The road to unity

Laurent Mazliak (LPSM)

Abstract : *The talk will present some aspects of Emile Borel's interest for the mathematics of randomness, leading him in the aftermath of the Great War to give an impulse for the creation of the Statistical Institute of the University of Paris in 1922 and the Henri Poincaré Institute in 1928. The new laboratory LPSM can be seen as a direct heir of both branches of Borel's institutional activity.*

My task in this talk is to prove that the fusion between our university's two old laboratories of probability and statistics represents a truly historic moment. After one century, we are now witnessing the convergence of two parallel ways that emerged in the 1920s from the overwhelming activity of a single man who we can rightly consider our common ancestor: the mathematician Emile Borel.

Here is our man, aged fifty at the moment of his election at the Paris Academy of Science in 1921. To understand why Borel has been so important to the mathematics of randomness, it is first necessary to revisit the 1890s, when he began his rocket trajectory as a young star of mathematics in France at the turn of the century.

In the beginning was Georg Cantor. Like all French mathematicians at the end of the 19th century, Borel had been educated by teachers who, after the harsh defeat of 1870, had discovered with amazement the extent of German mathematics. For Borel's generation, Cantor played a specific role among the German mathematicians with his treatment of sets and his studies of the transfinite, studies that were justly seen as something radically new. At first, Borel had been a fascinated follower of Cantor.

ANNALES
SCIENTIFIQUES
DE
L'ÉCOLE NORMALE SUPÉRIEURE.

SUR QUELQUES POINTS
DE LA
THÉORIE DES FONCTIONS,

PAR M. ÉMILE BOREL,
MAITRE DE CONFÉRENCES A LA FACULTÉ DES SCIENCES DE LILLE.

INTRODUCTION.

L'intégrale de Cauchy a été le premier exemple d'une expression analytique égale à zéro dans une certaine région du plan et à une fonction déterminée dans une autre région. Dès lors, on a dû se poser la question suivante : Étant données deux fonctions d'une variable complexe, définies, l'une lorsque la variable est dans un certain domaine, l'autre lorsqu'elle est dans un domaine différent, dans quels cas peut-on dire que *c'est la même fonction?* Avant les travaux de Cauchy, cette question aurait paru à peu près dénuée de sens ou tout au moins résolue immédiatement : on a la même fonction, pensait-on, lorsqu'on a la même expression analytique. Il est clair que l'existence d'expressions analytiques telles que l'intégrale de Cauchy ou certaines séries qui en ont été déduites par M. P. Appell (¹) conduirait,

(¹) P. APPELL, *Développements en série dans une aire limitée par des arcs de cercle* (*Acta mathematica*, t. I).

Ann. de l'Éc. Normale. 3e Série. Tome XII. — Janvier 1895.

In his PhD devoted to complex analysis, which he defended in 1894, he made use of the Cantorian approach to prove the compactness of the closed unit interval.

même une suite ininterrompue d'intervalles sur la droite. Nous continuerons de même, en passant à la limite lorsque cela sera nécessaire et montrant alors qu'on peut conserver seulement un nombre fini des intervalles déjà considérés. Je dis que nous atteindrons nécessairement l'extrémité B de la droite, car, si on ne l'atteignait pas, on définirait une série d'intervalles ayant pour extrémités

$$B_{i_1}, \quad B_{i_2}, \quad \ldots, \quad B_{i_\omega}, \quad B_{i_{2\omega}}, \quad \ldots, \quad B_{i_{\omega^2}}, \quad \ldots, \quad B_{i_{\omega^\omega}}, \quad \ldots,$$

les indices étant *tous* les nombres de la seconde classe de nombres (définis par M. Cantor). Mais ces indices sont aussi dans un certain ordre, les nombres naturels, en tout ou en partie. C'est là une contradiction puisque la seconde classe de nombres constitue un ensemble de seconde puissance.

See here the striking way in which he used Cantorian arithmetic as a final argument for his proof. In his thesis, Borel also began to apply sigma-additivity to measure a subset of the real line, thus opening the way to Lebesgue's theory of integration.

