2(X_t)\Big]\\
&\le 4\Big[\Big(\mathbb{E}^{(x,s)}e^{\frac12\int_s^t(K_2(r,X_r)-K_1(r,X_r))\,dr}-1\Big)|\nabla^s f|_s^2+\langle\nabla^s f,\nabla^s P_{s,t}f\rangle_s\\
&\qquad -\Big\langle\nabla^s f,\mathbb{E}^{(x,s)}\Big[e^{-\int_s^t K_1(r,X_r)\,dr}//_{s,t}^{-1}\nabla^t f(X_t)\Big]\Big\rangle_s\Big]\wedge 0;
\end{aligned}$$

Theorem–cont.

(iii) for $f \in C_0^\infty(M)$, $p \in (1,2]$ and $0 \le s \le t$ in I,

$$\frac{p(P_{s,t}f^2 - (P_{s,t}f^{2/p})^p)}{4(p-1)} - \mathbb{E}^{(x,s)}\left[\int_s^t e^{-2\int_r^t K_1(\tau,X_\tau)\,d\tau} dr \times |\nabla^t f|_t^2(X_t)\right]$$

$$\le 4\int_s^t \left[\mathbb{E}^{(x,s)} e^{\frac{1}{2}\int_r^t (K_2(\tau,X_\tau)-K_1(\tau,X_\tau))d\tau} - 1\right] P_{s,r}|\nabla^r f|_r^2$$

$$+ \mathbb{E}^{(x,s)}\left\langle \nabla^r f(X_r), \nabla^r P_{r,t} f(X_r) - e^{-\int_r^t K_1(\tau,X_\tau)d\tau} //_{r,t}^{-1} \nabla^t f(X_t)\right\rangle_r dr \wedge 0;$$

(iv) for $f \in C_0^\infty(M)$ and $0 \le s \le t$ in I,

$$\frac{1}{4}\left(P_{s,t}(f^2 \log f^2) - P_{s,t}f^2 \log P_{s,t}f^2\right)$$

$$- \mathbb{E}^{(x,s)}\left[\int_s^t e^{-2\int_r^t K_1(\tau,X_\tau)\,d\tau} dr \times |\nabla^t f|_t^2(X_t)\right]$$

$$\le 4\int_s^t \left[\mathbb{E}^{(x,s)} e^{\frac{1}{2}\int_r^t (K_2(\tau,X_\tau)-K_1(\tau,X_\tau))d\tau} - 1\right] P_{s,r}|\nabla^r f|_r^2$$

$$+ \mathbb{E}^{(x,s)}\left\langle \nabla^r f(X_r), \nabla^r P_{r,t} f(X_r) - e^{-\int_r^t K_1(\tau,X_\tau)d\tau} //_{r,t}^{-1} \nabla^t f(X_t)\right\rangle_r dr \wedge 0.$$

Corollary [Cheng-A.Th. 2017]

Let $(t,x) \mapsto K(t,x)$ be some continuous function on $I \times M$. The following statements are equivalent to each other:

(i) the family $(M, g_t)_{t \in I}$ evolves by

$$\frac{1}{2}\partial_t g_t = \mathrm{Ric}_t - \nabla^t Z_t - K(t,\cdot)g_t, \quad t \in I;$$

(ii) for $f \in C_0^\infty(M)$ and $0 \le s \le t$ in I,

$$|\nabla^s P_{s,t} f|_s^2 - \mathbb{E}^{(x,s)}\left[e^{-2\int_s^t K(r,X_r)dr} |\nabla^t f|_t^2(X_t)\right] \le 4\left[\langle \nabla^s f, \nabla^s P_{s,t} f\rangle_s - \left\langle \nabla^s f, \mathbb{E}^{(x,s)}\left[e^{-\int_s^t K(r,X_r)dr} //_{s,t}^{-1} \nabla^t f(X_t)\right]\right\rangle_s\right] \wedge 0;$$