However, at the turn of the 20th century, Borel's fascination with Cantor decreased and was gradually replaced by a kind of worry. From Borel's point of view, Cantor's theories were of course logically sound, but he asked: was logic really the ultimate basis on which mathematics must rely? A good example of Borel's concern appears in this comment on mathematical activity:

The mathematician who is absorbed in his dream is similar to the situation of the pupil for whom the francs of the problems are not real francs, used to buy objects; he lives in a world apart, built in his mind, having the feeling that this world often has nothing to do with the real world. One of the following two events usually occurs: either the mathematician builds an a priori real world, adequate to his world of ideas; it then leads to a metaphysical system that is not based on anything. Or, he establishes an absolute demarcation between his theoretical life and his practical life, and his science serves him nothing to understand the world; he accepts, almost without thinking, the beliefs of the environment in which he lives. (Borel, 1904)

Following a tradition he discovered in du Bois-Reymond, Borel became convinced of the necessity for the mathematician to base his work not on logic alone but on mathematical facts—such as models, even if they are ideal models or analogies. A specific problem under scrutiny in these years was Zermelo's axiom of choice.

CORRESPONDANCE.

CINQ LETTRES SUR LA THÉORIE DES ENSEMBLES.

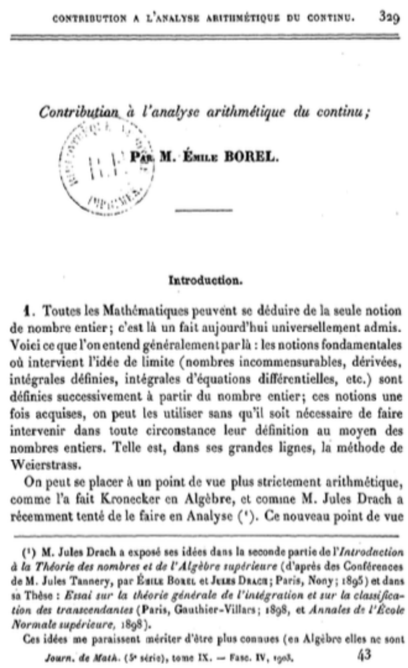
I. — *Lettre de M. Hadamard à M. Borel.*

J'ai lu avec intérêt les arguments que tu opposes (2e Cahier du tome LX des *Mathematische Annalen*) à la démonstration de M. Zermelo parue dans le Tome précédent. Je ne partage cependant pas ton opinion à ce sujet. Je n'admets pas, tout d'abord, l'assimilation que tu établis entre le fait qui sert de point de départ à M. Zermelo et le raisonnement qui consisterait à numéroter les éléments de l'ensemble les uns après les autres, ce numérotage étant poursuivi *transfiniment*. Il y a, en effet, une différence fondamentale entre les deux cas : le raisonnement qui vient d'être cité en dernier lieu comporte une série de choix successifs *dont chacun dépend des précédents;* c'est pour cela que son application transfinie est inadmissible. Je ne vois aucune analogie à établir, au point de vue qui nous occupe, entre les choix en question et ceux dont parle M. Zermelo, lesquels sont *indépendants les uns des autres.*

C'est d'ailleurs dans le cas d'une infinité *non dénombrable* de choix que tu récuses cette manière d'opérer; mais, à mon tour, je ne vois pas de différence, à cet égard, entre le cas d'une infinité non dénombrable et celui d'une infinité dénombrable. La diffé-

XXXIII. 17.

It gave rise to a famous correspondence exchanged between Borel and the French mathematicians Baire, Hadamard, and Lebesgue. Hadamard declared complete acceptance of Zermelo's axiom, but Borel expressed his doubts. Hadamard observed that in this difference of approach, one could detect two opposed conceptions of mathematics: whether or not to admit that an object or a concept (such as "to order a set") can be used in mathematics without being able to construct or realize it effectively. For Hadamard, Borel's worry was not new: some had for instance opposed the general definition of a function, which they found meaningless in the absence of an analytic expression. However, what was at stake for Borel was nothing less than the unity of the mathematical community. As he would later suggest, mathematicians should accept to deal only with objects that satisfy the following criterion: a mathematical being A is well defined when any two mathematicians who mention A are sure to mention the same being, without any possible ambiguity. In the first place when this being is a number. For Borel, the continuity of the real line should be examined alongside the consideration of constructible real numbers.



CONTRIBUTION A L'ANALYSE ARITHMÉTIQUE DU CONTINU. 329

Contribution à l'analyse arithmétique du continu;

PAR M. ÉMILE BOREL.

Introduction.

1. Toutes les Mathématiques peuvent se déduire de la seule notion de nombre entier; c'est là un fait aujourd'hui universellement admis. Voici ce que l'on entend généralement par là : les notions fondamentales où intervient l'idée de limite (nombres incommensurables, dérivées, intégrales définies, intégrales d'équations différentielles, etc.) sont définies successivement à partir du nombre entier; ces notions une fois acquises, on peut les utiliser sans qu'il soit nécessaire de faire intervenir dans toute circonstance leur définition au moyen des nombres entiers. Telle est, dans ses grandes lignes, la méthode de Weierstrass.

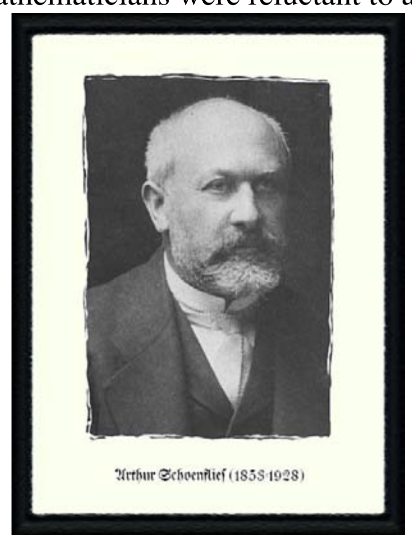
On peut se placer à un point de vue plus strictement arithmétique, comme l'a fait Kronecker en Algèbre, et comme M. Jules Drach a récemment tenté de le faire en Analyse (1). Ce nouveau point de vue

(1) M. Jules Drach a exposé ses idées dans la seconde partie de l'*Introduction à la Théorie des nombres et de l'Algèbre supérieure* (d'après des Conférences de M. Jules Tannery, par ÉMILE BOREL et JULES DRACH; Paris, Nony; 1895) et dans sa Thèse : *Essai sur la théorie générale de l'intégration et sur la classification des transcendantes* (Paris, Gauthier-Villars; 1898, et *Annales de l'École Normale supérieure*, 1898).

Ces idées me paraissent mériter d'être plus connues (en Algèbre elles ne sont

Journ. de Math. (5e série), tome IX. — Fasc. IV, 1903. 43

In a paper published in 1903, Borel illustrated how a geometrical vision based on the measure of sets techniques provides a new approach. By coating already built real numbers (such as rationals) with open intervals of arbitrarily small total length, one may prove that other points are necessary in order to fill the length. Borel's approach basically relied on countable additivity for the measure, a property some mathematicians were reluctant to accept.



German mathematician Arthur Schoenflies, for example, mentioned that this property could not legitimately result from a simple definition. Schoenflies wrote: "above all, it only has the nature of a postulate because we cannot decide if a property that can be verified by a finite sum can be extended over an infinite number of terms by an axiom." Moreover, Schoenflies criticized Borel for having no other application of his measure of sets than the problems of analytical continuation for which he introduced it.

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Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar 1900. N:o 7.
Stockholm.

Über eine Wahrscheinlichkeitsaufgabe bei Kettenbruchentwicklungen.

Von A. WIMAN.

(Mitgeteilt am 12 September 1900 durch D. G. LINDHAGEN.)