(iii) version of a Poincaré inequality

(iv) version of a log-Sobolev inequality

- If $Z_t \equiv 0$ and $K \equiv 0$, the results characterize solutions to the Ricci flow; see Haslhofer and Naber (2018) for characterizations on path space.
- We have
$$\frac{1}{2}\partial_t g_t = \mathrm{Ric}_t, \quad t \in I$$
if and only if for $f \in C_0^\infty(M)$ and $0 \le s \le t$ in I,
$$|\nabla^s P_{s,t} f|_s^2 - P_{s,t}|\nabla^t f|_t^2 \\ \le 4\Big[\langle \nabla^s f, \nabla^s P_{s,t} f\rangle_s - \Big\langle \nabla^s f, \mathbb{E}^{(x,s)}\big[//_{s,t}^{-1}\nabla^t f(X_t)\big]\Big\rangle_s\Big] \wedge 0;$$

- Consider the heat equation under Ricci flow:

$$\begin{cases} \dfrac{\partial}{\partial t}u - \Delta_{g_t}u = 0 \\ \dfrac{\partial}{\partial t}g_t - 2\,\mathrm{Ric}_{g_t} = 0 \end{cases}$$

  To deal with the forward Ricci flow, we reparametrize the metric:

$$\hat{g}_t := g_{T-t}$$

- As before, if $u(\cdot, s) = f$, we write

$$u(x,t) = (P_{s,t}f)(x), \quad 0 \le s \le t \;\text{ in } I.$$

Supersolutions to the Ricci flow

For a smooth family $(M, g(t))_{t\in I}$ of Riemannian metrics are equivalent:

- $(M, g(t))_{t\in I}$ is a supersolution to the Ricci flow, i.e.

$$2\,\mathrm{Ric}_{g(t)} - \frac{\partial}{\partial t} g(t) \geq 0.$$

- For each $f \in C_c^\infty(M)$ the heat flow on $(M, g(t))_{t\in I}$ satisfies

$$|\nabla^s P_{s,t} f|_{g(s)} \leq P_{s,t} |\nabla^t f|_{g(t)}, \quad 0 \leq s < t \text{ in } I.$$

- For each $f \in C_c^\infty(M)$ the heat flow on $(M, g(t))_{t\in I}$ satisfies

$$|\nabla^s P_{s,t} f|^2_{g(s)} \leq P_{s,t} |\nabla^t f|^2_{g(t)}, \quad 0 \leq s < t \text{ in } I.$$

- Let $\mathcal{P}^{(x,s)}M$ be the space of continuous paths on M, starting in x at time s and $\mathbb{P}^{(x,s)}$ the probability measure on it, induced by the (inhomogeneous) BM

$$X_t^{(x,s)}, \quad t \geq s.$$

- For a cylindrical function F on $\mathcal{P}^{(x,s)}M$ with

$$F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_r}), \quad s \leq t_1 < \ldots < t_r \leq t,$$

consider the intrinsic gradient defined as

$$D_s^{//} F(X_{[s,t]}) = \sum_{i=1}^{r} //_{s,t_i}^{-1} (\nabla_{g(t_i)}^i f)(X_{t_1}, \ldots, X_{t_r}),$$

where ∇^i denotes the gradient with respect to the i-th component.

Characterization of solutions to the Ricci flow

For a smooth family $(M, g(t))_{t\in I}$ of Riemannian metrics are equivalent:

- $(M, g(t))_{t\in I}$ is a solution to the Ricci flow, i.e.

$$\frac{\partial}{\partial t} g(t) - 2\,\mathrm{Ric}_{g(t)} = 0.$$

- For each cylindrical function $F\colon \mathcal{P}^{(x,s)}M \to \mathbb{R}$,

$$|\nabla_x^s \mathbb{E}^{(x,s)} F| \leq \mathbb{E}^{(x,s)} |D_s^{//} F|.$$

- For each cylindrical function $F\colon \mathcal{P}^{(x,s)}M \to \mathbb{R}$,

$$|\nabla_x^s \mathbb{E}^{(x,s)} F|^2 \leq \mathbb{E}^{(x,s)} |D_s^{//} F|^2.$$

  Here $|\cdot| = |\cdot|_{g(s)}$.