1. Wenn eine reelle Zahl μ zwischen 0 und 1 in einen Kettenbruch

$$\mu = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}} \qquad (1)$$

entwickelt wird, wo die a_ν durchweg ganze positive Zahlen bedeuten, so kann man die Frage nach der Wahrscheinlichkeit aufwerfen, dass irgend eine der Zahlen a_ν einen gewissen Werth k erhält. Auf diese Fragestellung wurde auch in der That GYLDÉN geführt, [1]) und zwar auf Grund ihres Zusammenhanges mit der Wahrscheinlichkeit für Convergenz gewisser in der Störungstheorie vorkommender Reihen. Eine Revision dieser GYLDÉN'schen Untersuchungen hat dann neuerdings BRODÉN vorgenommen, welcher dabei auch gewisse Anmerkungen an das von GYLDÉN gegebene Hauptresultat der erwähnten reihentheoretischen Frage knüpfte. [2])

[1]) Om sannolikheten af inträdande divergens vid användande af de hittills brukliga methoderna att analytiskt framställa planetariska störingar. Öfversigt af K. Sv. Vet.-Akad. Förh. (1888), p. 77; Om sannolikheten att påträffa stora tal vid utvecklingen af irrationela decimalbråk i kedjebråk, Ib. p. 349; Quelques remarques relativement à la représentation des nombres irrationels au moyen des fractions continues, C. R. (1888), p. 1584, 1777.

[2]) Wahrscheinlichkeitsbestimmungen bei der gewöhnlichen Kettenbruchentwickelung reeller Zahlen. Öfversigt af K. Sv. Vet.-Akad. Förh. (1900), p. 239.

Probably to his utmost surprise, in a paper written by the Swedish number theorist Anders Wiman, Borel discovered the solution to the following problem: if a real number is chosen at random between 0 and 1, how determining the distribution of the terms in its decomposition in continued fractions. In this work, Wiman made use of sigma-additivity to perform the calculations of the distribution function of the terms. Of course, Wiman outright ignored any concept of the measure of sets. Rather, his countable additivity was merely a straight extension of finite additivity. Wiman's paper about continued fraction expansions became a proof for Borel that the measure of sets was adequate to tackle probabilistic questions and, furthermore, that a probabilistic approach may be an efficient way to describe real numbers. With probability, Borel would find the intuitive framework he was looking for, in order to overtake Cantor's logical approach.

REMARQUES SUR CERTAINES QUESTIONS DE PROBABILITÉ;

Par M. ÉMILE BOREL.

1. On sait que les questions de probabilité où interviennent des variables continues ne peuvent acquérir de sens qu'en vertu de *conventions* précises. Comme le fait observer Joseph Bertrand, si une variable x est assujettie à rester comprise entre o et 1, son carré x^2 est assujetti aux mêmes conditions et la probabilité pour que x soit compris entre o et $\frac{1}{2}$ est égale à la probabilité pour que x^2 soit compris entre o et $\frac{1}{4}$. Cela serait absurde (1) si l'on supposait à chacune de ces probabilités une valeur *intrinsèque*, c'est-à-dire définie objectivement d'une manière indépendante de toute convention.

(1) *Voir* POINCARÉ, *Calcul des probabilités*, 8e Leçon.

A few months later, Borel published his first probabilistic article in which his aim was precisely to show that Lebesgue new measure and integral allowed for the formulation of probabilistic questions that were previously intractable. The example of the distribution of continued fractions was such a basic probabilistic situation.

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LES PROBABILITÉS DÉNOMBRABLES ET LEURS APPLICATIONS ARITHMÉTIQUES.

Par M. Émile Borel (Paris).

Adunanza dell'8 novembre 1908.

CHAPITRE I.

Les probabilités dénombrables.

1. On distingue généralement, dans les problèmes de probabilités, deux catégories principales, suivant que le nombre des cas possibles est fini ou infini: la première catégorie constitue ce que l'on appelle les *probabilités discontinues*, ou probabilités dans le domaine du discontinu, tandis que la seconde catégorie comprend les *probabilités continues* ou *probabilités géométriques*. Une telle classification apparait comme incomplète, lorsque l'on se reporte aux résultats acquis dans la théorie des ensembles; entre la puissance des ensembles finis et la puissance du continu se place la puissance des ensembles dénombrables; je me propose de montrer brièvement l'intérêt qui s'attache aux questions de probabilités dans l'énoncé desquelles interviennent de tels ensembles; je les appellerai, pour abréger, *probabilités dénombrables*.

2. Avant de définir plus précisément les probabilités dénombrables, je voudrais indiquer en quelques mots les raisons pour lesquelles il m'a semblé que leur étude ne devait pas être plus longtemps négligée. La principale de ces raisons est l'importance de la notion d'ensemble dénombrable; cette importance n'est contestée par aucun mathématicien; mais elle me parait être plus grande encore qu'on ne le croit généralement.

Beaucoup d'analystes, en effet, mettent au premier rang la notion du continu; c'est elle qui intervient d'une manière plus ou moins explicite dans leurs raisonnements. J'ai indiqué récemment [1]) en quoi cette notion du continu, considéré comme ayant une puissance supérieure à celle du dénombrable, me parait être une notion purement négative, la puissance des ensembles dénombrables étant la seule qui nous soit connue d'une manière positive, la seule qui intervienne *effectivement* dans nos raisonnements. Il est clair, en effet, que l'ensemble des éléments analytiques susceptibles d'être réellement dé-

[1]) BOREL, *Les paradoxes de la théorie des ensembles* [Annales scientifiques de l'École Normale supérieure, 3e série, tome XXV (1908)].

A few years later, in 1909, in an article for the journal *Rendiconti del Circolo matematico di Palermo,* Borel's ideas reached their deepest development with his application of denumerable probability to decimal and continued fractional expansions of real numbers. It was certainly one of the most influential probabilistic works of the 20th century. In this article, Borel introduces the notion of almost sure convergence and an initial version of the strong law of large numbers. He used these to provide for the first time a proof for the existence of an object, such as a normal number, by showing that its probability of existence equals 1. The strangeness of this kind of proof of existence, when contrasted with the classical methods, helps explain why the strong law of large numbers and denumerable probability "caught the mathematicians by surprise." This sort of semi-intuitionism, "half-axiomatic, and half-constructivist" as Brouwer would call it, became one of Borel's trademarks, but was sometimes received with a certain amount of skepticism.

As we have seen, probability theory became a significant topic in Borel's mathematical research after 1905, but there was another reason for his interest in probability.

La

Revue du Mois

PREMIÈRE ANNÉE

TOME I

JANVIER-JUIN 1906

PARIS

2, BOULEVARD ARAGO, XIIIe

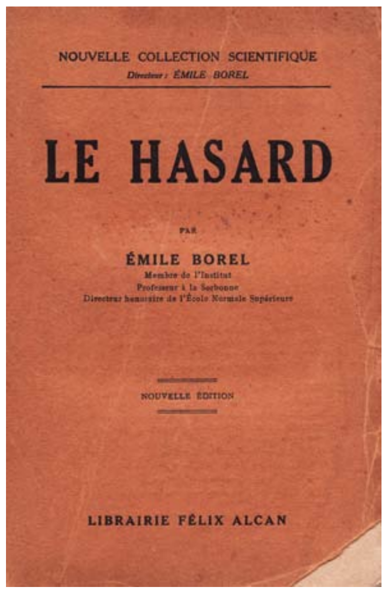
1906

Dépôt général de la Revue :

LIBRAIRIE H. LE SOUDIER

174-176, BOULEVARD SAINT-GERMAIN

In the same year of 1905, Borel and his wife founded the journal *Revue du Mois*, which became, for ten years, one of the major publications for intellectual debates in France. Borel asked numerous members of his huge network to contribute to his journal. In the typical atmosphere of the "radical-socialist" third republic, Borel's journal defended the idea that the scientific enlightenment of citizens was a basis to improve social conditions. Borel himself used the journal to present an evolving conception of the presence of probability in everyday life. From his perspective, calculus of probability and statistics provide citizens with tools allowing for the measurement of risks and are therefore the most useful part of mathematics. Probability calculations and their application to *social mathematics* go against the most antisocial aspects of a poorly thought-out individualism, which, as Borel would write, is generally only a "stupid egoism." Such calculations of risks constantly bring to the foreground the man as a citizen, belonging to a society and acting within it. It follows, then, that the study and practice of these different techniques, over and beyond scientific goals of analysis and prediction, have the virtue of limiting the "excesses of the individualistic mentality." Instead, they promote the values of social solidarity.



In 1914, Borel gathered the papers he wrote for the *Revue du Mois* to form his volume "le Hasard." One perceives in this book Borel's philosophy of "practical values" in science. Mathematics is at the heart of both the sciences and everyday life. This is an interpretation that is not wholly utilitarian, no more than it is axiomatic. The object is to find a middle path between an absolute faith in the results of mathematics, consisting of an application without any judgment, and a skepticism consisting in positing a radical rupture between mathematical equations and problems of everyday life. For Borel, the coefficient of probability was a clear answer to many questions corresponding to an absolutely tangible reality. Ironically, he commented that people who complain that they prefer certainty would probably also prefer that 2 plus 2 equals 5. As a commentator would summarize later, probabilities appear to be the only possible path to the future in a world that is no longer endowed with the sharp edges of certainty, but instead presents itself as the fuzzy realm of approximations.

Despite all this activity, Borel's support for a probabilistic approach to the world may have remained limited to the inner academic circle if an occasion had not provided a large-scale experience of the use of numbers, namely the outbreak of the Great War in 1914. Borel was deeply involved in the conflict at various levels.

LA DÉFENSE NATIONALE
et les Inventeurs

Un savant est chargé d'une direction spéciale

M. Painlevé a été nommé, tout à la fois, ministre de l'instruction publique et ministre des inventions intéressant la défense nationale. Grand maître de l'Université, il a pris la direction des services de l'enseignement au lendemain de la constitution du nouveau ministère. Il a, hier, organisé le nouveau service qui lui a été confié, et il a placé à sa tête M. Emile Borel, sous-directeur de l'Ecole normale supérieure, professeur à la Faculté des sciences.

M. EMILE BOREL

M. Borel a fait campagne comme sous-lieutenant d'artillerie territoriale. Sa brillante conduite lui a valu une citation à l'ordre du jour de l'armée.

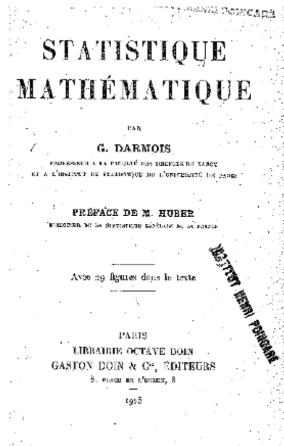
C'est un savant de premier ordre, dont la nomination à la tête du service des inventions sera accueillie avec une satisfaction unanime. Il est connu dans le monde scientifique pour ses travaux de mathématique pure et de mécanique. Enfin, il a collaboré déjà, avec le ministre actuel de l'instruction publique, dans les études que celui-ci a faites sur l'aviation.

D'autre part, le ministre de l'instruction publique a désigné comme chef de la section administrative de la nouvelle direction M. Bigeard, contrôleur général de la marine, ancien élève de l'Ecole polytechnique.

In 1915, he became the head of the *Commission des inventions intéressant la defense nationale* and in 1917, he was secretary of Painlevé's government. At each of these places, he realized that a sound statistical treatment of the enormous collections of data constantly provided to the government was urgently needed for a modern approach to governance. At the end of the war, Borel became convinced to enter a political career and to use his influence to implement his ideas for developing statistics and probability in France.

Borel's main institutional efforts were first directed toward statistics. Due to the old Napoleonic conception of educating administrators as jurists, he was extremely conscious of France's backwardness. In 1922, Borel accepted to be the first mathematician to become president of the *Société de Statistique de Paris*, where he began a campaign to emphasize the importance of a probabilistic approach to statistics. During the same year, he helped found the ISUP, where he soon asked Georges Darmois to organize the teaching of mathematical statistics.

In 1928, Darmois would write the first French textbook on probabilistic statistics in which he began to realize a transfer of statistical technology from foreign countries, especially from England with Pearson's biometrical research.


STATISTIQUE

MATHÉMATIQUE

PAR

G. DARMOIS

PRÉFACE DE M. HUBER

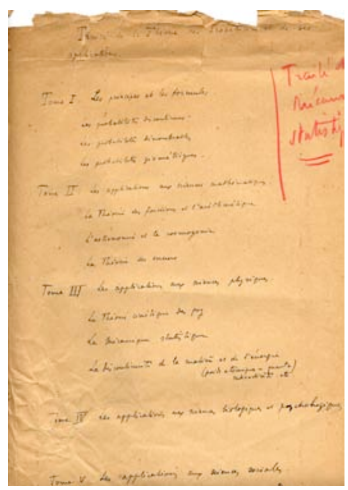
Avec 29 figures dans le texte

PARIS

LIBRAIRIE OCTAVE DOIN

GASTON DOIN & C^ie^, ÉDITEURS

Borel's involvement also concerned probability theory. His first step was to accept the *Chaire de Calcul des Probabilités et Physique Mathématique* at Paris University in 1920. Under Borel's direction and for the first time, the chair clearly took a mathematical turn to the detriment of physics. Probability theory became the main (and soon the only) topic of its syllabus.



In 1924, Borel launched a great project, a treatise collecting the probabilistic knowledge of the time, thereby proving that he was well aware of the explosive development of probability theory in those years. However, the treatise, which appeared in fascicles until 1939, was in many ways rather obsolete because Borel was skeptical of the use of too sophisticated mathematics in probability. This negative skepticism did not fail to attract the acidic criticism of the new generation of probabilists, such as Paul Lévy. After the publication of a little book by Borel and Deltheil in which they introduced the central limit theorem through only heuristic arguments, Lévy reacted with anger

In order to emphasize the role of Gaussian distribution in the theory of errors, it is possible to avoid constructing a precise mathematical theory and to limit oneself to commonsensical *reasoning. This is what MM. Borel and Deltheil have done in their little book and there is nothing to add to what they have written there. But, for the mathematician, this cannot be sufficient [though] M. Borel thinks that this result does not justify the mathematical apparatus required to achieve it*. (Lévy, 1925)

One must, however, honestly observe that Borel always remained strategically open toward modern probabilities.

This openness explains the 1928 founding of the Institut Henri Poincaré (IHP), which, thanks to Fréchet's help, became in the 1930s one of the major world centers for probability. At the IHP, a whole new generation of young probabilists—including Doeblin, Fortet, Ville, Dugué, Malécot, and so on—listened to the lectures of Borel, Fréchet, and Darmois and prepared their PhD. It is also from this past that we inherit today.

In conclusion, there remains one question: why did not Borel himself choose to gather statistics and probabilities under the same institutional roof in Paris? I propose a plausible hypothesis: it was already difficult to convince the timorous French mathematicians to accept probabilities…So, as far as statistics is concerned, Borel might have thought it prudent to wait a while…If this hypothesis is good, a century later, we have realized his hope…

For more details, see the following two papers and the references included

R. Catellier and L. Mazliak. The emergence of French probabilistic statistics. Borel and the Institut Henri Poincaré in the 1920s. Rev. Hist. Math. 18, No. 2, 271--335 (2012)

A. Durand and L. Mazliak. Revisiting the sources of Borel's interest in probability: continued fractions, social involvement, Volterra's Prolusione. Centaurus 53, No. 4, 306--332 (2011)